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Principal Investigator: Eugene R. Marshalek
Department of Physics
University of Notre Dame
Notre Dame, IN 46556

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2 SCIENTIFIC PROGRAM

2.1 Project Description

The general purview of the project is the theory of collective motion in atomic nuclei. The chief aim is to elucidate the phenomena of (1) anharmonic multiphonon excitations, and (2) collective tilted rotation, both of which are topics of considerable current interest. In the primary stage of an investigation it is often necessary to develop appropriate mathematical tools, as was the case here. In the next stage, the formalism must be tested on simple soluble models. The work described here is mainly concerned with these two stages. The final stage of realistic applications will require more time, manpower and, of course, the necessary funding. Some planning for this last stage has been carried out and anticipated problems are briefly discussed.

As it turns out, both of the above topics can be approached within the unified framework of a theorem that I developed, called the Cranking Bifurcation Theorem (CBT) to be described below. The CBT can be regarded as an outgrowth of the boson expansion method [1], which provides a general, and, in principal, exact formalism for treating collective excitations. We begin with a brief discussion of the CBT and then continue on to the applications.

2.2 The Cranking Bifurcation Theorem

The cranking method is a technique for describing a uniformly rotating system by following it in a corotating frame, wherein the physical effects arise from noninertial forces. Traditionally, this method has been applied to nuclei having a broken rotational symmetry in the ground-state mean field. The cranking axis is chosen such that the system is deformed in a plane perpendicular to this axis. The technique then generates the usual low-lying rotational bands found in permanently deformed nuclei. On the other hand, the common wisdom is that cranking a mean field about a symmetry axis just carries it into itself, leading to nothing new, except possibly a redefinition of the fermi surface (equivalent to introducing a "tilted fermi surface"). However, I discovered that this is not necessarily so — at certain critical cranking frequencies new solutions that break the symmetry can bifurcate from the symmetric one. In most cases such a bifurcation corresponds to a band of multphonon vibrational states rather than an ordinary rotational band. Therefore, the implementation of the cranking technique proposed here represents a novel application to nuclear mean fields. These considerations have led me to formulate what I call the Cranking Bifurcation Theorem (CBT), which may be stated as follows:

- **If a system of coupled anharmonic oscillators has an axially symmetric equilibrium configuration (vacuum), then self-consistent cranking about this axis yields families of symmetry-breaking solutions that bifurcate from the vacuum solution at the critical rotational frequencies \( \Omega_c = \omega_\mu / K_\mu \), where \( \omega_\mu \) is the small-oscillation (RPA) frequency (in the laboratory frame) of any mode carrying \( K_\mu \neq 0 \) units of angular momentum along the symmetry axis\(^1\). Each family describes the dynamics of the system on submanifolds of phase space characterized by a single nonvanishing action \( N_\mu = (I - I_0) / K_\mu \), where \( I \) is the total angular momentum carried by the mode and \( I_0 \) that of the vacuum.**

A general proof can be given that applies to any conservative Hamiltonian system of coupled anharmonic oscillators having a conserved component of angular momentum and a corresponding axially symmetric static equilibrium configuration [2]. This theorem has a broad range of applications, encompassing time-dependent mean-field approximations such as the time-dependent Hartree-Fock (TDHF) and Hartree-Fock-Bogoliubov (TDHFB) approximations, phenomenological mean-field models such as the Nilsson-Strutinsky model, empirical collective models (Bohr-Mottelson, IBM, etc.) and liquid-drop models. The main limitation of the method is that it can only be applied to nuclei having equilibrium mean-field solutions with an axis of rotational symmetry (spherical or axially symmetric deformed shapes) — although these can correspond to excited as well as ground-state configurations — and the vibrations built on these configurations must project an angular momentum \( K \neq 0 \) on the symmetry axis. This means that the method cannot describe monopole and \( \beta \) quadrupole vibrations, for which \( K = 0 \), but it can describe, for example, \( \gamma \) quadrupole vibrations.

\(^1\)This condition is equivalent to the statement that the corresponding RPA frequencies in the rotating frame must vanish at the critical point.
\(|K| = 2\), octupole \(|K| = 1, 2, 3\) and giant dipole \(|K| = 1\) vibrations in even-even nuclei. It can also describe collective vibrations in odd-\(A\) nuclei as will be discussed in Sec. (2.4.3) below.

The physical picture of such modes as a surface wave — an instantaneous distortion rotating about the symmetry axis — suggests the possibility of a stationary description in a frame rotating with the wave. Such considerations naturally lead to the cranking model, but with cranking about the (equilibrium) symmetry axis. The physical picture is presented in Fig. (1), which assumes that the broken symmetry is ellipsoidal as for a quadrupole mode, but, of course, higher multipoles can also be treated.

\[
\begin{align*}
&\Omega, \\
&(a) \quad i t_0, \\
&(b) \quad J_0, \\
&(c) \quad i t_0, \\
&(d) \quad J_0
\end{align*}
\]

Figure 1: The rotational motion of the bifurcating solution (solid ellipse) with angular speed \(\Omega\) compared with that of the vacuum state (dashed circle), which in general may carry angular momentum \(I_0\). The two rotations may be in the same sense \((a,c)\) or opposite senses \((b,d)\). The solutions \((c,d)\) are the time reverses of \((a,b)\).

A formal general proof of the CBT is given in Ref. [2], while briefer heuristic arguments together with simple applications were given in Refs. [3-5]. The derivation and significance of the theorem can easily be illustrated in the case of primary interest, namely, nuclear mean fields. This will clearly demonstrate that the cranking technique can be used to derive the RPA, and suggests the possibility for including higher-order anharmonic corrections. For simplicity, a separable multipole-multipole interaction will be assumed in the Hartree approximation, but the treatment can be extended in a straightforward fashion to arbitrary interactions in the Hartree-Fock (HF) or Hartree-Fock-Bogoliubov (HFB) approximations. The many-body cranked Hartree mean-field Hamiltonian \(H_{MF}\) is then given by (let \(\hbar = 1\) from now on)

\[
H_{MF} = H_0 - \Omega J_3 - \chi \sum_{K=-L}^{L} \delta Q_K^* \hat{Q}_K, \tag{1}
\]

where \(H_0\) is the uncranked axially symmetric mean-field Hamiltonian, the operator \(\hat{Q}_K\) is a component of an \(L\)-pole tensor, \(\delta Q_K^*\) is the change in the corresponding deformation parameter, and \(\chi\) is the strength of the interaction. Since \(H_0\) and the angular-momentum component \(J_3\) are assumed to commute, \(H_0 - \Omega J_3\) may be chosen as the zeroth-order Hamiltonian and the rest of the rhs of Eq. (1) treated as a perturbation. With \(|0\rangle\) denoting the unperturbed axially symmetric configuration (not necessarily the ground state) and \(|n\rangle\) the orthogonal excited (or deexcited) configurations, we have that \(H_0|0\rangle = E_0|0\rangle, H_0|n\rangle = E_n|n\rangle\). In addition, the symmetry implies that the basis may be chosen to satisfy

\[
J_3|0\rangle = I_0|0\rangle, \quad J_3|n\rangle = K_n|n\rangle, \tag{2}
\]

where it is assumed that the vacuum \(|0\rangle\) carries angular momentum \(I_0\). The perturbed eigenvector of \(H_{MF}\) to first order in the \(\delta Q_K^*\) is given by

\[
|\Psi_0\rangle = |0\rangle + \chi \sum_{K=-L}^{L} \delta Q_K^* \sum_{n \neq 0} \frac{|n\rangle \langle \hat{Q}_K|0\rangle}{E_n - E_0 - \Omega K|n\rangle}. \tag{3}
\]
The first-order changes $\delta Q_K$ in the deformation parameters are determined by the self-consistency conditions

$$\delta Q_K = (\Psi_0 | \hat{Q}_K | \Psi_0) - \delta_{K,0} Q_0,$$

where $Q_0 \equiv (0 | \hat{Q}_0 | 0)^2$, as follows:

$$\delta Q_K \left[ \chi \sum_{n \neq 0} \frac{|(n|\hat{Q}_K|0)^2|}{E_n - E_0 - \Omega K} + \chi \sum_{n \neq 0} \frac{|(n|\hat{Q}_K|0)^2|}{E_n - E_0 + \Omega K} - 1 \right] = 0. \tag{4}$$

It is immediately seen that there are two kinds of solutions of Eq. (4). First, there is the trivial solution with all the $\delta Q_K = 0$, for which $|\Psi_0) = |0)$. This is just the case of unproductive cranking, in which the axially symmetric field is carried into itself. However, there are also nontrivial solutions with one pair of deformation parameters $\delta Q_K = (-1)^K \delta Q_{-K} \neq 0$, for which the corresponding term in brackets vanishes, thereby giving rise to the eigenfrequency formula

$$\sum_{n \neq 0} \frac{|(n|\hat{Q}_K|0)^2|}{E_n - E_0 - \omega} + \sum_{n \neq 0} \frac{|(n|\hat{Q}_{-K}|0)^2|}{E_n - E_0 + \omega} = \frac{1}{\chi}, \tag{5}$$

provided that the cranking frequency is given by $\Omega = \Omega_c = \omega/K$ and $K \neq 0$, in agreement with the CBT. A second equation is obtained by letting $\Omega \to -\Omega$ and thus $\omega \to -\omega$. This pair of equations corresponds to what was dubbed the RPA for "nonhermitian fields" [7]. Equations of this type were first derived for odd-$A$ nuclei by Bès and Chung [8], which shows that the present cranking approach applies to odd nuclei as well as even-even ones [see Sec. (2.4.3) below]. If $|0)$ is an excited configuration, one must consider frequencies of both signs, with negative values of $\omega$ corresponding to deexcitations. Moreover, as long as $I_0 \neq 0$, there is no symmetry between modes with $\Delta I = +K$ and $\Delta I = -K$. If, however, $I_0 = 0$, the pair of equations collapses to the single well-known RPA dispersion formula

$$\sum_{n \neq 0} \frac{|(n|\hat{Q}_K + (-1)^K \hat{Q}_{-K}|0)^2|}{(E_n - E_0)^2 - \omega^2} = \frac{1}{2 \epsilon_0 K}, \tag{6}$$

for which the modes with $\pm K$ are degenerate. To determine the lowest-order deformation change $\delta Q_K$ for the mode in question, one must prescribe the spin using the condition

$$(\Psi_0 | J_3 | \Psi_0) = I. \tag{7}$$

Actually, this fixes $|\delta Q_K|$ as follows:

$$|\delta Q_K|^2 = \frac{|I - I_0|}{K \omega} \frac{1}{B_K(\omega)}, \tag{8}$$

where $B_K(\omega)$ is an RPA mass parameter, which will not be given here explicitly. What this result demonstrates is that the deformation associated with the symmetry breaking is proportional to the vibrational amplitude, which, in turn, is fixed by $|I - I_0|$. It is straightforward to show that to the given order of approximation the energy (laboratory frame) is given by $E = E_0 + \omega \frac{I - I_0}{K}$, for any chosen RPA frequency $\omega$ with a corresponding value of $K$. Since the cranking model is semiclassical, $I$ should be chosen to have quantized values, so that $I - I_0$ corresponds to the number of excitation phonons relative to the axially symmetric reference state.

The perturbative approach can be carried to higher orders for continuous values of $\Omega \neq \Omega_c$, by using an expansion in powers of $|\Omega - \Omega_c|^{1/2}$, where the square root may be regarded as a critical exponent. The energy (and other physical quantities) can then be converted to Taylor expansions in $I - I_0$, as follows:

$$E = E_0(I_0) + \omega \left( \frac{I - I_0}{K} \right) + \frac{1}{2} \epsilon_4 \left( \frac{I - I_0}{K} \right)^2 + \ldots \tag{9}$$

The quantity $\epsilon_4$ is the leading-order anharmonicity parameter, which can be derived microscopically. However, it is important to emphasize that the primary aim of this project is not to use perturbation theory but to

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2The condition of axial symmetry does not require that $Q_0$ vanish, although it may do so for a spherical nucleus.

3This situation differs from normal cranking, which requires an expansion in powers of $\Omega$. 

diagonalize the cranked mean field directly, thereby summing the anharmonicities to all orders. Indeed, perturbation theory can be quite tricky because of the possible occurrence of small resonance denominators. The cranking technique then provides a simple and elegant method for finding exact solutions of time-dependent mean-field equations.

The method just outlined applies not only to vibrational bands built on the ground states of even-even spherical and axially symmetric deformed nuclei \((I_0 = 0)\), but also to vibrations built on high-\(K\) isomers \((I_0 = K)\), including rotational band-termination states, and also to vibrations of odd-\(A\) nuclei, where \(I_0\) corresponds to the spin of the odd particle as discussed in Sec. (2.4.3). A second point to be emphasized is that since the \(\delta Q_K\) do not vanish for a nontrivial solution, not only is the axial symmetry broken, but the cranking axis may not coincide with a principal axis of the density distribution. Therefore, some of the solutions can describe tilted rotation, which involves broken signature symmetry as well. This is particularly obvious for the case of quadrupole deformation \((L = 2)\), where the \(|K| = 1\) mode corresponds to a tilted ellipsoid since it implies the existence of \(Y_{21}\) and \(Y_{2-1}\) components in the mean field. We return to this topic in Sec. (2.5).

2.2.1 Model applications of the CBT

The CBT has recently been applied to a number of simple models including the cranked volume-conserving harmonic oscillator (VCHO) model, the interacting boson model (IBM), the Bohr-Mottelson collective model and the Elliott SU(3) model. Not all of this work has yet been written up. However, a discussion of the quadrupole solutions of the VCHO with \(|K| = 2\) is given in Ref. [3–5,9], an application to the U(5) limit of the IBM is given in Ref. [10] and one to the SU(3) limit of the IBM in Ref. [3]. The VCHO model describes the motion of nucleons in a pure deformed harmonic oscillator whose equipotential volumes are independent of deformation (incompressibility). The cranked version of this model as applied to normal rotational bands has been discussed by numerous authors, but perhaps most exhaustively by Troudet and Arvieu [11]. It has been shown that the VCHO can be interpreted as the Hartree approximation applied to a special internucleon interaction that includes separable many-body components in addition to the quadrupole-quadrupole interaction [12–14]. With the aid of the CBT, new solutions of the cranked VCHO have recently been found [3, 4, 9] for nuclei with deformed axially symmetric ground states. These solutions can be interpreted as bands constituted of multiphonon giant quadrupole resonance rotational bandheads with spins \(I = 2, 4, \ldots, 2n\), where \(n\) is the number of \(|K| = 2\) aligned phonons. The anharmonicity of this giant resonance, which is calculated exactly (i.e., not perturbatively) within the model, is predicted to be extremely small for states with a few phonons. This result is consistent with other theoretical and experimental estimates of the anharmonicities of giant resonances.

An extension of the cranked VCHO model describes multiphonon vibrations built on an axially symmetric excited configuration with \(I_0 \neq 0\) as presented in Figs. (2) and (3), in which the angular momentum is labeled by \((J_z) = I\) (symmetry axis labelled as the \(x\)-axis). The vacuum configuration is the well-known ground-band termination state of \(^{20}\)Ne with spin \(I_0 = 8\). The RPA excitations built on this state were studied in detail some time ago within the framework of the VCHO by Kurasawa [15], who found that there are 3 modes with \(\Delta I = K = \pm 2\). As shown in Fig. (2), which is a spin vs. rotational frequency plot, cranking of the \(I_0 = 8\) configuration about the symmetry axis gives rise to 3 bifurcations (solid lines) at the cranking frequencies \(\omega/2\), where the values of \(\omega\) agree exactly in magnitude with the RPA frequencies. Also shown are the degenerate time-reversed solutions (dashed lines) obtained by cranking the time-reversed configuration with \(I_0 = -8\), with all rotational frequencies reversed in sign. The segment marked \(a\) represents a multiphonon anharmonic excitation branch with \(\Delta I = K = 2\), which corresponds to the mode type marked \((a)\) in Fig. (1). The segment marked \(b\) represents a multiphonon anharmonic excitation branch with \(\Delta I = K = -2\), which corresponds to the mode type marked \((b)\) in Fig. (1). The segment designated by \(g.b\.) is formally a \(\Delta I = K = -2\) deexcitation (negative-frequency) branch, but this mode, which is a “vibration” seen from the perspective of the \(I_0 = 8\) state, is really the ground rotational band, as already noted by Kurasawa in his RPA analysis. The time-reversed solutions are physically equivalent and give rise to the same energy vs. angular momentum plot Fig. (3), as long as one uses \(I = |(J_z)|\), which is the observed spin. The forking structure is quite interesting since it may be a general signature for band termination. It should be noted that Troudet and Arvieu [11] had found the trajectory \(a\) (but not \(b\)), but had no way of realizing that it could be interpreted as a vibrational band built on the band termination state. There also exist \(\Delta I = K = \pm 1\)
2.3 Anharmonic Multiphonon Excitations

Recently, there has been a revival of interest in multiphonon excitations, especially from the experimental side. For example, Aprahamian believes she has found evidence for at least 17 cases of 2-phonon $K = 4 + \gamma$ vibrations in rare-earth nuclei, many of which would have to be highly anharmonic, as judged by the ratio of energy spacings [16]. One of the best candidates yet for such a two-phonon $\gamma$ vibration was reported recently in the isotope $^{108}$Mo by Guessous et. al. [17]. In addition, there is growing experimental evidence for two-phonon octupole levels, and two-phonon giant dipole and quadrupole resonances in various spherical nuclei [18]. Another important recent development has been the discovery of remarkable regularities in the data on anharmonic multiphonon quadrupole vibrations in spherical nuclei by Casten and coworkers [19] as discussed in Sec. (2.4.2).

A large number of alternative microscopic methods are now available for the study of anharmonic multiphonon excitations, including boson expansions, nuclear field theory, adiabatic TDHF, generator coordinates. While the generator coordinate method is a powerful and fully quantal approach, it requires a formidable computational effort usually carried out by a team of collaborators [20]. Just as in the case of large-scale shell-model calculations, the final results may not always lead to physical insights. The cranking approach proposed here, which is really a technique for finding exact periodic solutions of time-dependent mean-field
equations (but without the adiabatic assumption) has some unique advantages, the main one being relative simplicity. The chief general disadvantage is that the cranking method is semiclassical, although possibilities exist for incorporation of additional quantal effects, as discussed below. On the other hand, to the extent that it is classical, the method is well suited for providing intuitive insights. For example, the crossings of the quasiparticle Routhian levels, which underlie any cranking approach, may provide insights into the peculiarities of collective vibrational bands. We now outline possible applications for which the CBT approach is especially suitable.

2.3.1 How repeatable is a phonon?

If an excitation mode has a pure particle-hole or two-quasiparticle nature, a double excitation is ruled out by the Pauli principle. In order for a “phonon” to be repeatable, it is obviously necessary that it be a collective mode. A fundamental question that this line of inquiry might be able to answer is: how many times can a given phonon excitation be repeated? A clear answer may possibly emerge from an exact, i.e., nonperturbative solution of the nonlinear cranking equations. Since the equations are nonlinear, it is possible, for example, for a solution to suddenly disappear for some value of the angular velocity or spin, thereby signaling a cutoff on repeatability. In fact, this could just be another way to view a bifurcation. An example of this is the segment marked “g.b.”, the terminal ground-state rotational band of Fig. (2). If its origin is taken as the \( I = 0 \) ground state (which could be interpreted as a special case of the CBT associated with the rotational Goldstone mode) then the terminus is the axially symmetric state with \( I = 8 \).

On the other hand, if the latter state is taken as the origin, then one has a CBT bifurcation at the cranking frequency \( \Omega = \omega/2 \), where \( \omega \) is one of the RPA frequencies found in Ref. [15], and this deexcitation band abruptly terminates at the ground state\(^4\).

2.3.2 \( \gamma \) and hexadecapole vibrations of deformed nuclei

Two-phonon excitations have been experimentally rather elusive, especially in deformed nuclei. Nevertheless, Prof. Ani Brahman of Notre Dame, using her recent data and those of others, has proposed 17 candidates for two-phonon \( \gamma \)-vibrational bandheads having spin \( I = K = 4 + \) in rare-earth nuclei [16]. Since each \( \gamma \)-vibrational phonon carries the spin projection of \( K = 2 + \), the resulting spin of \( K = 4 + \) would correspond to a parallel alignment of the two-phonons. Theoretically, there should also exist a not necessarily degenerate two-phonon \( \gamma \) vibration with \( I = K = 0 + \), corresponding to the antiparallel alignment, but very few convincing candidates have been found thus far. Since the energy ratios \( E_{4+}/E_{2+} \) of the proposed two-phonon to one-phonon states varies from 2.89 down to 1.29, significant deviations from the ideal harmonic limit of 2, the anharmonicity would appear to be quite large. However, the \( K = 4 + \) double \( \gamma \) vibration in \(^{106}\)Mo reported by Guessou et al. [17] appears to be very close to harmonic. The double-\( \gamma \) interpretation of the \( K = 4 + \) bands has been questioned by Akiyama [21] and by Burke [22], who raise the possibility that some or all of these states may really be examples of a one-phonon hexadecapole vibration, which is predicted by the \( sdg \)-boson version of the IBM [23], and has long been inherent in the Copenhagen-Frankfurt collective model [24]. While some microscopic calculations for \( \gamma \) and hexadecapole excitations have been carried out in Dubna [25] that lend support to the alternative interpretation, there is reason to not regard these calculations as definitive.

Calculations based on the CBT should naturally give rise to the sequence of aligned \( \gamma \)-vibrational bandheads with \( K = 2 +, 4 +, \ldots \), which lies on a trajectory that bifurcates at the cranking frequency \( \Omega = \omega_{2+}/2 \), where \( \omega_{2+} \) is the RPA frequency of the most collective \( \gamma \)-vibration (usually the lowest root). In the RPA limit, the eigenfrequency equations split into independent sectors for each possible value of \( |K| \) and parity. If the mean-field contains both quadrupole and hexadecapole deformations, then the \( |K| = 0 + \) and \( 2 + \) sectors each describe vibrations that are superpositions of quadrupole and hexadecapole modes. For \( |K| = 2 + \), one expects that one of the RPA roots has more of a quadrupole character than the others, and it is this root which is designated as the \( \gamma \) vibration with frequency \( \omega_{2+} \). If the mean field has no higher multipoles than the hexadecapole, then the \( |K| = 4 + \) branch should describe pure hexadecapole excitations. If the most collective root (usually the lowest) is designated by \( \omega_{4+} \), then the CBT predicts a solution branch bifurcating at the cranking frequency \( \Omega = \omega_{4+}/4 \), which covers the spins \( I = 4 +, 8 +, \ldots \) that corresponds\(^4\)

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\(^4\)One can also regard the terminus to be the \( I_0 = -8 \) state if solutions with \( \langle J_2 \rangle < 0 \) are admitted.
to the sequence of 1, 2, ..., phonon bandheads generated by the $K = 4^+$ hexadecapole phonon. In principle then, one should be able to distinguish the branch that contains the $(j_2) = I = 4^+$ double $\gamma$ vibration as defined here, and the one-phonon hexadecapole vibration with the same spin and parity. From the discussion in the previous subsection, it is possible that in some cases the $\gamma$-vibrational branch might terminate at a spin of less than 4 units, in which case the double-$\gamma$ excitation in question would not exist at all and the hexadecapole interpretation would win by default. Such a termination might occur if the strength of the double $\gamma$ excitation is washed out through coupling to many quasiparticle modes of comparable energy. Instead of speculating, of course, one needs to do the calculation and see what actually happens.

2.3.3 Prospects for developing realistic computer codes

Although most of the applications of the CBT thus far have concentrated on simple models in order to gain insights into the nature of the solutions, realistic computer codes can be developed in principle, although in practice certain complications can occur. Such codes can be either fully self-consistent cranked HFB codes using "realistic" internucleon forces or they can be based on phenomenological deformed potential models, such as the Nilsson or Saxon-Woods potentials, with the inclusion of pairing correlations. The latter approach would be the most feasible first step with limited manpower. While, in principle, standard codes designed for normal cranking can be trivially modified for cranking about a symmetry axis, this would only be the first step. In order to describe the hexadecapole excitations correctly, the phenomenological potentials must be augmented to include $Y_{4,\pm 1}$ and $Y_{4,\pm 3}$ deformations (normally only $Y_{4,0}$ is included). (These components are also essential for describing the hotly debated $C_{4b}$ staggering effects in some superdeformed bands.) Also, an RPA code is required to locate the bifurcation points. In the case of phenomenological deformed potential models, I have been able to show [2] that the RPA limit of the cranking equations exactly corresponds to the result of the vibrating potential model (VPM) [26, 27]. The results for the VCHO model discussed above is an example of this. Once a bifurcation associated with a particular RPA root (usually the lowest) is located, the corresponding solution must be followed as a function of the rotational frequency or the spin. Since for a given spin there may be a large number of self-consistent solutions, say, one for each RPA root, there is the problem of ascertaining that the desired solution is actually being followed. This may be the most difficult aspect of the computational procedure, since there seems to be no known algorithm for automatically discriminating among the various bifurcating branches. One therefore must follow a branch in small steps beginning at the bifurcation point.

Some of the problems that occur in the cranking model for normal rotational bands may also come into play in the application to certain vibrational modes in some nuclei. This includes well-known difficulties associated with avoided level crossings (ALC) [28] and with the phase-transitional collapse of pairing. While many of the interesting applications, such as to multiphonon $\gamma$ vibrations, involve low spins, it should be noted that the cranking frequencies are relatively high, making pairing collapse more likely. While on the one hand the same prescriptions used for normal rotation can be applied in such situations, for example, using particle-number projection before variation to forestall a pairing collapse, on the other hand, the anomalies themselves may represent valid physical mechanisms for terminating a vibrational band. A pairing collapse in such a situation might be interpreted as the dissolution of a vibrational mode by virtue of mixing with a large number of quasiparticle excitations that block the pairing correlations. Should this happen for a phonon number $< 2$, one might conclude that the mode in question is nonrepeatable.

While anomalies associated with single-particle (quasiparticle) ALC's do not seem to occur in connection with the high-lying giant resonances, they do appear too often in connection with low-lying quadrupole modes such as $\gamma$ vibrations. In fact, I consider this as a major obstacle to be overcome by a truly viable code. The appearance of ALC's is closely related to another classical problem in nonlinear physics, namely the problem of small divisors that arises when anharmonicities are treated by perturbation theory. Although exact diagonalization of the cranking equations provides a way to avoid small divisors in principle, in practice the problem is replaced by the anomalies that arise from ALC's. Indeed, small divisors and ALC's are just opposite sides of the same coin, a point that seems to have escaped previous notice. In the usual cranking theory of high spin states, the problems associated with ALC's are regarded as embarrassments to be swept away by an ad hoc prescription (e.g., "diabatic orbitals"). In my view, this is a very interesting but difficult problem that requires fundamental investigation. A possible key is the observation that mean-field equations are classical equations. Indeed, the density-matrix formulation of either the HF or HFB equations
have the typical form of classical Lax equations [29], where the density matrix and the single-(quasi-)particle "Hamiltonian" constitute a Lax pair. The ALC's are then associated with eigenvalues of the latter and therefore are purely classical quantities. It is conceivable that anomalies could be avoided if one understood how to properly quantize the Lax equations in the presence of ALC's, which seem to be manifestations of near-resonance.

2.4 Possible Future Applications

Depending on the nucleus and the collective modes of interest, the problem of avoided level crossings may or may not be a hindrance to the application of the CBT. With this proviso in mind there are a number of possible interesting applications for the future.

2.4.1 Octupole excitations in superdeformed nuclei

Since the supershells in superdeformed nuclei are composed of approximately equal numbers of single-particle levels of opposite parity, octupole correlations are expected to become prominent [30]. Indeed, early evidence for octupole excitations built on the superdeformed minimum has recently appeared [31]. Numerous calculations have been carried out for these nuclei, using the stretched octupole-octupole interaction [13] and either a pure deformed harmonic-oscillator potential [32-34] or a more realistic Nilsson potential with pairing [35]. In all cases, the RPA was used. However, since some of the octupole modes are expected to be quite soft, it is very important that anharmonic corrections be included. The cranking technique described here seems ideally suited for this purpose as long as one is content to study the \(|K| = 1, 2, 3\) octupole modes. Unfortunately, the \(K = 0\) mode is not directly accessible in this way.

2.4.2 Is there a universal anharmonic vibrator?

Recently, Casten, Zamfir and Brenner [19] on the basis of energy systematics proposed the "universal" quadrupole vibrator formula

\[ E(I) = nE(2\beta) + \frac{1}{2}n(n - 1)e_4 \]  \hspace{1cm} (10)

for the aligned \(n\) phonon levels with spins \(I = 2n\) in even-even spherical nuclei, a formula that is phenomenologically equivalent to the Ejiri [36] equation

\[ E(I) = aI + bI(I + 1). \]  \hspace{1cm} (11)

What is remarkable is not the expression itself, which has been known for a long time, but rather that the anharmonicity parameter \(e_4\) or \(b\) is nearly constant in each of three broad regions of the nuclear periodic table covering all even-even nuclei with \(Z \geq 38\) that are not well-deformed. With \(I_0 = 0\) for the ground state and \(K = 2\) for quadrupole vibrations, this is essentially a truncation of Eq. (9). Therefore, the anharmonicity parameter \(e_4\) can be calculated microscopically in lowest order within the CBT framework by extending the perturbation approach outlined in Sec. (2.2). In fact, a closed form can be obtained for \(e_4\), but this has resonance denominators that can become small. Therefore, the expression is very sensitive to the inputted single-particle levels. Nevertheless, it may be possible to meaningfully calculate an average value in a region of the nuclear periodic table.

2.4.3 Vibrations in odd-A nuclei

The CBT may provide a new approach to the old subject of the coupling of odd nucleons to collective vibrations. The (uncranked) vacuum state should be chosen as the mean-field equilibrium solution including the odd particle so that \(I_0 \neq 0\). If this is axially symmetric then the CBT can be applied. One may then anticipate two types of bifurcations: those that correspond to the scattering of the odd nucleon and those that correspond to excitations from the even-even core. The former take into account the effect on the single-particle (quasiparticle) energies of the coupling to the collective vibrations, while the latter includes the collective vibrations themselves. Since the vacuum state is at least two-fold degenerate, one must, of course, make a particular choice. For example, in a deformed nucleus one may choose the vacuum
\[ |\Psi_0\rangle = a_{1\beta}^+ |0\rangle, \]
where \(|0\rangle\) is the mean-field state of the even-even core and \(a_{1\beta}^+\) creates the odd nucleon, with angular momentum \(I_0 = K_\beta\). Corresponding to a collective excitation projecting \(K\) units of angular momentum on the symmetry axis in the even-even nucleus, one expects two modes in the odd nucleus that project \(K \pm K_\beta\) units of angular momentum for a one-phonon excitation. These modes are not expected to be degenerate since in one case the nucleon is rotating in the same sense as the collective wave, while in the other case it is rotating in the opposite sense. Of course, if the vacuum is chosen instead as the time-reversed state \(\langle a^\dagger \rangle = T a_{1\beta}^+ |0\rangle\), then \(I_0 = -K_\beta\), but the physics is the same, only the sense of rotations is reversed. The physical picture is basically that shown in Fig. (1). As mentioned earlier, in the RPA limit one recovers equations of the type proposed long ago by Bg and Chung [8] for describing \(\gamma\) vibrations in odd-A deformed nuclei, which certainly deserved more study. The proposal here goes much further since anharmonicity effects of all orders can be summed in the cranking approach, which is not limited to deformed nuclei, but can also be applied to spherical nuclei.

2.4.4 Electromagnetic transitions and moments

The discussion thus far has made no mention of the calculation of electromagnetic transitions and moments in the cranking approach. In fact, transition matrix elements can be calculated between levels lying along the same cranked trajectory, for example, in the case of \(\gamma\) vibrations, between the \(K = 4\) two-phonon and the \(K = 2\) one-phonon bandheads. The best derivation for the prescription, which uses only mean-field wave functions, is provided by the self-consistent cranking + RPA (SCC+RPA) method [37] briefly discussed below. An adaptation to multiphonon vibrations is given in Ref. [2]. The calculation of static moments has never presented a problem for the cranking model - one need only calculate the expectation values with respect to the cranked wave function.

2.4.5 Pairing vibrations with nonzero spin

Although the CBT applies to all spherical nuclei, one does not expect satisfactory results using the cranked HFB approximation for a nucleus with two particles or two holes outside of a closed shell for the simple reason that the BCS or HFB approximations are not very accurate in this case. However, the levels in such a nucleus may be regarded as pairing vibrations, i.e., pair-transfer modes built on the adjacent closed-shell system. Thus, for example, the levels in Te or Cd isotopes can be regarded as pairing vibrations built on a suitable Sn isotope. If the pairing vibration has nonzero angular momentum, such as the first \(2^+\) level of, for example, \(^{114}\)Cd, it can be obtained by cranking \(^{114}\)Sn. This provides a new approach for computing properties of nuclei with two particles or holes added to a closed shell. Of course, one can continue the cranking trajectory to reach \(4^+\) states with \(\pm 2\) pairs added to the closed shell, etc.

2.4.6 Further refinements

The cranking description of vibrations has two major drawbacks. First, while anharmonicities of all orders can be summed, the approximation is basically a classical one with a superimposed Bohr-Sommerfeld quantization condition on the angular momentum. Lacking are certain quantal-fluctuation corrections that cannot be incorporated in the quantization condition. Second, the cranking formalism usually describes levels that are harmonics of the same fundamental mode. For example, to return to the paradigm of the \(\gamma\) vibration, the cranking trajectory is generally expected to cover the one-phonon states \(B_{1\pm 2}^+ |0\rangle\), with \(I = |K| = 2\), and the aligned two-phonon states \(\left( B_{1\pm 2}^+ \right)^2 |0\rangle\), with \(I = |K| = 4\), but not the mixed two-phonon state \(B_2^+ B_{-2}^- |0\rangle\), with \(I = |K| = 0\), nor, for that matter, totally unrelated modes such as \(\beta\) vibrations. The possibility exists to compensate for both shortcomings by going beyond the cranked mean field approximation to include at least the quantized RPA fluctuations about the steady rotation. This is the SCC+RPA mentioned above [37, 38]. While the discussion there centers about normal high-spin states, there is no reason why the formalism should not also apply to cranked vibrations. It has the following advantages. First, it gives rise to zero-point correlation corrections along the cranked trajectory. Second, states that do not lie on

\(^5\)Of course, signature symmetry requires that the physical state be a linear combination of the states with \(K = \pm 2\).
the cranked trajectory may possibly be reached by applying the SCC+RPA phonon operators to the correlated vacuum state; i.e., the cranked trajectory with correlations, which acts as the reference. Third, the SCC+RPA provides a well-defined formalism for deriving transition matrix elements that goes beyond the cranked mean-field approximation. By using boson expansions about the cranked mean-field solution, it is possible in principle to include higher-order quantal correlation effects not included in the SCC+RPA. So far, this idea has been tested only in the U(5) limit of the IBM [10]. Finally, in a different approach, one may use angular-momentum projection from the cranked state vector to obtain quantal effects.

2.5 Tilted Rotation

The subject of tilted rotation is an emerging subfield of both theoretical and experimental high-spin physics. Although the approach to tilted rotation to be outlined here is closely connected to the CBT, it will be discussed as a separate topic since there is no reason to believe that all examples of this phenomenon have such a connection.

In the usual picture of nuclear rotation as implemented in the cranking model, a nucleus with an ellipsoidal or, at least, reflection-symmetric shape rotates about a principal axis of the density distribution, which is equivalent to rotation about a principal axis of the deformed self-consistent field. However, since the nucleus is not a rigid body, another mode of rotation is possible. This possibility was already pointed out in 1860 by Riemann [39], who showed that an ellipsoidal gravitating fluid with vorticity can rotate uniformly about an axis that does not coincide with a principal axis, i.e., it rotates about a tilted axis. He also proved the theorem that the rotation axis must always lie in a principal plane of the ellipsoid. A more modern account of Riemann's work is given in the classic treatise of Chandrasekhar [40], who refers to the tilted solutions as Riemann ellipsoids of types II and III. Normal principal-axis (PA) and tilted-axis (TA) rotations are contrasted in Fig. (4).

![Figure 4: Normal and tilted rotation.](image)

2.5.1 Tilted rotation in an irrotational fluid - a new ellipsoidal figure of equilibrium

Vorticity is not a requirement for a rotating liquid drop to have a tilted figure of equilibrium. I have recently calculated the ellipsoidal equilibria for ideal (inviscid, incompressible and homogeneous) rotating self-gravitating liquid drops with irrotational flow (in the laboratory frame). The results have been published in Physics of Fluids [42]. I found that an initially spherical drop of radius $R_0$ and density $\rho$ has two bifurcation branches, depicted on an angular momentum vs. frequency plot in Fig. (5). The results are in accord with the CBT. As the system has only one (degenerate) vibrational frequency given by $\omega_0 = (16\pi G \rho / 15)^{\frac{1}{2}}$ ($G$ is the gravitational constant and $\rho$ the density) [41], the CBT predicts for this quadrupole shape exactly two bifurcations with the critical rotational frequencies $\Omega_c = \omega_0/2$ and $\omega_0$, which is what is displayed. The
former bifurcation corresponds to a normal rotation about a principal axis also calculated by Chandrasekhar, while the latter is a tilted rotation apparently overlooked by him. The tilt angle is 45° in the neighborhood of the bifurcation point and then decreases to an asymptotic value of about 30.15° with one of the principal axes as the angular momentum goes to infinity. The behavior of the curves, with the slight forward bend in the neighborhood of the bifurcation points followed by the pronounced backbending, is very interesting but rather typical.

Figure 5: Scaled universal plot of the angular momentum $L$ [units: $\frac{16}{15} R_0^5 (\pi G \rho)^{\frac{3}{2}}$] vs the angular frequency $\Omega$ [units: $(\pi G \rho)^{\frac{3}{2}}$] for the ellipsoidal bifurcations of a self-gravitating rotating irrotational liquid drop of density $\rho$ and initial radius $R_0$. The curve marked “N” corresponds to normal rotation and that marked “T” to tilted rotation.

The reason that Chandrasekhar missed the tilted irrotational figure of equilibrium appears to have been his uncharacteristically lax statement of Riemann’s theorem which seemed to rule out the very existence of this figure. A closer examination of Riemann’s theorem shows that it is entirely compatible with the existence of this equilibrium. To be sure, it does not appear from the literature that Riemann or anyone else had found this solution.

I also carried out a study of the linear stability of the irrotational ellipsoids. It was found that the PA bifurcation branch (also studied by Chandrasekhar) is linearly stable everywhere while the new TA bifurcation branch is generally unstable except in a small neighborhood of the bifurcation point, where linear stability was established. On the other hand, a proof of nonlinear (Liapunov) stability, which requires more rigorous mathematics, has apparently not been given for any of the ellipsoidal solutions.

2.5.2 Tilted rotation in nuclei

Since a nucleus has a fluid structure with a moment of inertia intermediate between that of a rigid body and an irrotational fluid, one is led to wonder whether it can also exhibit tilted rotation, and, if so, whether the motion would be stable. Although Chandrasekhar’s account had been available for many years, the subject of tilted nuclear rotation developed for the most part in an independent way. Two equivalent formulations using the cranking model can be found in the literature. On the one hand, introducing a frame that always coincides with the principal axes of the deformed potential, one includes in the cranked Hamiltonian (Routhian) the term $-\overrightarrow{\Omega} \cdot \overrightarrow{J}$, where, for uniform rotation, $\overrightarrow{\Omega}$ is a constant angular velocity. If all three components of $\overrightarrow{\Omega}$ are nonzero, this is called three-dimensional cranking; if one component vanishes it is called two-dimensional cranking. Equivalently, one may transform to a frame having an axis coinciding with the axis of rotation, in which case the usual one-dimensional cranking applies, but the principal axes of the deformed potential are then tilted with respect to this frame.

While early examples of three-dimensional cranking exist in the nuclear literature [43–46], none dealt with uniform tilted rotation, but were concerned with the calculation of the moments of inertia of triaxial

\footnote{Perhaps triaxial cranking would be a better term.}
nuclei at low spin or with wobbling motion at high spin. Hamamoto [47] was one of the first to clearly discuss the possibility of uniform tilted (two-dimensional) rotation in the context of the classical orbits for \( \gamma \)-deformed single-\( j \) configurations. She showed that for a fixed value of \( \gamma \) the energy is minimal if the system is subject to (at least) two-dimensional cranking. This model of quasiparticles in a single-\( j \) potential was studied further by Bengtsson, Frisk and Wu [48], who showed that the tilted rotation can persist when the rotating core is included. They also found for the first time energy minima corresponding to rotation about a tilted axis that does not lie in a principal plane, i.e., a true three-dimensional cranking situation. Moreover, they also found such minima for more realistic two-quasiparticle configurations in a Nilsson potential for the odd-odd nucleus \(^{84}\)Y. However, since the deformation was kept fixed while the polar angles of the axis were varied, these solutions cannot be considered as fully self-consistent. In view of Riemann's theorem for the rotating fluid, this raises the interesting question of whether fully self-consistent solutions exist in nuclei exhibiting rotation about an axis that does not lie in a principal plane. Judging by his recent review talk [49], it appears that Bengtsson had not yet answered this question for his solution, while all other examples found since are either restricted to a principal plane or are not fully self-consistent.

Since 1992, Frauendorf [50] has been the most active proponent of tilted rotation, using for the most part two-dimensional cranking in a deformed shell model generated by the quadrupole-quadrupole force and attempting to explain or predict actual experimental data. In all cases thus far, the deformation was held fixed and only the tilting angle was varied. Approximately a dozen tilted bands have now been experimentally identified, these being one and three quasiparticle bands in odd nuclei and some cases of two-quasiparticle bands in even-even nuclei [51]. The tilted bands, which have come to be called "t-bands"\(^7\), are broken-signature cranking solutions, as emphasized by Frauendorf, and therefore correspond to bands with a \( \Delta I = 1 \) sequence, in contrast to the garden-variety solutions with good signature that occur, in general, as pairs of bands, each corresponding to a \( \Delta I = 2 \) sequence. Therefore, the t-bands have small signature splittings, and as shown by Frauendorf, the B(M1) values are relatively large. Since these are not unique identifiers, the assignments must, of course, be buttressed by a cranking analysis.

All of the examples of tilted rotation alluded to above are produced by one or a few quasiparticles that introduce asymmetries into the system. The question that has particularly interested me is whether collective tilted rotation - the analog of the fluid phenomena - is possible in nuclei. This question was first addressed in 1987 by Cuypers [52], who, using one-dimensional cranking, set up the nonlinear equations for tilted rotation within the volume-conserving harmonic-oscillator model. He found, as expected, that rotation about a principal axis is always a solution of these equations. Unfortunately, he erroneously jumped to the unwarranted conclusion that these are the only solutions\(^8\). Intrigued, I tried to find tilted solutions of his model in the equivalent three-dimensional cranking formulation using a numerical minimization code and found nothing but normal principal-axis rotation. Not wholly convinced that this was the entire story, I then looked for analytic solutions in the form of a truncated Taylor expansion using the computer-algebra program REDUCE. Indeed, I then succeeded in finding a tilted solution for closed-shell configurations. (Unfortunately, REDUCE choked on the open-shell calculation, so no results could be obtained there.) What was very interesting and especially clear using the original one-dimensional tilted cranking formulation of Cuypers is the connection of my solution with the collective vibrations of this model. I found that this solution bifurcated from the spherical ground state at the critical cranking frequency \( \sqrt{2} \omega_0 \), which coincides with the giant quadrupole resonance RPA frequency [53] and is what one would expect for a \( |K| = 1 \) mode. I later found an untilted solution bifurcating at one-half of this frequency \( (|K| = 2) \). These developments helped to crystallize the concept of the CBT. The fact that the tilted solution could not be found by minimization techniques suggests that it is either unstable or else an excited rather than yrast band. Of course, this question can be answered by an appropriate stability analysis, a task remaining to be done.

In addition to the cranked oscillator potential model, I also studied collective models in the cranking approximation. In both the Bohr-Mottelson model and the IBM U(5) limit [54], solutions similar to those found for the closed-shell oscillator were generated, the latter being discussed in Ref. [10], where the tilted band was referred to as the T band and the normal band as the Y band. While the T band is formally unstable in that the corresponding SCC+RPA Hamiltonian is not a positive quadratic form, the instability is a benign one, corresponding to the existence of a mode with a negative rather than imaginary frequency.

\(^7\)While the first usage of the term "t band" may have occurred in Ref. [10], it was clearly established by the subsequent advocacy of Frauendorf.

\(^8\)Since Cuypers was an unsupervised graduate student at the time, we might forgive him this lapse of logic.
This indicates that the T band is generally an excited band rather than an yrast one, which is borne out by the exact IBM solutions, whose energies were reproduced with accuracy of order $I^{-1}$. The energies become exact when higher-order correlations are added. The qualitative behavior of the T band is reminiscent of the tilted solution that I found for the rotating irrotational fluid.

In the meantime, Rosensteel [55] undertook a modern mathematical reformulation of the theory of Riemann rotors, opening the possibility of application to nuclei. In a novel generalization of the cranked anisotropic-oscillator model, Rosensteel [56] included vorticity by constraining not only the angular momentum, but also the Kelvin circulation, and then showed that for small angular and vortex velocities one can recover the Riemann rotor. Rosensteel and Goodman [57] then proved that the Riemann theorem holds for this version of the cranked oscillator, i.e., the angular velocity and the vortex velocity must lie in a principal plane. These are very interesting developments that deserve more study. Since the correspondence between the oscillator model and the classical Riemann rotor was demonstrated only for small angular frequencies, when the cranking terms can be treated perturbatively, it is not clear whether the theorem holds for higher angular frequencies. In particular, the bifurcating solutions provided by the CBT are not covered by this type of perturbation analysis. The solutions that I had found previously for the cranked oscillator happen to conform to Riemann’s theorem, but a general proof for all cranking frequencies is yet to be given, either for the Rosensteel or the conventional versions of the cranked oscillator model. Questions also remain about the validity of vorticity cranking in which the angular momentum and the Kelvin circulation are treated on an equal footing. The former is a rigorous constant of motion, while the latter in general is not. These are points that deserve further study.

Finally, to round out the account of recent developments in the area of tilted rotation, two papers should be mentioned. First, Nazarewicz and Szymański [58] presented an excellent overview of the subject and then discussed an exactly soluble model in which a search for tilted bands proved negative. Goodman [59] and Dodaro and Goodman [60] have employed three-dimensional cranking in a very different context, namely, in connection with the study of orientation fluctuations in a nucleus at nonzero temperatures. These authors also reinvestigated the single-$j$ model of tilted rotation that had been considered by Bengtsson. While conceding that their solutions were not fully self-consistent, they point out that their results appeared to be in accord with Riemann’s theorem.

From this survey, it is clear that the field of tilted rotation is still in its infancy, but definitely gathering steam from both the theoretical and experimental sides. Much work remains to be done to find fully self-consistent solutions in which both the deformation and the orientation are varied. This is quite important since a nucleus might drastically change its shape to avoid tilted rotation under certain circumstances. Before tackling realistic codes, I would first like to attempt to fully characterize all collective tilted solutions for simple models, including the cranked anisotropic oscillator model and various collective models, in particular the IBM. The IBM provides a nice opportunity to see how tilted solutions at the corners of the Casten triangle relate as one moves throughout the triangle. Since the motion in various parts of the triangle is believed to become chaotic [61], one may wonder what the effect is on the tilted solutions. Among the questions I would also like to answer is the relation of such solutions to the CBT. While the CBT allows one to find certain tilted solutions, are there others that are unrelated to it? Also, is there a fundamental distinction between single-particle tilted bands and collective tilted bands? Of course, there is still the pending question of whether fully self-consistent solutions exist that violate the Riemann condition restricting the angular velocity to a principal plane. Finally, there is the important question of how to treat quantal corrections associated with tunneling between degenerate tilted solutions. An important first step in this direction has very recently been taken by Horibata, Oi and Onishi [62], who employed a generator-coordinate calculation based on the tilted cranked HFB state.

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4 SCIENTIFIC PUBLICATIONS UNDER DE-FG02-91ER40640

The works listed below have already been published. Additional publications based on work done under the grant are expected in the near future.