Choas and Instabilities in Finite Difference Approximations to Nonlinear Differential Equations

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CHAOS AND INSTABILITIES IN FINITE DIFFERENCE APPROXIMATIONS TO NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

The numerical solution of time-dependent ordinary and partial differential equations by finite difference techniques is a common task in computational physics and engineering. The rate equations for chemical kinetics in combustion modeling are an important example. They not only are nonlinear, but they tend to be stiff, which makes their solution a challenge for transient problems. We show that one must be very careful how such equations are solved. In addition to the danger of large time-marching errors, there can be unphysical chaotic solutions that remain numerically stable for a range of time steps that depends on the particular finite difference method used. We point out that the solutions of the finite difference equations converge to those of the differential equations only in the limit as the time step approaches zero for *stable* and *consistent* finite difference approximations. The chaotic behavior observed for finite time steps in some nonlinear difference equations is unrelated to solutions of the differential equations, but is connected with the onset of numerical instabilities of the finite difference equations. This behavior suggests that the use of the theory of chaos in nonlinear iterated maps may be useful in stability analysis of finite difference approximations to nonlinear differential equations, providing more stringent time step limits than the formal linear stability analysis that tests only for unbounded solutions. This observation implies that apparently stable numerical solutions of nonlinear differential equations by finite difference techniques may in fact be contaminated (if not dominated) by nonphysical chaotic parasitic solutions that degrade the accuracy of the numerical solution. We demonstrate this phenomenon with some solutions of the logistic equation and a simple two-dimensional computational fluid dynamics example.

I INTRODUCTION

The natural language for describing most physical problems is systems of differential equations, both ordinary and partial. In combustion modeling, it is common practice to find numerical solutions of a system of coupled ordinary differential equations that describe the rates of the chemical reactions under study. A simulation may involve the interaction of hundreds of chemical species, each governed by its own rate equation. These equations are nonlinear and stiff (that is, the physical problem presents a wide range of time scales). Similar equations are encountered in nuclear astrophysics and in population dynamics studies. The nonlinearities force the use of numerical techniques such as finite differences to find approximate solutions to these systems of equations.

The question of stability and convergence of finite difference methods for linear problems is settled by the well-known Lax's Equivalence Theorem. *Given a properly posed initial-value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence* [1]. All of the commonly used numerical methods for computational physics have truncation errors that are at least first order in the numerical step size, so they are consistent. If the method is also stable, then the numerical solution will converge to that of the underlying differential equations as the step size approaches zero. If we go the other direction and increase the step size, accuracy is more or less smoothly degraded until a critical point is reached where the solution becomes unbounded (that is, unstable). It has been commonly assumed that nonlinear problems behave similarly since there is no rigorous theoretical framework for nonlinear problems. The point of this report is to show that nonlinear systems behave somewhat differently in that there may be several critical values of the step size separating different regimes of behavior of the numerical solution that are not found in the solutions of the underlying differential equations. This is not a new observation [2-6], but it is one that deserves more widespread attention among those who do numerical simulations.

In producing accurate numerical solutions to scientific and engineering problems, it is necessary to avoid numerical instabilities that introduce ill-behaved parasitic solutions of the difference equations that are unrelated to the solutions of the underlying differential equations. Most of the numerical methods in common usage are only conditionally stable,

which means the step size must be kept smaller than certain limits. (A notable exception is the Adams-Bashforth method, which can be unconditionally unstable while appearing to be stable [7]) What we demonstrate in this report is that certain parasitic solutions are related to the type of chaos exhibited by iterated algebraic maps. This means that some bounded, and therefore apparently stable, numerical solutions, may be physical nonsense. Lip service often is paid to performing grid and time step refinement studies, but this result emphasizes just how critical it is to actually perform these extra calculations to insure that the numerical solutions are even qualitatively correct.

The next section presents a specific simple example, the logistic equation. Section III explores the connection between finite difference approximations to the logistic equation and chaos in iterated maps. Section IV discusses extension of these ideas to computational fluid dynamics. Section V contains the summary and conclusions.

II THE LOGISTIC EQUATION

We begin by considering the logistic equation as a simple archetypal chemical rate equation for a single species. An admittedly contrived situation where the logistic equation is an approximate description is the nuclear stripping reaction

If we assume the gas is a nearly equimolar mixture of tritium and deuterium with a small concentration of neutrons ($[\text{n}] \ll [^2\text{H}], [^3\text{H}]$), and for a constant temperature T , the rate equation may be written

$$\frac{d[\text{n}]}{dt} = k_f(T)[^3\text{H}][\text{n}] - k_b(T)[^2\text{H}][\text{n}]^2 = \alpha[\text{n}] - \beta[\text{n}]^2, \quad (2)$$

where k_f and k_b are the forward and backward reaction rates. Under the assumptions given, α and β may be considered constant.

The same differential equation is the basis of a crude but popular and useful model of population dynamics [8]. If N is the population, then the logistic equation is

$$\frac{dN}{dt} = \alpha N - \beta N^2 = \alpha N \left(\frac{K - N}{K} \right), \quad (3)$$

where $K = \alpha/\beta$ is the carrying capacity of the environment. The first term on the right hand side represents the familiar exponential growth of the population via constant difference between birth and mortality rates. The nonlinear term is one plausible representation of an increase in mortality or decrease in fertility due to crowding.

The logistic equation has a very simple general solution. At first glance this equation appears to have two free parameters, but actually it has a self-similar solution with no parameters. Let us make the linear changes of variables $\mathcal{N} = N/K$ and $\tau = \alpha t + t_0$, where t_0 is a constant adjusted to make $\tau = 0$ at the start of the problem. Then Eq. 3 becomes

$$\frac{d\mathcal{N}}{d\tau} = \mathcal{N} - \mathcal{N}^2 \quad (4)$$

We assume that the value of $\mathcal{N}(0)$ is given. Equation 4 is easily solved by separation of variables

$$\int_{\mathcal{N}(0)}^{\mathcal{N}(\tau)} \frac{dn}{n^2 - n} = \ln \left(\frac{1-n}{n} \right) \Big|_{\mathcal{N}(0)}^{\mathcal{N}(\tau)} = - \int_0^\tau dt = -\tau, \quad (5)$$

which gives

$$\mathcal{N}(\tau) = \frac{\mathcal{N}(0)}{\mathcal{N}(0) + [1 - \mathcal{N}(0)] \exp(-\tau)} \quad (6)$$

This solution is valid provided the interval of integration over n does not include either zero or unity, where the integrand has singularities. These singularities are related to two constant equilibrium solutions, $\mathcal{N} = 0$ and 1.

The ecologically interesting case is $0 < \mathcal{N}(0) < 1$. In this case, the solution rises monotonically to unity at late times. The solution does not overshoot the carrying capacity. In chemical systems, $\mathcal{N}(0) > 1$ may occur also, in which case the solution decays monotonically to unity. The equilibrium solution $\mathcal{N}(\tau) = 1$ is stable. If the equilibrium is perturbed by decreasing the population, $\mathcal{N}$ grows monotonically back to unity. If the perturbation causes the population to exceed the carrying capacity, it decays monotonically back to equilibrium.

The equilibrium solution at $\mathcal{N} = 0$ is unstable. For a positive perturbation (that is, the creation of a new species), the solution follows Eq. 6 to unity. For a negative perturbation, the formal solution diverges to $-\infty$ in a finite time. The negative perturbation is not realizable physically for either chemistry or population dynamics.

Equation 6 shows two additional interesting features. First, the nature of the solution is independent of the parameters α and β . These parameters determine the carrying capacity

or equilibrium concentration and the time scale for population growth, but not the shape of the curve of N versus t . Second, populations obeying Eq. 3 do not exhibit chaos. This is not surprising since ordinary differential equations that are local in the independent variable and have no external forcing terms must be of at least third order to have chaotic solutions.

There is a second model called the logistic equation in the population dynamics literature. However, this model is not equivalent to Eq. 3. It is a recursion relation or iterated map that gives the population on step $j + 1$ in terms of the population from the previous step,

$$N_{j+1} = \alpha N_j - \beta N_j^2, \quad (7)$$

where again α and β are constants. This model approximates the population dynamics of a single species that propagates in discrete, nonoverlapping generations (such as some insects), lives in a homogeneous world, and whose birth and death rates depend only on the population density in such a way that we get the N_j^2 term. If we make the change of variables $\mathcal{M}_j = N_j\beta/\alpha$, Eq. 7 becomes

$$\mathcal{M}_{j+1} = \alpha(\mathcal{M}_j - \mathcal{M}_j^2) \quad (8)$$

Once again, α/β is the carrying capacity. This model has one parameter, and the behavior of the solution depends critically upon the value of that parameter [9, 10]. Assume some initial value $0 < \mathcal{M}_1 < 1$. If α is less than unity, the population approaches zero monotonically. If α is between one and three, there is a single nontrivial stable equilibrium solution. For slightly larger values of α , the solution settles into a period-two limit cycle. As α is increased further, the long-time solution undergoes a sequence of period doublings. For values of α between 3.57 and four, the solution has a complex chaotic behavior. For α greater than four, there will eventually be a value of $\mathcal{M}_j$ greater than unity, after which the sequence marches off to minus infinity. This mathematical behavior is well known and is described in more detail in references [9] and [10]. When the logistic map is applied to population dynamics [8, 9], this mathematical behavior suggests a connection between deterministic chaos and the observation of large, unpredictable population fluctuations in nature. We shall next consider another application in which chaos plays an important role.

III CHAOS AND INSTABILITY IN FINITE DIFFERENCE EQUATIONS

The simplest explicit finite difference approximation to Eq. 4 is

$$\frac{\mathcal{N}_{j+1} - \mathcal{N}_j}{\delta\tau} = \mathcal{N}_j - \mathcal{N}_j^2, \quad (9)$$

where $\delta\tau$ is the dimensionless time step. It is easy to show that this approximation is consistent. This equation may be rearranged to obtain

$$\mathcal{N}_{j+1} = (1 + \delta\tau)\mathcal{N}_j - \delta\tau \mathcal{N}_j^2. \quad (10)$$

If we apply the transformation $\mathcal{M}_j = \mathcal{N}_j \delta\tau / (1 + \delta\tau)$, we get

$$\mathcal{M}_{j+1} = (1 + \delta\tau) (\mathcal{M}_j - \mathcal{M}_j^2) \quad (11)$$

This equation has the same form as Eq. 8 with $\alpha = (1 + \delta\tau)$. Note that $\mathcal{M}$ has the value $\delta\tau / (1 + \delta\tau)$ for the nontrivial equilibrium solution $\mathcal{N} = 1$, which is the same as the stable equilibrium solution of Eq. 8, $1 - 1/\alpha$, for $1 < \alpha < 3$.

The difference equation 10 produces an accurate approximation to the solution to Eq. 4 only in the limit of sufficiently small $\delta\tau$. If $\delta\tau$ is made large enough to produce any kind of overshooting or oscillatory solutions, then Eq. 10 is not a good approximation to Eq. 4, and the numerical solution is nonphysical. We make the arbitrary choice of initial condition $\mathcal{N}(0) = 1 \times 10^{-3}$, which is of no significance except that the numerical solution starts off in the linear regime. Table 1¹ summarizes some numerical experiments performed with this approximation. The relevant numerical solution is in the column labeled “logistic,” and the column to the right of it is the analytical solution. Numerical experiments indicate that for $\delta\tau$ less than or equal to unity, the numerical solution approaches unity monotonically, and better accuracy is obtained for smaller time steps. For larger time steps, the finite difference equation produces a series of increasing complex solutions. For $\delta\tau$ between unity and two,

¹My apologies to the reader for the extensive tables. Actually, one can ignore the tables without sacrificing understanding of the point of this report. They were included as much for archival purposes as for enlightenment. Code listings were included to answer questions about precisely what was calculated (e.g., were there bugs?). The numerical tables show some details involving high order digits that would not show up in a graph. The five rows of double precision numbers are the value of the independent variable x , the numerical solution of the differential equation $dy/dx = -y$ (labelled `exp`), the analytic solution to that equation, the numerical solution of the logistic equation (labelled `logistic`), and the analytic solution of the logistic equation. The variable `delx` is $\delta\tau$.

the long term solution undergoes a damped oscillation about unity. For $\delta\tau$ between two and approximately 2.4, the long term solution is a limit cycle between two values, one of which is greater than unity and the other one less than unity. For larger $\delta\tau$ up to three, the solution first undergoes at least one period doubling and appears to become chaotic at sufficiently large values. For $\delta\tau = 2.45$ and 300 time steps, the solution appears to be settling down into a period four limit cycle. For $\delta\tau = 2.95$, the solution appears to be chaotic. $\delta\tau = 3.05$, the solution runs off to minus infinity, a classical form of numerical instability.

This complex behavior is precisely that of Eq. 8. One expects only conditional stability from explicit methods, with the existence of a critical value of $\delta\tau$ beyond which the numerical solution becomes unbounded. However, the nonphysical but bounded behavior of the solution for $1 < \delta\tau < 3$ was not expected and is not a property of the differential equation 4. Unfortunately, and this is a critical point, some of the pathological behavior might not be recognized as such if we had not known the analytic solution and had not refined the time step until we reached a grid-independent solution (not shown in the tables). The mild oscillations for $\delta\tau$ between unity and two might even be considered acceptable for some purposes. The more seriously flawed periodic solutions might not be recognized as such if output were taken only every n th time step, where n is a multiple of the period. Another way of stating this is that even if the time step is short enough that the solution does not blow up, the solution may still not be meaningful. This is the point that Yee, *et al.* [3-6] make in the context of finding false steady states with time marching methods. It is very important, therefore, when numerically solving nonlinear finite difference equations, to make sure the solution is truly step-size independent.

Next consider the fully implicit, first order method

$$\frac{\mathcal{N}_{j+1} - \mathcal{N}_j}{\delta\tau} = \mathcal{N}_{j+1} - \mathcal{N}_{j+1}^2 \quad (12)$$

This method appears to be unconditionally stable and its solution approaches unity monotonically for all values of $\delta\tau$ up to 10. No larger values were tested since accuracy of the transient was pretty poor by the time a value of unity was reached. Selected results are shown in Table 2.

The last two methods are second-order, two-step Runge-Kutta integrations,

$$\mathcal{N}_* - \mathcal{N}_j = \frac{\delta\tau}{2} (\mathcal{N}_j - \mathcal{N}_j^2) \quad (13)$$

$$\mathcal{N}_{j+1} - \mathcal{N}_j = \delta\tau (\mathcal{N}_* - \mathcal{N}_*^2) \quad (14)$$

and

$$\mathcal{N}_* - \mathcal{N}_j = \delta\tau (\mathcal{N}_j - \mathcal{N}_j^2) \quad (15)$$

$$\mathcal{N}_{j+1} - \mathcal{N}_j = \frac{\delta\tau}{2} [\mathcal{N}_j - \mathcal{N}_j^2 + \mathcal{N}_* - \mathcal{N}_*^2] \quad (16)$$

Selected results for these two methods are listed in Tables 3 and 4

Both methods have significant (< 32 percent) truncation errors for $\delta\tau = 1.05$. Both methods are well-behaved, and the mid-point rule, Eqs. 13-14 is slightly more accurate. However, both methods monotonically approach the wrong steady state values (0.9756098 and 0.8780488) for $\delta\tau = 2.05$. For $\delta\tau = 3.05$, the mid-point rule overshoots the equilibrium solution 0.6557377 and approaches it with damped oscillations. The trapezoidal rule, on the other hand, goes into a period-4 limit cycle. At 4.05, both methods produce solutions at minus infinity.

It is easy to show that the incorrect equilibrium solutions are real. If we algebraically eliminate $\mathcal{N}_*$ from Eq. 14 and look for equilibrium solutions ($\mathcal{N}_j = \mathcal{N}_{j+1}$), then the equilibrium solutions are roots of the fourth order polynomial

$$-\frac{\delta\tau}{4}\mathcal{N}_j^4 + \left(\frac{\delta\tau^2}{2} + \delta\tau\right)\mathcal{N}_j^3 - \left(1 + \frac{3\delta\tau}{2} + \frac{\delta\tau^2}{4}\right)\mathcal{N}_j^2 + \left(1 + \frac{\delta\tau}{2}\right)\mathcal{N}_j = 0 \quad (17)$$

Two of the roots are 0 and 1, the equilibrium solutions of the differential equation. The other two equilibrium solutions are the roots of

$$-\frac{\delta\tau^2}{4}\mathcal{N}_j^2 + \left(\delta\tau + \frac{\delta\tau^2}{4}\right)\mathcal{N}_j - \left(1 + \frac{\delta\tau}{2}\right) = 0 \quad (18)$$

The roots are always real and are $2/\delta\tau$ and $(2 + \delta\tau)/\delta\tau$. For time steps smaller than 2.0, both roots are greater than unity. For larger time steps, the first root lies between zero and unity, in quantitative agreement with our numerical examples. Apparently it is a stable equilibrium solution.

Often partially implicit methods are used in practice, and there are numerous ways the logistic equation can be approximated. Each particular difference approximation has

its own behavior, but typically there are a region of stability for small time steps, then oscillatory and periodic solutions, chaotic solutions, and eventually solutions that become infinite. One of the most interesting partially implicit methods is

$$\frac{\mathcal{N}_{j+1} - \mathcal{N}_j}{\delta\tau} = \mathcal{N}_{j+1} - \mathcal{N}_j^2 \quad (19)$$

For time steps less than 0.5, it is stable and the solution is monotonic. Since the explicit method has well-behaved solutions out to a time step of unity, this shows that implicitness does not always improve the size of the time step that one can use. Above a time step of 0.5, Eq. 19 has a rich and complex set of behaviors. We will not go through the list, but Table 5 shows the code and output for $\delta\tau = 0.74$ and 0.76 . What is remarkable is that at 0.74 , the solution reaches a period-6 limit cycle that is almost-periodic with period 3. At 0.75 , the solution is chaotic, and at 0.76 it blows up. The range of time step that produces chaotic solutions is quite narrow. For still larger time steps, other types of behavior can occur, including oscillations about zero.

IV. RELEVANCE TO COMPUTATIONAL FLUID DYNAMICS

Now we come to a natural question. If numerical solution of a single nonlinear ordinary differential equation can lead to spurious periodic and chaotic solutions at time steps smaller than the limit above which the solution becomes unbounded, can similar things happen in nonlinear partial differential equations, such as the Navier-Stokes equations? The answer, alas, is “yes.” The next obvious question is how this phenomenon manifests itself. Unfortunately for the engineer or scientist who merely wants an accurate solution of the Navier-Stokes equations for his or her specific application, there seems to be an incredible richness in the ways these pathologies appear. In this section, we shall examine briefly two examples.

The simplest example of a periodic solution in a reactive flow simulation is described by Ramshaw and Amsden [11]. The difficulty arose in a simulation of premixed combustion in a simple internal combustion engine using a partial equilibrium assumption. Some of the chemical reactions were assumed to be in equilibrium, and the equilibrium equations were solved in each computational zone on each time step by an algorithm that held the

temperature, and hence the equilibrium constants, fixed during the solution. Later in the time step, the temperature was updated to reflect the chemically-induced changes, and the new temperature was not used in the chemistry package until the following time step. What occasionally happened was similar to the following simplified scenario. Consider a zone with pure hydrogen atom (H) at low temperature. On the first time step, the equilibrium package would allow a lot of the H to recombine to form H_2 , with a large release of thermal energy. On the following cycle, the package would then dissociate the hot H_2 , cooling the fluid. The solution would alternate between these two states in a period-two oscillation. The problem was fixed by changing the algorithm to update the temperature and the equilibrium constant as the composition was being adjusted by the solution algorithm. Reference [11] describes this problem and its solution in more detail. In this case, it was possible to eliminate the unphysical solution by modifying the difference equations to more accurately model the physics.

An example of a bounded nonsense solution of a transient problem is provided by a two-dimensional calculation performed with the COYOTE [12] reactive flow program. The problem is a well-resolved sphere of xenon immersed in helium. In this particular case, the sphere has an initial radius of 12 μm , and both fluids have a temperature of 2000 K and a pressure of 10 bars. The helium and xenon mix via molecular diffusion, which causes the xenon sphere to expand subsonically as helium diffuses into it. Figures 1 and 2 show the xenon mass fraction at the same time for two calculations that differ only by a factor of two in the time step. Figure 1 shows the expected smooth solution for the short time step, and Fig. 2 shows a strong every-other-cell oscillation. Both problems were carried out to a time of approximately 200 ns. The short time step continued to produce a smooth solution, and the longer time step retained the oscillation at a more or less fixed amplitude. There was no indication that running the problem further would either damp the oscillation or cause it to begin growing toward infinity.

Additional more detailed, rigorous, and sophisticated discussion of the partial differential equation analog of the present logistic equation study is given by Yee, Sweby, and collaborators [3-6]. They showed in some detail how several methods, including some of those discussed in the previous section, have very complex, unphysical behavior for large

time steps as steady state is approached. They also suggest that similar spurious behavior may occur with inadequate spatial resolution

V CONCLUSIONS

The results of these numerical experiments may be summarized by the following four conclusions

1) Stability and accuracy are not the same. Grossly inaccurate but plausible solutions may be generated at large time steps, so it is essential to check for grid independence by trying a minimum of two values of the time step that differ by a factor of at least 1.5.

2) Nonlinear finite difference equations can generate a rich set of bounded but non-physical solutions. Some finite difference equations show oscillation of the solution around an equilibrium solution, which may or may not be related to a solution of the underlying differential equations. We found one case for the logistic equation where the numerical solution decayed to zero, which is an unstable equilibrium solution of the differential equation.

3) Explicit and partially implicit finite difference equations may exhibit chaos, depending on the particular finite difference equation. This chaos is the type generated by nonlinear iterated maps and has no connection to possible chaotic solutions of the underlying differential equations.

4) All of the pathologies described here could be detected and eliminated by the simple device of performing time step refinement studies.

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Y 2 cycle= 9837 t= 4.836754D-08 dt= 5.000000D-12
max = 9.204426D-01 min = 1.362180D-05 dq = 9.204290D-02

Figure 1: Xenon mass fractions for the non-coflowing case at $t = 48.37$ ns for $\delta t = 5$ ps.

Y 2 cycle= 5120 t= 4.837432D-08 dt= 1.000000D-11
max = 9.177379D-01 min = 4.114669D-05 dq = 9.176967D-02

Figure 2: Xenon mass fractions for the non-coflowing case at $t = 48.37$ ns for $\delta t = 10$ ps.

Table 1. Logistic Equation Solutions for the Explicit Method

```

program expl
c +++ explicit method testing
  implicit double precision (a-h,o-z)
  open (unit=12,file='explicit out',form='formatted')
c +++ initialize parameters
  delx = 1 05d+00
  npts = 200
  j     = 0
  yj    = 1 d+00
  yjp   = yj
  yex   = yj
  zj    = 1 d-03
  zjp   = zj
  zex   = zj
  z0    = zj

  write (12,10) delx
10 format (/'      Explicit tests for delx =',1p,d16 8//
1 '      j          x          exp          logistic')
  write (12,20) j, x, yjp, yex, zjp, zex
20 format (1x,i6,1p,5d14 6)

  do j = 1, npts
c +++ dy/dh = -y
    yjp = yj - delx * yj

c +++ dz/dh = z - z**2
    zjp = zj + delx * (zj - zj**2)

c +++ exact solutions
    x = real(j) * delx
    yex = exp(-x)
    zex = z0 / (z0 + (1.d+00 - z0) * yex)
    write (12,20) j, x, yjp, yex, zjp, zex

    yj = yjp
    zj = zjp

  enddo

  close(unit=12)
  stop
end

+++++

Explicit tests for delx = 9 00000000D-01 (Well behaved)

      j          x          exp          logistic
0 0 000000D+00  1.000000D+00  1 000000D+00  1 000000D-03  1.000000D-03
1 9 000000D-01  1 000000D-01  4 065697D-01  1 899100D-03  2 456018D-03
2 1 800000D+00  1 000000D-02  1 652989D-01  3 605044D-03  6.019252D-03
3 2 700000D+00  1.000000D-03  6 720551D-02  6 837887D-03  1 467603D-02

```

4	3	600000D+00	1	000000D-04	2	732372D-02	1.294990D-02	3	534019D-02	
5	4	500000D+00	1.	000000D-05	1.	110900D-02	2.445389D-02	8.	265906D-02	
6	5	400000D+00	1	000000D-06	4	516581D-03	4.592419D-02	1.	814202D-01	
7	6	300000D+00	1	000000D-07	1.	836305D-03	8.535784D-02	3	527998D-01	
8	7	200000D+00	1	000000D-08	7	465858D-04	1.556225D-01	5.	727904D-01	
9	8	100000D+00	1	000000D-09	3	035391D-04	2	738863D-01	7	673210D-01
10	9	000000D+00	1.	000000D-10	1	234098D-04	4	528716D-01	8	902449D-01
11	9	900000D+00	1	000000D-11	5.	017468D-05	6	758726D-01	9	522680D-01
12	1	080000D+01	1	000000D-12	2	039950D-05	8.	730346D-01	9	800279D-01
13	1	170000D+01	1	000000D-13	8	293819D-06	9	727953D-01	9	917826D-01
14	1.	260000D+01	1	000000D-14	3.	372015D-06	9	966134D-01	9	966427D-01
15	1	350000D+01	1	000000D-15	1	370959D-06	9	996510D-01	9	986323D-01
16	1.	440000D+01	1	000000D-16	5.	573904D-07	9	999650D-01	9	994435D-01
17	1	530000D+01	1	000000D-17	2	266180D-07	9	999965D-01	9	997737D-01
18	1.	620000D+01	1.	000000D-18	9.	213601D-08	9	999996D-01	9	999080D-01
19	1	710000D+01	1.	000000D-19	3	745971D-08	1.	000000D+00	9	999626D-01
20	1	800000D+01	1	000000D-20	1	522998D-08	1	000000D+00	9	999848D-01
21	1	890000D+01	1.	000000D-21	6.	192048D-09	1	000000D+00	9	999938D-01
22	1	980000D+01	1	000000D-22	2	517499D-09	1.	000000D+00	9	999975D-01
23	2.	070000D+01	1	000000D-23	1	023539D-09	1	000000D+00	9	999990D-01
24	2	160000D+01	1.	000000D-24	4.	161397D-10	1	000000D+00	9	999996D-01
25	2	250000D+01	1	000000D-25	1	691898D-10	1	000000D+00	9	999998D-01
26	2	340000D+01	1	000000D-26	6	878744D-11	1.	000000D+00	9	999999D-01
27	2	430000D+01	1	000000D-27	2	796688D-11	1	000000D+00	1	000000D+00
28	2	520000D+01	1	000000D-28	1	137049D-11	1	000000D+00	1	000000D+00
29	2	610000D+01	1.	000000D-29	4	622895D-12	1	000000D+00	1	000000D+00
30	2	700000D+01	1.	000000D-30	1.	879529D-12	1	000000D+00	1	000000D+00
31	2	790000D+01	1.	000000D-31	7.	641594D-13	1	000000D+00	1	000000D+00
32	2	880000D+01	1	000000D-32	3.	106840D-13	1	000000D+00	1.	000000D+00
33	2	970000D+01	1	000000D-33	1	263147D-13	1	000000D+00	1.	000000D+00
34	3	060000D+01	1	000000D-34	5	135572D-14	1	000000D+00	1.	000000D+00
35	3	150000D+01	1	000000D-35	2.	087968D-14	1.	000000D+00	1	000000D+00

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Explicit tests for delx = 1 05000000D+00 (Damped oscillations)

j	x	exp	logistic
0	0.000000D+00	1 000000D+00	1 000000D-03
1	1.050000D+00	-5.000000D-02	3 499377D-01
2	2 100000D+00	2 500000D-03	2 048950D-03
3	3.150000D+00	-1 250000D-04	4 195939D-03
4	4 200000D+00	6 250000D-06	8 583190D-03
5	5 250000D+00	-3 125000D-07	2 282622D-02
6	6 300000D+00	1.562500D-08	6 257595D-02
7	7.350000D+00	-7 812500D-10	1 601981D-01
8	8 400000D+00	3 906250D-11	3 527998D-01
9	9 450000D+00	-1 953125D-12	7 162961D-02
10	1 050000D+01	9 765625D-14	3 559005D-02
11	1.155000D+01	-4.882813D-15	1 414534D-01
12	1 260000D+01	2.441406D-16	6 090320D-01
13	1.365000D+01	-1 220703D-17	8 165649D-01
14	1 470000D+01	6 103516D-19	4 754262D-01
15	1 575000D+01	-3 051758D-20	2.753645D-05
			7 372922D-01
			9 406692D-01
			9 904654D-01
			9 966427D-01
			9 988226D-01
			9 995877D-01
			9 998557D-01
			1 000036D+00
			9 999982D-01
			1 444980D-07
			1 000000D+00

16	1.680000D+01	1	525879D-21	5	056531D-08	1	000000D+00	9	999495D-01
17	1.785000D+01	-7	629395D-23	1	769471D-08	1	000000D+00	9	999823D-01
18	1.890000D+01	3.814697D-24		6	192048D-09	1.000000D+00		9	999938D-01
19	1.995000D+01	-1	907349D-25	2.166831D-09		1	000000D+00	9	999978D-01
20	2.100000D+01	9	536743D-27	7	582560D-10	1	000000D+00	9.999992D-01	
21	2.205000D+01	-4	768372D-28	2	653424D-10	1.000000D+00		9.999997D-01	
22	2.310000D+01	2	384186D-29	9.285333D-11		1	000000D+00	9.999999D-01	
23	2.415000D+01	-1.192093D-30		3	249288D-11	1	000000D+00	1	000000D+00
24	2.520000D+01	5	960464D-32	1	137049D-11	1	000000D+00	1	000000D+00
25	2.625000D+01	-2	980232D-33	3.978963D-12		1	000000D+00	1	000000D+00
26	2.730000D+01	1	490116D-34	1	392389D-12	1	000000D+00	1	000000D+00
27	2.835000D+01	-7.450581D-36		4	872495D-13	1.000000D+00		1	000000D+00
28	2.940000D+01	3	725290D-37	1.705070D-13		1	000000D+00	1	000000D+00
29	3.045000D+01	-1.862645D-38		5	966684D-14	1	000000D+00	1	000000D+00
30	3.150000D+01	9	313226D-40	2	087968D-14	1.000000D+00		1	000000D+00

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Explicit tests for delx = 1.95000000D+00 (Damped oscillations)

j	x	exp	logistic
0	0.000000D+00	1 000000D+00	1 000000D+00
1	1.950000D+00	-9 500000D-01	1.422741D-01
2	3 900000D+00	9 025000D-01	2.024191D-02
3	5 850000D+00	-8 573750D-01	2 879899D-03
4	7 800000D+00	8 145062D-01	2 879899D-03
5	9 750000D+00	-7 737809D-01	2 545850D-02
6	1 170000D+01	7 350919D-01	2 545850D-02
7	1 365000D+01	-6 983373D-01	7 383871D-02
8	1 560000D+01	6 634204D-01	7 383871D-02
9	1 755000D+01	-6 302494D-01	2 071925D-01
10	1 950000D+01	5.987369D-01	2 071925D-01
11	2.145000D+01	-5 688001D-01	5 829466D-05
12	2 340000D+01	5.403601D-01	5 829466D-05
13	2 535000D+01	-5 133421D-01	8.293819D-06
14	2 730000D+01	4 876750D-01	8.293819D-06
15	2.925000D+01	-4 632912D-01	1.179995D-06
16	3 120000D+01	4 401267D-01	1.179995D-06
17	3 315000D+01	-4.181203D-01	1 678828D-07
18	3 510000D+01	3 972143D-01	1 678828D-07
19	3.705000D+01	-3 773536D-01	9 867881D-01
20	3 900000D+01	3 584859D-01	9 867881D-01
21	4.095000D+01	-3 405616D-01	1 012211D+00
22	4 290000D+01	3 235335D-01	1 012211D+00
23	4 485000D+01	-3 073569D-01	9 881089D-01
24	4 680000D+01	2.919890D-01	9 881089D-01
25	4 875000D+01	-2 773896D-01	1.011021D+00
26	5.070000D+01	2 635201D-01	1.011021D+00
27	5.265000D+01	-2 503441D-01	9 892934D-01
28	5 460000D+01	2.378269D-01	9 892934D-01
29	5 655000D+01	-2 259355D-01	1.009948D+00
30	5 850000D+01	2 146388D-01	1.009948D+00
31	6 045000D+01	-2 039068D-01	9 903566D-01
32	6.240000D+01	1.937115D-01	9 903566D-01

33	6.435000D+01	-1	840259D-01	1.130186D-28	1	003577D+00	1	000000D+00
34	6.630000D+01	1	748246D-01	1.607962D-29	9	965772D-01	1	000000D+00
35	6.825000D+01	-1.	660834D-01	2.287713D-30	1.	003229D+00	1	000000D+00
36	7.020000D+01	1	577792D-01	3.254823D-31	9.	969123D-01	1	000000D+00
37	7.215000D+01	-1	498903D-01	4.630769D-32	1.	002915D+00	1	000000D+00
38	7.410000D+01	1.	423957D-01	6.588384D-33	9	972144D-01	1	000000D+00
39	7.605000D+01	-1.	352760D-01	9.373562D-34	1	002631D+00	1	000000D+00
40	7.800000D+01	1	285122D-01	1.333615D-34	9.	974869D-01	1	000000D+00
41	7.995000D+01	-1	220865D-01	1.897388D-35	1	002375D+00	1.	000000D+00
42	8.190000D+01	1	159822D-01	2.699491D-36	9	977326D-01	1	000000D+00
43	8.385000D+01	-1	101831D-01	3.840676D-37	1.	002144D+00	1	000000D+00
44	8.580000D+01	1	046740D-01	5.464286D-38	9	979542D-01	1	000000D+00
45	8.775000D+01	-9	944026D-02	7.774263D-39	1	001935D+00	1	000000D+00
46	8.970000D+01	9	446824D-02	1.106076D-39	9	981541D-01	1	000000D+00
47	9.165000D+01	-8	974483D-02	1.573659D-40	1.	001747D+00	1	000000D+00
48	9.360000D+01	8.	525759D-02	2.238909D-41	9	983345D-01	1.	000000D+00
49	9.555000D+01	-8	099471D-02	3.185387D-42	1	001577D+00	1	000000D+00
50	9.750000D+01	7	694498D-02	4.531980D-43	9.	984972D-01	1	000000D+00

121	2.359500D+02	-2	016305D-03	3.374559-103	1	000039D+00	1	000000D+00
122	2.379000D+02	1	915490D-03	4.801122-104	9	999626D-01	1	000000D+00
123	2.398500D+02	-1.	819715D-03	6.830752-105	1	000035D+00	1.	000000D+00
124	2.418000D+02	1.	728730D-03	9.718389-106	9	999663D-01	1.	000000D+00
125	2.437500D+02	-1	642293D-03	1.382675-106	1	000032D+00	1	000000D+00
126	2.457000D+02	1	560178D-03	1.967188-107	9	999696D-01	1	000000D+00
127	2.476500D+02	-1	482169D-03	2.798798-108	1	000029D+00	1	000000D+00
128	2.496000D+02	1	408061D-03	3.981964-109	9	999725D-01	1	000000D+00
129	2.515500D+02	-1	337658D-03	5.665302-110	1	000026D+00	1	000000D+00
130	2.535000D+02	1	270775D-03	8.060256-111	9	999752D-01	1	000000D+00

191	3.724500D+02	-5	561713D-05	1.766120-162	1.	000001D+00	1	000000D+00
192	3.744000D+02	5.	283628D-05	2.512731-163	9	999990D-01	1.	000000D+00
193	3.763500D+02	-5	019446D-05	3.574965-164	1.	000001D+00	1.	000000D+00
194	3.783000D+02	4	768474D-05	5.086248-165	9	999991D-01	1.	000000D+00
195	3.802500D+02	-4	530050D-05	7.236412-166	1	000001D+00	1	000000D+00
196	3.822000D+02	4.	303548D-05	1.029554-166	9	999992D-01	1.	000000D+00
197	3.841500D+02	-4	088370D-05	1.464788-167	1	000001D+00	1	000000D+00
198	3.861000D+02	3	883952D-05	2.084014-168	9	999992D-01	1.	000000D+00
199	3.880500D+02	-3	689754D-05	2.965011-169	1.	000001D+00	1	000000D+00
200	3.900000D+02	3	505267D-05	4.218442-170	9.	999993D-01	1	000000D+00

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Explicit tests for delx = 2 05000000D+00 (Period-2 limit cycle)

j	x	exp	logistic
0	0 000000D+00	1.000000D+00	1.000000D+00
1	2 050000D+00	-1.050000D+00	1.287349D-01
2	4.100000D+00	1 102500D+00	1 657268D-02
3	6 150000D+00	-1 157625D+00	2 133482D-03
4	8.200000D+00	1.215506D+00	2.746536D-04
5	1 025000D+01	-1 276282D+00	3 535750D-05
6	1 230000D+01	1 340096D+00	4 551744D-06
7	1 435000D+01	-1 407100D+00	5 859684D-07

8	1	640000D+01	1	477455D+00	7	543458D-08	8	714170D-01	9	999246D-01
9	1	845000D+01	-1	551328D+00	9	711064D-09	1	101118D+00	9	999903D-01
10	2	050000D+01	1	628895D+00	1	250153D-09	8	728647D-01	9	999988D-01
11	2	255000D+01	-1	710339D+00	1	609383D-10	1	100357D+00	9	999998D-01
12	2	460000D+01	1	795856D+00	2	071838D-11	8	739783D-01	1	000000D+00
13	2	665000D+01	-1	885649D+00	2	667178D-12	1	099766D+00	1	000000D+00
14	2	870000D+01	1	979932D+00	3	433589D-13	8	748419D-01	1	000000D+00
15	3	075000D+01	-2	078928D+00	4	420228D-14	1	099304D+00	1	000000D+00
16	3	280000D+01	2	182875D+00	5	690376D-15	8	755156D-01	1	000000D+00
17	3	485000D+01	-2	292018D+00	7	325501D-16	1	098941D+00	1	000000D+00
18	3	690000D+01	2	406619D+00	9	430476D-17	8	760438D-01	1	000000D+00
19	3	895000D+01	-2	526950D+00	1	214031D-17	1	098655D+00	1	000000D+00
20	4	100000D+01	2	653298D+00	1	562882D-18	8	764593D-01	1	000000D+00
21	4	305000D+01	-2	785963D+00	2	011975D-19	1	098430D+00	1	000000D+00
22	4	510000D+01	2	925261D+00	2	590114D-20	8	767871D-01	1	000000D+00
23	4	715000D+01	-3	071524D+00	3	334381D-21	1	098252D+00	1	000000D+00
24	4	920000D+01	3	225100D+00	4	292512D-22	8	770463D-01	1	000000D+00
25	5	125000D+01	-3	386355D+00	5	525961D-23	1	098110D+00	1	000000D+00
26	5	330000D+01	3	555673D+00	7	113840D-24	8	772517D-01	1	000000D+00
27	5	535000D+01	-3	733456D+00	9	157996D-25	1	097998D+00	1	000000D+00
28	5	740000D+01	3	920129D+00	1	178954D-25	8	774146D-01	1	000000D+00
29	5	945000D+01	-4	116136D+00	1	517725D-26	1	097909D+00	1	000000D+00
30	6	150000D+01	4	321942D+00	1	953842D-27	8	775440D-01	1	000000D+00
31	6	355000D+01	-4	538039D+00	2	515276D-28	1	097838D+00	1	000000D+00
32	6	560000D+01	4	764941D+00	3	238038D-29	8	776468D-01	1	000000D+00
33	6	765000D+01	-5	003189D+00	4	168486D-30	1	097782D+00	1	000000D+00
34	6	970000D+01	5	253348D+00	5	366296D-31	8	777286D-01	1	000000D+00
35	7	175000D+01	-5	516015D+00	6	908296D-32	1	097737D+00	1	000000D+00
36	7	380000D+01	5	791816D+00	8	893388D-33	8	777937D-01	1	000000D+00
37	7	585000D+01	-6	081407D+00	1	144889D-33	1	097701D+00	1	000000D+00
38	7	790000D+01	6	385477D+00	1	473872D-34	8	778455D-01	1	000000D+00
39	7	995000D+01	-6	704751D+00	1	897388D-35	1	097673D+00	1	000000D+00
40	8	200000D+01	7	039989D+00	2	442601D-36	8	778867D-01	1	000000D+00
41	8	405000D+01	-7	391988D+00	3	144480D-37	1	097650D+00	1	000000D+00
42	8	610000D+01	7	761588D+00	4	048043D-38	8	779196D-01	1	000000D+00
43	8	815000D+01	-8	149667D+00	5	211244D-39	1	097632D+00	1	000000D+00
44	9	020000D+01	8	557150D+00	6	708690D-40	8	779458D-01	1	000000D+00
45	9	225000D+01	-8	985008D+00	8	636426D-41	1	097618D+00	1	000000D+00
46	9	430000D+01	9	434258D+00	1	111809D-41	8	779667D-01	1	000000D+00
47	9	635000D+01	-9	905971D+00	1	431287D-42	1	097606D+00	1	000000D+00
48	9	840000D+01	1	040127D+01	1	842566D-43	8	779833D-01	1	000000D+00
49	1	004500D+02	-1	092133D+01	2	372025D-44	1	097597D+00	1	000000D+00
50	1	025000D+02	1	146740D+01	3	053624D-45	8	779966D-01	1	000000D+00
51	1	045500D+02	-1	204077D+01	3	931080D-46	1	097590D+00	1	000000D+00
52	1	066000D+02	1	264281D+01	5	060672D-47	8	780072D-01	1	000000D+00
53	1	086500D+02	-1	327495D+01	6	514852D-48	1	097584D+00	1	000000D+00
54	1	107000D+02	1	393870D+01	8	386888D-49	8	780156D-01	1	000000D+00
55	1	127500D+02	-1	463563D+01	1	079685D-49	1	097579D+00	1	000000D+00
56	1	148000D+02	1	536741D+01	1	389932D-50	8	780223D-01	1	000000D+00
57	1	168500D+02	-1	613578D+01	1	789327D-51	1	097576D+00	1	000000D+00
58	1	189000D+02	1	694257D+01	2	303489D-52	8	780277D-01	1	000000D+00
59	1	209500D+02	-1	778970D+01	2	965394D-53	1	097573D+00	1	000000D+00
60	1	230000D+02	1	867919D+01	3	817497D-54	8	780320D-01	1	000000D+00
61	1	250500D+02	-1	961315D+01	4	914451D-55	1	097570D+00	1	000000D+00

62	1.271000D+02	2	059380D+01	6.326614D-56	8.780354D-01	1	000000D+00
63	1.291500D+02	-2	162349D+01	8.144561D-57	1.097568D+00	1	000000D+00
64	1.312000D+02	2	270467D+01	1.048489D-57	8.780381D-01	1	000000D+00
65	1.332500D+02	-2	383990D+01	1.349772D-58	1.097567D+00	1	000000D+00
66	1.353000D+02	2	503190D+01	1.737627D-59	8.780402D-01	1	000000D+00
67	1.373500D+02	-2	628349D+01	2.236933D-60	1.097566D+00	1	000000D+00
68	1.394000D+02	2	759766D+01	2.879713D-61	8.780420D-01	1	000000D+00
69	1.414500D+02	-2	897755D+01	3.707196D-62	1.097565D+00	1	000000D+00
70	1.435000D+02	3	042643D+01	4.772455D-63	8.780434D-01	1	000000D+00
71	1.455500D+02	-3	194775D+01	6.143815D-64	1.097564D+00	1	000000D+00
72	1.476000D+02	3	354513D+01	7.909235D-65	8.780445D-01	1	000000D+00
73	1.496500D+02	-3	522239D+01	1.018195D-65	1.097563D+00	1	000000D+00
74	1.517000D+02	3	698351D+01	1.310772D-66	8.780453D-01	1	000000D+00
75	1.537500D+02	-3	883269D+01	1.687421D-67	1.097563D+00	1	000000D+00
76	1.558000D+02	4	077432D+01	2.172300D-68	8.780460D-01	1	000000D+00
77	1.578500D+02	-4	281304D+01	2.796508D-69	1.097562D+00	1	000000D+00
78	1.599000D+02	4	495369D+01	3.600082D-70	8.780466D-01	1	000000D+00
79	1.619500D+02	-4	720137D+01	4.634562D-71	1.097562D+00	1	000000D+00
80	1.640000D+02	4	956144D+01	5.966298D-72	8.780470D-01	1	000000D+00
81	1.660500D+02	-5	203951D+01	7.680708D-73	1.097562D+00	1	000000D+00
82	1.681000D+02	5	464149D+01	9.887753D-74	8.780474D-01	1	000000D+00
83	1.701500D+02	-5	737356D+01	1.272899D-74	1.097562D+00	1	000000D+00
84	1.722000D+02	6	024224D+01	1.638665D-75	8.780477D-01	1	000000D+00
85	1.742500D+02	-6	325435D+01	2.109534D-76	1.097562D+00	1	000000D+00
86	1.763000D+02	6	641707D+01	2.715707D-77	8.780479D-01	1	000000D+00
87	1.783500D+02	-6	973792D+01	3.496062D-78	1.097561D+00	1	000000D+00
88	1.804000D+02	7	322482D+01	4.500652D-79	8.780481D-01	1	000000D+00
89	1.824500D+02	-7	688606D+01	5.793910D-80	1.097561D+00	1	000000D+00
90	1.845000D+02	8	073037D+01	7.458785D-81	8.780482D-01	1	000000D+00
91	1.865500D+02	-8	476688D+01	9.602060D-82	1.097561D+00	1	000000D+00
92	1.886000D+02	8	900523D+01	1.236120D-82	8.780483D-01	1	000000D+00
93	1.906500D+02	-9	345549D+01	1.591318D-83	1.097561D+00	1	000000D+00
94	1.927000D+02	9	812826D+01	2.048582D-84	8.780484D-01	1	000000D+00
95	1.947500D+02	-1	030347D+02	2.637240D-85	1.097561D+00	1	000000D+00
96	1.968000D+02	1	081864D+02	3.395048D-86	8.780485D-01	1	000000D+00
97	1.988500D+02	-1	135957D+02	4.370612D-87	1.097561D+00	1	000000D+00
98	2.009000D+02	1	192755D+02	5.626503D-88	8.780486D-01	1	000000D+00
99	2.029500D+02	-1	252393D+02	7.243274D-89	1.097561D+00	1	000000D+00
100	2.050000D+02	1	315013D+02	9.324621D-90	8.780486D-01	1	000000D+00
101	2.070500D+02	-1	380763D+02	1.200404D-90	1.097561D+00	1	000000D+00
102	2.091000D+02	1	449801D+02	1.545339D-91	8.780486D-01	1	000000D+00
103	2.111500D+02	-1	522291D+02	1.989391D-92	1.097561D+00	1	000000D+00
104	2.132000D+02	1	598406D+02	2.561041D-93	8.780487D-01	1	000000D+00
105	2.152500D+02	-1	678326D+02	3.296953D-94	1.097561D+00	1	000000D+00
106	2.173000D+02	1	762243D+02	4.244329D-95	8.780487D-01	1	000000D+00
107	2.193500D+02	-1	850355D+02	5.463933D-96	1.097561D+00	1	000000D+00
108	2.214000D+02	1	942872D+02	7.033989D-97	8.780487D-01	1	000000D+00
109	2.234500D+02	-2	040016D+02	9.055199D-98	1.097561D+00	1	000000D+00
110	2.255000D+02	2	142017D+02	1.165720D-98	8.780487D-01	1	000000D+00
111	2.275500D+02	-2	249118D+02	1.500689D-99	1.097561D+00	1	000000D+00
112	2.296000D+02	2	361574D+02	1.931910D-100	8.780487D-01	1	000000D+00
113	2.316500D+02	-2	479652D+02	2.487043D-101	1.097561D+00	1	000000D+00
114	2.337000D+02	2	603635D+02	3.201692D-102	8.780487D-01	1	000000D+00
115	2.357500D+02	-2	733817D+02	4.121695D-103	1.097561D+00	1	000000D+00

116	2	378000D+02	2	870508D+02	5	306061-104	8	780488D-01	1	000000D+00
117	2	398500D+02	-3	014033D+02	6	830752-105	1	097561D+00	1	.000000D+00
118	2	419000D+02	3	164735D+02	8	793562-106	8	780488D-01	1	000000D+00
119	2	439500D+02	-3	322971D+02	1	132038-106	1	097561D+00	1	000000D+00
120	2	460000D+02	3	489120D+02	1	457328-107	8	780488D-01	1	000000D+00
195	3	997500D+02	-1	354919D+04	2	459126-174	1	097561D+00	1	.000000D+00
196	4	018000D+02	1	422665D+04	3	165754-175	8	780488D-01	1	000000D+00
197	4	038500D+02	-1	493798D+04	4	075430-176	1	097561D+00	1	000000D+00
198	4	059000D+02	1	568488D+04	5	246501-177	8	780488D-01	1	000000D+00
199	4	079500D+02	-1	646912D+04	6	754079-178	1	097561D+00	1	000000D+00
200	4	100000D+02	1	729258D+04	8	694857-179	8	780488D-01	1	000000D+00

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Explicit tests for delx = 2 35000000D+00 (Period-2 limit cycle)

j	x	exp	logistic
0	0.000000D+00	1 000000D+00	1 000000D-03
1	2 350000D+00	-1.350000D+00	9.536916D-02
2	4 700000D+00	1 822500D+00	9.095277D-03
3	7 050000D+00	-2 460375D+00	8.674090D-04
4	9.400000D+00	3 321506D+00	8.272407D-05
5	1 175000D+01	-4.484033D+00	1.213255D-01
6	1 410000D+01	6.053445D+00	3.718486D-01
7	1 645000D+01	-8.172151D+00	9.207551D-01
8	1 880000D+01	1.103240D+01	9.992223D+00
9	2 115000D+01	-1.489375D+01	9.999283D-01
10	2 350000D+01	2.010656D+01	9.999932D-01
11	2 585000D+01	-2 714385D+01	6 526370D-10
12	2.820000D+01	3.664420D+01	1.145999D+00
13	3.055000D+01	-4.946967D+01	7.528102D-01
14	3 290000D+01	6 678405D+01	9 999999D-01
15	3 525000D+01	-9 015847D+01	5 935915D-12
16	3 760000D+01	1 217139D+02	1.190115D+00
17	3.995000D+01	-1 643138D+02	6 584078D-01
18	4 230000D+01	2 218236D+02	1 000000D+00
19	4.465000D+01	-2 994619D+02	6 655084D-01
20	4 700000D+01	4.042736D+02	1 000000D+00
21	4 935000D+01	-5 457693D+02	1 000000D+00
22	5.170000D+01	7 367886D+02	1 000000D+00
23	5 405000D+01	-9 946646D+02	1 000000D+00
24	5 640000D+01	1 342797D+03	1 000000D+00
25	5.875000D+01	-1 812776D+03	1 000000D+00
26	6 110000D+01	2 447248D+03	1 000000D+00
27	6 345000D+01	-3 303785D+03	1 000000D+00
28	6.580000D+01	4.460109D+03	1 000000D+00
29	6 815000D+01	-6 021148D+03	1 000000D+00
30	7 050000D+01	8 128550D+03	1 000000D+00
31	7.285000D+01	-1.097354D+04	1 000000D+00
32	7 520000D+01	1 481428D+04	1 000000D+00
33	7 755000D+01	-1.999928D+04	1 000000D+00
34	7 990000D+01	2 699903D+04	1 000000D+00
35	8 225000D+01	-3 644869D+04	1 000000D+00

36	8	460000D+01	4	920573D+04	1	814207D-37	6	629988D-01	1.000000D+00
37	8	695000D+01	-6.	642773D+04	1	730194D-38	1	188063D+00	1.000000D+00
38	8	930000D+01	8.	967744D+04	1.	650072D-39	6	630018D-01	1.000000D+00
39	9	165000D+01	-1	210645D+05	1	573659D-40	1	188063D+00	1.000000D+00
40	9	400000D+01	1	634371D+05	1	500786D-41	6	630003D-01	1.000000D+00
41	9	635000D+01	-2	206401D+05	1	431287D-42	1.	188063D+00	1.000000D+00
42	9	870000D+01	2.	978642D+05	1	365006D-43	6	630011D-01	1.000000D+00
43	1	010500D+02	-4	021166D+05	1.	301795D-44	1	188063D+00	1.000000D+00
44	1.	034000D+02	5	428575D+05	1.	241511D-45	6	630006D-01	1.000000D+00
45	1	057500D+02	-7.	328576D+05	1.	184019D-46	1	188063D+00	1.000000D+00
46	1.	081000D+02	9	893577D+05	1.	129189D-47	6	630009D-01	1.000000D+00
47	1.	104500D+02	-1	335633D+06	1	076898D-48	1	188063D+00	1.000000D+00
48	1	128000D+02	1	803104D+06	1	027028D-49	6	630008D-01	1.000000D+00
49	1	151500D+02	-2.	434191D+06	9	794684D-51	1	188063D+00	1.000000D+00
50	1	175000D+02	3.	286158D+06	9	341108D-52	6	630008D-01	1.000000D+00
51	1	198500D+02	-4.	436313D+06	8	908536D-53	1	188063D+00	1.000000D+00
52	1	222000D+02	5	989023D+06	8	495996D-54	6	630008D-01	1.000000D+00
53	1	245500D+02	-8.	085181D+06	8.	102560D-55	1.	188063D+00	1.000000D+00
54	1.	269000D+02	1	091499D+07	7.	727344D-56	6	630008D-01	1.000000D+00
55	1.	292500D+02	-1	473524D+07	7	369503D-57	1	188063D+00	1.000000D+00
56	1	316000D+02	1.	989258D+07	7	028233D-58	6	630008D-01	1.000000D+00
57	1.	339500D+02	-2	685498D+07	6	702767D-59	1	188063D+00	1.000000D+00
58	1	363000D+02	3.	625422D+07	6	392373D-60	6	630008D-01	1.000000D+00
59	1	386500D+02	-4.	894320D+07	6	096353D-61	1.	188063D+00	1.000000D+00
60	1	410000D+02	6.	607332D+07	5	814040D-62	6	630008D-01	1.000000D+00
61	1	433500D+02	-8.	919898D+07	5	544802D-63	1	188063D+00	1.000000D+00
62	1	457000D+02	1.	204186D+08	5	288031D-64	6.	630008D-01	1.000000D+00
63	1	480500D+02	-1.	625651D+08	5.	043151D-65	1.	188063D+00	1.000000D+00
64	1	504000D+02	2	194629D+08	4.	809611D-66	6.	630008D-01	1.000000D+00
65	1.	527500D+02	-2	962750D+08	4	586885D-67	1	188063D+00	1.000000D+00
66	1	551000D+02	3	999712D+08	4	374474D-68	6	630008D-01	1.000000D+00
67	1	574500D+02	-5	399611D+08	4.	171899D-69	1	188063D+00	1.000000D+00
68	1	598000D+02	7	289475D+08	3	978705D-70	6	630008D-01	1.000000D+00
69	1	621500D+02	-9.	840791D+08	3	794458D-71	1	188063D+00	1.000000D+00
70	1.	645000D+02	1	328507D+09	3	618743D-72	6	630008D-01	1.000000D+00
71	1	668500D+02	-1	793484D+09	3	451165D-73	1	188063D+00	1.000000D+00
72	1	692000D+02	2.	421204D+09	3.	291347D-74	6.	630008D-01	1.000000D+00
73	1.	715500D+02	-3	268625D+09	3.	138930D-75	1	188063D+00	1.000000D+00
74	1.	739000D+02	4	412644D+09	2.	993571D-76	6	630008D-01	1.000000D+00
75	1	762500D+02	-5.	957069D+09	2	854944D-77	1	188063D+00	1.000000D+00
76	1	786000D+02	8.	042043D+09	2	722736D-78	6	630008D-01	1.000000D+00
77	1	809500D+02	-1	085676D+10	2	596651D-79	1	188063D+00	1.000000D+00
78	1	833000D+02	1.	465662D+10	2	476404D-80	6	630008D-01	1.000000D+00
79	1	856500D+02	-1	978644D+10	2	361726D-81	1.	188063D+00	1.000000D+00
80	1	880000D+02	2.	671170D+10	2	252358D-82	6	630008D-01	1.000000D+00
195	4	582500D+02	-2	600668D+25	9	650586-200	1	188063D+00	1.000000D+00
196	4	606000D+02	3	510901D+25	9	203683-201	6	630008D-01	1.000000D+00
197	4	629500D+02	-4	739717D+25	8	777475-202	1	188063D+00	1.000000D+00
198	4	653000D+02	6	398618D+25	8.	371004-203	6.	630008D-01	1.000000D+00
199	4.	676500D+02	-8	638134D+25	7	983357-204	1	188063D+00	1.000000D+00
200	4	700000D+02	1	166148D+26	7.	613660-205	6	630008D-01	1.000000D+00

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Explicit tests for delx = 2 45000000D+00 (Almost periodic with period 4)

j	x	exp	logistic
0	0 000000D+00	1 000000D+00	1 000000D-03
1	2 450000D+00	-1.450000D+00	3 447550D-03
2	4 900000D+00	2 102500D+00	1.186493D-02
3	7 350000D+00	-3 048625D+00	4.058910D-02
4	9 800000D+00	4 420506D+00	1 359961D-01
5	1 225000D+01	-6 409734D+00	4 238739D-01
6	1 470000D+01	9 294114D+00	1 022176D+00
7	1 715000D+01	-1 347647D+01	9 666405D-01
8	1 960000D+01	1.954088D+01	1 045645D+00
9	2.205000D+01	-2 833427D+01	9 287106D-01
10	2 450000D+01	4 108469D+01	1 090918D+00
11	2 695000D+01	-5 957280D+01	1 000000D+00
12	2 940000D+01	8 638056D+01	1 000000D+00
13	3 185000D+01	-1 252518D+02	1.163854D+00
14	3 430000D+01	1.816151D+02	6.966334D-01
15	3 675000D+01	1 269695D-15	1 214405D+00
16	3.920000D+01	1 095665D-16	5 764877D-01
17	4 165000D+01	-2 633419D+02	1 000000D+00
18	4.410000D+01	3 818458D+02	1 000000D+00
19	4 655000D+01	-5 536764D+02	1 000000D+00
20	4 900000D+01	8.028308D+02	8 158960D-19
21	5 145000D+01	1 040660D-20	6 720161D-01
22	5 390000D+01	-1.164105D+03	1 212022D+00
23	5.635000D+01	1 687952D+03	5 824331D-01
24	5 880000D+01	-2 447530D+03	1 000000D+00
25	6 125000D+01	3 548919D+03	6 636126D-01
26	6 370000D+01	-5 145932D+03	1 210528D+00
27	6 615000D+01	3 369038D-25	1 000000D+00
28	6.860000D+01	7 461602D+03	5 861445D-01
29	7 105000D+01	-1 081932D+04	1 000000D+00
30	7 350000D+01	2 508782D-27	1 000000D+00
31	7.595000D+01	1 568802D+04	1 180463D+00
32	7 840000D+01	-2 274763D+04	6 585389D-01
33	8 085000D+01	3 298406D+04	1 209459D+00
34	8 330000D+01	-4 782688D+04	1 000000D+00
35	8 575000D+01	1.391160D-31	6 549981D-01
36	8 820000D+01	6.934898D+04	1 000000D+00
37	9 065000D+01	-1.005560D+05	1 208638D+00
38	9 310000D+01	1 458062D+05	5.908262D-01
39	9 555000D+01	-2 114190D+05	1 000000D+00
40	9 800000D+01	3 065576D+05	1 183115D+00
41	1 004500D+02	-4 445085D+05	6 523317D-01
42	1 029000D+02	6 656864D-37	1 000000D+00
43	1 053500D+02	5 744446D-38	1 207980D+00
44	1 078000D+02	4 957089D-39	5.924536D-01
45	1 102500D+02	4 277650D-40	1 000000D+00
46	1 127000D+02	4 345791D+05	6.502250D-01
47	1 151500D+02	1 355140D+06	1 207434D+00
48	1 176000D+02	-1 964953D+06	1 206973D+00
49	1 200500D+02	3 849181D+06	5 937988D-01
		2 748785D-43	1 000000D+00
		7 714204D-36	1 184743D+00
		8 939487D-35	6 523317D-01
		1 035939D-33	1 000000D+00
		2 508782D-27	1 000000D+00
		2 907264D-26	1 000000D+00
		3 369038D-25	1 000000D+00
		4 524274D-23	1 000000D+00
		5 242886D-22	1 000000D+00
		6 075638D-21	1 000000D+00
		7 040660D-20	1 000000D+00
		8 158960D-19	1 000000D+00
		9 454886D-18	1 000000D+00
		1 269695D-15	1 000000D+00
		1 471366D-14	1 000000D+00
		1 705070D-13	1 000000D+00
		1 975894D-12	1 000000D+00
		2 289735D-11	1 000000D+00
		2 653424D-10	1 000000D+00
		3 074880D-09	1 000000D+00
		3 563277D-08	1 000000D+00
		4 129249D-07	1 000000D+00
		4 785117D-06	1 000000D+00
		5 545160D-05	1 000000D+00
		6 425924D-04	1 000000D+00
		7 446583D-03	1 000000D+00
		8 629359D-02	1 000000D+00
		1 000000D+00	1 000000D+00

50	1.225000D+02	1	170577D+08	6	293989D-54	1	206228D+00	1.000000D+00
51	1.249500D+02	-1	.697337D+08	5	.431309D-55	5	.967710D-01	1.000000D+00
52	1.274000D+02	2	.461139D+08	4	686871D-56	1	186328D+00	1.000000D+00
53	1.298500D+02	-3	568651D+08	4	044469D-57	6	.447657D-01	1.000000D+00
54	1.323000D+02	5	174544D+08	3	490117D-58	1	205921D+00	1.000000D+00
55	1.347500D+02	-7	503089D+08	3	011748D-59	5	975266D-01	1.000000D+00
56	1.372000D+02	1	087948D+09	2	598945D-60	1	186724D+00	1.000000D+00
57	1.396500D+02	-1	577525D+09	2	242723D-61	6	438299D-01	1.000000D+00
58	1.421000D+02	2	.287411D+09	1	935326D-62	1	205647D+00	1.000000D+00
59	1.445500D+02	-3	316745D+09	1	.670062D-63	5	982005D-01	1.000000D+00
60	1.470000D+02	4	809281D+09	1	441157D-64	1	187074D+00	1.000000D+00
61	1.494500D+02	-6	973457D+09	1	.243626D-65	6	.430000D-01	1.000000D+00
62	1.519000D+02	1	.011151D+10	1	073169D-66	1	.205400D+00	1.000000D+00
63	1.543500D+02	-1	466169D+10	9	260762D-68	5	.988067D-01	1.000000D+00
64	1.568000D+02	2	125946D+10	7	991444D-69	1	187388D+00	1.000000D+00
65	1.592500D+02	-3	082621D+10	6	896103D-70	6	422577D-01	1.000000D+00
66	1.617000D+02	4	469801D+10	5	950895D-71	1	205176D+00	1.000000D+00
67	1.641500D+02	-6	481211D+10	5	135241D-72	5	993556D-01	1.000000D+00
68	1.666000D+02	9	.397756D+10	4	431383D-73	1	187670D+00	1.000000D+00
69	1.690500D+02	-1	362675D+11	3	824000D-74	6	415886D-01	1.000000D+00
70	1.715000D+02	1	975878D+11	3	.299866D-75	1	204973D+00	1.000000D+00
71	1.739500D+02	-2	.865023D+11	2	847573D-76	5	.998559D-01	1.000000D+00
72	1.764000D+02	4	.154284D+11	2	457273D-77	1	.187926D+00	1.000000D+00
73	1.788500D+02	-6	.023711D+11	2	120469D-78	6	409815D-01	1.000000D+00
74	1.813000D+02	8	734382D+11	1	829829D-79	1	204786D+00	1.000000D+00
75	1.837500D+02	-1	266485D+12	1	579025D-80	6	003143D-01	1.000000D+00
76	1.862000D+02	1	.836404D+12	1	362597D-81	1	188160D+00	1.000000D+00
77	1.886500D+02	-2	.662785D+12	1	175834D-82	6	404276D-01	1.000000D+00
78	1.911000D+02	3	.861039D+12	1	014669D-83	1	204614D+00	1.000000D+00
79	1.935500D+02	-5	598506D+12	8	755945D-85	6	007362D-01	1.000000D+00
80	1.960000D+02	8	117834D+12	7	555819D-86	1	188374D+00	1.000000D+00
81	1.984500D+02	-1	177086D+13	6	520187D-87	6	399197D-01	1.000000D+00
82	2.009000D+02	1	706775D+13	5	626503D-88	1	204455D+00	1.000000D+00
83	2.033500D+02	-2	474823D+13	4	855312D-89	6	011263D-01	1.000000D+00
84	2.058000D+02	3	.588494D+13	4	189822D-90	1	188571D+00	1.000000D+00
85	2.082500D+02	-5	.203316D+13	3	.615548D-91	6	394517D-01	1.000000D+00
86	2.107000D+02	7	544808D+13	3	119986D-92	1	204307D+00	1.000000D+00
87	2.131500D+02	-1	093997D+14	2	692348D-93	6	.014883D-01	1.000000D+00
88	2.156000D+02	1	.586296D+14	2	323324D-94	1	.188754D+00	1.000000D+00
89	2.180500D+02	-2	300129D+14	2	004879D-95	6	390189D-01	1.000000D+00
90	2.205000D+02	3	335187D+14	1	730082D-96	1	.204170D+00	1.000000D+00
91	2.229500D+02	-4	.836021D+14	1	.492950D-97	6	018254D-01	1.000000D+00
92	2.254000D+02	7	012231D+14	1	288320D-98	1	188923D+00	1.000000D+00
93	2.278500D+02	-1	016773D+15	1	111738D-99	6	.386170D-01	1.000000D+00
94	2.303000D+02	1	474322D+15	9	593583-101	1	204041D+00	1.000000D+00
95	2.327500D+02	-2	.137766D+15	8	278646-102	6	.021402D-01	1.000000D+00
96	2.352000D+02	3	099761D+15	7	143941-103	1	189080D+00	1.000000D+00
97	2.376500D+02	-4	494653D+15	6	164763-104	6	382428D-01	1.000000D+00
98	2.401000D+02	6	.517248D+15	5	.319795-105	1	203921D+00	1.000000D+00
99	2.425500D+02	-9	450009D+15	4	.590642-106	6	.024351D-01	1.000000D+00
100	2.450000D+02	1	370251D+16	3	961430-107	1	.189227D+00	1.000000D+00
101	2.474500D+02	-1	.986864D+16	3	418460-108	6	.378931D-01	1.000000D+00
102	2.499000D+02	2	880953D+16	2	.949911-109	1	203808D+00	1.000000D+00
103	2.523500D+02	-4	.177382D+16	2	.545584-110	6	027121D-01	1.000000D+00

104	2	548000D+02	6	057204D+16	2	196676-111	1	189365D+00	1.000000D+00
105	2	572500D+02	-8	.782946D+16	1	895591-112	6	375656D-01	1.000000D+00
106	2	.597000D+02	1	273527D+17	1	635773-113	1	203701D+00	1.000000D+00
107	2	621500D+02	-1	846614D+17	1	411567-114	6	029728D-01	1.000000D+00
108	2	646000D+02	2	677591D+17	1	218092-115	1	189494D+00	1.000000D+00
109	2	.670500D+02	-3	882507D+17	1	051135-116	6	372580D-01	1.000000D+00
110	2	695000D+02	5	629635D+17	9	070623-118	1	203601D+00	1.000000D+00
111	2	.719500D+02	-8	162971D+17	7	.827366-119	6	.032188D-01	1.000000D+00
112	2	744000D+02	1	.183631D+18	6	754515-120	1	189616D+00	1.000000D+00
113	2	.768500D+02	-1	.716265D+18	5	.828713-121	6	369685D-01	1.000000D+00
114	2	793000D+02	2	488584D+18	5	.029806-122	1	.203506D+00	1.000000D+00
115	2	817500D+02	-3	.608446D+18	4	340400-123	6	034513D-01	1.000000D+00
116	2	842000D+02	5	232247D+18	3	745487-124	1	.189731D+00	1.000000D+00
117	2	866500D+02	-7	.586759D+18	3	232115-125	6	366953D-01	1.000000D+00
118	2	891000D+02	1	100080D+19	2	789108-126	1	.203416D+00	1.000000D+00
119	2	915500D+02	-1	.595116D+19	2	406821-127	6	.036716D-01	1.000000D+00
120	2	940000D+02	2	.312918D+19	2	.076932-128	1	189839D+00	1.000000D+00
121	2	.964500D+02	-3	353731D+19	1	792259-129	6	364372D-01	1.000000D+00
122	2	989000D+02	4	.862910D+19	1	.546605-130	1	203330D+00	1.000000D+00
123	3	.013500D+02	-7	051220D+19	1	334621-131	6	038805D-01	1.000000D+00
124	3	038000D+02	1	022427D+20	1	151692-132	1	189942D+00	1.000000D+00
125	3	062500D+02	-1	.482519D+20	9	938364-134	6	361927D-01	1.000000D+00
126	3	087000D+02	2	149653D+20	8	576171-135	1	203249D+00	1.000000D+00
127	3	111500D+02	-3	.116996D+20	7	400686-136	6	040791D-01	1.000000D+00
128	3	.136000D+02	4	519645D+20	6	386317-137	1	190040D+00	1.000000D+00
129	3	.160500D+02	-6	553485D+20	5	.510982-138	6	359608D-01	1.000000D+00
130	3	185000D+02	9	502553D+20	4	.755624-139	1	203172D+00	1.000000D+00
131	3	209500D+02	-1	377870D+21	4	103799-140	6	042681D-01	1.000000D+00
132	3	234000D+02	1	997912D+21	3	541315-141	1	190132D+00	1.000000D+00
133	3	258500D+02	-2	896972D+21	3	055928-142	6	357404D-01	1.000000D+00
134	3	283000D+02	4	200609D+21	2	637070-143	1	203098D+00	1.000000D+00
135	3	307500D+02	-6	090884D+21	2	275622-144	6	044483D-01	1.000000D+00
136	3	332000D+02	8	.831781D+21	1	963716-145	1	190220D+00	1.000000D+00
137	3	.356500D+02	-1	280608D+22	1	694561-146	6	.355307D-01	1.000000D+00
138	3	381000D+02	1	856882D+22	1	.462297-147	1	203028D+00	1.000000D+00
139	3	405500D+02	-2	.692479D+22	1	261869-148	6	046202D-01	1.000000D+00
140	3	430000D+02	3	904094D+22	1	088912-149	1	.190304D+00	1.000000D+00
141	3	454500D+02	-5	660937D+22	9	396611-151	6	353310D-01	1.000000D+00
142	3	.479000D+02	8	.208359D+22	8	108672-152	1	202961D+00	1.000000D+00
143	3	503500D+02	-1	.190212D+23	6	997264-153	6	.047845D-01	1.000000D+00
144	3	528000D+02	1	.725807D+23	6	.038190-154	1	190384D+00	1.000000D+00
145	3	552500D+02	-2	502421D+23	5	210571-155	6	351403D-01	1.000000D+00
146	3	.577000D+02	3	628510D+23	4	496388-156	1	202896D+00	1.000000D+00
147	3	601500D+02	-5	261340D+23	3	880095-157	6	049417D-01	1.000000D+00
148	3	626000D+02	7	628942D+23	3	348273-158	1	.190460D+00	1.000000D+00
149	3	.650500D+02	-1	.106197D+24	2	889345-159	6	349582D-01	1.000000D+00
150	3	.675000D+02	1	.603985D+24	2	.493319-160	1	.202835D+00	1.000000D+00
227	5	.561500D+02	0	000000D+00	2	931729-242	6	071021D-01	1.000000D+00
228	5	586000D+02	0	000000D+00	2	529894-243	1	191499D+00	1.000000D+00
229	5	610500D+02	0	000000D+00	2	.183137-244	6	.324815D-01	1.000000D+00
230	5	.635000D+02	0	000000D+00	1	883907-245	1	201981D+00	1.000000D+00
231	5	659500D+02	0	000000D+00	1	625691-246	6	071773D-01	1.000000D+00
232	5	684000D+02	0	.000000D+00	1	.402867-247	1	.191534D+00	1.000000D+00

233	5	708500D+02	0.000000D+00	1.210584-248	6	323963D-01	1.000000D+00
234	5	733000D+02	0 000000D+00	1.044657-249	1	201951D+00	1.000000D+00
291	7	129500D+02	0.000000D+00	2 342875-310	6	080910D-01	1 000000D+00
292	7	154000D+02	0.000000D+00	2 021751-311	1	191966D+00	1 000000D+00
293	7	178500D+02	0 000000D+00	1 744641-312	6	313645D-01	1 000000D+00
294	7	203000D+02	0 000000D+00	1 505514-313	1	201586D+00	1 000000D+00
295	7	227500D+02	0 000000D+00	1 299162-314	6	081405D-01	1 000000D+00
296	7	252000D+02	0 000000D+00	1 121093-315	1	191989D+00	1 000000D+00
297	7	276500D+02	0 000000D+00	9 674315-317	6	313088D-01	1 000000D+00
298	7	301000D+02	0 000000D+00	8 348316-318	1	201566D+00	1 000000D+00
299	7	325500D+02	0 000000D+00	7.204070-319	6	081889D-01	1.000000D+00
300	7	350000D+02	0 000000D+00	6 216828-320	1	192012D+00	1 000000D+00

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Explicit tests for delx = 2 95000000D+00 (Appears to be chaotic)

j	x	exp	logistic	
0	0 000000D+00	1.000000D+00	1.000000D+00	1 000000D-03
1	2.950000D+00	-1 950000D+00	5.233971D-02	3 947050D-03
2	5 900000D+00	3 802500D+00	2 739445D-03	1 554489D-02
3	8 850000D+00	-7 414875D+00	1 433817D-04	6.068946D-02
4	1 180000D+01	1 445901D+01	7 504558D-06	2 288579D-01
5	1 475000D+01	-2 819506D+01	3 927864D-07	7 494797D-01
6	1.770000D+01	5.498037D+01	2 055832D-08	1 303371D+00
7	2.065000D+01	-1.072117D+02	1 076017D-09	1.369251D-01
8	2.360000D+01	2.090629D+02	5 631839D-11	4 855460D-01
9	2 655000D+01	-4 076726D+02	2.947688D-12	1 222430D+00
10	2 950000D+01	7 949615D+02	1.542811D-13	4 203109D-01
11	3 245000D+01	-1 550175D+03	8.075028D-15	1 139077D+00
12	3 540000D+01	3 022841D+03	4 226446D-16	6 717389D-01
13	3 835000D+01	-5 894540D+03	2.212109D-17	1 322231D+00
14	4.130000D+01	1 149435D+04	1 157812D-18	6 534326D-02
15	4 425000D+01	-2.241399D+04	6 059952D-20	2.455102D-01
16	4 720000D+01	4 370728D+04	3 171761D-21	7 919532D-01
17	5.015000D+01	-8.522920D+04	1.660090D-22	1 278005D+00
18	5 310000D+01	1 661969D+05	8.688864D-24	2 298941D-01
19	5 605000D+01	-3 240840D+05	4 547726D-25	7.521703D-01
20	5.900000D+01	6 319638D+05	2.380266D-26	1 302080D+00
21	6.195000D+01	-1 232329D+06	1 245824D-27	1.417488D-01
22	6 490000D+01	2 403042D+06	6 520608D-29	5 006343D-01
23	6 785000D+01	-4 685933D+06	3.412867D-30	1.238133D+00
24	7 080000D+01	9 137569D+06	1.786285D-31	3.683536D-01
25	7.375000D+01	-1.781826D+07	9 349362D-33	1 054728D+00
26	7 670000D+01	3 474561D+07	4 893428D-34	8.844450D-01
27	7.965000D+01	-6.775393D+07	2 561206D-35	1 185941D+00
28	8 260000D+01	1.321202D+08	1 340528D-36	5 354215D-01
29	8 555000D+01	-2 576343D+08	7.016283D-38	1.269220D+00
30	8 850000D+01	5 023869D+08	3 672302D-39	2.612060D-01
31	9.145000D+01	-9.796545D+08	1 922072D-40	8 304894D-01
32	9 440000D+01	1 910326D+09	1 006007D-41	1.245781D+00
33	9.735000D+01	-3 725136D+09	5 265410D-43	3.425231D-01
34	1 003000D+02	7.264016D+09	2.755900D-44	1 006866D+00

35	1	032500D+02	-1	416483D+10	1.442430D-45	9	864718D-01	1	000000D+00	
36	1	062000D+02	2	.762142D+10	7	549636D-47	1	025840D+00	1	000000D+00
37	1	.091500D+02	-5	.386177D+10	3	951457D-48	9	.476420D-01	1	.000000D+00
38	1	121000D+02	1	050305D+11	2	.068181D-49	1	094011D+00	1	000000D+00
39	1	150500D+02	-2	048094D+11	1	082480D-50	7	906061D-01	1	.000000D+00
40	1	.180000D+02	3	993783D+11	5	665668D-52	1	.278973D+00	1	.000000D+00
41	1	.209500D+02	-7	.787877D+11	2	.965394D-53	2	.264161D-01	1	000000D+00
42	1	.239000D+02	1	518636D+12	1	552079D-54	7	431142D-01	1	000000D+00
43	1	.268500D+02	-2	.961340D+12	8	.123533D-56	1	306256D+00	1	000000D+00
44	1	.298000D+02	5	.774613D+12	4	251833D-57	1	261126D-01	1	000000D+00
45	1	.327500D+02	-1	126050D+13	2	225397D-58	4	.512269D-01	1	.000000D+00
46	1	.357000D+02	2	.195797D+13	1	164766D-59	1	181709D+00	1	000000D+00
47	1	.386500D+02	-4	.281804D+13	6	096353D-61	5	482626D-01	1	.000000D+00
48	1	.416000D+02	8	.349517D+13	3	190813D-62	1	278891D+00	1	000000D+00
49	1	.445500D+02	-1	628156D+14	1	670062D-63	2	267101D-01	1	000000D+00
50	1	.475000D+02	3	174904D+14	8	741056D-65	7	438824D-01	1	000000D+00
51	1	.504500D+02	-6	191062D+14	4	575043D-66	1	305920D+00	1	.000000D+00
52	1	.534000D+02	1	207257D+15	2	.394564D-67	1	.273725D-01	1	.000000D+00
53	1	.563500D+02	-2	354151D+15	1	253308D-68	4	552613D-01	1	000000D+00
54	1	.593000D+02	4	590595D+15	6	559776D-70	1	186857D+00	1	000000D+00
55	1	.622500D+02	-8	951661D+15	3	.433368D-71	5	326289D-01	1	000000D+00
56	1	.652000D+02	1	.745574D+16	1	797015D-72	1	266988D+00	1	000000D+00
57	1	.681500D+02	-3	403869D+16	9	405521D-74	2	690890D-01	1	000000D+00
58	1	.711000D+02	6	637545D+16	4	922822D-75	8	492954D-01	1	000000D+00
59	1	.740500D+02	-1	294321D+17	2	576591D-76	1	226874D+00	1	.000000D+00
60	1	.770000D+02	2	523926D+17	1	348580D-77	4	057541D-01	1	000000D+00
61	1	.799500D+02	-4	921656D+17	7	.058428D-79	1	117051D+00	1	.000000D+00
62	1	.829000D+02	9	.597230D+17	3	694360D-80	7	.313320D-01	1	000000D+00
63	1	.858500D+02	-1	871460D+18	1	.933617D-81	1	310964D+00	1	000000D+00
64	1	.888000D+02	3	.649347D+18	1	.012050D-82	1	083584D-01	1	000000D+00
65	1	.917500D+02	-7	.116226D+18	5	297038D-84	3	933783D-01	1	000000D+00
66	1	.947000D+02	1	.387664D+19	2	772454D-85	1	097342D+00	1	.000000D+00
67	1	.976500D+02	-2	705945D+19	1	451094D-86	7	822302D-01	1	.000000D+00
68	2	006000D+02	5	276593D+19	7	594985D-88	1	.284751D+00	1	000000D+00
69	2	.035500D+02	-1	028936D+20	3	.975193D-89	2	055395D-01	1	000000D+00
70	2	.065000D+02	2	.006424D+20	2	.080604D-90	6	.872539D-01	1	000000D+00
71	2	.094500D+02	-3	912528D+20	1	.088982D-91	1	.321315D+00	1	000000D+00
72	2	.124000D+02	7	.629429D+20	5	.699701D-93	6	886781D-02	1	000000D+00
73	2	.153500D+02	-1	.487739D+21	2	983207D-94	2	580367D-01	1	.000000D+00
74	2	.183000D+02	2	901090D+21	1	561402D-95	8	228252D-01	1	000000D+00
75	2	.212500D+02	-5	657126D+21	8	172330D-97	1	252888D+00	1	000000D+00
76	2	.242000D+02	1	.103140D+22	4	.277373D-98	3	182100D-01	1	000000D+00
77	2	.271500D+02	-2	151122D+22	2	238765D-99	9	.582196D-01	1	000000D+00
78	2	.301000D+02	4	.194688D+22	1	171763-100	1	076322D+00	1	000000D+00
79	2	.330500D+02	-8	.179642D+22	6	132972-102	8	.339877D-01	1	.000000D+00
80	2	.360000D+02	1	.595030D+23	3	209980-103	1	242422D+00	1	000000D+00
81	2	.389500D+02	-3	110309D+23	1	.680094-104	3	539113D-01	1	000000D+00
82	2	.419000D+02	6	065102D+23	8	793562-106	1	028453D+00	1	000000D+00
83	2	.448500D+02	-1	182695D+24	4	602524-107	9	421293D-01	1	000000D+00
84	2	.478000D+02	2	.306255D+24	2	.408948-108	1	102968D+00	1	000000D+00
85	2	.507500D+02	-4	.497198D+24	1	260836-109	7	.679345D-01	1	000000D+00
86	2	.537000D+02	8	769535D+24	6	599179-111	1	293657D+00	1	.000000D+00
87	2	.566500D+02	-1	.710059D+25	3	453991-112	1	729764D-01	1	000000D+00
88	2	.596000D+02	3	334616D+25	1	807809-113	5	949903D-01	1	000000D+00

89	2	625500D+02	-6	502501D+25	9.462018-115	1.305872D+00	1.000000D+00
90	2	655000D+02	1	267988D+26	4 952392-116	1 275546D-01	1.000000D+00
91	2	684500D+02	-2	472576D+26	2 592068-117	4 558435D-01	1 000000D+00
92	2	714000D+02	4	821523D+26	1 356681-118	1.187592D+00	1 000000D+00
93	2	743500D+02	-9	401970D+26	7.100826-120	5 303841D-01	1 000000D+00
94	2	773000D+02	1	833384D+27	3 716552-121	1 265161D+00	1 000000D+00
95	2	802500D+02	-3	575099D+27	1.945232-122	2 755217D-01	1 000000D+00
96	2	832000D+02	6	971443D+27	1 018129-123	8.643697D-01	1 000000D+00
97	2	861500D+02	-1	359431D+28	5 328856-125	1 210212D+00	1 000000D+00
98	2	891000D+02	2	650891D+28	2 789108-126	4 597284D-01	1 000000D+00
99	2	920500D+02	-5	169238D+28	1.459811-127	1 192444D+00	1 000000D+00
100	2	950000D+02	1	008001D+29	7 640607-129	5 154815D-01	1 000000D+00
101	2	979500D+02	-1	965603D+29	3.999071-130	1 252274D+00	1 000000D+00
102	3	009000D+02	3	832925D+29	2.093102-131	3 203196D-01	1 000000D+00
103	3	038500D+02	-7	474204D+29	1 095523-132	9 625788D-01	1 000000D+00
104	3	068000D+02	1	457470D+30	5 733937-134	1 068840D+00	1 000000D+00
105	3	097500D+02	0	000000D+00	3.001126-135	8 517814D-01	1 000000D+00
106	3	127000D+02	0	000000D+00	1 570781-136	1 224218D+00	1 000000D+00
107	3	156500D+02	0	000000D+00	8.221419-138	4 144658D-01	1 000000D+00
108	3	186000D+02	0	000000D+00	4 303067-139	1.130383D+00	1 000000D+00
109	3	215500D+02	0	000000D+00	2 252212-140	6.956030D-01	1 000000D+00
110	3	245000D+02	0	000000D+00	1 178801-141	1 320234D+00	1 000000D+00
111	3	274500D+02	0	000000D+00	6 169812-143	7 302013D-02	1 000000D+00
112	3	304000D+02	0	000000D+00	3 229261-144	2 727003D-01	1 000000D+00
113	3	333500D+02	0	000000D+00	1 690186-145	8 577880D-01	1 000000D+00
114	3	363000D+02	0	000000D+00	8 846383-147	1.217652D+00	1 000000D+00
115	3	392500D+02	0	000000D+00	4 630171-148	4 358307D-01	1 000000D+00
116	3	422000D+02	0	000000D+00	2 423418-149	1.161183D+00	1 000000D+00
117	3	451500D+02	0	000000D+00	1 268410-150	6.090510D-01	1 000000D+00
118	3	481000D+02	0	000000D+00	6 638819-152	1.311469D+00	1 000000D+00
119	3	510500D+02	0	000000D+00	3.474739-153	1.064464D-01	1 000000D+00
120	3	540000D+02	0	000000D+00	1 818668-154	3.870372D-01	1 000000D+00
121	3	569500D+02	0	000000D+00	9 518854-156	1 086893D+00	1 000000D+00
122	3	599000D+02	0	000000D+00	4.982140-157	8 082841D-01	1.000000D+00
123	3	628500D+02	0	000000D+00	2 607638-158	1 265419D+00	1 000000D+00
124	3	658000D+02	0	000000D+00	1.364830-159	2.746142D-01	1 000000D+00
125	3	687500D+02	0	000000D+00	7.143479-161	8.622578D-01	1 000000D+00
126	3	717000D+02	0	000000D+00	3 738876-162	1.212627D+00	1 000000D+00
127	3	746500D+02	0	000000D+00	1 956917-163	4 520066D-01	1.000000D+00
128	3	776000D+02	0	000000D+00	1.024244-164	1 182712D+00	1 000000D+00
129	3	805500D+02	0	000000D+00	5.360866-166	5 452309D-01	1.000000D+00
130	3	835000D+02	0	000000D+00	2 805861-167	1 276696D+00	1 000000D+00
131	3	864500D+02	0	000000D+00	1.468580-168	2 345901D-01	1 000000D+00
132	3	894000D+02	0	000000D+00	7 686502-170	7.642848D-01	1 000000D+00
133	3	923500D+02	0	000000D+00	4.023093-171	1 295738D+00	1.000000D+00
134	3	953000D+02	0	000000D+00	2.105675-172	1 653021D-01	1.000000D+00
135	3	982500D+02	0	000000D+00	1 102104-173	5 723351D-01	1 000000D+00
136	4	012000D+02	0	000000D+00	5 768380-175	1.294400D+00	1 000000D+00
137	4	041500D+02	0	000000D+00	3 019153-176	1 702410D-01	1 000000D+00
138	4	071000D+02	0	000000D+00	1 580216-177	5 869551D-01	1.000000D+00
139	4	100500D+02	0	000000D+00	8.270803-179	1 302150D+00	1 000000D+00
140	4	130000D+02	0	000000D+00	4 328914-180	1 414900D-01	1 000000D+00
141	4	159500D+02	0	000000D+00	2 265741-181	4 998281D-01	1.000000D+00
142	4	189000D+02	0	000000D+00	1.185882-182	1 237328D+00	1 000000D+00

143	4.218500D+02	0.000000D+00	6 206872-184	3.710528D-01	1.000000D+00
144	4.248000D+02	0.000000D+00	3.248659-185	1 059502D+00	1 000000D+00
145	4.277500D+02	0.000000D+00	1 700338-186	8 735267D-01	1 000000D+00
146	4.307000D+02	0.000000D+00	8 899521-188	1 199436D+00	1.000000D+00
147	4 336500D+02	0 000000D+00	4 657983-189	4.937636D-01	1 000000D+00
148	4 366000D+02	0 000000D+00	2 437975-190	1 231149D+00	1 000000D+00
149	4 395500D+02	0 000000D+00	1 276029-191	3 916417D-01	1 000000D+00
150	4 425000D+02	0 000000D+00	6 678697-193	1 094504D+00	1 000000D+00
151	4 454500D+02	0.000000D+00	3 495611-194	7 893700D-01	1 000000D+00
152	4 484000D+02	0 000000D+00	1 829592-195	1 279852D+00	1 000000D+00
153	4 513500D+02	0 000000D+00	9 576032-197	2 232539D-01	1 000000D+00
154	4 543000D+02	0 000000D+00	5 012067-198	7 348182D-01	1 000000D+00
155	4 572500D+02	0 000000D+00	2 623301-199	1 309656D+00	1 000000D+00
156	4 602000D+02	0 000000D+00	1 373028-200	1 133031D-01	1.000000D+00
157	4 631500D+02	0 000000D+00	7 186389-202	4 096763D-01	1.000000D+00
158	4 661000D+02	0 000000D+00	3 761335-203	1 123109D+00	1 000000D+00
159	4 690500D+02	0 000000D+00	1 968672-204	7 152274D-01	1 000000D+00
160	4 720000D+02	0 000000D+00	1 030397-205	1 316075D+00	1.000000D+00
161	4 749500D+02	0.000000D+00	5 393067-207	8 893857D-02	1 000000D+00
162	4 779000D+02	0 000000D+00	2 822715-208	3 279727D-01	1.000000D+00
163	4 808500D+02	0 000000D+00	1 477401-209	9 781721D-01	1.000000D+00
164	4 838000D+02	0 000000D+00	7 732673-211	1 041159D+00	1 000000D+00
165	4 867500D+02	0.000000D+00	4 047258-212	9 147428D-01	1.000000D+00
166	4 897000D+02	0 000000D+00	2 118323-213	1 144809D+00	1 000000D+00
167	4 926500D+02	0 000000D+00	1.108724-214	6 557629D-01	1 000000D+00
168	4 956000D+02	0 000000D+00	5.803029-216	1.321690D+00	1.000000D+00
169	4 985500D+02	0 000000D+00	3.037289-217	6.742627D-02	1.000000D+00
170	5 015000D+02	0 000000D+00	1 589708-218	2.529222D-01	1.000000D+00
171	5 044500D+02	0 000000D+00	8 320484-220	8 103322D-01	1.000000D+00
172	5 074000D+02	0 000000D+00	4 354917-221	1 263729D+00	1 000000D+00
173	5 103500D+02	0 000000D+00	2 279351-222	2 805462D-01	1 000000D+00
174	5.133000D+02	0 000000D+00	1.193005-223	8 759744D-01	1 000000D+00
175	5 162500D+02	0 000000D+00	6 244156-225	1 196472D+00	1 000000D+00
176	5 192000D+02	0 000000D+00	3.268173-226	5 030059D-01	1 000000D+00
177	5 221500D+02	0 000000D+00	1.710552-227	1.240479D+00	1.000000D+00
178	5 251000D+02	0.000000D+00	8.952979-229	3 604663D-01	1 000000D+00
179	5 280500D+02	0 000000D+00	4 685963-230	1 040531D+00	1 000000D+00
180	5 310000D+02	0 000000D+00	2.452619-231	9.161187D-01	1 000000D+00
181	5 339500D+02	0 000000D+00	1 283694-232	1 142812D+00	1 000000D+00
182	5 369000D+02	0 000000D+00	6 718815-234	6 613502D-01	1 000000D+00
183	5 398500D+02	0.000000D+00	3.516608-235	1 322050D+00	1 000000D+00
184	5 428000D+02	0 000000D+00	1.840582-236	6 603880D-02	1 000000D+00
185	5.457500D+02	0 000000D+00	9 633553-238	2 479880D-01	1 000000D+00
186	5 487000D+02	0 000000D+00	5.042173-239	7 981332D-01	1 000000D+00
187	5 516500D+02	0 000000D+00	2.639059-240	1.273427D+00	1 000000D+00
188	5 546000D+02	0 000000D+00	1.381276-241	2 462681D-01	1 000000D+00
189	5 575500D+02	0 000000D+00	7 229556-243	7.938473D-01	1.000000D+00
190	5 605000D+02	0 000000D+00	3.783928-244	1.276626D+00	1.000000D+00
191	5.634500D+02	0 000000D+00	1.980497-245	2 348400D-01	1 000000D+00
192	5 664000D+02	0.000000D+00	1.036586-246	7 649261D-01	1 000000D+00
193	5 693500D+02	0 000000D+00	5.425462-248	1.295378D+00	1.000000D+00
194	5 723000D+02	0 000000D+00	2.839671-249	1 666312D-01	1 000000D+00
195	5 752500D+02	0 000000D+00	1 486275-250	5 762837D-01	1 000000D+00
196	5.782000D+02	0 000000D+00	7 779122-252	1 296617D+00	1 000000D+00

197	5	811500D+02	0	000000D+00	4	071569-253	1	620507D-01	1	000000D+00
198	5	841000D+02	0	000000D+00	2	131047-254	5	626320D-01	1	000000D+00
199	5	870500D+02	0	000000D+00	1	115384-255	1	288560D+00	1	000000D+00
200	5	900000D+02	0	000000D+00	5	837887-257	1	916713D-01	1	000000D+00

+++++

Explicit tests for delx = 3 05000000D+00 (Runs off to -infinity)

j	x	exp	logistic
0	0 000000D+00	1.000000D+00	1.000000D-03
1	3.050000D+00	-2.050000D+00	4 735892D-02
2	6 100000D+00	4.202500D+00	2 242868D-03
3	9 150000D+00	-8.615125D+00	1 062198D-04
4	1 220000D+01	1.766101D+01	5 030456D-06
5	1 525000D+01	-3 620506D+01	2.382370D-07
6	1.830000D+01	7 422038D+01	1 128265D-08
7	2 135000D+01	-1 521518D+02	5 343340D-10
8	2.440000D+01	3 119111D+02	2 530548D-11
9	2 745000D+01	-6 394178D+02	1 198440D-12
10	3 050000D+01	1 310807D+03	5 675685D-14
11	3.355000D+01	-2 687153D+03	2 687943D-15
12	3.660000D+01	5 508665D+03	1 272981D-16
13	3 965000D+01	-1 129276D+04	6 028702D-18
14	4 270000D+01	2 315016D+04	2 855128D-19
15	4 575000D+01	-4 745783D+04	1.352158D-20
16	4 880000D+01	9 728856D+04	6.403675D-22
17	5 185000D+01	-1 994415D+05	3.032712D-23
18	5 490000D+01	4.088552D+05	1.436260D-24
19	5 795000D+01	-8.381531D+05	6 801971D-26
20	6.100000D+01	1 718214D+06	3 221340D-27
21	6.405000D+01	-3 522338D+06	1.525592D-28
22	6.710000D+01	7 220794D+06	7 225040D-30
23	7.015000D+01	-1.480263D+07	3 421701D-31
24	7 320000D+01	3 034539D+07	1.620481D-32-Inf
25	7.625000D+01	-6 220804D+07	7 674423D-34-Inf

Table 2 Logistic Equation Solutions for the Implicit Method

```

program impl
c +++ implicit method testing
  implicit double precision (a-h,o-z)
  open (unit=12,file='implicit out',form='formatted')
c +++ initialize parameters
  delx = 0 95d+00
  npts = 300
  j     = 0
  yj    = 1 d+00
  yjp   = yj
  yex   = yj
  zj    = 1.d-03
  zjp   = zj
  zex   = zj
  z0    = zj

  write (12,10) delx
10 format (/ '      Implicit tests for delx =',1p,d16 8//
1 '      j      x      exp      logistic')
  write (12,20) j, x, yjp, yex, zjp, zex
20 format (1x,i6,1p,5d14 6)

  do j = 1, npts
c +++ dy/dh = -y
  yjp = yj / (1 d+00 + delx)

c +++ dz/dh = z - z**2
c      zjp = zj + delx * (zj - zj**2)
      zjp = (delx**2 - 2.d+00 * delx + 1 d+00) + 4.d+00 * delx * zj
      zjp = ( -1 d+00 + delx + sqrt(zjp) ) / ( 2 d+00 * delx )

c +++ exact solutions
  x = real(j) * delx
  yex = exp(-x)
  zex = z0 / (z0 + (1 d+00 - z0) * yex)
  write (12,20) j, x, yjp, yex, zjp, zex

  yj = yjp
  if (abs(yj) gt 1 d+30) yj = 0 d+00
  zj = zjp

  enddo

  close(unit=12)
  stop
  end

+++++

Implicit tests for delx = 9 50000000D-01 (Well behaved)

      j      x      exp      logistic
0 0 000000D+00 1 000000D+00 1 000000D+00 1 000000D-03 1 000000D-03

```

1	9	500000D-01	5.128205D-01	3	867410D-01	1	545923D-02	2	581616D-03	
2	1	900000D+00	2.629849D-01	1	495686D-01	1	039355D-01	6	648094D-03	
3	2.	850000D+00	1.348640D-01	5	784432D-02	3	054948D-01	1	701072D-02	
4	3.	800000D+00	6	916105D-02	2	237077D-02	5	413690D-01	4	282949D-02
5	4	750000D+00	3	546720D-02	8.	651695D-03	7	290349D-01	1.	037017D-01
6	5	700000D+00	1	818831D-02	3.	345965D-03	8	500960D-01	2.	302758D-01
7	6	650000D+00	9	327338D-03	1	294022D-03	9.	200089D-01	4	361616D-01
8	7.	600000D+00	4	783250D-03	5	004514D-04	9	581246D-01	6	666885D-01
9	8	550000D+00	2.	452949D-03	1	935451D-04	9	782960D-01	8	379760D-01
10	9	500000D+00	1	257923D-03	7	485183D-05	9	888087D-01	9	304256D-01
11	1.	045000D+01	6	450885D-04	2	894827D-05	9.	942447D-01	9	718935D-01
12	1	140000D+01	3	308146D-04	1	119548D-05	9	970443D-01	9	889394D-01
13	1	235000D+01	1	696485D-04	4.	329753D-06	9.	984831D-01	9.	956932D-01
14	1	330000D+01	8	699924D-05	1.	674493D-06	9.	992218D-01	9	983300D-01
15	1.	425000D+01	4.	461499D-05	6	475952D-07	9.	996009D-01	9	993535D-01
16	1	520000D+01	2	287948D-05	2	504516D-07	9	997953D-01	9	997499D-01
17	1.	615000D+01	1	173307D-05	9.	685992D-08	9	998950D-01	9	999032D-01
18	1	710000D+01	6	016958D-06	3	745971D-08	9	999462D-01	9	999626D-01
19	1.	805000D+01	3	085620D-06	1	448720D-08	9	999724D-01	9	999855D-01
20	1.	900000D+01	1.	582369D-06	5	602796D-09	9	999858D-01	9	999944D-01
21	1	995000D+01	8	114713D-07	2.	166831D-09	9.	999927D-01	9	999978D-01
22	2	090000D+01	4	161391D-07	8	380025D-10	9.	999963D-01	9.	999992D-01
23	2	185000D+01	2.	134047D-07	3	240900D-10	9	999981D-01	9	999997D-01
24	2	280000D+01	1	094383D-07	1	253389D-10	9	999990D-01	9	999999D-01
25	2	375000D+01	5	612220D-08	4	847369D-11	9	999995D-01	1	000000D+00
26	2	470000D+01	2	878062D-08	1	874676D-11	9	999997D-01	1	000000D+00
27	2	565000D+01	1	475929D-08	7	250142D-12	9	999999D-01	1.	000000D+00
28	2	660000D+01	7	568867D-09	2.	803928D-12	9.	999999D-01	1.	000000D+00
29	2	755000D+01	3.	881470D-09	1	084394D-12	1	000000D+00	1.	000000D+00
30	2	850000D+01	1	990498D-09	4	193796D-13	1	000000D+00	1	000000D+00

+++++

Implicit tests for delx = 3 05000000D+00 (Well behaved)

j	x	exp	logistic
0	0	000000D+00	1.000000D+00
1	3	050000D+00	2.469136D-01
2	6.	100000D+00	6.096632D-02
3	9	150000D+00	1.505341D-02
4	1	220000D+01	3.716892D-03
5	1	525000D+01	9.177510D-04
6	1.	830000D+01	2.266052D-04
7	2.	135000D+01	5.595190D-05
8	2	440000D+01	1.381528D-05
9	2.	745000D+01	3.411181D-06
10	3	050000D+01	8.422670D-07
11	3	355000D+01	2.079672D-07
12	3.	660000D+01	5.134991D-08
13	3	965000D+01	1.267899D-08
14	4	270000D+01	3.130615D-09
15	4	575000D+01	7.729914D-10

+++++

Implicit tests for delx = 1 00500000D+01 (Well behaved)

j	x	exp		logistic	
0	0.000000D+00	1 000000D+00	1 000000D+00	1 000000D-03	1 000000D-03
1	1.005000D+01	9 049774D-02	4 318575D-05	9 006080D-01	9 586417D-01
2	2.010000D+01	8.189841D-03	1 865009D-09	9 909304D-01	9 999981D-01
3	3.015000D+01	7 411620D-04	8 054181D-14	9.991786D-01	1 000000D+00
4	4 020000D+01	6 707349D-05	3 478258D-18	9.999257D-01	1 000000D+00
5	5.025000D+01	6.069999D-06	1.502112D-22	9 999933D-01	1 000000D+00
6	6.030000D+01	5 493212D-07	6 486983D-27	9 999994D-01	1 000000D+00
7	7 035000D+01	4 971232D-08	2 801452D-31	9 999999D-01	1 000000D+00
8	8.040000D+01	4 498853D-09	1 209828D-35	1 000000D+00	1 000000D+00
9	9 045000D+01	4 071360D-10	5.224733D-40	1 000000D+00	1 000000D+00
10	1.005000D+02	3 684489D-11	2 256340D-44	1 000000D+00	1.000000D+00

Table 3. Logistic Equation Solutions for the Midpoint Rule

```

program rk2a
c +++ Second order Runge-Kutta (midpoint rule) method testing
implicit double precision (a-h,o-z)
open (unit=12,file='rk2a out',form='formatted')
c +++ initialize parameters
delx = 1 05d+00
npts = 300
j = 0
yj = 1.d+00
yjp = yj
yex = yj
zj = 1 d-03
zjp = zj
zex = zj
z0 = zj

write (12,10) delx
10 format (/ '      RK-Midpoint tests for delx =',1p,d16 8//
1 '      j      x      exp      logistic')
write (12,20) j, x, yjp, yex, zjp, zex
20 format (1x,i6,1p,5d14 6)

do j = 1, npts
dt1 = 0 5d+00 * delx
dt2 = delx
c +++ dy/dh = -y
yj1 = yj - dt1 * yj
yjp = yj - dt2 * yj1

c +++ dz/dh = z - z**2
zj1 = zj + dt1 * (zj - zj**2)
zjp = zj + dt2 * (zj1 - zj1**2)

c +++ exact solutions
x = real(j) * delx
yex = exp(-x)
zex = z0 / (z0 + (1 d+00 - z0) * yex)
write (12,20) j, x, yjp, yex, zjp, zex

yj = yjp
if (abs(yj) .gt 1 d+30) yj = 0 d+00
zj = zjp

enddo

close(unit=12)
stop
end

```

+++++

RK-Midpoint tests for delx = 1.05000000D+00 (Well behaved)

j	x	exp	logistic
0	0.000000D+00	1 000000D+00	1 000000D-03
1	1.050000D+00	5 012500D-01	2 598259D-03
2	2 100000D+00	2.512516D-01	6 738543D-03
3	3 150000D+00	1.259398D-01	1 739324D-02
4	4 200000D+00	6 312735D-02	4 434747D-02
5	5.250000D+00	3 164258D-02	1 096177D-01
6	6 300000D+00	1.586084D-02	2 513500D-01
7	7 350000D+00	7 950248D-03	4 902694D-01
8	8 400000D+00	3 985062D-03	7 372768D-01
9	9.450000D+00	1.997512D-03	8 791317D-01
10	1 050000D+01	1 001253D-03	9 430206D-01
11	1 155000D+01	5.018781D-04	9 723597D-01
12	1 260000D+01	2 515664D-04	9 863742D-01
13	1.365000D+01	1 260977D-04	9 932271D-01
14	1 470000D+01	6 320645D-05	9.966193D-01
15	1 575000D+01	3.168223D-05	9 983090D-01
16	1 680000D+01	1 588072D-05	9 991533D-01
17	1 785000D+01	7 960211D-06	9.995758D-01
18	1.890000D+01	3 990056D-06	9 997874D-01
19	1 995000D+01	2 000015D-06	9 998935D-01
20	2 100000D+01	1 002508D-06	9 999466D-01
21	2 205000D+01	5 025070D-07	9 999732D-01
22	2 310000D+01	2 518816D-07	9.999866D-01
23	2 415000D+01	1 262557D-07	9 999933D-01
24	2 520000D+01	6.328565D-08	9 999966D-01
25	2 625000D+01	3 172193D-08	9 999983D-01
26	2 730000D+01	1 590062D-08	9 999992D-01
27	2 835000D+01	7 970185D-09	9 999996D-01
28	2 940000D+01	3 995055D-09	9 999998D-01
29	3 045000D+01	2 002521D-09	9 999999D-01
30	3.150000D+01	1 003764D-09	9 999999D-01
31	3 255000D+01	5.031367D-10	1 000000D+00
32	3.360000D+01	2 521972D-10	1 000000D+00
33	3 465000D+01	1 264139D-10	1 000000D+00
34	3 570000D+01	6 336495D-11	1 000000D+00
35	3 675000D+01	3.176168D-11	1 000000D+00

+++++

RK-Midpoint tests for delx = 2 05000000D+00 (Increases monotonically
to the wrong steady state)

j	x	exp	logistic
0	0 000000D+00	1 000000D+00	1 000000D-03
1	2 050000D+00	1 051250D+00	5 140751D-03
2	4 100000D+00	1 105127D+00	2 620476D-02
3	6 150000D+00	1 161764D+00	1 279240D-01
4	8 200000D+00	1.221305D+00	5 042558D-01
5	1 025000D+01	1 283897D+00	8.776559D-01
6	1 230000D+01	1 349696D+00	9 025281D-01
7	1 435000D+01	1 418868D+00	9 173869D-01
8	1 640000D+01	1 491585D+00	9 274440D-01
9	1 845000D+01	1.568029D+00	9 347610D-01

10	2.050000D+01	1 648390D+00	1 250153D-09	9 403454D-01	9 999988D-01
11	2 255000D+01	1 732870D+00	1.609383D-10	9 447562D-01	9.999998D-01
12	2 460000D+01	1 821680D+00	2.071838D-11	9 483315D-01	1 000000D+00
13	2 665000D+01	1.915041D+00	2 667178D-12	9.512887D-01	1 000000D+00
14	2 870000D+01	2 013187D+00	3 433589D-13	9 537751D-01	1 000000D+00
15	3.075000D+01	2 116363D+00	4 420228D-14	9.558937D-01	1 000000D+00
16	3 280000D+01	2 224826D+00	5 690376D-15	9 577193D-01	1 000000D+00
17	3.485000D+01	2 338849D+00	7 325501D-16	9 593075D-01	1 000000D+00
18	3 690000D+01	2 458715D+00	9 430476D-17	9 607005D-01	1 000000D+00
19	3 895000D+01	2 584724D+00	1.214031D-17	9 619309D-01	1 000000D+00
20	4.100000D+01	2 717191D+00	1 562882D-18	9 630246D-01	1 000000D+00
21	4.305000D+01	2 856447D+00	2 011975D-19	9 640019D-01	1 000000D+00
22	4.510000D+01	3 002840D+00	2 590114D-20	9 648795D-01	1 000000D+00
23	4 715000D+01	3 156736D+00	3 334381D-21	9 656711D-01	1 000000D+00
24	4.920000D+01	3 318518D+00	4 292512D-22	9 663878D-01	1 000000D+00
25	5 125000D+01	3 488592D+00	5.525961D-23	9 670389D-01	1 000000D+00
26	5 330000D+01	3 667383D+00	7 113840D-24	9 676323D-01	1.000000D+00
27	5.535000D+01	3 855336D+00	9.157996D-25	9 681747D-01	1 000000D+00
28	5 740000D+01	4 052922D+00	1 178954D-25	9 686718D-01	1 000000D+00
29	5 945000D+01	4 260634D+00	1 517725D-26	9 691284D-01	1 000000D+00
30	6 150000D+01	4 478992D+00	1.953842D-27	9 695488D-01	1 000000D+00
31	6 355000D+01	4 708540D+00	2 515276D-28	9 699365D-01	1 000000D+00
32	6 560000D+01	4.949853D+00	3 238038D-29	9 702948D-01	1.000000D+00
33	6 765000D+01	5.203533D+00	4 168486D-30	9 706265D-01	1 000000D+00
34	6.970000D+01	5 470214D+00	5 366296D-31	9.709341D-01	1 000000D+00
35	7 175000D+01	5.750562D+00	6 908296D-32	9 712196D-01	1 000000D+00
36	7 380000D+01	6 045279D+00	8.893388D-33	9 714850D-01	1 000000D+00
37	7 585000D+01	6 355099D+00	1.144889D-33	9 717322D-01	1 000000D+00
38	7.790000D+01	6 680798D+00	1.473872D-34	9.719625D-01	1 000000D+00
39	7.995000D+01	7 023189D+00	1.897388D-35	9.721773D-01	1 000000D+00
40	8 200000D+01	7 383127D+00	2 442601D-36	9 723780D-01	1 000000D+00
41	8 405000D+01	7 761512D+00	3 144480D-37	9 725655D-01	1 000000D+00
42	8 610000D+01	8 159290D+00	4 048043D-38	9 727410D-01	1 000000D+00
43	8 815000D+01	8.577454D+00	5 211244D-39	9 729053D-01	1 000000D+00
44	9.020000D+01	9 017048D+00	6 708690D-40	9.730593D-01	1 000000D+00
45	9.225000D+01	9 479172D+00	8 636426D-41	9 732036D-01	1.000000D+00
46	9 430000D+01	9.964979D+00	1 111809D-41	9.733391D-01	1 000000D+00
47	9 635000D+01	1 047568D+01	1 431287D-42	9 734663D-01	1.000000D+00
48	9.840000D+01	1 101256D+01	1 842566D-43	9 735858D-01	1 000000D+00
49	1 004500D+02	1.157696D+01	2.372025D-44	9 736981D-01	1 000000D+00
50	1 025000D+02	1.217028D+01	3.053624D-45	9 738038D-01	1 000000D+00
51	1 045500D+02	1 279400D+01	3 931080D-46	9.739032D-01	1 000000D+00
52	1 066000D+02	1.344970D+01	5 060672D-47	9 739968D-01	1 000000D+00
53	1 086500D+02	1 413899D+01	6 514852D-48	9 740849D-01	1 000000D+00
54	1 107000D+02	1 486362D+01	8 386888D-49	9.741679D-01	1 000000D+00
55	1.127500D+02	1 562538D+01	1 079685D-49	9 742462D-01	1 000000D+00
56	1 148000D+02	1 642618D+01	1 389932D-50	9 743200D-01	1 000000D+00
57	1 168500D+02	1 726802D+01	1 789327D-51	9.743896D-01	1 000000D+00
58	1 189000D+02	1.815300D+01	2 303489D-52	9 744552D-01	1 000000D+00
59	1 209500D+02	1 908335D+01	2.965394D-53	9 745172D-01	1 000000D+00
60	1.230000D+02	2.006137D+01	3.817497D-54	9.745757D-01	1.000000D+00
61	1 250500D+02	2 108951D+01	4 914451D-55	9.746309D-01	1 000000D+00
62	1 271000D+02	2 217035D+01	6 326614D-56	9 746831D-01	1 000000D+00
63	1 291500D+02	2.330658D+01	8 144561D-57	9.747324D-01	1 000000D+00

64	1	312000D+02	2.450104D+01	1	048489D-57	9	747790D-01	1	000000D+00	
65	1	332500D+02	2	575672D+01	1.349772D-58	9	748230D-01	1	000000D+00	
66	1.	353000D+02	2	707675D+01	1	737627D-59	9	748646D-01	1	000000D+00
67	1	373500D+02	2	846444D+01	2	236933D-60	9	749040D-01	1	000000D+00
68	1	394000D+02	2.	992324D+01	2	879713D-61	9	749412D-01	1	000000D+00
69	1.	414500D+02	3	145680D+01	3	707196D-62	9	749764D-01	1	000000D+00
70	1	435000D+02	3	306897D+01	4	772455D-63	9	750097D-01	1	000000D+00
71	1.	455500D+02	3	476375D+01	6	143815D-64	9	750412D-01	1	000000D+00
72	1.	476000D+02	3	654539D+01	7.	909235D-65	9	750710D-01	1	000000D+00
73	1.	496500D+02	3	841834D+01	1.	018195D-65	9	750992D-01	1	000000D+00
74	1	517000D+02	4	038728D+01	1	310772D-66	9	751260D-01	1	000000D+00
75	1	537500D+02	4.	245713D+01	1.	687421D-67	9	751512D-01	1.	000000D+00
76	1	558000D+02	4	463306D+01	2	172300D-68	9.	751752D-01	1.	000000D+00
77	1	578500D+02	4.	692051D+01	2.	796508D-69	9.	751979D-01	1	000000D+00
78	1	599000D+02	4	932518D+01	3	600082D-70	9	752193D-01	1.	000000D+00
79	1	619500D+02	5	185310D+01	4.	634562D-71	9	752396D-01	1	000000D+00
80	1	640000D+02	5	451057D+01	5.	966298D-72	9.	752589D-01	1	000000D+00
201	4	120500D+02	2	306257D+04	1	119332-179	9	756092D-01	1.	000000D+00
202	4.	141000D+02	2	424453D+04	1	440970-180	9	756092D-01	1	000000D+00
203	4	161500D+02	2	548706D+04	1	855032-181	9	756092D-01	1	000000D+00
204	4	182000D+02	2	679327D+04	2.	388073-182	9	756092D-01	1	000000D+00
205	4	202500D+02	2	816643D+04	3	074284-183	9	756093D-01	1	000000D+00
296	6	068000D+02	2	660594D+06	2	951946-264	9	756098D-01	1	000000D+00
297	6.	088500D+02	2.	796949D+06	3	800185-265	9.	756098D-01	1	000000D+00
298	6.	109000D+02	2	940293D+06	4	892164-266	9.	756098D-01	1	000000D+00
299	6	129500D+02	3	090983D+06	6	297923-267	9	756098D-01	1	000000D+00
300	6	150000D+02	3.	249396D+06	8	107625-268	9	756098D-01	1	000000D+00

+++++

RK-Midpoint tests for delx = 3 05000000D+00 (Damped oscillations about
the wrong steady state)

j	x	exp	logistic
0	0 000000D+00	1.000000D+00	1 000000D+00 1 000000D-03 1 000000D-03
1	3 050000D+00	2 601250D+00	4 735892D-02 8.677177D-03 2 069898D-02
2	6.100000D+00	6 766502D+00	2 242868D-03 7 370325D-02 3 085825D-01
3	9 150000D+00	1.760136D+01	1 062198D-04 5.196068D-01 9 040663D-01
4	1 220000D+01	4.578554D+01	5 030456D-06 7 934464D-01 9 949997D-01
5	1 525000D+01	1.190996D+02	2 382370D-07 6 554063D-01 9.997621D-01
6	1 830000D+01	3 098080D+02	1.128265D-08 6.559374D-01 9.999887D-01
7	2 135000D+01	8 058879D+02	5 343340D-10 6.556178D-01 9 999995D-01
8	2 440000D+01	2.096316D+03	2 530548D-11 6 558098D-01 1.000000D+00
9	2.745000D+01	5 453042D+03	1.198440D-12 6 556943D-01 1 000000D+00
10	3 050000D+01	1 418473D+04	5 675685D-14 6.557638D-01 1 000000D+00
11	3 355000D+01	3 689802D+04	2 687943D-15 6.557220D-01 1 000000D+00
12	3 660000D+01	9 598097D+04	1 272981D-16 6 557471D-01 1 000000D+00
13	3 965000D+01	2 496705D+05	6 028702D-18 6 557320D-01 1 000000D+00
14	4 270000D+01	6.494554D+05	2 855128D-19 6 557411D-01 1 000000D+00
15	4 575000D+01	1 689396D+06	1.352158D-20 6.557357D-01 1 000000D+00
16	4 880000D+01	4 394541D+06	6 403675D-22 6.557389D-01 1 000000D+00
17	5.185000D+01	1 143130D+07	3 032712D-23 6 557370D-01 1 000000D+00

18	5	490000D+01	2.973567D+07	1	436260D-24	6	557382D-01	1	000000D+00
19	5	795000D+01	7 734990D+07	6	801971D-26	6	557374D-01	1	000000D+00
20	6	100000D+01	2 012064D+08	3	221340D-27	6	557379D-01	1	000000D+00
21	6	405000D+01	5 233882D+08	1	525592D-28	6	557376D-01	1	000000D+00
22	6	710000D+01	1.361464D+09	7	225040D-30	6	557378D-01	1	000000D+00
23	7	015000D+01	3 541507D+09	3	421701D-31	6	557377D-01	1	000000D+00
24	7	320000D+01	9 212346D+09	1	620481D-32	6	557377D-01	1	000000D+00
25	7	625000D+01	2 396361D+10	7	674423D-34	6	557377D-01	1	000000D+00
26	7	930000D+01	6 233535D+10	3	634524D-35	6	557377D-01	1	000000D+00
27	8	235000D+01	1.621498D+11	1	721272D-36	6	557377D-01	1	000000D+00
28	8	540000D+01	4 217923D+11	8	151757D-38	6	557377D-01	1	000000D+00
29	8	845000D+01	1.097187D+12	3	860585D-39	6	557377D-01	1	000000D+00
30	9	150000D+01	2 854058D+12	1	828331D-40	6	557377D-01	1	000000D+00
31	9	455000D+01	7 424118D+12	8	658781D-42	6	557377D-01	1	000000D+00
32	9	760000D+01	1 931199D+13	4	100705D-43	6	557377D-01	1	000000D+00
33	1	006500D+02	5 023531D+13	1	942050D-44	6	557377D-01	1	000000D+00
34	1	037000D+02	1 306746D+14	9	197340D-46	6	557377D-01	1	000000D+00
35	1	067500D+02	3 399173D+14	4	355761D-47	6	557377D-01	1	000000D+00
36	1	098000D+02	8 842099D+14	2	062842D-48	6	557377D-01	1	000000D+00
37	1	128500D+02	2 300051D+15	9	769396D-50	6	557377D-01	1	000000D+00
38	1	159000D+02	5 983007D+15	4	626681D-51	6	557377D-01	1	000000D+00
39	1	189500D+02	1 556330D+16	2	191146D-52	6	557377D-01	1	000000D+00
40	1	220000D+02	4 048403D+16	1	037703D-53	6	557377D-01	1	000000D+00

+++++

RK-Midpoint tests for delx = 4 05000000D+00 (Solution is unstable)

j		x		exp		logistic				
0	0	000000D+00	1	000000D+00	1	000000D+00	1	000000D-03	1	000000D-03
1	4	050000D+00	5	151250D+00	1	742237D-02	1	320604D-02	5	433320D-02
2	8	100000D+00	2	653538D+01	3	035391D-04	1	672167D-01	7	673210D-01
3	1	215000D+01	1	366904D+02	5	288373D-06	1	169269D+00	9	947447D-01
4	1	620000D+01	7	041262D+02	9	213601D-08	1	889840D+00	9	999080D-01
5	2	025000D+01	3	627130D+03	1	605228D-09	-1	354992D+01	9	999984D-01
6	2	430000D+01	1	868425D+04	2	796688D-11	-6	917507D+05	1	000000D+00
7	2	835000D+01	9	624726D+04	4	872495D-13	-3	802817D+24	1	000000D+00
8	3	240000D+01	4	957937D+05	8	489044D-15	-3	473177D+99	1	000000D+00
9	3	645000D+01	2	553957D+06	1	478993D-16	-Inf		1	000000D+00
10	4	050000D+01	1	315607D+07	2	576757D-18	-Inf		1	000000D+00

Table 4 Logistic Equation Solutions for the Trapezoidal Rule

```

program rk2b
c +++ Second order Runge-Kutta (trapezoidal rule) method testing
implicit double precision (a-h,o-z)
open (unit=12,file='rk2b out',form='formatted')
c +++ initialize parameters
delx = 1 05d+00
npts = 300
j = 0
yj = 1 d+00
yjp = yj
yex = yj
zj = 1 d-03
zjp = zj
zex = zj
z0 = zj

write (12,10) delx
10 format (/ ' RK-Trapezoid tests for delx =',1p,d16 8//
1 ' j x exp logistic')
write (12,20) j, x, yjp, yex, zjp, zex
20 format (1x,i6,1p,5d14 6)

do j = 1, npts
dt2 = 0 5d+00 * delx
dt1 = delx
c +++ dy/dh = -y
yj1 = yj - dt1 * yj
yjp = yj - dt2 * (yj1 + yj)

c +++ dz/dh = z - z**2
zj1 = zj + dt1 * (zj - zj**2)
zjp = zj + dt2 * (zj1 - zj1**2 + zj - zj**2)

c +++ exact solutions
x = real(j) * delx
yex = exp(-x)
zex = z0 / (z0 + (1 d+00 - z0) * yex)
write (12,20) j, x, yjp, yex, zjp, zex

yj = yjp
if (abs(yj) gt. 1 d+30) yj = 0 d+00
zj = zjp

enddo

close(unit=12)
stop
end

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RK-Trapezoid tests for delx = 1.05000000D+00 (Well behaved)

j	x	exp	logistic
0	0 000000D+00	1 000000D+00	1 000000D-03
1	1 050000D+00	5 012500D-01	2 597970D-03
2	2 100000D+00	2.512516D-01	6 735853D-03
3	3.150000D+00	1 259398D-01	4 285213D-02
4	4 200000D+00	6 312735D-02	1 737339D-02
5	5 250000D+00	3 164258D-02	4 421354D-02
6	6 300000D+00	1 586084D-02	6 257595D-02
7	7 350000D+00	5.247518D-03	1 087867D-01
8	8 400000D+00	1 836305D-03	2 469625D-01
9	9.450000D+00	7 950248D-03	3 527998D-01
10	1.050000D+01	6 425924D-04	6 090320D-01
11	1.155000D+01	2.248673D-04	7 070302D-01
12	1.260000D+01	2.248673D-04	8 165649D-01
13	1.365000D+01	1 997512D-03	9 271184D-01
14	1.470000D+01	1 001253D-03	9 732276D-01
15	1.575000D+01	5 018781D-04	9 904654D-01
16	1.680000D+01	2 515664D-04	9 966427D-01
17	1.785000D+01	3 372015D-06	9 988226D-01
18	1.890000D+01	1 179995D-06	9 995877D-01
19	1.995000D+01	6 320645D-05	9 998557D-01
20	2 100000D+01	3 168223D-05	9 999495D-01
21	2.205000D+01	1 588072D-05	9 999823D-01
22	2 310000D+01	7 960211D-06	9 999938D-01
23	2 415000D+01	3 990056D-06	9 999978D-01
24	2 520000D+01	2 000015D-06	9 999992D-01
25	2 625000D+01	1 002508D-06	9 999997D-01
26	2 730000D+01	5 025070D-07	9 999999D-01
27	2 835000D+01	2 518816D-07	1.000000D+00
28	2 940000D+01	1 262557D-07	1.000000D+00
29	3 045000D+01	6.328565D-08	1.000000D+00
30	3.150000D+01	3.172193D-08	1.000000D+00
31	3 255000D+01	1.590062D-08	1.000000D+00
32	3 360000D+01	1 392389D-12	1.000000D+00
33	3 465000D+01	7.970185D-09	1.000000D+00
34	3 570000D+01	4.872495D-13	1.000000D+00
35	3 675000D+01	1.705070D-13	1.000000D+00

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RK-Trapezoid tests for delx = 2 05000000D+00 (Slow monotonic convergence to wrong steady state)

j	x	exp	logistic
0	0 000000D+00	1.000000D+00	1 000000D-03
1	2.050000D+00	1 051250D+00	5 138602D-03
2	4 100000D+00	1.105127D+00	7.715682D-03
3	6 150000D+00	1.161764D+00	2.613763D-02
4	8 200000D+00	1.221305D+00	1.262184D-01
5	1 025000D+01	1 283897D+00	3 193513D-01
6	1 230000D+01	2 746536D-04	7 846960D-01
7	1.435000D+01	3 535750D-05	9.658829D-01
8	1.640000D+01	4.551744D-06	7 831987D-01
9	1 845000D+01	5.859684D-07	9.954734D-01

10	2	050000D+01	1	648390D+00	1	250153D-09	8	376910D-01	9	999988D-01
11	2	255000D+01	1	732870D+00	1	609383D-10	8.438334D-01	9.999998D-01		
12	2.460000D+01	1.821680D+00	2	071838D-11	8	487614D-01	1.000000D+00			
13	2	665000D+01	1	915041D+00	2	667178D-12	8	527905D-01	1	000000D+00
14	2	870000D+01	2	013187D+00	3	433589D-13	8	561338D-01	1	000000D+00
15	3.075000D+01	2	116363D+00	4	420228D-14	8	589409D-01	1	000000D+00	
16	3	280000D+01	2	224826D+00	5	690376D-15	8.613205D-01	1	000000D+00	
17	3	485000D+01	2	338849D+00	7	325501D-16	8	633539D-01	1	000000D+00
18	3	690000D+01	2	458715D+00	9.430476D-17	8	651029D-01	1	000000D+00	
19	3	895000D+01	2	584724D+00	1	214031D-17	8.666157D-01	1	000000D+00	
20	4	100000D+01	2	717191D+00	1	562882D-18	8	679304D-01	1	000000D+00
21	4	305000D+01	2	856447D+00	2	011975D-19	8.690777D-01	1.000000D+00		
22	4	510000D+01	3	002840D+00	2	590114D-20	8	700823D-01	1	000000D+00
23	4	715000D+01	3.156736D+00	3	334381D-21	8.709647D-01	1	000000D+00		
24	4	920000D+01	3	318518D+00	4	292512D-22	8.717418D-01	1	000000D+00	
25	5.125000D+01	3.488592D+00	5.525961D-23	8.724277D-01	1	000000D+00				
26	5	330000D+01	3	667383D+00	7.113840D-24	8	730344D-01	1	000000D+00	
27	5	535000D+01	3.855336D+00	9	157996D-25	8	735720D-01	1	000000D+00	
28	5	740000D+01	4	052922D+00	1.178954D-25	8	740491D-01	1	000000D+00	
29	5.945000D+01	4.260634D+00	1	517725D-26	8.744730D-01	1	000000D+00			
30	6	150000D+01	4.478992D+00	1.953842D-27	8	748502D-01	1.000000D+00			
31	6	355000D+01	4	708540D+00	2	515276D-28	8.751861D-01	1	000000D+00	
32	6	560000D+01	4	949853D+00	3.238038D-29	8	754857D-01	1	000000D+00	
33	6.765000D+01	5.203533D+00	4.168486D-30	8	757530D-01	1	000000D+00			
34	6.970000D+01	5.470214D+00	5	366296D-31	8.759916D-01	1	000000D+00			
35	7	175000D+01	5.750562D+00	6	908296D-32	8	762049D-01	1	000000D+00	
36	7	380000D+01	6.045279D+00	8	893388D-33	8	763956D-01	1	000000D+00	
37	7	585000D+01	6.355099D+00	1.144889D-33	8	765662D-01	1	000000D+00		
38	7	790000D+01	6.680798D+00	1	473872D-34	8.767189D-01	1	000000D+00		
39	7	995000D+01	7.023189D+00	1.897388D-35	8	768557D-01	1	000000D+00		
40	8.200000D+01	7.383127D+00	2.442601D-36	8	769781D-01	1	000000D+00			
41	8	405000D+01	7.761512D+00	3	144480D-37	8	770879D-01	1	000000D+00	
42	8	610000D+01	8	159290D+00	4	048043D-38	8.771863D-01	1	000000D+00	
43	8	815000D+01	8.577454D+00	5.211244D-39	8.772745D-01	1	000000D+00			
44	9	020000D+01	9.017048D+00	6	708690D-40	8	773536D-01	1	000000D+00	
45	9	225000D+01	9	479172D+00	8.636426D-41	8.774245D-01	1.000000D+00			
46	9	430000D+01	9.964979D+00	1	111809D-41	8.774882D-01	1	000000D+00		
47	9.635000D+01	1.047568D+01	1	431287D-42	8.775453D-01	1	000000D+00			
48	9.840000D+01	1.101256D+01	1.842566D-43	8.775966D-01	1	000000D+00				
49	1	004500D+02	1	157696D+01	2	372025D-44	8.776426D-01	1	000000D+00	
50	1	025000D+02	1	217028D+01	3	053624D-45	8.776839D-01	1	000000D+00	
51	1	045500D+02	1	279400D+01	3	931080D-46	8	777210D-01	1.000000D+00	
52	1	066000D+02	1	344970D+01	5	060672D-47	8.777544D-01	1	000000D+00	
53	1	086500D+02	1	413899D+01	6	514852D-48	8.777843D-01	1	000000D+00	
54	1	107000D+02	1	486362D+01	8	386888D-49	8.778111D-01	1	000000D+00	
55	1	127500D+02	1.562538D+01	1.079685D-49	8.778353D-01	1	000000D+00			
56	1	148000D+02	1	642618D+01	1	389932D-50	8	778569D-01	1	000000D+00
57	1	168500D+02	1	726802D+01	1	789327D-51	8.778764D-01	1	000000D+00	
58	1	189000D+02	1	815300D+01	2	303489D-52	8.778939D-01	1	000000D+00	
59	1	209500D+02	1	908335D+01	2	965394D-53	8.779096D-01	1	000000D+00	
60	1	230000D+02	2	006137D+01	3	817497D-54	8	779237D-01	1	000000D+00
61	1.250500D+02	2	108951D+01	4	914451D-55	8	779364D-01	1	000000D+00	
62	1.271000D+02	2	217035D+01	6	326614D-56	8.779478D-01	1	000000D+00		
63	1.291500D+02	2	330658D+01	8	144561D-57	8	779580D-01	1	000000D+00	

64	1	312000D+02	2	450104D+01	1.048489D-57	8	779672D-01	1	000000D+00	
65	1	332500D+02	2	575672D+01	1	349772D-58	8	779755D-01	1.000000D+00	
66	1	353000D+02	2	707675D+01	1	737627D-59	8	779829D-01	1.000000D+00	
67	1	373500D+02	2	846444D+01	2	236933D-60	8	779896D-01	1.000000D+00	
68	1	394000D+02	2	992324D+01	2	879713D-61	8	779956D-01	1	000000D+00
69	1	414500D+02	3	145680D+01	3.707196D-62	8	780010D-01	1	000000D+00	
70	1	435000D+02	3	306897D+01	4	772455D-63	8	780058D-01	1	000000D+00
71	1	455500D+02	3	476375D+01	6	143815D-64	8	780102D-01	1	000000D+00
72	1	476000D+02	3	654539D+01	7.909235D-65	8	780141D-01	1	000000D+00	
73	1	496500D+02	3	841834D+01	1	018195D-65	8	780176D-01	1	000000D+00
74	1	517000D+02	4	038728D+01	1	310772D-66	8	780208D-01	1	000000D+00
75	1	537500D+02	4	245713D+01	1	687421D-67	8	780236D-01	1	000000D+00
76	1	558000D+02	4	463306D+01	2	172300D-68	8	780261D-01	1	000000D+00
77	1	578500D+02	4	692051D+01	2	796508D-69	8	780284D-01	1	000000D+00
78	1	599000D+02	4	932518D+01	3	600082D-70	8	780305D-01	1	000000D+00
79	1	619500D+02	5	185310D+01	4	634562D-71	8	780323D-01	1	000000D+00
80	1	640000D+02	5	451057D+01	5	966298D-72	8	780340D-01	1	000000D+00
81	1	660500D+02	5	730423D+01	7	680708D-73	8	780355D-01	1	000000D+00
82	1	681000D+02	6	024108D+01	9.887753D-74	8	780369D-01	1	000000D+00	
83	1	701500D+02	6	332843D+01	1	272899D-74	8	780381D-01	1	000000D+00
84	1	722000D+02	6	657401D+01	1	638665D-75	8	780391D-01	1	000000D+00
85	1	742500D+02	6	998593D+01	2	109534D-76	8	780401D-01	1	000000D+00
86	1	763000D+02	7	357271D+01	2.715707D-77	8	780410D-01	1	000000D+00	
87	1	783500D+02	7	734331D+01	3	496062D-78	8	780418D-01	1	000000D+00
88	1	804000D+02	8	130716D+01	4	500652D-79	8	780425D-01	1	000000D+00
89	1	824500D+02	8	547415D+01	5	793910D-80	8	780431D-01	1	000000D+00
90	1	845000D+02	8	985470D+01	7	458785D-81	8	780437D-01	1	000000D+00
91	1	865500D+02	9	445975D+01	9	602060D-82	8	780442D-01	1	000000D+00
92	1	886000D+02	9	930081D+01	1	236120D-82	8	780447D-01	1	000000D+00
93	1	906500D+02	1	043900D+02	1	591318D-83	8	780451D-01	1	000000D+00
94	1	927000D+02	1	097400D+02	2	048582D-84	8	780455D-01	1	000000D+00
95	1	947500D+02	1	153641D+02	2	637240D-85	8	780458D-01	1	000000D+00
96	1	968000D+02	1	212766D+02	3	395048D-86	8	780461D-01	1	000000D+00
97	1	988500D+02	1	274920D+02	4	370612D-87	8	780464D-01	1	000000D+00
98	2	009000D+02	1	340259D+02	5	626503D-88	8	780466D-01	1	000000D+00
99	2	029500D+02	1	408948D+02	7.243274D-89	8	780468D-01	1	000000D+00	
100	2	050000D+02	1	481156D+02	9	324621D-90	8	780470D-01	1	000000D+00
296	6	068000D+02	2	660594D+06	2	951946-264	8	780488D-01	1	000000D+00
297	6	088500D+02	2	796949D+06	3.800185-265	8	780488D-01	1	000000D+00	
298	6	109000D+02	2	940293D+06	4.892164-266	8	780488D-01	1	000000D+00	
299	6	129500D+02	3	090983D+06	6	297923-267	8	780488D-01	1	000000D+00
300	6	150000D+02	3	249396D+06	8.107625-268	8	780488D-01	1	000000D+00	

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RK-Trapezoid tests for delx = 3 05000000D+00 (Period-4 limit cycle)

j	x	exp	logistic
0	0 000000D+00	1 000000D+00	1.000000D-03
1	3 050000D+00	2 601250D+00	4 735892D-02
2	6 100000D+00	6 766502D+00	7 312059D-02
3	9 150000D+00	1 760136D+01	4.838026D-01
4	1 220000D+01	4 578554D+01	5 030456D-06

5	1	525000D+01	1	190996D+02	2	382370D-07	5	411014D-01	9	997621D-01
6	1	830000D+01	3	098080D+02	1	128265D-08	3	288060D-01	9	999887D-01
7	2	135000D+01	8	058879D+02	5	343340D-10	6	624309D-01	9	999995D-01
8	2	440000D+01	2	096316D+03	2	530548D-11	2	971980D-01	1	000000D+00
9	2	745000D+01	5	453042D+03	1	198440D-12	7	093955D-01	1	000000D+00
10	3	050000D+01	1	418473D+04	5	675685D-14	3	336890D-01	1	000000D+00
11	3	355000D+01	3	689802D+04	2	687943D-15	6	545075D-01	1	000000D+00
12	3	660000D+01	9	598097D+04	1	272981D-16	2	937844D-01	1	000000D+00
13	3	965000D+01	2	496705D+05	6	028702D-18	7	139244D-01	1	000000D+00
14	4	270000D+01	6	494554D+05	2	855128D-19	3	386620D-01	1	000000D+00
15	4	575000D+01	1	689396D+06	1	352158D-20	6	462932D-01	1	000000D+00
16	4	880000D+01	4	394541D+06	6	403675D-22	2	910836D-01	1	000000D+00
17	5	185000D+01	1	143130D+07	3	032712D-23	7	174198D-01	1	000000D+00
18	5	490000D+01	2	973567D+07	1	436260D-24	3	426720D-01	1	000000D+00
19	5	795000D+01	7	734990D+07	6	801971D-26	6	395728D-01	1	000000D+00
20	6	100000D+01	2	012064D+08	3	221340D-27	2	895067D-01	1	000000D+00
21	6	405000D+01	5	233882D+08	1	525592D-28	7	194236D-01	1	000000D+00
22	6	710000D+01	1	361464D+09	7	225040D-30	3	450379D-01	1	000000D+00
23	7	015000D+01	3	541507D+09	3	421701D-31	6	355701D-01	1	000000D+00
24	7	320000D+01	9	212346D+09	1	620481D-32	2	888373D-01	1	000000D+00
25	7	625000D+01	2	396361D+10	7	674423D-34	7	202660D-01	1	000000D+00
26	7	930000D+01	6	233535D+10	3	634524D-35	3	460470D-01	1	000000D+00
27	8	235000D+01	1	621498D+11	1	721272D-36	6	338549D-01	1	000000D+00
28	8	540000D+01	4	217923D+11	8	151757D-38	2	886119D-01	1	000000D+00
29	8	845000D+01	1	097187D+12	3	860585D-39	7	205485D-01	1	000000D+00
30	9	150000D+01	2	854058D+12	1	828331D-40	3	463874D-01	1	000000D+00
31	9	455000D+01	7	424118D+12	8	658781D-42	6	332754D-01	1	000000D+00
32	9	760000D+01	1	931199D+13	4	100705D-43	2	885440D-01	1	000000D+00
33	1	006500D+02	5	023531D+13	1	942050D-44	7	206334D-01	1	000000D+00
34	1	037000D+02	1	306746D+14	9	197340D-46	3	464899D-01	1	000000D+00
35	1	067500D+02	3	399173D+14	4	355761D-47	6	331007D-01	1	000000D+00
36	1	098000D+02	8	842099D+14	2	062842D-48	2	885244D-01	1	000000D+00
37	1	128500D+02	2	300051D+15	9	769396D-50	7	206579D-01	1	000000D+00
38	1	159000D+02	5	983007D+15	4	626681D-51	3	465195D-01	1	000000D+00
39	1	189500D+02	1	556330D+16	2	191146D-52	6	330502D-01	1	000000D+00
40	1	220000D+02	4	048403D+16	1	037703D-53	2	885188D-01	1	000000D+00
41	1	250500D+02	1	053091D+17	4	914451D-55	7	206650D-01	1	000000D+00
42	1	281000D+02	2	739352D+17	2	327431D-56	3	465280D-01	1	000000D+00
43	1	311500D+02	7	125741D+17	1	102246D-57	6	330358D-01	1	000000D+00
44	1	342000D+02	1	853583D+18	5	220121D-59	2	885172D-01	1	000000D+00
45	1	372500D+02	4	821633D+18	2	472193D-60	7	206670D-01	1	000000D+00
46	1	403000D+02	1	254227D+19	1	170804D-61	3	465304D-01	1	000000D+00
47	1	433500D+02	3	262559D+19	5	544802D-63	6	330317D-01	1	000000D+00
48	1	464000D+02	8	486732D+19	2	625958D-64	2	885167D-01	1	000000D+00
49	1	494500D+02	2	207611D+20	1	243626D-65	7	206675D-01	1	000000D+00
50	1	525000D+02	5	742548D+20	5	889677D-67	3	465311D-01	1	000000D+00
51	1	555500D+02	1	493780D+21	2	789288D-68	6	330305D-01	1	000000D+00
52	1	586000D+02	3	885696D+21	1	320977D-69	2	885166D-01	1	000000D+00
53	1	616500D+02	1	010767D+22	6	256004D-71	7	206677D-01	1	000000D+00
54	1	647000D+02	2	629257D+22	2	962776D-72	3	465313D-01	1	000000D+00
55	1	677500D+02	6	839355D+22	1	403139D-73	6	330302D-01	1	000000D+00
56	1	708000D+02	1	779087D+23	6	645115D-75	2	885166D-01	1	000000D+00
57	1	738500D+02	4	627850D+23	3	147055D-76	7	206677D-01	1	000000D+00
58	1	769000D+02	1	203820D+24	1	490411D-77	3	465313D-01	1	000000D+00

59	1	799500D+02	3.131436D+24	7.058428D-79	6	330301D-01	1	000000D+00
60	1	.830000D+02	8.145647D+24	3.342796D-80	2	885165D-01	1	.000000D+00
61	1	860500D+02	2.118886D+25	1 583112D-81	7	206677D-01	1	000000D+00
62	1	891000D+02	5.511753D+25	7.497448D-83	3	465314D-01	1	.000000D+00
63	1	921500D+02	1 433745D+26	3 550711D-84	6	330300D-01	1	000000D+00
64	1	952000D+02	3.729529D+26	1 681578D-85	2	885165D-01	1	000000D+00
65	1	982500D+02	9 701437D+26	7 963775D-87	7	206677D-01	1	000000D+00
66	2	013000D+02	2 523586D+27	3.771558D-88	3	465314D-01	1	000000D+00
67	2	043500D+02	6 564479D+27	1 786169D-89	6	.330300D-01	1	000000D+00
68	2	.074000D+02	1 707585D+28	8 459106D-91	2	885165D-01	1	000000D+00
69	2	104500D+02	4 441856D+28	4.006141D-92	7	206677D-01	1	.000000D+00
70	2	.135000D+02	1 155438D+29	1 897266D-93	3	.465314D-01	1	000000D+00
71	2	.165500D+02	3 005582D+29	8 985245D-95	6	.330300D-01	1	.000000D+00
72	2	.196000D+02	7 818271D+29	4 255316D-96	2	885165D-01	1	000000D+00
73	2	.226500D+02	2 033728D+30	2 015272D-97	7	.206677D-01	1	.000000D+00
74	2	.257000D+02	0.000000D+00	9 544110D-99	3	465314D-01	1	000000D+00
75	2	287500D+02	0 000000D+00	4 519988-100	6	.330300D-01	1	000000D+00
76	2	.318000D+02	0 000000D+00	2 140618-101	2	885165D-01	1	.000000D+00
77	2	348500D+02	0 000000D+00	1.013773-102	7	206677D-01	1	000000D+00
78	2	379000D+02	0.000000D+00	4 801122-104	3	465314D-01	1	000000D+00
79	2	.409500D+02	0.000000D+00	2.273760-105	6	330300D-01	1	000000D+00
80	2	440000D+02	0.000000D+00	1.076828-106	2	.885165D-01	1	000000D+00
81	2	470500D+02	0 000000D+00	5.099742-108	7	206677D-01	1	000000D+00
82	2	501000D+02	0 000000D+00	2 415183-109	3	465314D-01	1	000000D+00
83	2	531500D+02	0.000000D+00	1 143805-110	6	330300D-01	1	000000D+00
84	2	562000D+02	0 000000D+00	5 416936-112	2	885165D-01	1	000000D+00
85	2	.592500D+02	0 000000D+00	2 565403-113	7	206677D-01	1	.000000D+00
86	2	.623000D+02	0 000000D+00	1 214947-114	3	465314D-01	1	000000D+00
87	2	.653500D+02	0 000000D+00	5 753859-116	6	330300D-01	1	000000D+00
88	2	684000D+02	0 000000D+00	2 724966-117	2	.885165D-01	1	000000D+00
89	2	.714500D+02	0 000000D+00	1 290514-118	7	.206677D-01	1	000000D+00
90	2	.745000D+02	0 000000D+00	6 111738-120	3	.465314D-01	1	.000000D+00
291	8	875500D+02	0 000000D+00	0 000000D+00	6	330300D-01	1	000000D+00
292	8	.906000D+02	0 000000D+00	0.000000D+00	2	885165D-01	1	000000D+00
293	8	936500D+02	0 000000D+00	0.000000D+00	7	206677D-01	1	000000D+00
294	8	.967000D+02	0 000000D+00	0 000000D+00	3	465314D-01	1	.000000D+00
295	8	997500D+02	0 000000D+00	0 000000D+00	6	.330300D-01	1	000000D+00
296	9	028000D+02	0.000000D+00	0 000000D+00	2	885165D-01	1	000000D+00
297	9	058500D+02	0 000000D+00	0 000000D+00	7	206677D-01	1	000000D+00
298	9	089000D+02	0 000000D+00	0 000000D+00	3	465314D-01	1	000000D+00
299	9	119500D+02	0 000000D+00	0 000000D+00	6	330300D-01	1	000000D+00
300	9	150000D+02	0 000000D+00	0 000000D+00	2	.885165D-01	1	000000D+00

+++++

RK-Trapezoid tests for delx = 4 05000000D+00 (Unstable solution)

j	x	exp	logistic
0	0 000000D+00	1 000000D+00	1 000000D+00
1	4 050000D+00	5 151250D+00	1 742237D-02
2	8 100000D+00	2 653538D+01	3 035391D-04
3	1 215000D+01	1 366904D+02	5 288373D-06
4	1 620000D+01	7.041262D+02	9 213601D-08

5	2	025000D+01	3.627130D+03	1.605228D-09	7	740049D-01	9	999984D-01		
6	2	430000D+01	1.868425D+04	2	796688D-11	-3	200206D-01	1	000000D+00	
7	2	835000D+01	9.624726D+04	4	872495D-13	-1	364002D+01	1	000000D+00	
8	3	240000D+01	4.957937D+05	8	489044D-15	-1	371625D+06	1	000000D+00	
9	3	645000D+01	2.553957D+06	1	478993D-16	-1	175647D+26	1	000000D+00	
10	4	050000D+01	1	315607D+07	2	576757D-18	-6	345173+105	1	000000D+00
11	4	455000D+01	6	777022D+07	4	489323D-20-Inf		1	000000D+00	
12	4	860000D+01	3.491014D+08	7	821466D-22-Inf			1	000000D+00	

Table 5 Logistic Equation Solutions for the Partially Implicit Method

```

program a7
c +++ partially implicit method testing
      implicit double precision (a-h,o-z)
      open (unit=12,file='a-7 out',form='formatted')
c +++ initialize parameters
      delx = 0 76d+00
      npts = 300
      j     = 0
      yj    = 1 d+00
      yjp   = yj
      yex   = yj
      zj    = 1 d-03
      zjp   = zj
      zex   = zj
      z0    = zj

      write (12,10) delx
10 format (/,'      A-7 tests for delx =',1p,d16 8//
1 '      j      x      exp      logistic')
      write (12,20) j, x, yjp, yex, zjp, zex
20 format (1x,i6,1p,5d14 6)

      do j = 1, npts
c +++ dy/dh = -y
      yjp = yj - delx * yj

c +++ dz/dh = z - z**2
      zjp = (zj - delx * zj**2) / (1 d+00 - delx)

c +++ exact solutions
      x = real(j) * delx
      yex = exp(-x)
      zex = z0 / (z0 + (1 d+00 - z0) * yex)
      write (12,20) j, x, yjp, yex, zjp, zex

      yj = yjp
      if (abs(yj) gt 1 d+30) yj = 0 d+00
      zj = zjp
      enddo

      close(unit=12)
      stop
      end

+++++
A-7 tests for delx = 7.40000000D-01 (Period-6 limit cycle)

      j      x      exp      logistic
0 0 000000D+00 1 000000D+00 1 000000D+00 1.000000D-03 1 000000D-03
1 7 400000D-01 2.600000D-01 4 771139D-01 3.843308D-03 2 093641D-03
2 1 480000D+00 6 760000D-02 2 276377D-01 1.473991D-02 4 378091D-03
3 2.220000D+00 1 757600D-02 1 086091D-01 5 607360D-02 9 132378D-03

```


4	2	960000D+00	4	569760D-03	5	181892D-02	2	067187D-01	1	895120D-02
5	3	700000D+00	1	188138D-03	2	472353D-02	6	734482D-01	3	891232D-02
6	4	440000D+00	3	089158D-04	1	179594D-02	1	299362D+00	7	822191D-02
7	5	180000D+00	8	031810D-05	5	628006D-03	1	922656D-01	1	510031D-01
8	5	920000D+00	2	088271D-05	2	685200D-03	6	342719D-01	2	715535D-01
9	6	660000D+00	5	429504D-06	1	281146D-03	1	294497D+00	4	386224D-01
10	7	400000D+00	1	411671D-06	6	112528D-04	2	094699D-01	6	208706D-01
11	8	140000D+00	3	670344D-07	2	916372D-04	6	807709D-01	7	743861D-01
12	8	880000D+00	9	542896D-08	1	391442D-04	1	299302D+00	8	779593D-01
13	9	620000D+00	2	481153D-08	6	638762D-05	1	924777D-01	9	378037D-01
14	1	036000D+01	6	450997D-09	3	167446D-05	6	348556D-01	9	693278D-01
15	1	110000D+01	1	677259D-09	1	511232D-05	1	294634D+00	9	851273D-01
16	1	184000D+01	4	360874D-10	7	210300D-06	2	089886D-01	9	928484D-01
17	1	258000D+01	1	133827D-10	3	440134D-06	6	794931D-01	9	965751D-01
18	1	332000D+01	2	947951D-11	1	641336D-06	1	299335D+00	9	983630D-01
19	1	406000D+01	7	664673D-12	7	831043D-07	1	923627D-01	9	992183D-01
20	1	480000D+01	1	992815D-12	3	736299D-07	6	345391D-01	9	996269D-01
21	1	554000D+01	5	181319D-13	1	782640D-07	1	294560D+00	9	998219D-01
22	1	628000D+01	1	347143D-13	8	505226D-08	2	092488D-01	9	999150D-01
23	1	702000D+01	3	502571D-14	4	057961D-08	6	801839D-01	9	999595D-01
24	1	776000D+01	9	106686D-15	1	936110D-08	1	299318D+00	9	999807D-01
25	1	850000D+01	2	367738D-15	9	237450D-09	1	924208D-01	9	999908D-01
26	1	924000D+01	6	156120D-16	4	407316D-09	6	346990D-01	9	999956D-01
27	1	998000D+01	1	600591D-16	2	102792D-09	1	294597D+00	9	999979D-01
28	2	072000D+01	4	161537D-17	1	003271D-09	2	091171D-01	9	999990D-01
29	2	146000D+01	1	082000D-17	4	786746D-10	6	798344D-01	9	999995D-01
30	2	220000D+01	2	813199D-18	2	283823D-10	1	299327D+00	9	999998D-01
31	2	294000D+01	7	314317D-19	1	089644D-10	1	923902D-01	9	999999D-01
32	2	368000D+01	1	901722D-19	5	198843D-11	6	346148D-01	9	999999D-01
33	2	442000D+01	4	944478D-20	2	480440D-11	1	294578D+00	1	000000D+00
34	2	516000D+01	1	285564D-20	1	183453D-11	2	091864D-01	1	000000D+00
35	2	590000D+01	3	342467D-21	5	646417D-12	6	800183D-01	1	000000D+00
36	2	664000D+01	8	690415D-22	2	693984D-12	1	299323D+00	1	000000D+00
37	2	738000D+01	2	259508D-22	1	285337D-12	1	924060D-01	1	000000D+00
38	2	812000D+01	5	874721D-23	6	132523D-13	6	346582D-01	1	000000D+00
39	2	886000D+01	1	527427D-23	2	925912D-13	1	294588D+00	1	000000D+00
40	2	960000D+01	3	971311D-24	1	395993D-13	2	091506D-01	1	000000D+00
41	3	034000D+01	1	032541D-24	6	660478D-14	6	799234D-01	1	000000D+00
42	3	108000D+01	2	684606D-25	3	177807D-14	1	299325D+00	1	000000D+00
43	3	182000D+01	6	979977D-26	1	516176D-14	1	923977D-01	1	000000D+00
44	3	256000D+01	1	814794D-26	7	233886D-15	6	346355D-01	1	000000D+00
45	3	330000D+01	4	718464D-27	3	451388D-15	1	294583D+00	1	000000D+00
46	3	404000D+01	1	226801D-27	1	646705D-15	2	091693D-01	1	000000D+00
47	3	478000D+01	3	189682D-28	7	856659D-16	6	799729D-01	1	000000D+00
48	3	552000D+01	8	293173D-29	3	748521D-16	1	299324D+00	1	000000D+00
49	3	626000D+01	2	156225D-29	1	788472D-16	1	924020D-01	1	000000D+00
50	3	700000D+01	5	606185D-30	8	533048D-17	6	346473D-01	1	000000D+00
51	3	774000D+01	1	457608D-30	4	071236D-17	1	294585D+00	1	000000D+00
52	3	848000D+01	3	789781D-31	1	942443D-17	2	091596D-01	1	000000D+00
53	3	922000D+01	9	853430D-32	9	267667D-18	6	799472D-01	1	000000D+00
54	3	996000D+01	2	561892D-32	4	421733D-18	1	299324D+00	1	000000D+00
55	4	070000D+01	6	660919D-33	2	109670D-18	1	923998D-01	1	000000D+00
56	4	144000D+01	1	731839D-33	1	006553D-18	6	346412D-01	1	000000D+00
57	4	218000D+01	4	502781D-34	4	802405D-19	1	294584D+00	1	000000D+00

58	4	292000D+01	1	170723D-34	2	291294D-19	2	091646D-01	1	000000D+00
59	4	366000D+01	3	043880D-35	1	093208D-19	6	799606D-01	1	000000D+00
60	4	440000D+01	7	914088D-36	5	215849D-20	1	299324D+00	1	000000D+00
61	4	514000D+01	2	057663D-36	2	488554D-20	1	924009D-01	1	000000D+00
62	4	588000D+01	5	349924D-37	1	187324D-20	6	346444D-01	1	000000D+00
63	4	662000D+01	1	390980D-37	5	664887D-21	1	294585D+00	1	000000D+00
64	4	736000D+01	3	616548D-38	2	702796D-21	2	091620D-01	1	000000D+00
65	4	810000D+01	9	403026D-39	1	289542D-21	6	799536D-01	1	000000D+00
66	4	884000D+01	2	444787D-39	6	152583D-22	1	299324D+00	1	000000D+00
67	4	958000D+01	6	356445D-40	2	935483D-22	1	924003D-01	1	000000D+00
68	5	032000D+01	1	652676D-40	1	400560D-22	6	346427D-01	1	000000D+00
69	5	106000D+01	4	296957D-41	6	682266D-23	1	294584D+00	1	000000D+00
70	5	180000D+01	1	117209D-41	3	188202D-23	2	091634D-01	1	000000D+00
71	5	254000D+01	2	904743D-42	1	521136D-23	6	799572D-01	1	000000D+00
72	5	328000D+01	7	552332D-43	7	257549D-24	1	299324D+00	1	000000D+00
73	5	402000D+01	1	963606D-43	3	462678D-24	1	924007D-01	1	000000D+00
74	5	476000D+01	5	105376D-44	1	652092D-24	6	346436D-01	1	000000D+00
75	5	550000D+01	1	327398D-44	7	882360D-25	1	294584D+00	1	000000D+00
76	5	624000D+01	3	451234D-45	3	760784D-25	2	091627D-01	1	000000D+00
77	5	698000D+01	8	973209D-46	1	794322D-25	6	799554D-01	1	000000D+00
78	5	772000D+01	2	333034D-46	8	560961D-26	1	299324D+00	1	000000D+00
79	5	846000D+01	6	065889D-47	4	084553D-26	1	924005D-01	1	000000D+00
80	5	920000D+01	1	577131D-47	1	948797D-26	6	346431D-01	1	000000D+00
81	5	994000D+01	4	100541D-48	9	297983D-27	1	294584D+00	1	000000D+00
82	6	068000D+01	1	066141D-48	4	436197D-27	2	091630D-01	1	000000D+00
83	6	142000D+01	2	771966D-49	2	116571D-27	6	799563D-01	1	000000D+00
84	6	216000D+01	7	207111D-50	1	009846D-27	1	299324D+00	1	000000D+00
85	6	290000D+01	1	873849D-50	4	818114D-28	1	924006D-01	1	000000D+00
86	6	364000D+01	4	872007D-51	2	298789D-28	6	346434D-01	1	000000D+00
87	6	438000D+01	1	266722D-51	1	096784D-28	1	294584D+00	1	000000D+00
88	6	512000D+01	3	293477D-52	5	232911D-29	2	091628D-01	1	000000D+00
89	6	586000D+01	8	563040D-53	2	496695D-29	6	799558D-01	1	000000D+00
90	6	660000D+01	2	226390D-53	1	191208D-29	1	299324D+00	1	000000D+00
91	6	734000D+01	5	788615D-54	5	683418D-30	1	924005D-01	1	000000D+00
92	6	808000D+01	1	505040D-54	2	711638D-30	6	346432D-01	1	000000D+00
93	6	882000D+01	3	913104D-55	1	293760D-30	1	294584D+00	1	000000D+00
94	6	956000D+01	1	017407D-55	6	172710D-31	2	091629D-01	1	000000D+00
95	7	030000D+01	2	645258D-56	2	945086D-31	6	799561D-01	1	000000D+00
96	7	104000D+01	6	877671D-57	1	405141D-31	1	299324D+00	1	000000D+00
97	7	178000D+01	1	788194D-57	6	704125D-32	1	924006D-01	1	000000D+00
98	7	252000D+01	4	649306D-58	3	198631D-32	6	346433D-01	1	000000D+00
99	7	326000D+01	1	208819D-58	1	526111D-32	1	294584D+00	1	000000D+00
100	7	400000D+01	3	142931D-59	7	281290D-33	2	091629D-01	1	000000D+00
101	7	474000D+01	8	171620D-60	3	474005D-33	6	799560D-01	1	000000D+00
102	7	548000D+01	2	124621D-60	1	657496D-33	1	299324D+00	1	000000D+00
103	7	622000D+01	5	524015D-61	7	908144D-34	1	924005D-01	1	000000D+00
104	7	696000D+01	1	436244D-61	3	773086D-34	6	346433D-01	1	000000D+00
105	7	770000D+01	3	734234D-62	1	800192D-34	1	294584D+00	1	000000D+00
106	7	844000D+01	9	709009D-63	8	588965D-35	2	091629D-01	1	000000D+00
107	7	918000D+01	2	524342D-63	4	097915D-35	6	799560D-01	1	000000D+00
108	7	992000D+01	6	563290D-64	1	955172D-35	1	299324D+00	1	000000D+00
109	8	066000D+01	1	706455D-64	9	328398D-36	1	924006D-01	1	000000D+00
110	8	140000D+01	4	436784D-65	4	450709D-36	6	346433D-01	1	000000D+00
111	8	214000D+01	1	153564D-65	2	123495D-36	1	294584D+00	1	000000D+00

112	8	288000D+01	2.999266D-66	1 013149D-36	2 091629D-01	1.000000D+00
113	8	362000D+01	7.798091D-67	4.833875D-37	6.799560D-01	1 000000D+00
114	8	436000D+01	2.027504D-67	2 306309D-37	1.299324D+00	1 000000D+00
115	8	510000D+01	5 271510D-68	1 100372D-37	1 924005D-01	1.000000D+00
116	8	584000D+01	1 370593D-68	5 250029D-38	6 346433D-01	1 000000D+00
117	8	658000D+01	3.563541D-69	2 504862D-38	1 294584D+00	1 000000D+00
118	8	732000D+01	9 265206D-70	1 195104D-38	2 091629D-01	1 000000D+00
119	8	806000D+01	2 408953D-70	5 702009D-39	6 799560D-01	1 000000D+00
120	8	880000D+01	6 263279D-71	2.720508D-39	1.299324D+00	1 000000D+00
283	2	094200D+02	2.738152-166	1.122147D-91	1 924006D-01	1.000000D+00
284	2	101600D+02	7 119194-167	5 353918D-92	6.346433D-01	1.000000D+00
285	2	109000D+02	1 850990-167	2 554429D-92	1 294584D+00	1 000000D+00
286	2	116400D+02	4 812575-168	1 218753D-92	2 091629D-01	1 000000D+00
287	2	123800D+02	1 251270-168	5 814842D-93	6 799560D-01	1 000000D+00
288	2	131200D+02	3.253301-169	2.774342D-93	1 299324D+00	1 000000D+00
289	2	138600D+02	8 458582-170	1 323677D-93	1.924006D-01	1 000000D+00
290	2	146000D+02	2.199231-170	6 315448D-94	6 346433D-01	1 000000D+00
291	2	153400D+02	5 718001-171	3 013188D-94	1.294584D+00	1.000000D+00
292	2	160800D+02	1.486680-171	1 437634D-94	2 091629D-01	1.000000D+00
293	2	168200D+02	3 865369-172	6 859152D-95	6 799560D-01	1 000000D+00
294	2	175600D+02	1 004996-172	3 272597D-95	1 299324D+00	1 000000D+00
295	2	183000D+02	2 612989-173	1 561402D-95	1 924006D-01	1 000000D+00
296	2	190400D+02	6 793772-174	7.449664D-96	6 346433D-01	1 000000D+00
297	2	197800D+02	1.766381-174	3.554338D-96	1 294584D+00	1 000000D+00
298	2	205200D+02	4 592590-175	1 695824D-96	2 091629D-01	1 000000D+00
299	2	212600D+02	1 194073-175	8 091014D-97	6 799560D-01	1 000000D+00
300	2	220000D+02	3 104591-176	3 860335D-97	1 299324D+00	1 000000D+00

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A-7 tests for delx = 7.60000000D-01 (Unstable)

j	x	exp	logistic
0	0 000000D+00	1 000000D+00	1 000000D-03
1	7.600000D-01	2 400000D-01	4.676664D-01
2	1 520000D+00	5 760000D-02	2.187119D-01
3	2.280000D+00	1 382400D-02	1 022842D-01
4	3 040000D+00	3.317760D-03	4 783489D-02
5	3 800000D+00	7 962624D-04	2 237077D-02
6	4.560000D+00	1 911030D-04	1.046206D-02
7	5 320000D+00	4 586471D-05	4 892754D-03
8	6 080000D+00	1.100753D-05	2.288177D-03
9	6.840000D+00	2 641808D-06	1 070103D-03
10	7 600000D+00	6 340338D-07	5 004514D-04
11	8 360000D+00	1.521681D-07	2.340443D-04
12	9 120000D+00	3 652035D-08	1 094547D-04
13	9 880000D+00	8.764883D-09	5.118828D-05
14	1 064000D+01	2.103572D-09	2 393904D-05
15	1 140000D+01	5 048573D-10	1.119548D-05
16	1 216000D+01	1 211657D-10	5 235752D-06
17	1 292000D+01	2.907978D-11	2.448586D-06
18	1.368000D+01	6 979147D-12	1.145121D-06
19	1 444000D+01	1 674995D-12	5.355348D-07-Inf

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