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Abstract

The growth rate of Toroidal Alfvén Eigenmodes (TAE) driven unstable by resonant coupling of energetic charged particles is evaluated in the 'ballooning' limit over a wide range of parameters. All damping effects are ignored. Variations in orbit width, aspect ratio, and the ratio of Alfvén velocity to energetic particle 'birth' velocity, are explored. The relative contribution of passing and trapped particles, and finite Larmor radius effects, are also examined. The phase space location of resonant particles which interact strongly with the modes is described. The accuracy of the analytic results with respect to growth rate magnitude and parametric dependence is investigated by comparison with numerical results.

I. INTRODUCTION

In toroidal plasma confinement devices, toroidal Alfvén eigenmodes[1] (TAE modes) can be destabilized by the presence of energetic charged particles produced by fusion reactions, ICRF heating, or neutral beam injection. It has been observed in recent experiments[2] that the excitation of TAE instabilities by ICRH heating or neutral beam injection result in the rapid loss of energetic particles. This suggests that energetic particles interact with a spectrum of unstable modes and experience rapid diffusion on a global scale.

The instabilities are due to the resonant coupling of energetic particles with TAE modes. The energetic particles which can be in 'resonance' interact strongly and diffuse in phase space with a net transfer of particle energy to the growing unstable modes. Such 'resonant' particles will lie on surfaces in phase space. At low amplitudes, the 'resonance surfaces' will overlap (resonance overlap) only if the frequency interval between them is less than the linear growth rate. Thus for a discrete mode spectrum in which the frequency interval between resonance surfaces is greater than the linear growth rate, there will be no resonance overlap, and the particle phase space diffusion will be local.

However, at finite mode amplitudes, the width of the 'resonance surfaces' is nonlinearly broadened when the particle 'bounce frequency' is larger than the linear growth rates. Thus 'resonance surfaces' which do not overlap at low mode amplitudes may nevertheless eventually overlap if the unstable modes can grow to high enough amplitudes before the onset of nonlinear saturation.

Under certain circumstances, this nonlinear broadening can be effective in inducing resonance overlap of all modes even in circumstances where resonance overlap at low amplitudes occurs for only a small subset of unstable modes. Such unstable modes can grow to significantly larger amplitudes when there is resonance overlap than otherwise due to the increased

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free energy which can be extracted from the particles before the modes are stabilized by quasilinear relaxation. This small subset of modes grows to amplitudes for which the bounce frequency exceeds the linear growth rate, and the intrinsic width of the corresponding resonance surfaces is thereby broadened so that resonance overlap of nearby resonance surfaces can now occur. The mode amplitudes continue to increase, triggering a cascade in which more and more resonance surfaces successively overlap in a manner analogous to a 'domino effect.' The result will be global phase space diffusion and enhanced loss of energetic particles from the containment device.

This 'domino effect' has been demonstrated by Berk *et al.*[3] It is therefore of some interest to investigate the relevance of the 'domino effect' as a possible explanation for the energetic particle losses observed in TAE instability experiments. The relevance of the 'domino effect' depends on the distribution of unstable modes and the extent to which TAE growth rates can or cannot establish resonance overlap for a few or many modes. To address the issues of resonance overlap, we have initiated an investigation to locate the regions in phase space where the resonant particles interact strongly with unstable TAE modes, and to explore the dependence of the TAE growth rates over a wide range of the parameters.

The linear growth rates have been calculated by many authors.[4–7] Analytic results[4,5] were obtained in the limit in which: (1) the inverse aspect ratio ε is small, $\varepsilon \sim a_0/R_0 \ll 1$; (2) the particle 'banana width' $\Delta_b \sim q r_L$ is small compared to the separation $\Delta \sim \left(n_0 \frac{\partial q}{\partial r}\right)^{-1}$ between mode rational surfaces, $\Delta \gg \Delta_b$. a_0 is the minor radius, R_0 the major radius, q the safety factor, r_L the particle Larmor radius, and n_0 the toroidal mode number.

The regimes of interest, however, are often outside these limits. Because of the complicated nature of the spatial mode structure and the particle phase space trajectories, these instabilities have been investigated primarily by numerical methods. It is therefore a tedious task to establish the growth rate dependence on the large number of parameters involved. Consequently numerical surveys often cover a limited range of parameters, and it is a useful supplement to such surveys to determine the accuracy of the analytic results with respect to growth rate magnitudes and parametric dependences.

In this paper, we calculate the growth rates perturbatively using the lowest order 'ballooning' eigenfunction. In Sec. II, we outline the formalism employed. In Sec. III, we describe the particle phase space trajectories in the equilibrium fields and in Sec. IV we integrate the linearized Vlasov equation to obtain the perturbed energetic particle distribution function. In Sec. V, we map the regions in phase space where the resonant particles are located, evaluate the resonant particle energy transfer, and determine the TAE growth rates. In Sec. VI, we discuss the growth rate scaling with respect to the parameters $\Delta_b/\varepsilon\Delta$, $v_A/v_0, \varepsilon$, and r_L. $\Delta_b/\varepsilon\Delta$ is the ratio of the 'banana width' to the 'internal mode width,' and v_A/v_0 the ratio of the Alfvén speed to the 'birth' speed of the energetic particles.

It is found that the numerically calculated growth rates increase proportionately with $\Delta_b/\varepsilon\Delta$ when $\Delta_b/\varepsilon\Delta < 1$, in agreement with the analytic theory , and its magnitude is within 20 percant of the analytic result when $\varepsilon \lesssim .1$. As $\Delta_b/\varepsilon\Delta$ approaches and increases above unity, the growth rate levels off,eventually reaching a maximum at $\Delta_b/\Delta \sim 1$. This maximum is lower than the predicted analytic maximum by 20 percent when $\varepsilon = .025$, and can be as much as a factor of four lower when $\varepsilon = .2$. The growth rate decreases thereafter as Δ_b/Δ exceeds unity. The effect of finite Larmor radius is to suppress the growth rate even further. When $\Delta_b/\varepsilon\Delta < 1$, the growth rates for $v_A/v_0 < 1$ are larger than the growth rates for $v_A/v_0 > 1$. However, their maximum values occurring at $\Delta_b/\Delta \sim 1$ are approximately the same when $v_A/v_0 \lesssim 2$ but begin to decrease when $v_A/v_0 > 2$.

These results are consistent with and complementary to the growth rate scalings previously obtained by Fu *et al.*[6]

II. FORMALISM

We describe Alfvén modes in terms of the perturbed perpendicular magnetic potential $A_\perp = \widetilde{A}_\perp \exp(-i\omega t)$. In equilibria with small plasma beta, the sources of the perturbed plasma currents are the ion polarization drift and the perturbed equilibrium current

$$\widetilde{J}_\perp = \omega^2 \frac{N_p m_p c}{B^2} \widetilde{A}_\perp - \frac{1}{B^2} B \times \left[J \times (\nabla \times \widetilde{A}_\perp) \right].$$

We will neglect the perpendicular components of the equilibrium current (J) and we take $J = \frac{J \cdot B}{B^2} B$ to be in the direction of the equilibrium magnetic field B.

The eigenmode equation is then given by the following variational quadratic form

$$\int d^3r (\widetilde{A}_\perp^\dagger \cdot L_0 \widetilde{A}_\perp) \equiv$$

$$\int d^3r \left[(\nabla \times \widetilde{A}_\perp^\dagger) \cdot (\nabla \times \widetilde{A}_\perp) - \frac{\omega^2 4\pi N_p m_p}{B^2} \widetilde{A}_\perp^\dagger \cdot \widetilde{A}_\perp - \frac{4\pi B \cdot J}{c B^2} \widetilde{A}_\perp^\dagger \cdot (\nabla \times \widetilde{A}_\perp) \right] = 0 \quad (1)$$

where we have introduced the adjoint function $\widetilde{A}_\perp^\dagger$. Since $\nabla \cdot J = 0$, this quadratic form is self-adjoint:

$$\int d^3r (\widetilde{A}_\perp^\dagger \cdot L_0 \widetilde{A}_\perp) = \int d^3r (\widetilde{A}_\perp \cdot L_0 \widetilde{A}_\perp^\dagger).$$

For shear Alfvén modes, involving field line bending and negligible compression, we may take the field variable $\widetilde{A}_\perp$ to be $\widetilde{A}_\perp = \nabla_\perp \Phi \equiv \nabla\Phi - b\, b \cdot \nabla\Phi$, where $b = \frac{B}{B}$. The field $\Phi(r, \theta, \zeta)$ of toroidal eigenmodes can be expressed as a sum of Fourier modes in the generalized poloidal angle θ

$$\Phi = \sum_m \Phi_m(r) \exp(im\theta - in_0\zeta) \quad (2)$$

where $\theta = \int^{\theta_1} d\theta_1 \left(1 + (r/R_0)\cos\theta_1\right)^{-1}$, ς is the toroidal angle, R_0 the major radius, and r, θ_1 the radial, azimuthal angular coordinates of the minor cross-section. We assume circular equilibrium magnetic flux surfaces. The equilibrium magnetic field is $B = \frac{B_0 r}{q(r)} \nabla r \times \nabla(q\theta - \varsigma)$ with safety factor $q(r) = \frac{\nabla\varsigma \cdot B}{\nabla\theta \cdot B}$

TAE-modes have the feature that the Fourier amplitudes $\Phi_m(r)$ have similar radial dependencies, localized in the neighborhood of mode rational surfaces defined by $m - n_0 q(r) = 0$. In the limit of high toroidal mode number n_0, there is approximate translational symmetry (ballooning symmetry):

$$\Phi_m(r) \approx \phi(n_0 q(r) - m) \exp(-im\theta_0) \approx \phi((r-r_0)/\Delta - \ell) \exp(-i(m_0+\ell)\theta_0) \equiv \phi_\ell(r) \exp(-im_0\theta_0) \tag{3}$$

where $m = m_0 + \ell$, r_0 is the location of a reference mode rational surface ($n_0 q(r_0) = m_0$) corresponding to a reference poloidal mode number m_0, $q(r)$ is considered to be an increasing function of r with $\frac{\partial q}{\partial r}$ approximately constant, $\Delta = \left(n_0 \frac{\partial q}{\partial r_0}\right)^{-1} > 0$ is the separation between mode rational surfaces, and θ_0 is the phase shift between successive Fourier amplitudes. Hereafter we consider the high-n_0 limit and we assume exact ballooning symmetry.

Let $\widehat{\phi}(\eta)$ be the Fourier integral transform of $\phi(n_0 q(r) - m)$

$$\phi(n_0 q(r) - m) = \int \frac{d\eta}{2\pi}\, \widehat{\phi}(\eta) \exp(-i\eta(m - n_0 q(r))).$$

We can therefore represent the field variable Φ in the form [8]

$$\begin{aligned}\Phi(r,\theta,\zeta) &= \sum_m \exp\left(im(\theta-\theta_0) - in_0\zeta\right) \int \frac{d\eta}{2\pi}\, \widehat{\phi}(\eta) \exp\left(-i\eta(m - n_0 q(r))\right) \\ &= \sum_N \widehat{\phi}(\theta + 2\pi N - \theta_0) \exp\left(in_0 q(\theta + 2\pi N - \theta_0) - in_0\zeta\right).\end{aligned} \tag{4}$$

The variational quadratic form simplifies to

$$\int d^3r \sum_N \left[(k_\perp \cdot k_\perp) \left\{ (b\cdot\nabla\theta)^2 \frac{\partial \widehat{\phi}^+}{\partial\theta} \frac{\partial\widehat{\phi}}{\partial\theta} - \frac{\omega^2 4\pi N_p m_p}{B^2}\, \widehat{\phi}^+ \widehat{\phi} \right\} \right] = 0 \tag{5}$$

$$k_\perp = n_0(\theta + 2\pi N - \theta_0) \frac{\partial q}{\partial r}\, \nabla r + n_0 q\, \nabla\theta - n_0\, \nabla\zeta \tag{6}$$

where higher order terms in $1/n_0$ have been neglected. The associated eigenmode equation can then be approximated by the following second order differential equation[8,9] for $\widehat{\phi}(\eta)$

$$\frac{\partial}{\partial\eta}(1 + S_0^2\eta^2) \frac{\partial\widehat{\phi}(\eta)}{\partial\eta} + \left(\frac{\omega q_0 R_0}{v_A}\right)^2 (1 + S_0^2\eta^2)\,(1 + 2\widehat{\varepsilon}_0 \cos(\eta + \theta_0))\, \widehat{\phi}(\eta) = 0 \tag{7}$$

where $+\infty > \eta > -\infty$, $q_0 = q(r_0)$, $S_0 = \frac{r_0}{q_0}\frac{\partial q_0}{\partial r_0}$, $v_A^2 = \frac{B_0^2}{4\pi N_p m_p}$, $\widehat{\varepsilon} = 2\,(\varepsilon_0 + \Delta_s)$, $\varepsilon_0 = r_0/R_0$, and $\Delta_s \approx \varepsilon_0/4$ is a parameter introduced to account for the Shafranov shift of the equilibrium flux surfaces. The local dispersion relation for ω is obtained by solving this equation subject to the boundary conditions $\widehat{\phi}(\eta) \to 0$ as $|\eta| \to \infty$.

For $S_0 \sim 1$ and $|\eta| \gtrsim (1/\widehat{\varepsilon}) \gg 1$,

$$\widehat{\phi}^{\pm} = \alpha^{\pm}\frac{1}{|\eta|}\exp\left(i(\eta+\theta_0)/2 \mp \widehat{\Gamma}\eta\right) + \alpha^{\pm *}\frac{1}{|\eta|}\exp\left(-i(\eta+\theta_0)/2 \mp \widehat{\Gamma}\eta\right) \tag{8}$$

$$\alpha^{\pm} = a^{\pm}\left[\frac{(-\Gamma_-)^{1/2}}{2} \pm i\,\frac{(\Gamma_+)^{1/2}}{2}\right] \qquad \Gamma_{\pm} = \left(\frac{\omega q_0 R_0}{v_A}\right)^2 - \frac{1}{4} \pm \widehat{\varepsilon}\left(\frac{\omega q_0 R_0}{v_A}\right)^2$$

$$\widehat{\Gamma} = (-\Gamma_-\Gamma_+)^{1/2} \tag{9}$$

where the $\pm$ sign refers to positive and negative values of η, $\alpha^{\pm}*$ are the complex conjugate of $\alpha^{\pm}$, $a^{(\pm)}$ are arbitrary constants. For exponentially decaying solutions, we require $\widehat{\Gamma}$ real.

In order to discuss the destabilization of shear Alfvén modes due to the presence of a species of energetic particles, we add to the perturbed plasma current the contribution of the perturbed particle current $\widetilde{J}_i = e_i \int d^3v\;\delta f\;\mathbf{v}$, where δf is the perturbed particle distribution function. We assume the particle density to be a small fraction of the main plasma density and we treat the presence of energetic particles perturbatively.

The first order correction $\delta\widetilde{A}_\perp$ to the eigenfunction $\widetilde{A}_\perp$ is determined by

$$L_0\delta\widetilde{A}_\perp = -\frac{2i\omega\gamma}{v_A^2}\widetilde{A}_\perp - \frac{4\pi e_i}{c}\int d^3v\;\delta f\;\mathbf{v}_\perp$$

where we assume exponential growth $\exp(\gamma\,t)$ of the mode amplitude with growth rate γ. We take the scalar product of this equation with $\widetilde{A}_\perp^{+}$ and we integrate over space to annihilate the left-hand side. We then obtain the following explicit expression for the growth rate

$$\frac{2\omega\gamma}{v_A^2}\int d^3r\widetilde{A}_\perp^{+}\cdot\widetilde{A}_\perp = -i4\pi\int d^3r\int d^3v\delta f\;\mathrm{H}_1^{+} \tag{10}$$

where $\mathrm{H}_1^{+} = -\frac{e_i}{c}\,\widetilde{A}_\perp^{+}\cdot\mathbf{v}_\perp$.

The perturbed distribution function δf is determined by the linearized Liouville equation

$$\frac{\partial \delta f}{\partial t} + [\delta f, \mathrm{H}_0] + [F, \mathrm{H}_1] = 0 \tag{11}$$

where H_0 is the equilibrium Hamiltonian and H_1 the perturbed Hamiltonian.

III. EQUILIBRIUM HAMILTONIAN AND PARTICLE TRAJECTORIES

Since we consider frequencies ω much less than the cyclotron frequency $\Omega_i = eB_0/m_i c$, we approximate H_0 with the following Hamiltonian for guiding center motion:

$$\mathrm{H}_0 = \frac{P_\theta^2}{2m_i \bar{q}^2 R_0^2} + \mu B_0 \left(1 - \frac{\bar{r}}{R_0} \cos Q_\theta\right) \tag{12}$$

where $\bar{r} = \bar{r}(P_\zeta)$ is determined by $P_\zeta + \frac{e}{c} \int^{\bar{r}} dr \, \frac{rB_0}{q} = P_0 =$ constant, and the coordinates r, θ, ζ of the particle guiding center are related to the canonically conjugate coordinates $P_\theta, Q_\theta, P_\zeta, Q_\zeta$ by

$$\theta = Q_\theta - \frac{\bar{r}}{R_0} \sin Q_\theta + \cdots \tag{13}$$

$$r - \bar{r} = \frac{1}{m_i \bar{r} \Omega_i} \left\{ P_\theta \left(1 + \frac{\bar{r}}{R_0} \cos Q_\theta\right) - \bar{q}\, P_0 \right\} \tag{14}$$

$$\zeta - q(r)\theta = Q_\zeta - \left(r - \bar{r} + \frac{\bar{q}\, P_0}{m_i \bar{r} \Omega_i}\right) \left(\frac{\partial \bar{q}}{\partial \bar{r}} Q_\theta - \frac{1}{R_0} \frac{\partial \bar{r}\bar{q}}{\partial \bar{r}} \sin Q_\theta\right) + \cdots \tag{15}$$

The magnetic moment μ is a constant of the guiding center motion, the canonical momentum P_ζ is constant due to axisymmetry, and $\bar{q} = q(\bar{r})$. We have neglected terms higher order in the Larmor radius $\frac{r_L}{\bar{r}} < 1$, the inverse aspect ratio $\frac{\bar{r}}{R_0} < 1$, and the orbit width parameter $\frac{r - \bar{r}}{\bar{r}} < 1$. The particle orbit is closed and periodic in the $P_\theta - Q_\theta$ plane. The constant P_0 is chosen so that the time average of $r - \bar{r}$ over a periodic trajectory is zero.

Motion along the field line on a magnetic flux surface at $\bar{r}$ is described in terms of the variables P_θ, Q_θ.

The Hamiltonian equations of motion are

$$\frac{dQ_\theta}{dt} = \frac{\partial \mathrm{H}_0}{\partial P_\theta} = \pm \frac{1}{\bar{q}\,R_0}\left[\frac{2}{m_i}\left\{\mathrm{H}_0 - \mu B_0\left(1 - \frac{\bar{r}}{R_0}\cos Q_\theta\right)\right\}\right]^{1/2}$$

$$\frac{dP_\theta}{dt} = -\frac{\partial \mathrm{H}_0}{\partial Q_\theta}$$

$$\frac{dQ_\zeta}{dt} = \frac{\partial \mathrm{H}_0}{\partial P_\zeta} = -\frac{\bar{q}}{m_i \bar{r}\Omega_i}\frac{\partial \mathrm{H}_0}{\partial \bar{r}}$$

It may be verified that the guiding center drifts in the radial direction and across magnetic field lines are:

$$\frac{dr}{dt} = v_D \cdot \nabla r + \cdots$$

$$\frac{d}{dt}\left(q(r)\theta - \zeta\right) = Q_\theta \frac{\partial \bar{q}}{\partial \bar{r}}\frac{dr}{dt} + \bar{q}\, v_D \cdot \nabla\theta + \cdots\cdots$$

where the guiding center drift velocity v_D is

$$v_D \cdot \nabla r = -\frac{1}{m_i \Omega_i R_0}\left(\frac{P_\theta^2}{m_i \bar{q}^2 R_0^2} + \mu B_0\right)\sin Q_\theta$$

$$v_D \cdot \nabla \theta = -\frac{1}{\bar{r} m_i \Omega_i R_0}\left(\frac{P_\theta^2}{m_i \bar{q}^2 R_0^2} + \mu B_0\right)\cos Q_\theta$$

The particles may be grouped into two categories: (1) passing particles with energies $\mathrm{H}_0 > \mu B_0(1 + \bar{r}/R_0)$ which encircle the magnetic axis; (2) trapped particles with energies $\mu B_0(1 + \bar{r}/R_0) > \mathrm{H}_0 > \mu B_0(1 - \bar{r}/R_0)$, bounded in Q_θ, which 'reflect' and form 'banana' orbits.

A. Passing particles

We solve the equation for Q_θ by quadrature. For passing particles, with initial condition $Q_\theta = 0$ when $t = 0$, we obtain:

$$\int_0^{Q_\theta} \frac{dQ_\theta}{\left(1 - \kappa^2 \sin^2(Q_\theta/2)\right)^{1/2}} = 2u$$

where

$$\kappa^2 = \frac{2\mu B_0(\overline{r}/R_0)}{\{\mathrm{H}_0 - \mu B_0(1-\overline{r}/R_0)\}} \qquad u = \sigma\,\frac{\{\mathrm{H}_0 - \mu B_0(1-\overline{r}/R_0)\}^{1/2}\,t}{(2m_i)^{1/2}\overline{q}\,R_0}$$

The parameter $\sigma = +1$ or $\sigma = -1$, depending on the direction of circulation of the passing particles about the magnetic axis.

In terms of Jacobian elliptic functions sn, cn, dn and their Fourier series representation, we have:

$$\sin(Q_\theta/2) = sn(u,\kappa)$$

$$\sin Q_\theta = -\frac{2}{\kappa^2}\frac{\partial}{\partial u}dn(u,\kappa) = \frac{4\pi^2}{\kappa^2K^2}\sum_{n=1}^{\infty}\frac{n\widehat{q}^n}{(1+\widehat{q}^{2n})}\sin n\Theta$$

$$\cos Q_\theta = (1-2sn^2(u,\kappa)) = 1 + \frac{2}{\kappa^2}\left(\frac{E}{K}-1\right) + \frac{4\pi^2}{\kappa^2K^2}\sum_{n=1}^{\infty}\frac{n\widehat{q}^n}{(1-\widehat{q}^{2n})}\cos n\Theta$$

$$P_\theta = 2m_i\overline{q}^2R_0^2\frac{du}{dt}dn(u,\kappa) = m_i\overline{q}^2R_0^2\frac{d\Theta}{dt}\left(1+4\sum_{n=1}^{\infty}\frac{\widehat{q}^n}{(1+\widehat{q}^{2n})}\cos n\Theta\right)$$

$$\frac{dQ_\theta}{dt} \equiv \frac{d\Theta}{dt} + \frac{d\widetilde{Q}_\theta}{dt} = \frac{P_\theta}{m_i\overline{q}^2R_0^2} \tag{16}$$

$$\frac{dQ_\varsigma}{dt} \equiv \frac{d\overline{Q}_\varsigma}{dt} + \frac{d\widetilde{Q}_\varsigma}{dt} = \frac{\partial \mathrm{H}_0}{\partial P_\varsigma} \tag{17}$$

$$\frac{d\Theta}{dt} = \frac{\pi}{K}\frac{du}{dt} \equiv \sigma\omega_b \qquad P_0 = m_i\overline{q}\,R_0^2\frac{d\Theta}{dt}$$

where $\widetilde{Q}_\theta$, $\widetilde{Q}_\varsigma$ are oscillatory in Θ, $E = E(\kappa^2)$ and $K = K(\kappa^2)$ are the complete elliptic integrals of the first and second kind respectively, and $\widehat{q} = \widehat{q}(\kappa^2) = \exp\left(-K(1-\kappa^2)/K(\kappa^2)\right)$.

For particles with small 'pitch angle' $\kappa^2 < 1$,far from the separatrix boundary between passing and trapped particles,

$$\omega_b \approx \frac{\{\mathrm{H}_0 - \mu B_0\,(1-\overline{r}/R_0)\}^{1/2}}{(2m_i)^{1/2}\overline{q}\,R_0}$$

$$r - \overline{r} \approx \Delta_b\cos\Theta \qquad \Delta_b = \frac{\overline{q}(2\mathrm{H}_0 - \mu B_0)}{(2m_i)^{1/2}\Omega_i(\mathrm{H}_0-\mu B_0)^{1/2}}$$

where Δ_b is the orbit excursion from a flux surface (orbit width).

B. Trapped particles

Similarly, for trapped particles, we obtain:

$$\int_0^{Q_\theta} \frac{dQ_\theta}{\left(\kappa'^2 - \sin^2(Q_\theta/2)\right)^{1/2}} = 2u$$

where

$$\kappa'^2 = \frac{\{\mathrm{H}_0 - \mu B_0 (1 - \bar{r}/R_0)\}}{2\mu B_0 (\bar{r}/R_0)} = \frac{1}{\kappa^2} \qquad u = \frac{1}{\bar{q}\, R_0}\left(\frac{\mu B_0 \bar{r}}{m_i R_0}\right)^{1/2} t$$

$$K' = K(\kappa'^2), \qquad E' = E(\kappa'^2), \qquad \hat{q}' = \hat{q}(\kappa'^2)$$

and

$$\sin(Q_\theta/2) = \kappa'\, sn(u, \kappa')$$

$$\sin Q_\theta = -2\kappa' \frac{d}{du} cn(u, \kappa') = \frac{4\pi^2}{K'^2} \sum_{n=0}^{\infty} \frac{(n+1/2)\hat{q}'^{n+1/2}}{(1+\hat{q}'^{2n+1})} \sin(2n+1)\Theta$$

$$\cos Q_\theta = 1 - 2\kappa'^2 sn^2(u, \kappa') = \frac{2E'}{K'} - 1 + \frac{4\pi^2}{K'^2} \sum_{n=1}^{\infty} \frac{n\hat{q}'^n}{(1-\hat{q}'^{2n})} \cos 2n\Theta$$

$$P_\theta = 2m_i\bar{q}^2 R_0^2 \kappa' \frac{du}{dt} cn(u, \kappa') = 8m_i\bar{q}^2 R_0^2 \frac{d\Theta}{dt} \sum_{n=0}^{\infty} \frac{\hat{q}'^{n+1/2}}{(1+\hat{q}'^{2n+1})} \cos(2n+1)\Theta$$

$$\frac{dQ_\theta}{dt} \equiv \frac{d\tilde{Q}_\theta}{dt} = \frac{P_\theta}{m_i\bar{q}^2 R_0^2} \tag{18}$$

$$\frac{dQ_\varsigma}{dt} \equiv \frac{d\overline{Q}_\varsigma}{dt} + \frac{d\tilde{Q}_\varsigma}{dt} = \frac{\partial \mathrm{H}_0}{\partial P_\varsigma} \tag{19}$$

$$\frac{d\Theta}{dt} = \frac{\pi}{2K'} \frac{du}{dt} \equiv \omega_b' \qquad P_0 = \frac{1}{2\pi q} \oint d\Theta\, P_\theta = 0.$$

IV. PERTURBED HAMILTONIAN AND DISTRIBUTION FUNCTION

The perturbed Hamiltonian is

$$\mathrm{H}_1 = -\frac{e}{c}\sum_N i\, k_\perp \cdot v_D\, J_0\left(\frac{k_\perp v_\perp}{\Omega_i}\right)\widehat{\phi}(\theta + 2\pi N - \theta_0)\;\exp\left\{in_0 q\left(\theta + 2\pi N - \theta_0\right) - in_0\zeta - i\omega t\right\} \tag{20}$$

$$k_\perp = |k_\perp|, \qquad v_\perp = \left(\frac{2\mu B_0}{m_i}\right)^{1/2}$$

The Bessel function $J_0\left(\frac{k_\perp v_\perp}{\Omega_i}\right)$ takes account of finite Larmor radius effects.

Substituting the solutions of the equilibrium trajectories, we express H_1 in terms of the variables $\mathrm{H}_0, \mu, \overline{r}, \overline{r} - r_0, \Theta$.

For passing particles

$$\mathrm{H}_1 = \widetilde{\mathrm{H}}^{(1)}_{\mathrm{pas}}\left(\mathrm{H}_0, \mu, \overline{r}, \overline{r} - r_0, \Theta\right)\exp\left(-i\,\Omega\, t - in_0 q_0\theta_0\right) \tag{21}$$

where $\widetilde{\mathrm{H}}^{(1)}_{\mathrm{pas}}$ is periodic in Θ

$$\widetilde{\mathrm{H}}^{(1)}_{\mathrm{pas}} = \sum_\ell \widetilde{\mathrm{H}}_\ell \exp(i\ell\Theta) \tag{22}$$

and

$$\Omega = \omega + \frac{(\overline{r} - r_0)}{\Delta}\sigma\omega_b + n_0\left\langle \dot{Q}_\varsigma\right\rangle_{\mathrm{pas}}$$

$$\left\langle \dot{Q}_\varsigma\right\rangle_{\mathrm{pas}} = \frac{d\overline{Q}_\varsigma}{dt} - \frac{\overline{q}^2 R_0^2\omega_b^2}{\overline{r}\Omega_i}\frac{\partial\overline{q}}{\partial\overline{r}} = -\frac{\mu B_0\overline{q}}{m_i\overline{r}\Omega_i R_0\kappa^2}\left\{2 - \kappa^2 - 2E/K + 4\overline{S}\left(\pi^2/4K^2 - E/K\right)\right\}.$$

The Fourier coefficients $\widetilde{\mathrm{H}}_\ell$ are given by

$$\widetilde{\mathrm{H}}_\ell = \frac{1}{2\pi}\int_0^{2\pi} d\Theta\; \widetilde{\mathrm{H}}^{(1)}_{\mathrm{pas}}\exp(-i\ell\Theta) = \frac{(2H_0 - \mu B_0)}{B_0 R_0\Delta}\widehat{\mathrm{H}}_\ell$$

$$\widehat{\mathrm{H}}_\ell = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\Theta\; J_0\left(\frac{k_\perp v_\perp}{\Omega_i}\right)\widehat{\phi}(\theta')\left\{\theta'\sin Q_\theta + \frac{\cos Q_\theta}{\overline{S}}\right\}\exp\left(-i\ell\Theta - i\widetilde{\chi}_{\mathrm{pas}}\right) \tag{23}$$

where $\theta' = \theta - \theta_0$ and

$$\widetilde{\chi}_{\rm pas} = n_0 \widetilde{Q}_\varsigma - n_0 (r - r_0) \frac{\partial q}{\partial r_0} (\Theta - \theta_0) - n_0 \left(r - \overline{r} + \frac{\overline{q}\, P_0}{m_i \overline{r} \Omega_i} \right) \left(\frac{\partial \overline{q}}{\partial \overline{r}} \widetilde{Q}_\theta - \frac{1}{R_0} \frac{\partial \overline{r}\overline{q}}{\partial \overline{r}} \sin Q_\theta \right)$$

We have made use of the fact that $n_0 q(r_0) = m_0 =$ integer, we have assumed that $\frac{\partial \overline{q}}{\partial \overline{r}} \approx \frac{\partial q_0}{\partial r_0}$, and $\overline{S} = \frac{\overline{r}}{\overline{q}} \frac{\partial \overline{q}}{\partial \overline{r}}$.

We determine the perturbed distribution δf by integrating along the equilibrium phase space trajectories

$$\delta f = - \int^t dt \, [F(\mathrm{H}_0, \overline{r}), \mathrm{H}_1] = \mathrm{H}_1 \frac{\partial F}{\partial \mathrm{H}_0} + \int^t dt \, \mathrm{H}_1 \left(i\omega \frac{\partial F}{\partial \mathrm{H}_0} + \frac{i n_0 \overline{q}}{m \overline{r} \Omega_i} \frac{\partial F}{\partial \overline{r}} \right)$$

and we obtain

$$\delta f = \mathrm{H}_1 \frac{\partial F}{\partial \mathrm{H}_0} - \sum_\ell \widetilde{\mathrm{H}}_\ell \frac{\left(\omega \frac{\partial F}{\partial \mathrm{H}_0} + \frac{n_0 \overline{q}}{m_i \overline{r} \Omega_i} \frac{\partial F}{\partial \overline{r}} \right)}{\Omega - \sigma \ell \omega_b} \exp\left(-i\Omega t + i\ell\Theta - i m_0 \theta_0 \right). \qquad (24)$$

For trapped particles

$$\mathrm{H}_1 = \widetilde{\mathrm{H}}^{(1)}_{\rm trp} (\mathrm{H}_0, \mu, \overline{r}, \overline{r} - r_0, \Theta) \exp\left(-i\,\Omega\, t - i n_0 q_0 \theta_0 \right) \qquad (25)$$

$$\widetilde{\mathrm{H}}^{(1)}_{\rm trp} = \sum_{\ell, N} \widetilde{\mathrm{H}}_{\ell, N} \exp(i\ell\Theta) \qquad (26)$$

where

$$\Omega = \omega + n_0 \left\langle \dot{Q}_\varsigma \right\rangle_{\rm trp}$$

$$\left\langle \dot{Q}_\varsigma \right\rangle_{\rm trp} = \frac{d\overline{Q}_\varsigma}{dt} = \frac{\mu B_0 \overline{q}}{m_i \overline{r} \Omega_i R_0} \left\{ 2E'/K' - 1 + 4\overline{S} \left(\kappa'^2 - 1 + E'/K' \right) \right\}$$

The Fourier coefficients $\widetilde{\mathrm{H}}_{\ell,N}$ are given by

$$\widetilde{\mathrm{H}}_{\ell,N} = \frac{1}{2\pi} \int_0^{2\pi} d\Theta \, \widetilde{\mathrm{H}}^{(1)}_{\rm trp} \exp(-i\ell\Theta) = \frac{(2H_0 - \mu B_0)}{B_0 R_0 \Delta} \widehat{\mathrm{H}}_{\ell,N} \exp\left(i \frac{(\overline{r} - r_0)}{\Delta} (2\pi N - \theta_0) \right)$$

$$\widehat{\mathrm{H}}_{\ell,N} = \frac{i}{2\pi} \int_0^{2\pi} d\Theta J_0 \left(\frac{k_\perp v_\perp}{\Omega_i} \right) \widehat{\phi}(\theta'_N) \left\{ \theta'_N \sin Q_\theta + \frac{\cos Q_\theta}{\overline{S}} \right\} \exp\left(-i\ell\Theta - i\widetilde{\chi}_{\rm trp} \right) \qquad (27)$$

where $\theta'_N = \theta + 2\pi N - \theta_0$ and

$$\widetilde{\chi}_{\rm trp} = n_0 \widetilde{Q}_\varsigma - n_0(r - \overline{r}) \left(\frac{\partial \overline{q}}{\partial \overline{r}} (Q_\theta + 2\pi N - \theta_0) - \frac{1}{R_0} \frac{\partial \overline{r}\overline{q}}{\partial \overline{r}} \sin Q_\theta \right).$$

The perturbed distribution is

$$\delta f = \mathrm{H}_1 \frac{\partial F}{\partial \mathrm{H}_0} - \sum_{\ell,N} \widetilde{\mathrm{H}}_{\ell,N} \frac{\left(\omega \frac{\partial F}{\partial \mathrm{H}_0} + \frac{n_0 \overline{q}}{m_i \overline{r} \Omega_i} \frac{\partial F}{\partial \overline{r}_0}\right)}{\Omega - \ell \omega'_b} \exp\left(-i\Omega t + i\ell\Theta - i m_0 \theta_0\right). \tag{28}$$

V. GROWTH RATE

We consider a 'slowing down' distribution function for the energetic particles of the form:

$$F = \frac{3B_0^2 \widehat{\beta}(\overline{r}) h(v_0 - v)}{16\pi^2 m_i v_0^5} G(\overline{v}, \overline{\mu}, \overline{r}) \tag{29}$$

$$G(\overline{v}, \overline{\mu}, \overline{r}) = \frac{2\pi v_0^3 \left\{ g(\overline{v}, \overline{\mu}, \overline{r}) / (\overline{v}^3 + \overline{v}_I^3) \right\}}{\frac{1}{2\pi} \int_0^{2\pi} d\theta \int d^3 v\, \overline{v}^2 \left\{ g(\overline{v}, \overline{\mu}, \overline{r}) / (\overline{v}^3 + \overline{v}_I^3) \right\} h(1 - \overline{v})}$$

$$g(\overline{v}, \overline{\mu}, \overline{r}) = \frac{\exp\left[-(1-\lambda)/\delta\lambda\right] + \exp\left[-(1+\lambda)/\delta\lambda\right]}{\delta\lambda \left[1 - \exp(-2/\delta\lambda\right]}$$

$$\lambda(\overline{\mu}, \overline{r}) = \left(1 - \overline{\mu}(1 - \overline{r}/R_0)\right)^{1/2}$$

$$\delta\lambda(\overline{v}) = \delta\lambda_0 + \frac{1}{3} A \ln\left(\frac{1 + \overline{v}_I^3/\overline{v}^3}{1 + \overline{v}_I^3} \right)$$

where $\overline{\mu} = \mu B_0/\mathrm{H}_0$, $v = (2\mathrm{H}_0/m_i)^{1/2}$, $\overline{v} = v/v_0$, $\overline{v}_I = v_I/v_0$, $h(v)$ is the step function, $m_i v_0^2/2$ is the birth kinetic energy, and the physical meaning of the constants $\delta\lambda_0$, A, v_I is discussed in Ref. 10. The distribution function F is normalized so that $\widehat{\beta}(\overline{r}) = (8\pi m_i/3B_0^2)\,(\frac{1}{2\pi})\int_0^{2\pi} d\theta \int d^3 v v^2 F$ is the mean beta of the energetic particles.

For $\delta\lambda(v) \gg 1$, the function $g(\overline{v}, \overline{\mu}, \overline{r}) \to 1$ and the distribution function is isotropic $G(\overline{v}, \overline{\mu}, \overline{r}) \approx 1/(\overline{v}^3 + \overline{v}_I^3)$, $\overline{v}_I \ll 1$.

To calculate the growth rate of TAE-modes, we substitute the solutions for δf in(see Eq. (10)):

$$\frac{\omega\gamma}{2\pi v_A^2} \int dr^3 \widetilde{A}_\perp^* \cdot \widetilde{A}_\perp = -i \left\langle \mathrm{H}_1^* \delta f \right\rangle \tag{30}$$

where $\widetilde{A}_\perp^\dagger$ is replaced by the complex conjugate $\widetilde{A}_\perp^*$.

It is sufficient to consider only the response of the 'resonant' particles to the perturbing fields. The 'resonant' particles lie on a 'resonant' surface, determined by:

$$\overline{\Omega}\left(\mathrm{H}_0 = E_{\mathrm{res}}, \mu, \overline{r}\right) = \begin{cases} \left[\Omega - \sigma\ell\omega_b\right]_{\mathrm{H}_0=E_{\mathrm{res}}} = 0 & \text{passing} \\ \left[\Omega - \ell\omega_b'\right]_{\mathrm{H}_0=E_{\mathrm{res}}} = 0 & \text{trapped} \end{cases} \tag{31}$$

in the three-dimensional particle phase space $\mathrm{H}_0, \mu, \overline{r}$.

For passing particles, the resonance condition can be written

$$\sigma\left\{\frac{(\overline{r}-r_0)}{\Delta} - \ell\right\} \equiv \sigma\overline{x}(\ell) \quad \approx \quad X_{\mathrm{pas}}(\overline{v}_{\mathrm{res}}, \overline{\mu}) \equiv \left[\frac{v_A\omega_{0b}}{2v_0\omega_b} - \frac{n_0\left\langle\dot{Q}_\varsigma\right\rangle_{\mathrm{pas}}}{\omega_b}\right]_{\overline{v}=\overline{v}_{\mathrm{res}}\overline{r}=r_0} \tag{32}$$

where $\omega \approx -v_A/2q_0R_0$, $E_{\mathrm{res}} = m_i v_{\mathrm{res}}^2/2$, and $\overline{x}(\ell)$ is the dimensionless distance from the mode rational surface of the $m_0+\ell$ azimuthal component.

In Fig. 1, we plot X_{pas} as a function of the pitch angle parameter $\kappa^2(\overline{\mu}) = 2\overline{\mu}\varepsilon_0/\left(1-\overline{\mu}(1-\varepsilon_0)\right)$ for constant values of resonant velocity $\overline{v}_{\mathrm{res}}$ and of magnetic moment $\langle\mu B_0\rangle = \overline{\mu}\overline{v}_{\mathrm{res}}^2$. The 'toroidal' drift is $\left\langle\dot{Q}_\varsigma\right\rangle_{\mathrm{pas}} = 0$ at $\kappa^2 = 0$, and $\left\langle\dot{Q}_\varsigma\right\rangle_{\mathrm{pas}} = -\left(\mathrm{H}_0 q_0\right)/\left(m_i\overline{r}\Omega_i R_0(1+\varepsilon_0)\right)$ at $\kappa^2 = 1$. Thus X_{pas} increases monotonically from $v_A/\left(2v_0\overline{v}_{\mathrm{res}}\right)$ as κ^2 increases from zero to unity. Resonant particles with $\overline{v}_{\mathrm{res}} < 1$ are in the region $|\overline{x}| > v_A/\left(2v_0\right)$ and have values of κ^2 bounded by the lines $\kappa^2 = 0$ and $\overline{v}_{\mathrm{res}} = 1$. The resonant energy $\overline{v}_{\mathrm{res}}^2$ of passing particles at $\overline{x}$ with 'pitch angle' κ^2 is determined by the line of constant $\overline{v}_{\mathrm{res}}$ passing through the point $(X_{\mathrm{pas}} = \overline{x}, \kappa^2)$. Since the particle magnetic moment is conserved, the particles are constrained to move along lines of constant $\langle\mu B_0\rangle$.

For trapped particles, the resonance condition can be written

$$-\ell = X_{\mathrm{trp}}(\overline{v}_{\mathrm{res}}, \overline{\mu}) \equiv \left[\frac{v_A\omega_{0b}}{2v_0\omega_b'} - \frac{n_0\left\langle\dot{Q}_\varsigma\right\rangle_{\mathrm{trps}}}{\omega_b'}\right]_{\overline{v}=\overline{v}_{\mathrm{res}},\overline{r}=r_0} . \tag{33}$$

In Fig. 2, we plot X_{trp} as a function of $\kappa'^2(\overline{\mu}) = 1/\kappa^2(\overline{\mu})$ for constant values of resonant velocity $\overline{v}_{\mathrm{res}}$ and of magnetic moment $\langle\mu B_0\rangle = \overline{\mu}\overline{v}_{\mathrm{res}}^2$. The 'toroidal' drift is $\left\langle\dot{Q}_\varsigma\right\rangle_{\mathrm{trp}} =$

$(\mathrm{H}_0 q_0) \,/\, (m_i \bar{r} \Omega_i R_0 (1-\varepsilon_0))$ at $\kappa'^2 = 0$, and $\left\langle \dot{Q}_\varsigma \right\rangle_{\rm trp} = -\,(\mathrm{H}_0 q_0) \,/\, (m_i \bar{r} \Omega_i R_0 (1+\varepsilon_0))$ at $\kappa'^2 = 1$, and hence its direction changes at an intermediate value of κ'^2. Thus at $\kappa'^2 \to 1$, $X_{\rm trp}$ is large and positive, while at $\kappa'^2 = 0$, $X_{\rm trp} = \{v_A(1-\varepsilon_0)/v_0 \bar{v}_{\rm res} - q_0 r_L \bar{v}_{\rm res}/S_0 \Delta\} \Big/ \left\{(2\varepsilon_0)^{1/2}(1-\varepsilon_0)^{1/2}\right\}$. Resonant particles have values of κ'^2 which are bounded by the lines $\kappa'^2 = 0$ and $\bar{v}_{\rm res} = 1$. Resonances occur at integer values of $X_{\rm trp}$, $\ell = -X_{\rm trp}$. The resonant energy $\bar{v}^2_{\rm res}$ of trapped particles with 'pitch angle' κ'^2 and in resonance with bounce harmonic number ℓ is determined by the line of constant $\bar{v}_{\rm res}$ passing through the point $(X_{\rm trp} = -\ell, \kappa'^2)$.

The perturbed distribution function has a simple pole on this surface. We follow the Landau prescription to evaluate the 'resonant' particle contribution to the perturbed current by making the substitution:

$$\frac{1}{\overline{\Omega}} \longrightarrow -i\pi \, \frac{\delta\left(\mathrm{H}_0 - E_{\rm res}\right)}{\left|\frac{\partial \overline{\Omega}}{\partial \mathrm{H}_0}\right|}.$$

A. Passing particles

For passing particles, we obtain the growth rate:

$$\frac{\gamma_{\rm passing}}{|\omega|} = i8\pi q_0^2 R_0^2 \frac{\langle \delta f \mathrm{H}_1^* \rangle_{\rm passing}}{\int d^3r \widetilde{A}_\perp^* \cdot \widetilde{A}_\perp} \approx -\frac{3\,\pi\, n_0 v_0 q_0^4 R_0}{2\, r_0 \Omega_i} \frac{\partial \widehat{\beta}}{\partial r_0} \frac{\sum_{\sigma,\ell} \int d\bar{r}\; \Pi_{\rm pas}}{\sum_{\ell} \int dr\; W}$$

$$\approx -\frac{3\,\pi\, q_0^3 r_L R_0}{2\, S_0 \Delta} \frac{\partial \widehat{\beta}}{\partial r_0} \frac{\sum_{\sigma} \int d\bar{x}\; \Pi_{\rm pas}}{\int dx\; W} \tag{34}$$

where

$$\Pi_{\rm pas} = \int_0^{1/(1+\bar{r}/R_0)} d\bar{\mu}(1-\bar{\mu}/2)^2 \int_0^\infty d\bar{v}\, \bar{v}^7\, \widehat{\mathrm{H}}_\ell^* \, \widehat{\mathrm{H}}_\ell \, \frac{\omega_{0b}^2}{\omega_b} \frac{\delta\left(\bar{v} - \bar{v}_{\rm res}\right)}{\left|\frac{\partial \overline{\Omega}}{\partial \bar{v}_{\rm res}}\right|} \left\{1 + \frac{\omega}{\omega^* \bar{v}} \frac{\partial}{\partial \bar{v}}\right\}$$
$$\times \; G(\bar{v}, \bar{\mu}, \bar{r}) h(1-\bar{v}) \tag{35}$$

$$\int d^3r \widetilde{A}_\perp^* \cdot \widetilde{A}_\perp \approx \frac{4\pi^2 r_0 R_0}{\Delta^2} \sum_\ell \int dr \left\{\frac{\partial \phi_\ell^*}{\partial r} \frac{\partial \phi_\ell}{\partial r} + \frac{m_0^2}{r_0^2} \phi_\ell^* \phi_\ell\right\} \equiv \frac{4\pi^2 r_0 R_0}{\Delta^2} \sum_\ell \int dr\, W$$

and $\bar{x} = (\bar{r} - r_0)/\Delta - \ell, \quad r_L = v_0/\Omega_i, \quad \omega^* = \frac{n_0 q v_0 r_L}{\bar{r}\beta}\frac{\partial \widehat{\beta}}{\partial \bar{r}}, \quad \omega_{0b} = \frac{v_0}{q_0 R_0}$
$\Pi_{\text{pas}} = \Pi_{\text{pas}}(\sigma\bar{x}, \bar{r}) \approx \Pi_{\text{pas}}(\sigma\bar{x}, r_0), x = (r - r_0)/\Delta - \ell, W = W(x, r_0).$

If we take the $\kappa^2 < 1$ approximation to characterize the particle orbits, we can obtain an approximate analytic expression for γ_{passing} in the limit of: (1) small $\widehat{\varepsilon} \ll 1$ where $\frac{\partial \phi_\ell(r)}{\partial r}$ is sharply peaked at values of $(r - r_0)/\Delta = \ell \pm \frac{1}{2}$ with a spatial width of $\widehat{\varepsilon}\Delta$; (2) small orbit width satisfying the inequality $\Delta_b/\Delta \ll 1$. We then have:

$$\omega_b \approx \bar{v}(1 - \bar{\mu})^{1/2}\omega_{0b}, \qquad \bar{v}_{\text{res}} \approx \frac{v_A}{2v_0\sigma\bar{x}(1 - \bar{\mu})^{1/2}}$$

$$\widetilde{X}_{\text{pas}} \approx -\frac{(\bar{r} - r_0)}{\Delta}(\Theta - \theta_0) - \frac{\Delta_b}{\Delta}(\Theta - \theta_0)\cos\Theta$$

and approximating $\widehat{\phi}(\eta)$ by its asymptotic expansion (Eq. (8))

$$\widehat{\mathrm{H}}_\ell \approx \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\eta\, \widehat{\phi}(\eta)\, \eta \sin(\eta + \theta_0) \exp\left(-i\ell(\eta + \theta_0) - i\widetilde{X}_{\text{pas}}\right)$$

$$\approx \frac{i\Delta}{2\pi\Delta_b}\sum_n \left\{\alpha^+ h_n(\bar{x} + \frac{1}{2}, \bar{v}, \bar{\mu}) + \alpha^- h_n^*(\bar{x} + \frac{1}{2}, \bar{v}, \bar{\mu})\right\} \exp\left(-i(\ell + n - \frac{1}{2})\theta_0\right)$$

$$+ \frac{i\Delta}{2\pi\Delta_b}\sum_n \left\{\alpha^{+*} h_n(\bar{x} - \frac{1}{2}, \bar{v}, \bar{\mu}) + \alpha^{-*} h_n^*(\bar{x} - \frac{1}{2}, \bar{v}, \bar{\mu})\right\} \exp\left(-i(\ell + n + \frac{1}{2})\theta_0\right)$$

$$h_n(\bar{x} + \frac{1}{2}, \bar{v}, \bar{\mu}) = \int_0^\infty d\eta\, n J_n\left(\frac{\Delta_b \eta}{\Delta}\right)\eta^{-1} \exp\left(i(\bar{x} + \frac{1}{2} - n)\eta - \widehat{\Gamma}\eta + in\frac{\pi}{2}\right)$$

$$= z^n$$

$$z = \left\{\left(\bar{x} + \frac{1}{2} - n + i\widehat{\Gamma}\right)^2 \left(\frac{\Delta}{\Delta_b}\right)^2 - 1\right\}^{1/2} - \left(\bar{x} + \frac{1}{2} - n + i\widehat{\Gamma}\right)\frac{\Delta}{\Delta_b}$$

where the branch of the square root is determined by $|z| < 1$. $\widehat{\mathrm{H}}_\ell$ has maxima at $\bar{x} = n \pm 1/2$ with the largest maxima occurring at $\bar{x} = \pm\frac{1}{2}, \pm\frac{3}{2}$. For an isotropic distribution function with $\bar{v}_I \ll 1$ (ignoring the contributions proportional to $\frac{\partial F}{\partial v}$), we obtain

$$\sum_\sigma \int d\bar{x}\, \Pi_{\text{pas}} \approx 2\int_0^1 d\bar{\mu}\, \frac{(1 - \bar{\mu}/2)^2}{(1 - \bar{\mu})} \int_0^\infty d\bar{v}\, \bar{v}^3 \delta\left(\bar{v} - \xi/(1 - \bar{\mu})^{1/2}\right) \int d\bar{x} \widehat{\mathrm{H}}_\ell^* \widehat{\mathrm{H}}_\ell$$

$$\int d\overline{x}\ \widehat{\mathrm{H}}_\ell^*\,\widehat{\mathrm{H}}_\ell \approx \frac{\Delta^2}{4\pi^2\Delta_b^2}\int d\overline{x}\ \left[\left(|\alpha^+|^2+\left|\alpha^-\right|^2\right)\left(\left|h_1(\overline{x}+\frac{1}{2},\overline{v},\overline{\mu})\right|^2+\left|h_{-1}(\overline{x}-\frac{1}{2},\overline{v},\overline{\mu})\right|^2\right)\right]$$

$$= \frac{\Delta^2}{\pi\Delta_b^2}\left(|\alpha^+|^2+|\alpha^-|^2\right)\int_0^\infty d\eta\ J_1^2\left(\frac{\Delta_b\eta}{\Delta}\right)\eta^{-2}\exp(-2\widehat{\Gamma}\eta).$$

where we assume $\xi = v_A/v_0 < 1$, and we take into account only the dominant contribution (due to the energy weighting) in the neighborhood of $\overline{x} = \pm\frac{1}{2}$.

Since

$$\int dxW \approx \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\eta|\widehat{\phi}(\eta)|^2\eta^2 \approx \frac{|\alpha^+|^2+|\alpha^-|^2}{2\pi\widehat{\Gamma}}$$

we have

$$\frac{\int d\overline{x}\,\widehat{\mathrm{H}}_\ell^*\,\widehat{\mathrm{H}}_\ell}{\int dxW} \approx \frac{2\Delta^2\widehat{\Gamma}}{\Delta_b^2}\int_0^\infty d\eta\ J_1^2\left(\frac{\Delta_b\eta}{\Delta}\right)\eta^{-2}\exp(-2\widehat{\Gamma}\eta) \approx \begin{cases} \dfrac{1}{4}, & 1>\widehat{\Gamma}>\dfrac{\Delta_b}{\Delta} \\ \\ \dfrac{8\widehat{\Gamma}}{3\pi}\,\dfrac{\Delta}{\Delta_b},, & 1>\dfrac{\Delta_b}{\Delta}>\Gamma. \end{cases}$$

For $1 > \widehat{\Gamma} > \Delta_b/\Delta$, the growth rate may be approximated by[5]

$$\frac{\gamma_{\mathrm{pas}}}{|\omega|} \approx \frac{\pi\,q_0^3 r_L R_0}{8\,S_0\Delta}\left(-\frac{\partial\widehat{\beta}}{\partial r_0}\right)\left(1+6\,\xi^2-4\,\xi^3-3\,\xi^4\right) \tag{36}$$

and for $1 > \Delta_b/ > \widehat{\Gamma}$

$$\frac{\gamma_{\mathrm{pas}}}{|\omega|} \approx \frac{8\,q_0^2R_0\widehat{\Gamma}}{S_0}\left(-\frac{\partial\widehat{\beta}}{\partial r_0}\right)\xi(1-\xi^2) \approx \frac{5\,q_0^2r_0}{S_0}\left(-\frac{\partial\widehat{\beta}}{\partial r_0}\right)\xi(1-\xi^2) \tag{37}$$

where $\widehat{\Gamma} \approx 5\varepsilon_0/8$ (Eq. (9)).

B. Trapped particles

For trapped particles, we obtain the growth rate $\gamma_{\mathrm{trapped}}$:

$$\frac{\gamma_{\mathrm{trapped}}}{|\omega|} = i8\pi q_0^2R_0^2\frac{\langle\delta f\mathrm{H}_1^*\rangle_{\mathrm{trp}}}{\int d^3r\widetilde{A}_\perp^*\cdot\widetilde{A}_\perp} = -\frac{3\,\pi\,n_0v_0q_0^4R_0}{2\,r_0\Omega_i}\,\frac{\partial\widehat{\beta}}{\partial r_0}\,\frac{\sum_{\ell,N}\int d\overline{r}\ \Pi_{\mathrm{trp}}}{\sum_{\ell'}\int dr\ W}$$

$$= -\frac{3\,\pi\,q_0^3r_LR_0}{2\,S_0\Delta}\,\frac{\partial\widehat{\beta}}{\partial r_0}\,\frac{\sum_{\ell,N}\Pi_{\mathrm{trp}}}{\int dx\,W} \tag{38}$$

where

$$\Pi_{\rm trp} = \int_{1/(1+\overline{r}/R_0)}^{1/(1-\overline{r}/R_0)} d\overline{\mu}(1-\overline{\mu}/2)^2 \int_0^\infty d\overline{v}\, \overline{v}^7 \widehat{\rm H}^*_{\ell,N}\widehat{\rm H}_{\ell,N}\, \frac{\omega_{0b}^2}{\omega_b}\, \frac{\delta\left(\overline{v}-\overline{v}_{\rm res}\right)}{\left|\frac{\partial\Omega}{\partial\overline{v}_{\rm res}}\right|} \left\{1+\frac{\omega}{\omega^*\overline{v}}\frac{\partial}{\partial\overline{v}}\right\} G(\overline{v},\overline{\mu},\overline{r})h(1-\overline{v}) \tag{39}$$

$$\sum_{\ell,N}\int d\overline{r}\,\Pi_{\rm trp}(\overline{r},\ell,N) \approx \sum_{\ell,N}\int d\overline{r}\,\Pi_{\rm trp}(r_0,\ell,N) = \sum_{\ell'}\Delta\sum_{\ell,N}\Pi_{\rm trp}(r_0,\ell,N).$$

VI. NUMERICAL RESULTS

We determine the eigenfunction $\widehat{\phi}(\eta)$ by numerically solving Eq. (7) subject to the boundary condition $\widehat{\phi}(\eta) \to 0$ as $|\eta| \to \infty$, and we evaluate the Fourier coefficients $\widehat{\rm H}_\ell$, $\widehat{\rm H}_{\ell,N}$ by numerical integration after substitution of $\widehat{\phi}(\eta)$ and the solutions of the particle phase space trajectories in Eqs. (23) and (27). In Fig. 3, we plot the magnitude of the resonant particle coupling $|\widetilde{\rm H}_\ell/(m_i v_0^2)|$ as a function of position $\overline{x}$ for passing particles with zero pitch angle $\kappa^2 = 0$. For small orbit width $q_0 r_L < \varepsilon_0\Delta$, $\widehat{\rm H}_\ell$ has finite peaked maxima at $\overline{x} = \pm\frac{1}{2}$ and $\overline{x} = \pm\frac{3}{2}$. $q_0 r_L$ measures the typical 'banana width' of the passing particles and $\varepsilon_0\Delta$ the 'internal radial width' of $\frac{\partial}{\partial x}\phi(x)$. However (assuming $v_A/v_0 < 1$) the coupling strength is largest in the neighborhood of $\overline{x} = \pm\frac{1}{2}$ due to the resonant energy weighting. As the parameter $q_0 r_L/\varepsilon_0\Delta$ approaches and exceeds unity, the particle 'banana width' becomes larger than the 'internal radial width,' the particle interacts strongly over a smaller fraction of its periodic trajectory, and the resonant particle coupling to the mode progressively weakens. At the same time there is orbit broadening of the region of finite resonant particle coupling. For $1 > q_0 r_L/\Delta > \frac{1}{2}(1 - v_A/v_0)$, $v_A/v_0 < 1$, $\varepsilon_0 \to 0$, the width $\delta\overline{x}_{\rm res}$ of the 'resonance region' about $|\overline{x}| = \frac{1}{2}$ is approximately $\delta\overline{x}_{\rm res} = \left\{(1 + 8q_0 r_L v_A/v_0\Delta)^{1/2} + 1\right\}\Big/4 - v_A/2v_0$.

In our numerical investigation, we ignore the typically damping (negative) contribution to γ of the terms proportional to $\frac{\partial F}{\partial v}$ which are negligible when $|\omega/\omega^*| < 1$. This enables us to focus on the growth (positive) contribution to γ (the term proportional to $\frac{\partial F}{\partial \overline{r}}$) without having to specify a value for the mean beta $\left| r_0 \frac{\partial \widehat{\beta}}{\partial r_0} \right|$ of the energetic particles. We introduce the notation γ^* to denote the contribution proportional to $\frac{\partial F}{\partial \overline{r}}$

$$\frac{\gamma^*}{|\omega|} = \frac{q_0^2}{S_0} \left(-a_0 \frac{\partial \widehat{\beta}}{\partial r_0} \right) \left(\Gamma_p^* + \Gamma_t^* \right) \tag{40}$$

where

$$\Gamma_p^* = \frac{3\pi q_0 r_L R_0}{2 a_0 \Delta} \sum_\sigma \frac{\int d\overline{x}\, \Pi_{\text{pas}}^*}{\int dx\, W} \tag{41}$$

$$\Gamma_t^* = \frac{3\pi q_0 r_L R_0}{2 a_0 \Delta} \frac{\sum_{\ell',N} \Pi_{\text{trp}}^*}{\int dx\, W} \tag{42}$$

and Π_{pas}^* and Π_{trp}^* are determined by Eqs. (35) and (39) excluding the term proportional to $\frac{\partial}{\partial \overline{v}} G(\overline{v}, \overline{\mu}, \overline{r}) h(1-\overline{v})$. We use Eqs. (41) and (42) to numerically calculate the TAE growth rates in terms of the parameters $\frac{v_A}{v_0} q_0$, S_0, $\frac{r_0}{R_0}$, $\frac{\partial \widehat{\beta}}{\partial r_0}$, $\frac{n_0 v_0}{\Omega_i}$, θ_0.

Unless otherwise specified, we consider energetic particles having an isotropic slowing down distribution function. In Fig. 4, we plot Γ_p^* for passing particles as a function of $q_0 r_L/\varepsilon_0 \Delta$ for different values of ε_0. When $q_0 r_L/\varepsilon_0 \Delta < 1$, Γ_p^* increases linearly with $q_0 r_L/\varepsilon_0 \Delta$, reflecting primarily the increase in the diamagnetic drift frequency $|\omega^*|$, and in agreement with the analytic theory. For $\varepsilon_0 \lesssim .1$, the magnitude of Γ_p^* is within 20% of the predicted magnitude (Eq. (36)). The rate of increase of Γ_p^* decreases when $q_0 r_L/\varepsilon_0 \Delta \sim 1$ and the resonant particle coupling begins to weaken. Γ_p^* reaches a maximum when $q_0 r_L/\varepsilon_0 \Delta$ is larger than unity and the orbit width is of the order of the separation between mode rational surfaces $q_0 r_L/\Delta \sim .55$). This maximum is close to the magnitude given by Eq. (37), namely $\frac{5 r_0}{a_0} \xi(1-\xi^2)$, when $\varepsilon_0 \to 0$ and the condition for its validity can be well satisfied. For finite values of ε_0, Γ_p^* can be lower than the above limit by as much as a factor of $4 (\varepsilon_0 = 0.2)$. Thereafter, Γ_p^* decreases as $q_0 r_L/\Delta$ increases above unity.

In Fig. 5, we compare the relative contribution to the growth rates of trapped particles Γ_t^* and passing particles Γ_p^*. Γ_t^* is typically lower than Γ_p^* for isotropic distribution functions.

The structure displayed by the plot of Γ_t^* is due to the appearance of additional trapped particle bounce frequency resonances as $q_0r_L/\varepsilon_0\Delta$ increases. When $q_0r_L/S_0\Delta \ll v_A/v_0$, the toroidal drifts are negligible and resonances occur for integer values of ℓ satisfying $\ell < -\{v_A(1-\varepsilon_0)^{1/2}\}/\{v_0(2\varepsilon_0)^{1/2}\}$. Thus for $v_A/v_0 = 2/3$ and $\varepsilon_0 = .2, \ell = -1, -2, \cdots$ etc. When $q_0r_L/S_0\Delta = v_A(1-\varepsilon_0)/v_0$, the resonance at $\ell = 0$ is now possible, and it is the onset of this resonance which is responsible for the increase in Γ_t^* seen in Fig. 5 at $q_0r_L/\varepsilon_0\Delta = 8/3$.

In Fig. 6, we plot $\Gamma^* = \Gamma_p^* + \Gamma_t^*$ as a function of $q_0r_L/\varepsilon_0\Delta$ with and without $\left(J_0\left(\frac{k_\perp v_\perp}{\Omega_i}\right) = 1\right)$ finite Larmor radius (FLR) effects for several values of ε_0. FLR effects weakens the resonance particle coupling. Γ^* with FLR is smaller than Γ^* without FLR, the difference increasing in magnitude as q_0r_L/Δ increases above unity.

In Fig. 7, we plot Γ^* as a function of $q_0r_L/\varepsilon_0\Delta$ for different values of v_A/v_0. For small orbit widths $q_0r_L/\varepsilon_0\Delta < 1$, Γ^* is significantly lower in magnitude when $v_A/v_0 > 1$ since there are no particles in the neighborhood of the strong 'resonance' at $\bar{x} = \pm.5$. However, the maximum values of Γ^* are comparable when $v_A/v_0 \leq 2$ due to the effects of finite orbit width, but begin to decrease when $v_A/v_0 > 2$. Qualitatively similar results are obtained for 'beam-like' distribution functions as may be seen in Fig. 8.

VII. SUMMARY

Starting with a perturbed field representation satisfying the required periodicities in the toroidal angle ζ and the generalized poloidal angle θ, and treating the presence of energetic particles perturbatively, we derived expressions for TAE growth rates which reduce to the conventional expressions in the ballooning limit.[6] We calculated the TAE growth rates numerically with all damping effects neglected, and we discussed their behavior over a wide range of parameter space.

The instabilities are due to the resonant coupling of energetic particles with TAE modes.

The resonance condition for passing particles with small pitch angles ($\kappa^2 \to 0$) is $|x| = \dfrac{v_A}{2v_0\overline{v}_{\rm res}}$. Assuming $\dfrac{v_A}{v_0} < 1$, the resonance particle coupling is largest in the neighborhood of $|\overline{x}| = \frac{1}{2}$. The effective spatial width $\delta\overline{x}_{\rm res}$ of this resonance is $\delta\overline{x}_{\rm res} \sim \varepsilon_0\Delta$ for small orbit widths $q_0 r_L < \varepsilon_0\Delta$. As $\dfrac{q_0 r_L}{\varepsilon_0\Delta}$ approaches and exceeds unity, the strength of the resonance coupling is reduced. At the same time, the effective resonance width $\delta\overline{x}_{\rm res}$ is broadened by finite orbit effects, and for $1 > q_0 r_L/\Delta > \frac{1}{2}(1 - v_A/v_0)$, $\delta\overline{x}_{\rm res}$ is approximately

$$\delta\overline{x}_{\rm res} \sim \left\{(1 + 8q_0 r_L v_A/v_0\Delta)^{1/2} + 1\right\} \Big/ 4 - v_A/2v_0.$$

For trapped particles with large pitch angles $\kappa^2 \to \infty$, in resonance with the ℓ^{th} harmonic of the bounce frequency, the resonance condition is

$$\ell = -\left\{v_A(1-\varepsilon_0)/v_0\overline{v}_{\rm res} - q_0 r_L \overline{v}_{\rm res}/s_0\Delta\right\} \Big/ \left\{(2\varepsilon_0)^{1/2}(1-\varepsilon_0)^{1/2}\right\}.$$

Many harmonics contribute to the growth rate with the lowest harmonics dominant.

Our numerical results on TAE growth rates may be summarized as follows:

1. The 'growth rate' Γ_p^* for passing particles is in agreement with the analytic theories only in the limit of $\varepsilon_0 \to 0$. For $q_0 r_L/\varepsilon_0\Delta < 1$, Γ_p^* is within 20% of the analytic result when $\varepsilon_0 \lesssim .1$. Γ_p^* reaches a maximum at $q_0 r_L/\Delta \sim 1$, and this maximum is lower than the predicted maximum by $\sim$ 20% when $\varepsilon_0 = .025$ and as much as 4 times lower when $\varepsilon_0 = .2$.

2. The 'growth rates' Γ^* decrease as $q_0 r_L/\Delta$ increases above unity.

3. For isotropic distribution functions, the trapped particle 'growth rate' Γ_t^* is lower than the passing particle 'growth rate' Γ_p^*.

4. Finite Larmor radius effects further suppress the growth rates when $q_0 r_L/\Delta > 1$.

5. Although the 'growth rates' Γ^* in the regime $q_0 r_L/\varepsilon_0\Delta < 1$ are largest when $\dfrac{v_A}{v_0} < 1$, their maximum values occurring at $q_0 r_L/\Delta \sim 1$ are comparable when $v_A/v_0 \lesssim 2$ but begin to decrease when $v_A/v_0 > 2$.

These results provide information on resonant particle coupling and TAE growth rates. With additional information on TAE mode distribution which must be obtained from an analysis of the eigenvalues of global Alfvén eigenmodes, we will be able to assess whether the 'domino effect' is a viable mechanism for producing rapid diffusion on a global scale when TAE instabilities are excited.

Acknowledgments

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Figure Captions

[1] Contours of constant $\overline{v}_{\rm res}$ and $\langle\mu B_0\rangle$; $\overline{v}_{\rm res}$ of resonant passing particles at $\overline{x}$ with 'pitch angle' κ^2 is determined by line of constant $\overline{v}_{\rm res}$ passing through the point $(X_{\rm pas} = \overline{x}, \kappa^2)$.

[2] Contours of constant $\overline{v}_{\rm res}$ and $\langle\mu B_0\rangle$; $\overline{v}_{\rm res}$ of trapped particles with 'pitch angle' κ'^2 and in resonance with bounce harmonic number ℓ is determined by line of constant $\overline{v}_{\rm res}$ passing through the point $(X_{\rm trp} = -\ell, \kappa'^2)$.

[3] Variation of resonant particle coupling $|\widetilde{\rm H}_\ell/m_i v_0^2|$ (arbitrary units) with $\overline{x}$ for several values of $q_0 r_L/\Delta$.

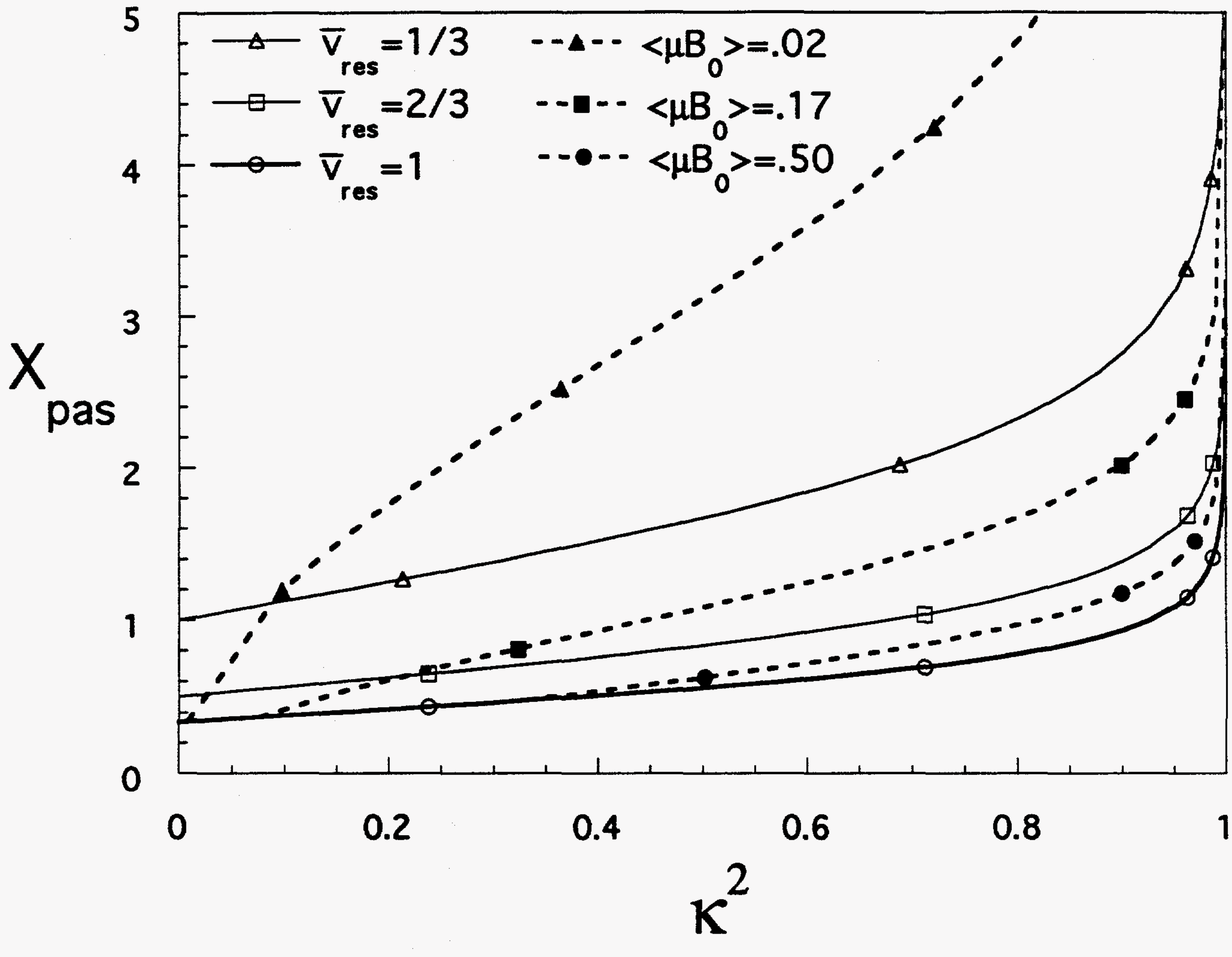
[4] Variation of passing particle 'growth rate' Γ_p^* with $q_0 r_L/\varepsilon_0\Delta$ for several values of ε_0;without FLR; straight solid lines correspond to analytic results.

[5] Relative contributions to 'growth rate' Γ^* of passing particles Γ_p^* and trapped particles Γ_t^*;with FLR — and without FLR $\cdots$; variation with $q_0 r_L/\varepsilon_0\Delta$.

[6] 'Growth rates' Γ^* with FLR — and without FLR $\cdots$; variation with $q_0 r_L/\varepsilon_0\Delta$ for several values of ε_0.

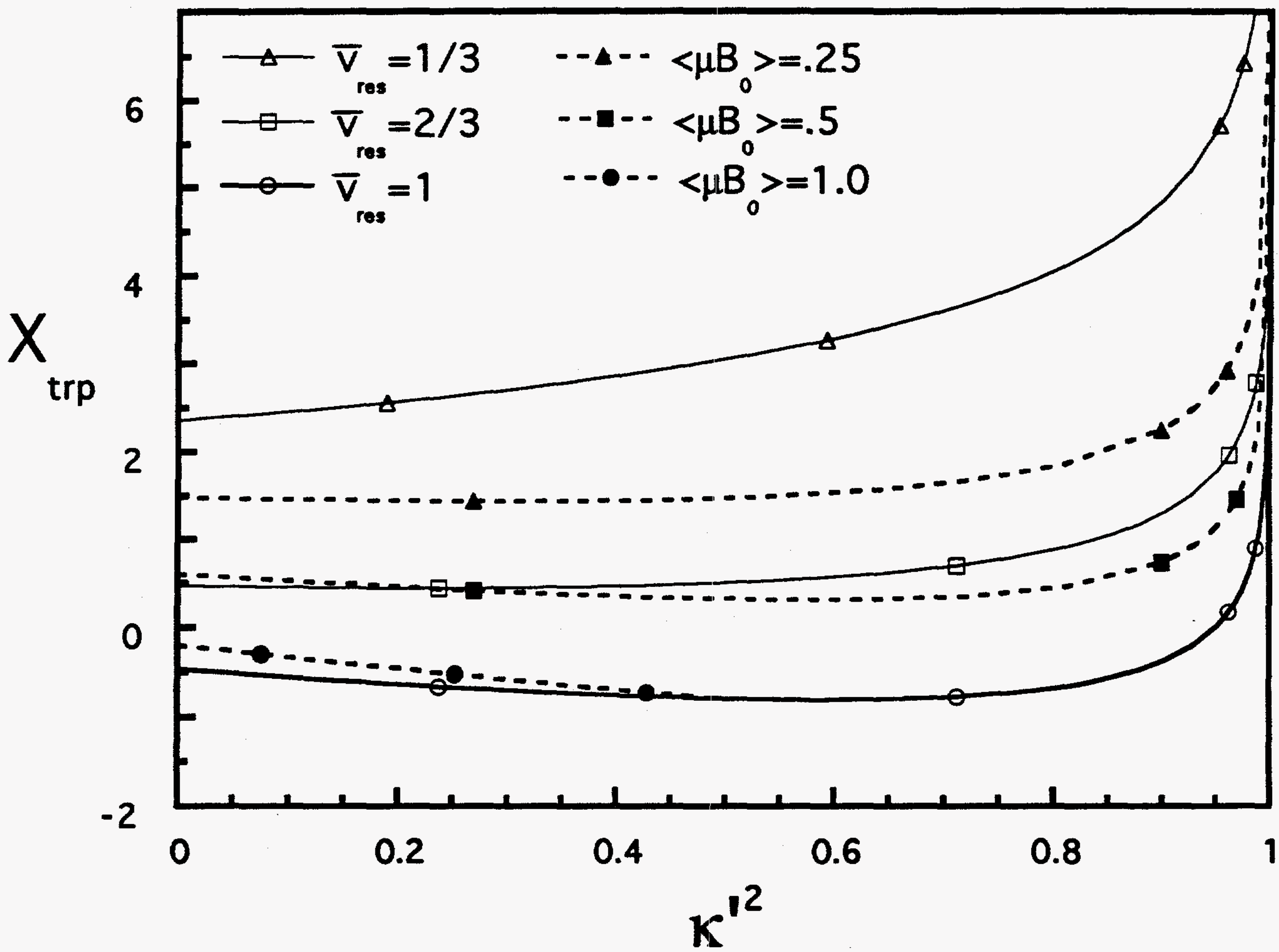
[7] 'Growth rate' Γ^* with FLR; variation with $q_0 r_L/\varepsilon_0\Delta$ for several values of v_A/v_0; isotropic distribution.

[8] 'Growth rate' Γ^* with FLR; variation with $q_0 r_L/\varepsilon_0\Delta$ for several values of v_A/v_0; 'beam-like' distribution: $\delta\lambda_0 = .1, A = 2, \overline{v}_I = .61$.



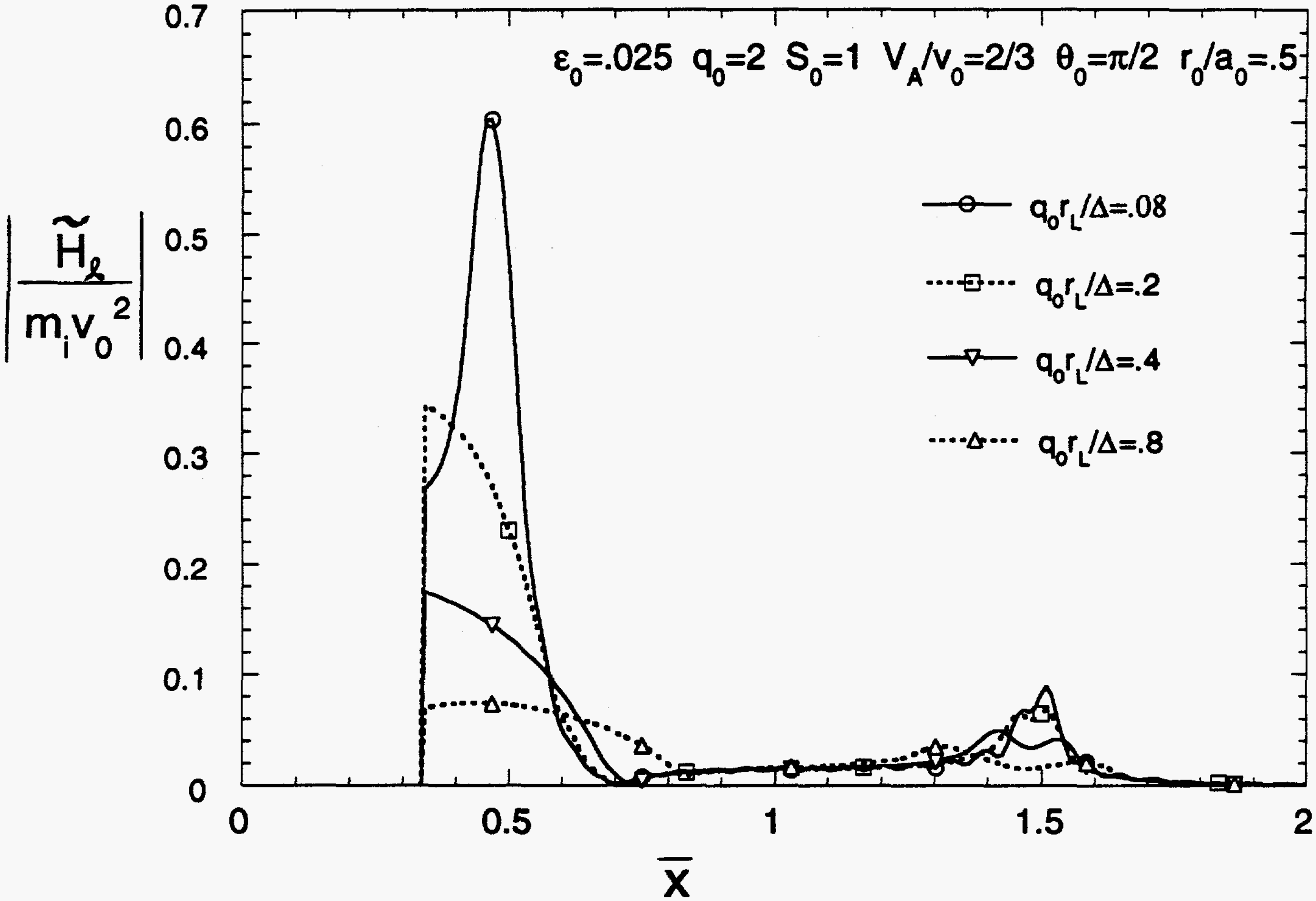
5
4
3
2
1
0
0
0.2
0.4
0.6
0.8
1
X_{pas}
κ^2
$\bar{v}_{res}$=1/3
$\bar{v}_{res}$=2/3
$\bar{v}_{res}$=1
$<\mu B_0>$=.02
$<\mu B_0>$=.17
$<\mu B_0>$=.50

Fig. 1



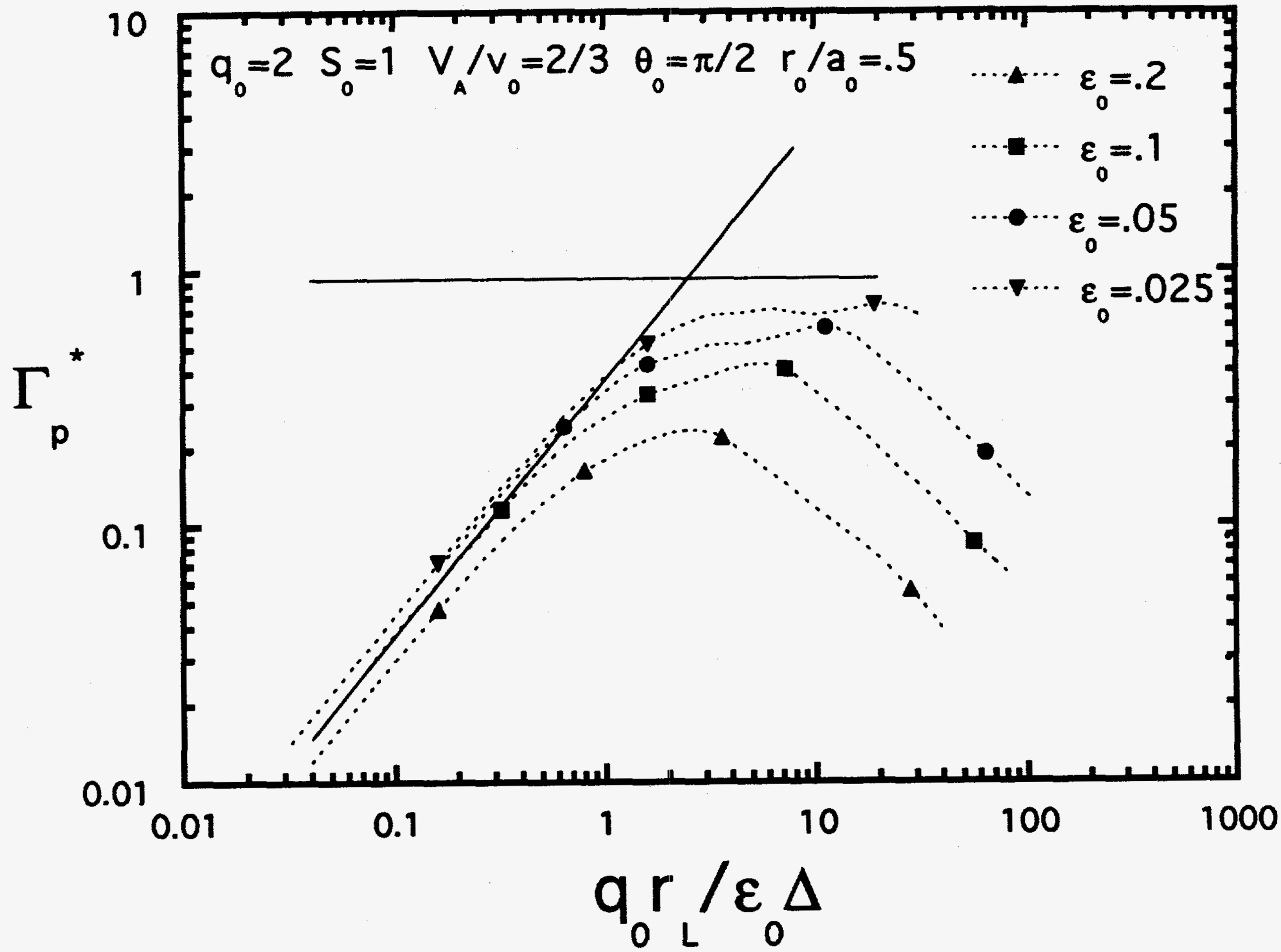
$\bar{V}_{res}$=1/3
$\bar{V}_{res}$=2/3
$\bar{V}_{res}$=1
<μB$_o$>=.25
<μB$_o$>=.5
<μB$_o$>=1.0
X_{trp}
κ'^2
6
4
2
0
-2
0
0.2
0.4
0.6
0.8
1

Fig. 2



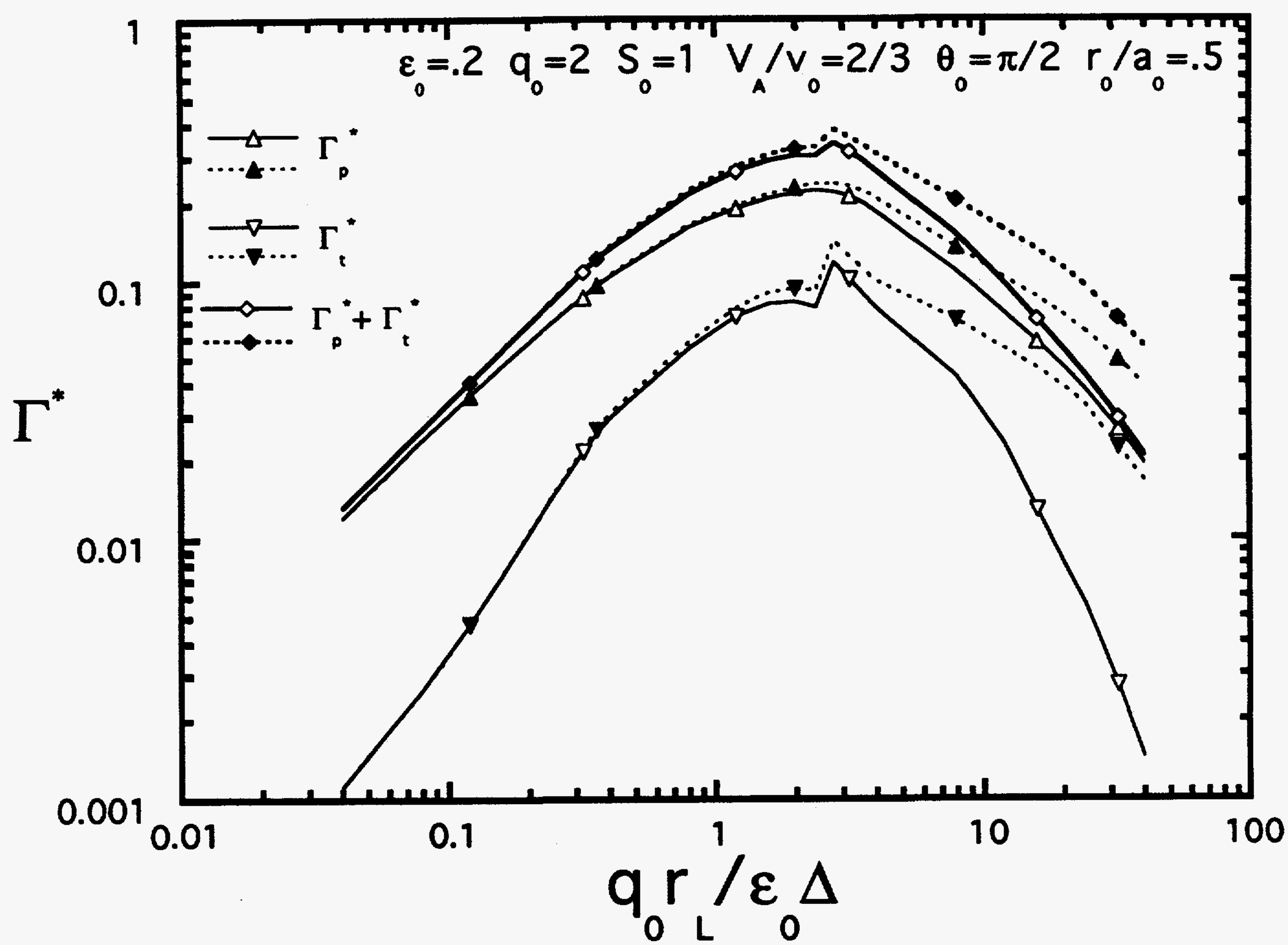
ε_0=.025 q_0=2 S_0=1 V_A/v_0=2/3 θ_0=π/2 r_0/a_0=.5
$q_0 r_L/\Delta$=.08
$q_0 r_L/\Delta$=.2
$q_0 r_L/\Delta$=.4
$q_0 r_L/\Delta$=.8
$\left|\frac{\tilde{H}_\ell}{m_i v_0^2}\right|$
$\bar{x}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
0.5
1
1.5
2

Fig. 3



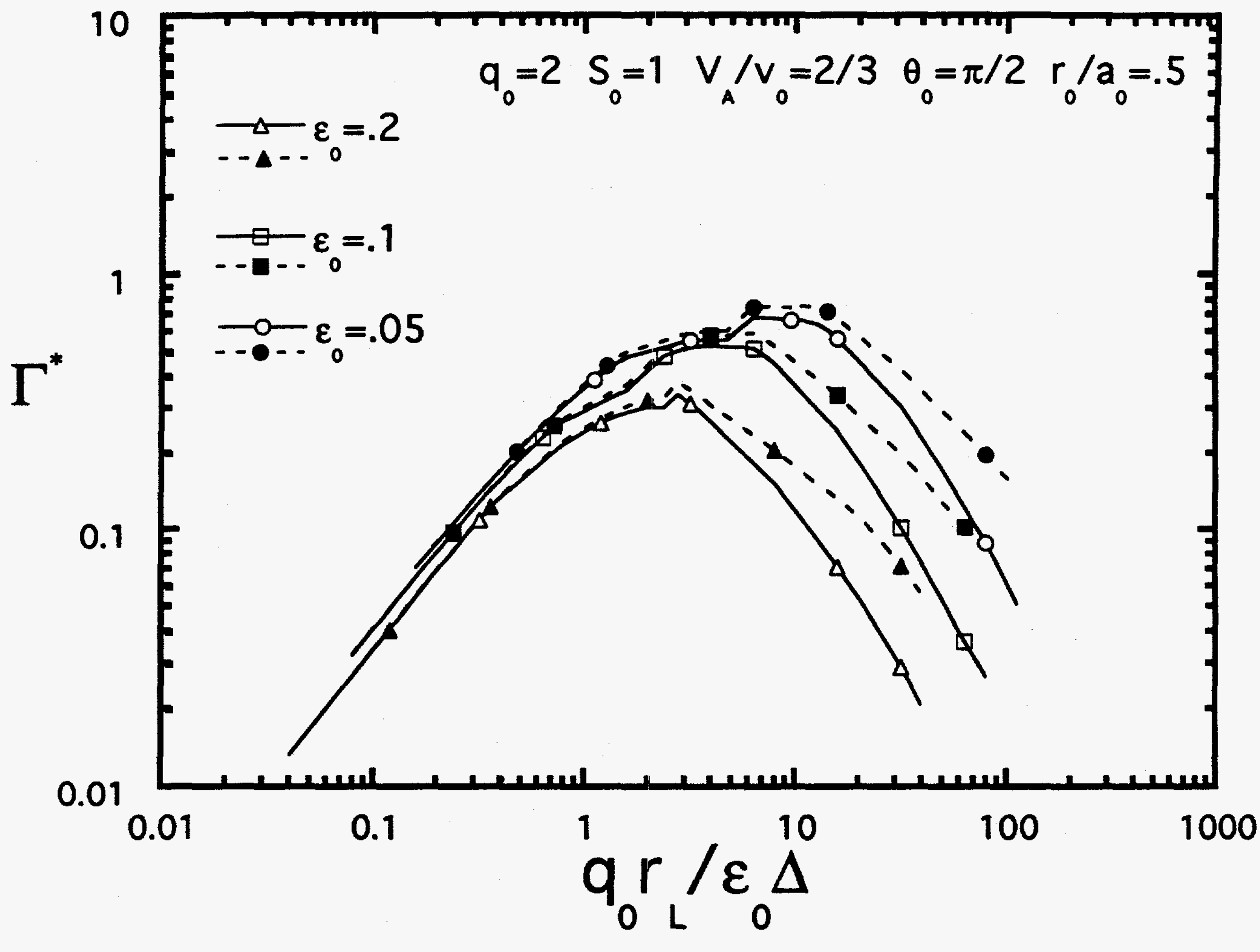
q0=2 S0=1 VA/v0=2/3 θ0=π/2 r0/a0=.5
ε0=.2
ε0=.1
ε0=.05
ε0=.025
10
1
0.1
0.01
Γp*
0.01
0.1
1
10
100
1000
q0 rL/ε0 Δ

Fig. 4



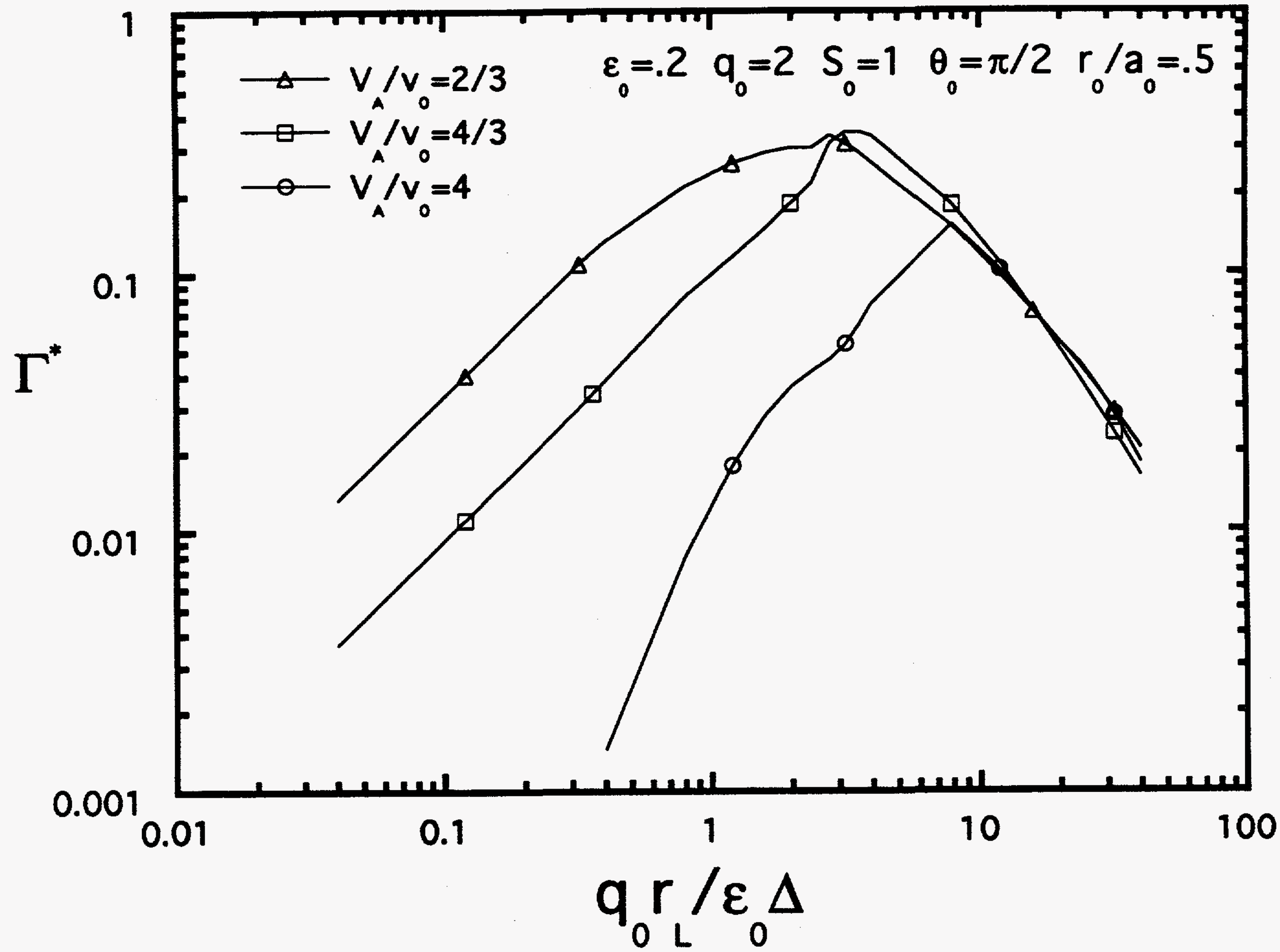
ε0=.2 q0=2 S0=1 VA/v0=2/3 θ0=π/2 r0/a0=.5
Γp*
Γt*
Γp*+Γt*
Γ*
1
0.1
0.01
0.001
0.01
0.1
1
10
100
q0rL/ε0Δ

Fig. 5



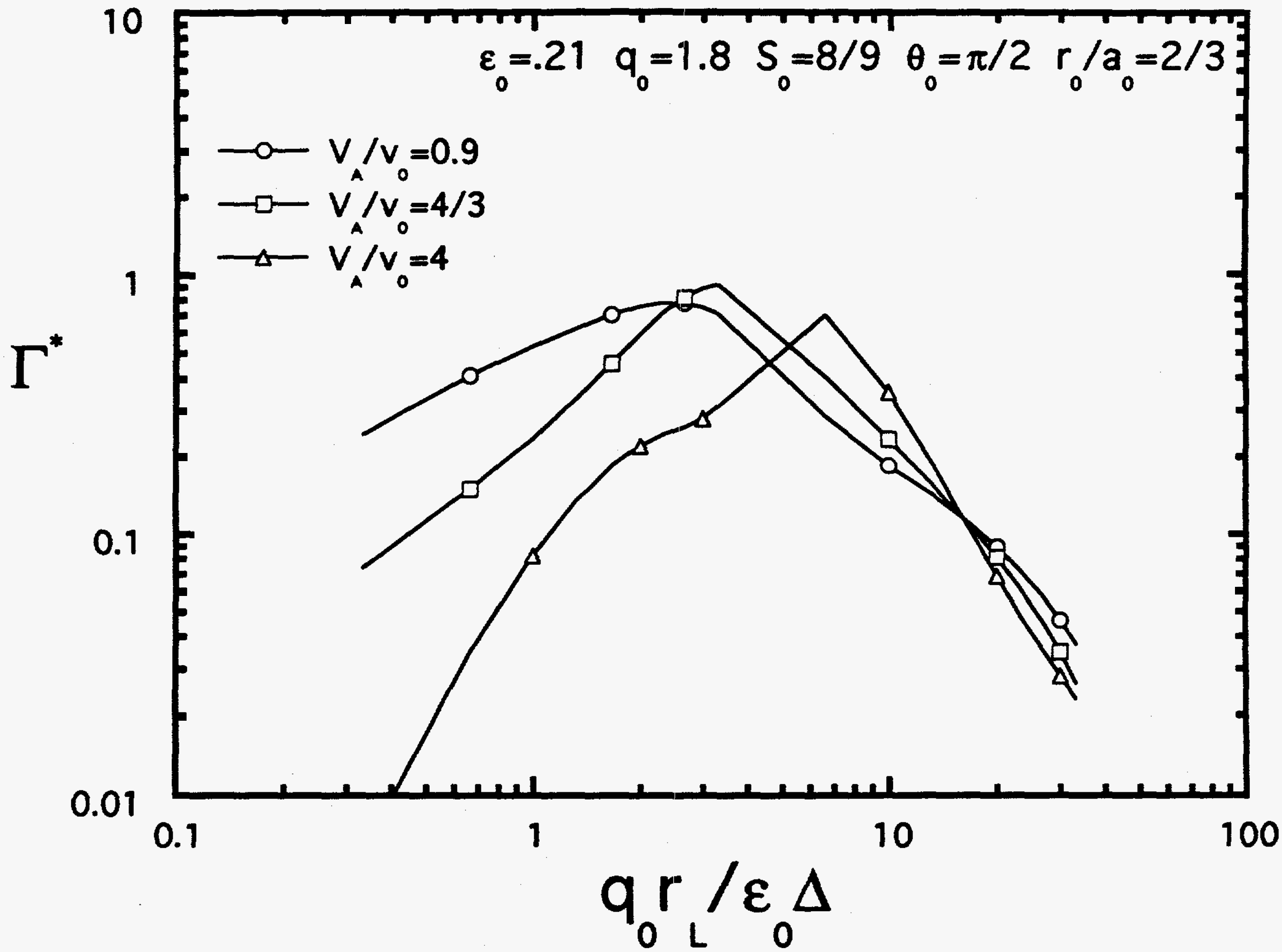
$q_0=2$ $S_0=1$ $V_A/v_0=2/3$ $\theta_0=\pi/2$ $r_0/a_0=.5$
$\varepsilon_0=.2$
$\varepsilon_0=.1$
$\varepsilon_0=.05$
Γ^*
10
1
0.1
0.01
0.01
0.1
1
10
100
1000
$q_0 r_L/\varepsilon_0 \Delta$

Fig. 6



1
0.1
0.01
0.001
0.01
0.1
1
10
100
Γ*
$q_0 r_L/\varepsilon_0 \Delta$
V_A/v_0=2/3
V_A/v_0=4/3
V_A/v_0=4
ε_0=.2 q_0=2 S_0=1 θ_0=π/2 r_0/a_0=.5

Fig. 7



10
1
0.1
0.01
0.1
1
10
100
ε_0=.21 q_0=1.8 S_0=8/9 $\theta_0=\pi/2$ r_0/a_0=2/3
V_A/v_0=0.9
V_A/v_0=4/3
V_A/v_0=4
Γ^*
$q_0 r_L/\varepsilon_0\Delta$

Fig. 8