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S.B. Zheng*, A.J. Wootton and E. R. Solano

Fusion Research Center
University of Texas at Austin
Austin, TX,

*Institute of Physics
Chinese Academy of Sciences
Beijing, China

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A tokamak equilibrium with arbitrary aspect ratio

S. B. Zheng*, A. J. Wootton and E. R. Solano

Fusion Research Center, Univ. of Texas at Austin, Austin, TX, USA.

*Institute of Physics, Chinese Academy of Sciences, Beijing, China.

Abstract:

A general analytical solution of the Grad-Shafranov equation is presented. It allows the simulation of plasmas with elongation and triangularity, with an independent choice of pressure and plasma current. A numerically computed fit to families of such solutions allows the direct computation of the poloidal flux function $\Psi(R,Z)$ from a parametric description of the plasma given by aspect ratio A , elongation κ , triangularity δ , size (i.e. major radius), plasma beta poloidal β_{pol} , and plasma current I_p .

Introduction:

Analytic solutions of the Grad-Shafranov [1,2] equation are useful for theoretical studies of plasma equilibrium, transport and MHD stability. Our specific requirement was to find an analytic solution valid at arbitrary aspect ratio. This was needed to allow efficient mapping techniques to be used to follow fast injected particle orbits in low aspect ratio tokamaks, and to derive the parametric dependence of the safety factor on aspect ratio, shape and size. Initially the Solov'ev equilibrium [3] was used, but these solutions are over constrained, either in shape (ellipticity) or in plasma current (which is commonly determined by the choice of poloidal beta, β_{pol}).

In an axisymmetric system a convenient representation of the magnetic field is:

$$\mathbf{B} = F\nabla\phi + \nabla\Psi \times \nabla\phi \quad (1)$$

where ϕ is the ignorable angle in the cylindrical coordinate system (R, ϕ, Z) , and F and Ψ are axisymmetric scalar functions. The function F is a flux function, associated with the poloidal current in the system, while Ψ is the poloidal flux divided by 2π . MKS units will be used throughout this paper.

The Grad-Shafranov[1,2] equation can be written as follows:

$$R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \Psi}{\partial R} \right) + \frac{\partial^2 \Psi}{\partial Z^2} = -\mu_0 R J_\phi = -\mu_0 R^2 \frac{\partial P}{\partial \Psi} - F \frac{\partial F}{\partial \Psi} \quad (2)$$

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Here J_ϕ is the toroidal plasma current density. P is the thermal pressure, which is a flux function. The toroidal field is not determined by the Grad-Shafranov equation; its choice will only re-scale the values of the safety factor q . Its value will not change the shape of the flux surfaces, nor the shape of the profiles of pressure, current density, q , etc.

The simplest solution to equation (1) can be found by assuming that

$$-\mu_0 \frac{\partial P}{\partial \Psi} = A_1, \quad F \frac{\partial F}{\partial \Psi} = A_2, \quad (3)$$

with A_1 and A_2 constant. This reduces the set of possible current density profile shapes to $J_\phi \propto R A_1 - A_2/R$. With this restriction the Grad-Shafranov equation becomes

$$R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \Psi}{\partial R} \right) + \frac{\partial^2 \Psi}{\partial Z^2} = R^2 A_1 - A_2 \quad (4)$$

with a solution

$$\Psi = \Psi_0 + \frac{A_1}{8} R^4 - \frac{A_2}{2} Z^2, \quad (5)$$

Ψ_0 is the solution of the homogeneous equation:

$$R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \Psi_0}{\partial R} \right) + \frac{\partial^2 \Psi_0}{\partial Z^2} = 0. \quad (6)$$

The usual solution is based on the expansion

$$\Psi_0 = \sum_{n=0,1,\dots} f_n(R) Z^n \quad (7)$$

with each expansion term satisfying the equation:

$$R \frac{d}{dR} \left(\frac{1}{R} \frac{df_n(R)}{dR} \right) = -(n+1)(n+2) f_{n+2}, \quad n = 0, 2, \dots \quad (8)$$

The expansion is frequently truncated by assuming $f_n(R) = 0$ for $n \geq 3$. A more convenient expression for Ψ_0 is given by a rearranged series expansion as follows:

$$\Psi_0 = \sum_{n=0} \sum_{k=0}^{k=n/2} G(n,k,R) Z^{n-2k} \quad (9)$$

Here the functions $G(n,k,R)$ satisfy the following equations:

$$R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial G(n,0,R)}{\partial R} \right) = 0. \quad (10)$$

$$R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial G(n,k,R)}{\partial R} \right) = - (n-2k+1) (n-2k+2) G(n,k-1,R), \quad (n/2 \geq k > 0) \quad (11)$$

whose solution is given by:

$$G(n,k,R) = g_{n1} G1(n,k,R) + g_{n2} G2(n,k,R) \quad (12)$$

with

$$G1(n,0,R) = 1$$

$$\begin{aligned} G1(n,k>0,R) &= (-1)^k \frac{n!}{(n-2k)!} \frac{1}{2^{2k} k! (k-1)!} R^{2k} \left(2 \ln(R) + \frac{1}{k} - 2 \sum_{j=1}^k \frac{1}{j} \right) \\ G2(n,k,R) &= (-1)^k \frac{n!}{(n-2k)!} \frac{1}{2^{2k} k! (k+1)!} R^{2k+2} \end{aligned} \quad (13)$$

The constants g_{n1} , g_{n2} , A_1 and A_2 are determined by imposing external constraints.

A simple solution:

If the plasma is assumed to be up-down symmetric, its shape can be described by 4 parameters: the equatorial innermost and outermost points, R_{in} and R_{out} , and the coordinates the highest point, (R_t, Z_t) (or equivalently, the major radius $R_0 = (R_{in} + R_{out})/2$, minor radius $a = (R_{out} - R_{in})/2$, elongation $\kappa = Z_t/a$, and triangularity $\delta = (R_0 - R_t)/a$). Specified values of plasma current and pressure at the magnetic axis (or, equivalently, β_{pol}) provide two additional constraints. The simplest solution with that much freedom is given by:

$$\Psi = c_1 + c_2 R^2 + c_3 (R^4 - 4 R^2 Z^2) + c_4 (R^2 \ln(R) - Z^2) + \frac{A_1}{8} R^4 - \frac{A_2}{2} Z^2 \quad (14)$$

which is equivalent to choosing $f_n = 0$ for $n \geq 4$ in the traditional expansion given in equation (7) (with even terms only). The advantage of the representation given in equation (9) lies in the explicit form of the general solution $\forall n,k$ presented in equations (12).

With equation (14), the boundary conditions become:

$$c_1 + c_2 R_{in}^2 + c_3 R_{in}^4 + c_4 R_{in}^2 \ln(R_{in}) = -\frac{A_1}{8} R_{in}^4 \quad (15)$$

$$c_1 + c_2 R_{out}^2 + c_3 R_{out}^4 + c_4 R_{out}^2 \ln(R_{out}) = -\frac{A_1}{8} R_{out}^4 \quad (16)$$

$$c_1 + c_2 R_t^2 + c_3 R_t^2 (R_t^2 - 4 Z_t^2) + c_4 [R_t^2 \ln(R_t) - Z_t^2] = -\frac{A_1}{8} R_t^4 + \frac{A_2}{2} Z_t^2 \quad (17)$$

$$2 c_2 + 4 c_3 (R_t^2 - 2 Z_t^2) + c_4 [2 \ln(R_t) + 1] = -\frac{A_1}{2} R_t^2 \quad (18)$$

$$I_p = \int j_\phi dR dZ = -\frac{\int (R^2 A_1 - A_2) dR dZ}{\mu_0 R} \quad (19)$$

$$\beta_{pol} = \frac{8\pi}{\mu_0} \frac{\int P dR dZ}{I_p^2} = -\frac{8\pi A_1}{\mu_0^2 I_p^2} \int \Psi dR dZ \quad (20)$$

In previous studies of spherical tokamak equilibria [4] only three shape coefficients are used ($c_4 = 0$). Furthermore the constant A_2 is set to zero for simplicity. This has the inconvenient consequence of over-constraining the plasma current or β_{pol} .

Scaling relations:

Multiplying the coefficients ($c_1, c_2, c_3, c_4, A_1, A_2$) by a constant α_I produces a new solution with the same shape and poloidal beta, but with a re-scaled value of the plasma current $I_p' = \alpha_I I_p$. In the same manner the spatial dimension can be re-scaled. Defining

$$R' = \alpha_L R, Z' = \alpha_L Z,$$

the new function $\Psi'(R', Z')$ is defined by

$$\begin{aligned} \Psi'(R', Z') &= \Psi'(\alpha_L R, \alpha_L Z) = \Psi(R, Z) \\ &= d_1 + d_2 (R')^2 + d_3 ((R')^4 - (R')^2 (Z')^2) + d_4 ((R')^2 \ln(R') - (Z')^2) \\ &\quad + A_1' (R')^4 / 8 + A_2' (Z')^2 / 2 \end{aligned} \quad (21)$$

with

$$\begin{aligned} d_1 &= c_1; & d_2 &= (c_2 - c_4 \ln(\alpha_L)) / \alpha_L^2; & d_3 &= c_3 / \alpha_L^4; \\ d_4 &= c_4 / \alpha_L^2; & A_1' &= A_1 / \alpha_L^4; & A_2' &= A_2 / \alpha_L^2 \end{aligned}$$

$\Psi'(R, Z)$ satisfies equation (4), with coefficients A_1' and A_2' replacing A_1 and A_2 , respectively. The corresponding plasma current is given by

$$I_p' = -\frac{\int [(R')^2 A_1' - A_2'] dR' dZ'}{\mu_0 R'} = \frac{I_p}{\alpha_L} \quad (22)$$

and the poloidal beta by

$$\beta'_{pol} = - \frac{8 \pi A_1'}{(\mu_0 I_p')^2} \int \Psi' dR' dZ' = \beta_{pol} \quad (23)$$

Hence, given the coefficients c_{10} , c_{20} , c_{30} , c_{40} , A_{10} and A_{20} for a normalized plasma current $I_p = 1$ A and major radius $R_{00} = 1$ m, the coefficients for a plasma with the same shape and poloidal beta (but different I_p and size) are calculated by similarity:

$$\begin{aligned} c_1 &= (I_p R_0) c_{10}; & c_2 &= (I_p / R_0) (c_{20} - c_{40} \ln(R_0)); & c_3 &= (I_p / R_0^3) c_{30}; \\ c_4 &= (I_p / R_0) c_{40}; & A_1 &= (I_p / R_0^3) A_{10}; & A_2 &= (I_p / R_0) A_{20}; \end{aligned}$$

It is simplest to first solve for a plasma with unity current and unit major radius and use the scaling relations described above to find the final desired equilibrium. However even in the simplest case only numerical solutions to equations (15) to (20) have been found. A program has been written to numerically compute the coefficients given a desired plasma description. For example, two different equilibria are presented in Figures 1 and 2. The first corresponds to parameters typical of TEXT, a medium size, medium aspect ratio tokamak: $I_p = .250$ MA, $R_0 = 1$ m, $a = .27$ m, $\kappa = 1.0$, $\delta = 0$, $\beta_{pol} = 0.1$. A low aspect ratio plasma corresponding to the USTX point design [5] is shown in Figure 2. The plasma is characterized by $I_p = 1$ MA, $R_0 = .70$ m, $a = .49$ m, $\kappa = 1.7$, $\delta = .125$, $\beta_{pol} = 0.4$.

Fitted solution

In most cases, a desired tokamak equilibrium is prescribed by a suitable choice of β_{pol} , aspect ratio $A = R_0/a$, elongation κ and triangularity δ , and size (usually given by R_0). It is most convenient to express the coefficients in equation (14) or equation (21) as a function of these parameters. This can not be done analytically. Instead, numerical solutions were computed for a range of normalized plasmas (with $I_{p0} = 1$ A, $R_{00} = 1$ m) given by

$$1.05 < A < 5, 1 \leq \kappa \leq 2, 0 \leq \delta \leq 0.5, 0 \leq \beta_{pol} \leq 0.6,$$

Because equations (15) to (18) are linear in A_1 and A_2 , it is sufficient to numerically find fitting functions for these two coefficients and then solve equations (15) to (18) to find the remaining coefficients c_1 , c_2 , c_3 , c_4 with $R_{in} = 1 - 1/A$, $R_{out} = 1 + 1/A$, $Z_t = \kappa/A$ and $R_t = 1 - \delta/A$.

The fitting function for A_{10} is given by:

$$\begin{aligned}
A_{10} = & -5.386 \cdot 10^{-8} + (u_1 \beta_{pol} + u_2 \beta_{pol}^2 + u_3 \beta_{pol} \kappa + u_4 \beta_{pol} \delta) \cdot 10^{-7} \\
& + 7.3 \cdot 10^{-8} \kappa - 2.334 \cdot 10^{-8} \kappa^2 - 2.56 \cdot 10^{-8} \delta - 5.45 \cdot 10^{-8} \delta^2 \\
& + 2.261 \cdot 10^{-8} \kappa \delta + 1.823 \cdot 10^{-7} \beta_{pol} \kappa \delta
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
u_1 &= -1.342 - 1.198 A - 4.970 A^2 \\
u_2 &= 4.690 - 1.330 A \\
u_3 &= -1.310 + 1.316 A + 1.275 A^2 \\
u_4 &= -5.145 + 2.183 A - 1.283 A^2
\end{aligned}$$

The average relative fitting error of the above function is 4%, the maximum relative fitting error is 12%. Fitting results are shown in Figure 3.

The fitting function for A_{20} is:

$$\begin{aligned}
A_{20} = & (v_1 + v_2 \kappa + v_3 \kappa^2 + v_4 \beta_{pol} + v_5 \beta_{pol} \kappa) \cdot 10^{-7} \\
& - 1.63 \cdot 10^{-7} \delta + 5.0 \cdot 10^{-8} \kappa \delta + 1.3 \cdot 10^{-7} \beta_{pol}^2 + 9.0 \cdot 10^{-8} \beta_{pol} \delta,
\end{aligned} \tag{25}$$

with

$$\begin{aligned}
v_1 &= -1.387 - 0.4166 A + 7.213 A^2 - 0.16637 \sqrt{A-1} \\
v_2 &= 0.8526 + 0.3107 A - 3.818 A^2 + 7.819 \cdot 10^{-2} \sqrt{A-1} \\
v_3 &= -0.1787 - 5.880 \cdot 10^{-2} A + 0.5935 A^2 - 5.986 \cdot 10^{-3} \sqrt{A-1} \\
v_4 &= -0.3967 - 2.463 \cdot 10^{-2} A - 5.554 A^2 + 0.2457 \sqrt{A-1} \\
v_5 &= -0.1846 + 4.744 \cdot 10^{-2} A + 1.5668 A^2 - 8.646 \cdot 10^{-2} \sqrt{A-1}
\end{aligned}$$

The average relative fitting error is 2%, the maximum relative error is 8%. Fitting results are shown in Figure 4.

Numerical computation:

A numerical code has been developed based on the above description of the equilibrium solutions. The code computes an exact equilibrium solution as given by equation (14), with a given parametric plasma specification, by finding the appropriate expansion coefficients. The code is available for general use on a VAX. The source code is in FRC3:[ZHESB.PARTICLE]PSI8.FOR. Potential users should consult with the authors to access it.

Conclusions:

A general analytical solution of the Grad-Shafranov equation. It is shown that, given the usual parametric description of the plasma (shape and size, β_{pol} and I_p) an equilibrium can be computed with enough freedom to independently control pressure and

plasma current. A fit to numerically computed solutions is presented, which allows the direct computation of a equilibrium solution $\Psi(R,Z)$ from the parametric plasma description.

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References:

- [1] Grad, H., Rubin, H., Proceedings of the Second United Nations Conference on the Peaceful uses of Atomic Energy, United Nations, Geneva, Vol. 31, pg. 190, 1958.
- [2] Shafranov, V. D., Soviet Physics JETP Vol. 6, p. 545 (1958);
Zh. Eksp. Teor. Fiz. 33, p. 710 (1957)
- [3] Solov'ev, L. S., Soviet Physics JETP Vol. 26, No. 2, p. 400 (1968);
Zh. Eksp. Teor. Fiz. 53, p. 626 (1967).
- [4] J. P. Freidberg, "Ideal Magneto hydrodynamics", Plenum Press, New York and London, 1985.
- [5] S. C. McCool, P. H. Edmonds, H. He, et. al., "USTX- The University Spherical Tokamak Experiment", Fusion Research Center Report #468, The University of Texas at Austin.

Fig.1 Flux Surface for TEXT-U

R0(CM)= 100.00 AP(CM)= 27.00
IP(KA)= 250.00 BP(CM)= 27.00
BETA_p= 0.100 RB(CM)= 100.00

Fig.2 Flux Surface for USTX

RO(CM)= 70.00 AP(CM)= 49.00
IP(KA)= 1000.00 BP(CM)= 83.30
BETA_p= 0.400 RB(CM)= 50.00

Fig.3 Fitting Result of $-A1$

Fig.4 Fitting Result of A2