The Size of a Polymer of String-Bits:
A Numerical Investigation*

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ABSTRACT

In string-bit models, string is described as a polymer of point-like constituents. We attempt to use string-bit ideas to investigate how the size of string is affected by string interactions in a non-perturbative context. Lacking adequate methods to deal with the full complications of bit rearrangement interactions, we study instead a simplified analog model with only "direct" potential interactions among the bits. We use the variational principle in an approximate calculation of the mean-square size of a polymer as a function of the number of constituents/bits for various interaction strengths g in three specific models.

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1 Introduction

A long known, perhaps disturbing, fact about string is its infinite physical extent[1, 2]. Working in the light-cone gauge, one finds that the mean-square transverse size of a string is given by

$$R_\perp^2 \equiv \int_0^{P^+/T_0} \frac{T_0 d\sigma}{P^+} \langle 0|[\mathbf{x}(\sigma) - \mathbf{x}(0)]^2|0\rangle = \frac{d}{\pi T_0} \sum_{n=1}^\infty \frac{1}{n} \longrightarrow \infty \ . \tag{1.1}$$

Any experiment designed to measure this quantity will necessarily involve a finite resolution time ϵ. This means that modes with frequency $> 1/\epsilon$ will get averaged over, and the observed transverse size will actually be given by:

$$R_\perp^2 = 2\alpha' d \sum_{n=1}^{\alpha' P^+/\epsilon} \frac{1}{n} \sim 2\alpha' d \ln \frac{\alpha' P^+}{\epsilon} \quad \text{for} \quad P^+ \gg \epsilon/\alpha' \ , \tag{1.2}$$

where $\alpha' = 1/2\pi T_0$ is the Regge slope parameter, and d denotes the number of transverse dimensions. As the resolution is improved (smaller ϵ), the observable transverse extent of string grows logarithmically. Alternatively, for a fixed ϵ the transverse size grows with increasing longitudinal momentum. Since the growth is logarithmic it will barely be noticeable unless the string experiences very high longitudinal boosts, such as when it falls into a black hole [3, 4].

This peculiar property of perturbative strings, known as branching diffusion, is well known from the days of dual resonance models [1]. The form of the scattering amplitude at high energy and fixed momentum transfer (Regge regime),

$$A \sim s^{1-t} = e^{(1-t)\ln s} \ , \tag{1.3}$$

implies a transverse target size $R_\perp^2 \sim \ln s$. Consider a process in which one of the strings carries most of the longitudinal momentum, $P_1^+ \gg P_2^+$. Then in a frame where the transverse momenta are equal and opposite, $\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}$, the c.o.m. (energy)^2 is given by

$$s = 2(P_1^+ + P_2^+)(P_1^- + P_2^-) \sim |\mathbf{p}|^2 \frac{P_1^+}{P_2^+} = 2P_1^+ E_2 \ . \tag{1.4}$$

The transverse size is therefore given by

$$R_\perp^2 \sim \ln 2P_1^+ E_2 \sim \ln \frac{P_1^+}{P_2^+} \ . \tag{1.5}$$

Comparing with (1.2), we see that the resolution time ϵ is given by the longitudinal momentum of the small string, $\epsilon = \alpha' P_2^+$. Since the energy of this "probe" string is given by $\mathbf{p}^2/(2P_2^+)$, the transverse size of the "target" string will appear to grow as the energy of the probe string is increased. In string theory P^+ is a continuous variable, so in particular the probe string can have $P_2^+ = 0$, in which case the target string's size will be infinite.

Consider discretizing a piece of string into M segments, such that each segment carries a longitudinal momentum ΔP^+. The total longitudinal momentum is then $M\Delta P^+$. This means that the smallest probe string has $P_2^+ = \Delta P^+$, and that given this probe, the target string grows logarithmically with $P_1^+ \equiv P^+$. String-bit models give precisely such a dicretization, where ΔP^+ is given by the bit mass m. This provides a *physical* cutoff, resulting in a transverse size given by

$$R_\perp^2 \sim \frac{d}{\pi T_0} \ln \frac{P^+}{m} = \frac{d}{\pi T_0} \ln M \ , \tag{1.6}$$

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where M is the number of bits in the target string. Comparison with the previous discussion shows that we can identify $\epsilon = m\alpha'$. As the transverse size is a measure of the transverse volume occupied by the string, the transverse bit number density of string is given by

$$\rho_\perp = \frac{M}{R_\perp^d} \sim \left(\frac{\pi T_0}{d}\right)^{d/2} \frac{M}{(\ln M)^{d/2}} \,. \tag{1.7}$$

String interactions arise in string-bit models through the rearrangement interaction of string-bits. In perturbation theory a longitudinal dimension (x^-) emerges dynamically. Thus the effective interaction of string is measured by the product of the string coupling and the bit number density in this higher dimension space. The longitudinal size of noninteracting string is given by

$$R_L^2 \equiv \int_0^{P^+/T_0} \frac{T_0 d\sigma}{P^+} \langle 0|[x^-(\sigma) - x^-(0)]^2|0\rangle \sim \frac{d}{3P^{+2}} \sum_{n=1}^{\alpha'P^+/\epsilon} n \sim \frac{d}{6}\left(\frac{\alpha'}{\epsilon}\right)^2 \,, \tag{1.8}$$

from which we see that R_L doesn't grow with P^+ at fixed ϵ, but is nonetheless very large $O(\alpha'/\epsilon = 1/m)$. Thus the effective bit number density relevant to weakly interacting string is

$$\rho_{eff} = \frac{\rho_\perp}{R_L} \sim \sqrt{\frac{6}{d}} \frac{P^+}{(2d\alpha' \ln(\alpha'P^+/\epsilon))^{d/2}}. \tag{1.9}$$

For *fixed* ϵ (m in string bit models) this density grows essentially linearly with increasing P^+. It will eventually become comparable to $1/g^2$, where g is the string coupling constant. At this point interactions become important and perturbation theory must break down. Since the above formulas rely on a perturbative picture, they cannot be correct at arbitrarily high P^+. At such energies the integrity of string is lost and a more appropriate description would presumably be in terms of a fluid of string-bits.

It has been suggested that one effect of string interactions is to spread string out, in essence to push bits away from each other [4]. In fact it has been conjectured that the transverse size should grow just rapidly enough to yield a constant transverse bit number density of m_{Planck}^d as $P^+ \to \infty$. This would be necessary if string theory is to account for the Beckenstein-Hawking black-hole entropy [5, 6], and is to provide a realization of 't Hooft's holographic principle [7]. This limiting transverse density translates to a lower bound on the transverse size:

$$R_\perp^2 \gtrsim \frac{M^{2/d}}{m_{Planck}^2} \,. \tag{1.10}$$

For $d = 2$ ($D = 4$ space-time) this lower bound is simply linear in bit number.

The only indication in perturbative string theory that the net effect of the interactions is repulsive is the underlying supersymmetry, which guarantees a positive semidefinite energy. In the context of supersymmetric string-bit models, since bits must be held together by attractive interactions, it follows that residual interactions between composites of bits (*i.e.* string) should on the average be repulsive. It is clear that further insight into this issue requires information beyond perturbation theory. We propose to study string size in the context of $d = 2$ string-bit models, which are particular composite formulations of $D = 4$ string theory, that (conveniently) incorporate perturbative string interactions in the larger setting of microscopic bit interactions.

The complete dynamics of string-bit models are extremely complicated. The interactions among bits include, among other complications, rearrangements of the bits among different polymers as well as on a single polymer. We have not yet developed methods to handle these complications,

so as a preliminary exploration, we consider a simple many-body quantum mechanical system of interacting polymers, motivated by the general principles of string-bit models. This simplified model contains only "direct" (potential energy) interactions, i.e. no exchange effects. Admittedly, this system is too simple to capture the quantitative details of non-perturbative string growth, but we can hope that it (or a slight improvement of it) can capture some of the qualitative physics. As we shall see, the growth predicted by this model is actually too rapid: quadratic in bit number rather than linear. It is too early to tell whether this "overshoot" signals a problem with the bit model itself, or it simply indicates the crudeness with which our analog system imitates a string-bit model.

The rest of the paper is organized as follows. In section 2 we discuss the size of generic extended many-body bound states in such potential models, and show that the condition that these objects follow a relativistic dispersion law ($P^- = (\mathbf{p}^2 + \mathcal{M}^2)/2P^+$) implies an upper bound on the growth rate. In section 3 we rederive the logarithmic transverse growth of free string as the growth of a polymer of string-bits in the $N_c \to \infty$ limit. In section 4 we consider the effect of finite $g \sim 1/N_c$, and propose quantum mechanical models which mimic it. In section 5 we use a variational approach to compute, in three specific models, the mean square size of a polymer and the bond length as functions of the number of bits. Section 6 is devoted to a discussion of the results and future directions.

2 Size of Extended Bound States

Consider a many particle system in d space dimensions, with dynamics governed by the Hamiltonian

$$h = \sum_{k=1}^{M} \frac{\mathbf{p}_k^2}{2m} + V(\mathbf{x}_1, \ldots, \mathbf{x}_M) \,. \tag{2.1}$$

Assuming that this system admits extended bound states of many particles (e.g. chains), what is their rate of growth with the number of constituents? We will discover that for these bound states to become relativistic objects in $d + 2$ space-time dimensions in the continuum limit, the mean-square size can grow at most linearly.

The mean-square distance between the $(k + l)$'th constituent and the l'th constituent ("k-separated" constituents) on the object in its ground state is given by:

$$R_k^2 = \langle 0|(\mathbf{x}_{k+l} - \mathbf{x}_l)^2|0\rangle = \frac{4}{M} \sum_{n=1}^{M-1} \sin^2 \frac{\pi n k}{M} \langle 0|\hat{\mathbf{x}}_n \cdot \hat{\mathbf{x}}_{M-n}|0\rangle \,, \tag{2.2}$$

where $\hat{\mathbf{x}}_n$ are the Fourier components of $\mathbf{x}_k$:

$$\hat{\mathbf{x}}_n = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \mathbf{x}_k e^{2\pi i n k/M} \,. \tag{2.3}$$

We define the mean-square size as the average of the mean-square distances

$$
\begin{aligned}
R^2 &\equiv \frac{1}{M} \sum_{k=1}^{M} R_k^2 \\
&= \frac{2}{M} \sum_{n=1}^{M-1} \langle 0|\hat{\mathbf{x}}_n \cdot \hat{\mathbf{x}}_{M-n}|0\rangle \,.
\end{aligned}
\tag{2.4}
$$

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Note that the variable conjugate to $\hat{x}_n$ is $\hat{p}_{M-n}$, so

$$-i\sum_i \langle 0|[\hat{x}_n^i, \hat{p}_{M-n}^i]|0\rangle = d \ . \tag{2.5}$$

Using $\hat{p}_n^i = -im[\hat{x}_n^i, h]$ and inserting complete sets of states we get

$$d = m\sum_\lambda (E_\lambda - E_0)\Big[\langle 0|\hat{\mathbf{x}}_n|\lambda\rangle \cdot \langle\lambda|\hat{\mathbf{x}}_{M-n}|0\rangle + \langle 0|\hat{\mathbf{x}}_{M-n}|\lambda\rangle \cdot \langle\lambda|\hat{\mathbf{x}}_n|0\rangle\Big] \ . \tag{2.6}$$

Since each term in the sum is positive definite, and $E_{\lambda_1} \geq E_{\lambda_2}$ for $\lambda_1 > \lambda_2$, we get the following inequality:

$$d \geq 2m(E_1 - E_0)(\langle 0|\hat{\mathbf{x}}_n \cdot \hat{\mathbf{x}}_{M-n}|0\rangle - \langle 0|\hat{\mathbf{x}}_n|0\rangle\langle 0|\hat{\mathbf{x}}_{M-n}|0\rangle) \ . \tag{2.7}$$

If the Hamiltonian possesses at least a cyclic symmetry with respect to permutations of the particles, $\langle 0|\mathbf{x}_k|0\rangle$ will be independent of particle label k, in which case (2.3) shows that $\langle 0|\hat{\mathbf{x}}_n|0\rangle = 0$ for $n \neq 0$, and the second term vanishes. We then find

$$d \geq 2m(E_1 - E_0)\langle 0|\hat{\mathbf{x}}_n \cdot \hat{\mathbf{x}}_{M-n}|0\rangle \ . \tag{2.8}$$

From eq. (2.4) we therefore get an upper bound on the size:

$$R^2 \leq \frac{d}{m}\frac{M-1}{M}\frac{1}{E_1 - E_0} \sim \frac{d}{m(E_1 - E_0)} \ . \tag{2.9}$$

The only question remaining is how the gap $E_1 - E_0$ scales with M. If the extended object is to describe a discretization of a *relativistic* object in the light-cone frame (e.g. chain of bits $\rightarrow$ light-cone string), then the gap should scale as $1/mM$. This is because in the continuum limit $mM \rightarrow P^+$, so

$$\Delta E = \Delta P^- = \frac{\mathcal{M}^2}{2P^+} \sim \frac{\mathcal{M}^2}{2mM} \ , \tag{2.10}$$

where $\mathcal{M}$ is the rest mass of the lightest massive particle. Consequently the bound becomes:

$$R^2 \lesssim \frac{2d}{\mathcal{M}^2}M = \frac{2d}{m\mathcal{M}^2}P^+ \ . \tag{2.11}$$

Note that we have not used a harmonic (or any specific) interaction (nearest-neighbor or not) to derive this bound. It is simply a consequence of the fact that the potential depends only on the positions and not on the momenta.

A related upper bound on the growth of relativistic strings was derived in a completely different context by Susskind [8]. This bound is due to causality in the presence of a background black hole geometry. The information carried by a spreading string on the stretched horizon is limited in its speed of propagation by the requirement that it lie inside the light cone. The bound on the spread of information is found to be

$$R_\perp^2 \lesssim e^{t/4M_{BH}G} \ , \tag{2.12}$$

where t is time as measured by an external (Schwarzschild) observer, and G is Newton's constant. Since the longitudinal momentum of the string in this situation grows as

$$P^+ \sim e^{t/4M_{BH}G} \ , \tag{2.13}$$

it follows that the maximum rate of growth of the mean-square size with longitudinal momentum is linear. Here too, the bound is independent of the number of dimensions, and happens to agree with the conjectured growth rate in four space-time dimensions ($d = 2$).

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In the next section we shall see that a bare polymer of string-bits, i.e. a free string, grows logarithmically, and therefore satisfies the bound (2.11) for any d. Moreover, for $d \geq 2$ this upper bound is compatible with the lower bound in eq. (1.10). Note that for $d = 1$, i.e. three space-time dimensions, the two bounds are incompatible. From the point of view of extended bound states the $d = 1$ growth pattern is trivially understood, since the size of a one dimensional chain is simply its length. From the above argument it follows that the continuum limit cannot correspond to a relativistic string. The black-hole causality argument is also problematic for $d = 1$.

3 Size of a Bare Polymer of String-Bits

In the $N_c \rightarrow \infty$ limit, the energy eigenstates of string-bit models [9, 10, 11, 12] are non-interacting (bare) multi-polymer states. A single bare polymer containing M bits has, for $M \rightarrow \infty$, physical properties identical to a relativistic string in light-cone gauge. The dynamics of a bare polymer is governed by a Hamiltonian with only "nearest-neighbor" interactions:

$$h = \frac{1}{2m} \sum_{k=1}^{M} \left[\mathbf{p}_k^2 + V(\mathbf{x}_{k+1} - \mathbf{x}_k) \right] , \tag{3.1}$$

where $V(\mathbf{x})$ is an attractive (and binding) potential. For the special case of harmonic interactions this becomes

$$\begin{aligned} h &= \frac{1}{2m} \sum_{k=1}^{M} \left[\mathbf{p}_k^2 + T_0^2 (\mathbf{x}_{k+1} - \mathbf{x}_k)^2 \right] \\ &= \frac{1}{2m} \sum_{n=1}^{M-1} \left[\hat{\mathbf{p}}_n \cdot \hat{\mathbf{p}}_{M-n} + \omega_n^2 \hat{\mathbf{x}}_n \cdot \hat{\mathbf{x}}_{M-n} \right] , \end{aligned} \tag{3.2}$$

and the Hamiltonian can be diagonalized in terms of the creation and annihilation operators given by

$$\begin{aligned} \mathbf{a}_n &= \frac{1}{\sqrt{2\omega_n}} (\hat{\mathbf{p}}_n - i\omega_n \hat{\mathbf{x}}_n) \\ \mathbf{a}_n^\dagger &= \frac{1}{\sqrt{2\omega_n}} (\hat{\mathbf{p}}_{M-n} + i\omega_n \hat{\mathbf{x}}_{M-n}) . \end{aligned} \tag{3.3}$$

Using these in the equations of the previous section gives

$$R_k^2 = \langle 0| (\mathbf{x}_{k+l} - \mathbf{x}_l)^2 |0\rangle = \frac{2d}{M} \sum_{n=1}^{M-1} \frac{\sin^2 \pi nk/M}{\omega_n} , \tag{3.4}$$

and

$$R^2 = \frac{1}{M} \sum_{k=1}^{M} R_k^2 = \frac{d}{M} \sum_{n=1}^{M-1} \frac{1}{\omega_n} , \tag{3.5}$$

for the mean-square distance between k-separated bits and the mean-square size of the chain, respectively. The above formulas hold for *any* harmonic system with normal mode frequencies ω_n. For the nearest-neighbor harmonic system of a bare chain (3.2): $\omega_n = 2T_0 \sin n\pi/M$, so

$$R^2 \sim \frac{d}{\pi T_0} \ln M , \tag{3.6}$$

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in agreement with the free string result.

For a general potential, an exact solution is impossible and we turn to the approximation methods developed in [13]. Technically it is easier to deal in the general case with an open polymer with the Hamiltonian

$$h_{\text{open}} = \frac{1}{2m} \sum_{i=1}^{M} \mathbf{p_i}^2 + \frac{1}{2m} \sum_{i=1}^{M-1} V(\mathbf{x}_{i+1} - \mathbf{x}_i). \tag{3.7}$$

Since the size of open polymers with a large number of bits is easily related to the size of closed polymers with the same large number of bits, the conclusions will be similar. Because of the nearest-neighbor interaction pattern, a particularly neat way of separating the center of mass motion is to define internal coordinates $\mathbf{y}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$, with corresponding conjugate momenta $\mathbf{q}_k$, in which case the Hamiltonian for internal dynamics becomes

$$h'_{\text{open}} \equiv \frac{1}{m} \left[\sum_{r=2}^{M} \mathbf{q}_r^2 - \sum_{r=2}^{M-1} \mathbf{q}_r \cdot \mathbf{q}_{r+1} + \frac{1}{2} \sum_{r=2}^{M} V(\mathbf{y}_r) \right]. \tag{3.8}$$

The correlation functions $\langle G|T[y_k(t)y_l(0)]|G \rangle$ contain information about the excitation spectrum of the model as well as information about the system's size. Spectral information is inferred by a Fourier analysis of the time dependence, while size information is given by the limit $t \to 0$. Define the Fourier components of this correlator by

$$\langle G|T[y_r^i(t)y_s^j(0)]|G \rangle \equiv -\int \frac{d\omega}{2\pi i} \mathcal{E}_{rs}^{ij}(\omega) e^{-i\omega t}. \tag{3.9}$$

As discussed in [13], it is convenient to introduce a certain "irreducible" two point correlator $\Pi_{rs}^{ij}(\omega)$ which facilitates the analysis of the low energy spectrum in the limit of large bit number. The irreducibility is with respect to a graphical description of time dependent perturbation theory, taking the term $\sum q_r q_{r+1}$ as the perturbation. For example the lowest order contribution to Π is just $\mathcal{E}$ for the system with this perturbation set to zero. In general Π is the sum of all connected graphs contributing to $\mathcal{E}$ which can not be disconnected by cutting only one line (which graphically represents the interaction). The all orders relation of $\mathcal{E}$ to Π is then, assuming rotation invariance:

$$\mathcal{E}_{rs}^{ij}(\omega) = \delta_{ij} \left[\Pi^{-1} - \frac{m^2 \omega^2}{2} \left(1 - \frac{2}{k^2} \right) \right]_{rs}^{-1}. \tag{3.10}$$

Here $(k^2)_{rs} = 2\delta_{rs} - \delta_{rs+1} - \delta_{rs-1}$. The eigenvalues of the matrix $(k^2)_{rs}$ are $4\sin^2(n\pi/2M)$, some of which scale as $1/M^2$ at large M. Thus if Π_{rs} is proportional to δ_{rs} and non-vanishing at $\omega = 0$, $\mathcal{E}$ will display poles at $\omega = O(1/M)$, as required for a stringy continuum limit. This is certainly the case for a harmonic potential for which Π_{rs} is exactly given by $\Pi_{rs} = 2\delta_{rs}/(m^2\omega^2 - 2T_0^2)$. Of course for a general potential one can't solve exactly for Π, but one can argue that it will generically have this low frequency behavior for $M \to \infty$ for a wide class of potentials.

On the other hand the integral of $\mathcal{E}$ over all ω gives the limit $t = 0$, which contains size information. Doing the integral by contours gives a contribution to the size from the above mentioned low energy poles similar to the purely harmonic nearest-neighbor case, i.e. a contribution scaling as $\ln M$. But there are of course other contributions which could conceivably alter this behavior. For the nearest-neighbor harmonic case this doesn't happen; the integral can be exactly performed and the conclusion of logarithmic growth is unaltered. We suspect this conclusion is more general, but our lack of detailed knowledge of the behavior of Π at high frequency leaves the matter open to future study.

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4 Bits With Elbows – A Toy Model

In order to carry the discussion of size beyond bare polymers (free string) we must deal with the complete string-bit Hamiltonian at finite N_c. The simplest (bosonic) string bit model [9] is described by the Hamiltonian:

$$H = \int d\mathbf{x} \frac{1}{2m} \mathrm{Tr} \nabla \phi^\dagger \cdot \nabla \phi - \frac{1}{4mN_c} \int d\mathbf{x} d\mathbf{y} V(\mathbf{x} - \mathbf{y}) : \mathrm{Tr}[\phi^\dagger(\mathbf{x}), \phi(\mathbf{x})][\phi^\dagger(\mathbf{y}), \phi(\mathbf{y})] : , \qquad (4.1)$$

where $\phi_\alpha^{\dagger \beta}(\mathbf{x})$ creates a bit at the position $\mathbf{x}$ with color (α, β), and lower (upper) color indices transform in the N_c ($\bar{N}_c$) representations of $SU(N_c)$.

The variational principle provides a natural approach to the problem of discovering how string interactions ($1/N_c$ corrections) affect the size of string. Since string-bit dynamics is standard many body non-relativistic quantum mechanics, exact energy eigenstates are given by states for which the functional

$$E[\psi] \equiv \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$$

is stationary with respect to arbitrary infinitesimal changes $\delta|\psi\rangle$. Furthermore the lowest energy eigenstate is the absolute minimum of this functional. Since the Hamiltonian is a color singlet and commutes with bit number, one loses no generality in restricting $|\psi\rangle$ to an irreducible representation of the color group and to contain a fixed number M of string-bits. Focusing on the color singlet sector, a general trial state $|\psi\rangle$ is a linear combination of multi-polymer states of the form (assuming $M \geq k_{n-1} \geq \cdots \geq k_2 \geq k_1 \geq 1$)

$$|\psi\rangle_M^{(k_1, \cdots, k_{n-1})} = N_c^{-M/2} \int d\mathbf{x}_1 \cdots d\mathbf{x}_M \mathrm{Tr}[\phi^\dagger(\mathbf{x}_1) \cdots \phi^\dagger(\mathbf{x}_{k_1})] \, \mathrm{Tr}[\phi^\dagger(\mathbf{x}_{k_1+1}) \cdots \phi^\dagger(\mathbf{x}_{k_2})] \cdots$$

$$\cdots \mathrm{Tr}[\phi^\dagger(\mathbf{x}_{k_{n-1}+1}) \cdots \phi^\dagger(\mathbf{x}_M)]|0\rangle \psi_M^{(k_1, \cdots, k_{n-1})}(\mathbf{x}_1, \ldots, \mathbf{x}_M) . \qquad (4.2)$$

At $N_c = \infty$ the Hamiltonian acts independently on each trace, showing that in this limit the above state describes n noninteracting bare polymers, whose energy eigenstates are described by the chain dynamics explained in the previous section.

Of course, making unrestricted variations is tantamount to solving the problem exactly, a presumably intractable task. To proceed in an approximate way, we seek a minimum of $E[\psi]$ within a restricted class of states. In this paper we restrict $|\psi\rangle$ to be a single bare polymer

$$|\psi_M\rangle^{(1)} = N_c^{-M/2} \int d\mathbf{x}_1 \cdots d\mathbf{x}_M \mathrm{Tr}[\phi^\dagger(\mathbf{x}_1) \cdots \phi^\dagger(\mathbf{x}_M)]|0\rangle \psi_M(\mathbf{x}_1, .., \mathbf{x}_M) ,$$

with ψ_M restricted to a class of functions for which integrals can be readily performed. The approximate Schrödinger dynamics that follows by using this trial in the variational principle assumes the form

$$\langle 0| \mathrm{Tr}[\phi(\mathbf{x}_M) \cdots \phi(\mathbf{x}_1)](H - E[\psi])|\psi_M\rangle^{(1)} = 0 \qquad (4.3)$$

The matrix element of H may be reduced by letting H act to the right. Let H' be the interaction term in (4.1). Its action on the one-polymer state is given by

$$
\begin{aligned}
H'|\psi_M\rangle^{(1)} \;=\; & -\frac{N_c^{-M/2}}{4mN_c}\int d\mathbf{y}_1\cdots d\mathbf{y}_M \sum_{i<j}[V(\mathbf{y}_i-\mathbf{y}_j)+V(\mathbf{y}_j-\mathbf{y}_i)] \\
& \Big\{ \; \mathrm{Tr}[\phi^\dagger(\mathbf{y}_i)\cdots\phi^\dagger(\mathbf{y}_{j-1})]\mathrm{Tr}[\phi^\dagger(\mathbf{y}_j)\cdots\phi^\dagger(\mathbf{y}_{i-1})] \\
& + \; \mathrm{Tr}[\phi^\dagger(\mathbf{y}_{i+1})\cdots\phi^\dagger(\mathbf{y}_j)]\mathrm{Tr}[\phi^\dagger(\mathbf{y}_{j+1})\cdots\phi^\dagger(\mathbf{y}_i)] \\
& - \; \mathrm{Tr}[\phi^\dagger(\mathbf{y}_i)\cdots\phi^\dagger(\tilde{\mathbf{y}}_j)]\mathrm{Tr}[\phi^\dagger(\mathbf{y}_{j+1})\cdots\phi^\dagger(\mathbf{y}_{i-1})] \\
& - \; \mathrm{Tr}[\phi^\dagger(\mathbf{y}_{i+1})\cdots\phi^\dagger(\mathbf{y}_{j-1})]\mathrm{Tr}[\phi^\dagger(\mathbf{y}_j)\cdots\phi^\dagger(\mathbf{y}_i)] \; \Big\}|0\rangle\,\psi_M(\mathbf{y}_1,..,\mathbf{y}_M)\,. \qquad (4.4)
\end{aligned}
$$

When one of the traces in (4.4) is empty, that trace simply provides a factor of $\mathrm{Tr}\,I = N_c$ which cancels the $1/N_c$ out front thus providing a term that survives the $N_c \to \infty$ limit. Simple inspection shows that this only happens for the last term in braces when $j = i+1$ and for the third term in braces when $i = 1$ and $j = M$. We combine these special terms with the terms coming from the matrix element of the kinetic term of H to give the first quantized Hamiltonian $\hat{h}$ of a bare polymer, defined by:

$$
\langle 0|\mathrm{Tr}[\phi(\mathbf{x}_M)\cdots\phi(\mathbf{x}_1)]|\hat{h}\psi_M\rangle^{(1)} \equiv \lim_{N_c\to\infty}\langle 0|\mathrm{Tr}[\phi(\mathbf{x}_M)\cdots\phi(\mathbf{x}_1)]H|\psi_M\rangle^{(1)}\,, \qquad (4.5)
$$

and therefore given by

$$
\hat{h} = \frac{1}{2m}\sum_{k=1}^{M}\left[\mathbf{p}_k^2 + V(\mathbf{x}_{k+1}-\mathbf{x}_k)\right]\,, \qquad (4.6)
$$

where we have, without loss of generality, taken the potential to be even $V(\mathbf{x}) = V(-\mathbf{x})$.

All the remaining terms in (4.4) vanish in the large N_c limit. At finite N_c, they introduce several new physical features into the dynamics. First of all, they describe interactions between non-nearest-neighbors on the bare polymer. At the same time, they allow non-cyclic bit rearrangement on the polymer. Finally, there are contributions of both an attractive and a repulsive character. Because the exact Hamiltonian is positive, we can expect that on average the non-nearest-neighbor interactions are repulsive. We can roughly confirm this by counting the number of repulsive and attractive contributions to the matrix element of H' at leading order in $1/N_c$. For example, the contribution of the first term in (4.4) is given by

$$
N_c^{-M-1}\langle 0|\mathrm{Tr}[\phi(\mathbf{x}_M)\cdots\phi(\mathbf{x}_1)]\mathrm{Tr}[\phi^\dagger(\mathbf{y}_i)\cdots\phi^\dagger(\mathbf{y}_{j-1})]\mathrm{Tr}[\phi^\dagger(\mathbf{y}_j)\cdots\phi^\dagger(\mathbf{y}_{i-1})]|0\rangle\,.
$$

To evaluate the above matrix element we simply contract the annihilation and creation operators. The leading order contribution corresponds to contractions which preserve the cyclic ordering in the traces. There are therefore $M(j-i)(M-j+i)$ such terms in the above matrix element, each giving an additional factor of N_c^{M-1}, for a total of N_c^{-2}. From eq. (4.4) it then follows that there are $2M(j-i)(M-j+i)$ repulsive terms and $M(j-i+1)(M-j+i-1)+M(j-i-1)(M-j+i+1) = 2M(j-i)(M-j+i)-2M$ attractive terms; an excess of repulsive over attractive of $2M$. (Note that the factor of M in these countings is common to all terms in the Hamiltonian matrix element and does not represent an undue enhancement.)

Handling bit rearrangement effects in the context of a variational calculation is daunting if not intractable. We defer such a direct attack and instead, in a first attempt, replace our string-bit system with an effective analog system in which rearrangement is not allowed. This strategy is

somewhat reminiscent of the Hartree as opposed to Hartree-Fock approximation to many electron atoms. We wish to suppress rearrangement of bits altogether and simulate $1/N_c$ effects by adding a simple non-nearest-neighbor potential term to $\hat{h}$:

$$h = \frac{1}{2m} \sum_{k=1}^{M} \left[\mathbf{p}_k^2 + V(\mathbf{x}_{k+1} - \mathbf{x}_k) \right] - \frac{\xi}{N_c^2} \sum_{k \neq l} V(\mathbf{x}_k - \mathbf{x}_l). \tag{4.7}$$

The last term is supposed to represent, in an average way, all of the terms in (4.4) with nonempty traces. We have taken it to be repulsive because although there are a large number of terms with both signs, there is a slight preponderance of repulsive ones. We have used the same potential function for the nearest-neighbor attraction as the non-nearest-neighbor repulsions, as both effects originate from the same quartic term in the Hamiltonian. But we have introduced the parameter ξ to absorb the renormalization effects of the averaging procedure.

Furthermore, for the sake of numerical study we would like to relax the condition that the non-nearest-neighbor interaction be tied to the nearest-neighbor interaction. Thus we shall study the "bits with elbows" system given by

$$h = \frac{1}{2m} \sum_{k=1}^{M} \left[\mathbf{p}_k^2 + V(\mathbf{x}_{k+1} - \mathbf{x}_k) \right] + g^2 \sum_{k \neq l} U(\mathbf{x}_k - \mathbf{x}_l) , \tag{4.8}$$

where $U(\mathbf{x})$ is an independent repulsive potential, and g^2 represents $1/N_c^2$ together with all renormalization effects. This will allow us in particular to use a long-range (harmonic) nearest-neighbor potential in conjunction with a short-range non-nearest-neighbor potential.

Consider first the harmonic chain of the previous section, $V(\mathbf{x}) \sim \mathbf{x}^2$. For small g, one might be tempted to treat the non-nearest-neighbor terms as a perturbation. The resulting first order correction to the size of the chain is found to be

$$\Delta R^2 \sim g^2 M^3 [\ln M]^\alpha , \tag{4.9}$$

where α depends on the precise form of $U(\mathbf{x})$. Comparing with the zeroth order result (3.6), we see that perturbation theory breaks down quickly with increasing number of bits. To treat a large number of bits we will instead use a variational approach.

5 Variational Approach and Numerical Results

Although we have used the variational principle to motivate our analog model, the model itself cannot be solved without approximation. Thus in this section we shall use the variational method to learn about the size properties of the analog system. If we make simple choices for V and U, the integrals involved in computing $E[\psi]$ can be readily done once we restrict the trial wave functions to be gaussians. A convenient way to parameterize such trial functions is to let them be the ground state wave functions of various many body systems with arbitrary harmonic forces. The ground state of such a system is defined by[1]

$$A_n |\phi) = 0 , \tag{5.1}$$

where

$$A_n = \frac{1}{\sqrt{2\omega_n}} (\hat{\mathbf{p}}_n - i\omega_n \hat{\mathbf{x}}_n) . \tag{5.2}$$

---

[1] First-quantized states are denoted $|\cdots)$, to distinguish them from second-quantized (Fock) states $|\cdots\rangle$.

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We take the normal mode frequencies ω_n of the trial harmonic system as the variational parameters, and minimize the expectation value of the energy

$$E(\omega_n) = \frac{(\phi|h|\phi)}{(\phi|\phi)} \tag{5.3}$$

with respect to them. The size of the chain will then be given by eq. (3.5). In addition we will want to keep track of the bond length, given by eq. (3.4) with $k = 1$:

$$R_1^2 = \frac{2d}{M} \sum_{n=1}^{M-1} \frac{\sin^2 \pi n/M}{\omega_n} . \tag{5.4}$$

This will determine the string scale in the stringy continuum limit ($M \to \infty$). Three specific models are considered:

1. Harmonic chain with δ-function elbows:

$$V(\mathbf{x}) = T_0^2 \mathbf{x}^2 \quad , \quad U(\mathbf{x}) = \delta^{(d)}(\mathbf{x}) . \tag{5.5}$$

Setting $m = 1$ and absorbing numerical factors into g^2, the expectation value for the energy is

$$E_{h\delta}(\omega_n) = \frac{1}{2} \sum_{n=1}^{M-1} \omega_n + \frac{1}{2} T_0^2 M R_1^2 + g^2 M \sum_{k=1}^{M} \frac{1}{R_k^d} . \tag{5.6}$$

2. Harmonic chain with gaussian elbows:

$$V(\mathbf{x}) = T_0^2 \mathbf{x}^2 \quad , \quad U(\mathbf{x}) = e^{-\mathbf{x}^2/a^2} , \tag{5.7}$$

$$E_{hg}(\omega_n) = \frac{1}{2} \sum_{n=1}^{M-1} \omega_n + \frac{1}{2} T_0^2 M R_1^2 + g^2 M \sum_{k=1}^{M} \left[\frac{a^2 d}{2} + R_k^2 \right]^{-d/2} . \tag{5.8}$$

3. Gaussian chain with gaussian elbows:

$$V(\mathbf{x}) = -\lambda e^{-\mathbf{x}^2/a^2} \quad , \quad U(\mathbf{x}) = e^{-\mathbf{x}^2/a^2} , \tag{5.9}$$

$$E_{gg}(\omega_n) = \frac{1}{2} \sum_{n=1}^{M-1} \omega_n - \frac{1}{2} \lambda M \left[\frac{a^2 d}{2} + R_1^2 \right]^{-d/2} + g^2 M \sum_{k=1}^{M} \left[\frac{a^2 d}{2} + R_k^2 \right]^{-d/2} . \tag{5.10}$$

The coefficient λ in the third (gg) model is fixed by requiring that the bond length (R_1) for the bare chain ($g = 0$) be the same in all three models,

$$\lambda = \frac{2a^2 T_0^2}{d} \left(1 + \frac{4}{\pi a^2 T_0} \right)^{1+d/2} . \tag{5.11}$$

The energies of the three models were minimized numerically for various values of M and g. The results for R^2 and R_1^2 as functions of the number of bits M for various coupling constants g are shown in fig. 1.

The R^2 plots clearly exhibit a change in the growth pattern for $g \neq 0$ at some number of bits M. For small M the growth is logarithmic, in agreement with the fact that perturbation theory is valid there, and the zeroth order result (3.6) dominates. For large M the growth is quadratic with M. This result seems to contradict the bound of linear growth found in section 2. Recall however that the linear bound was a consequence of the gap scaling as $1/M$ at large M. From fig. 2 it is clear that for $g \neq 0$ the gap in these models scales roughly as $1/M^3$ at large M. These polymers therefore do not correspond to relativistic strings in the continuum limit.
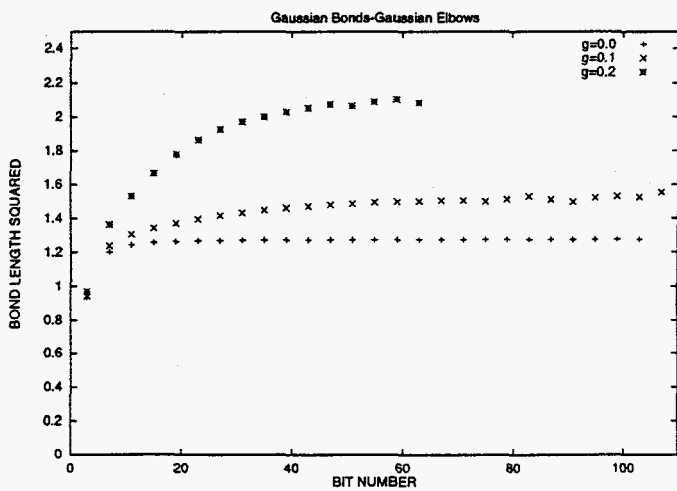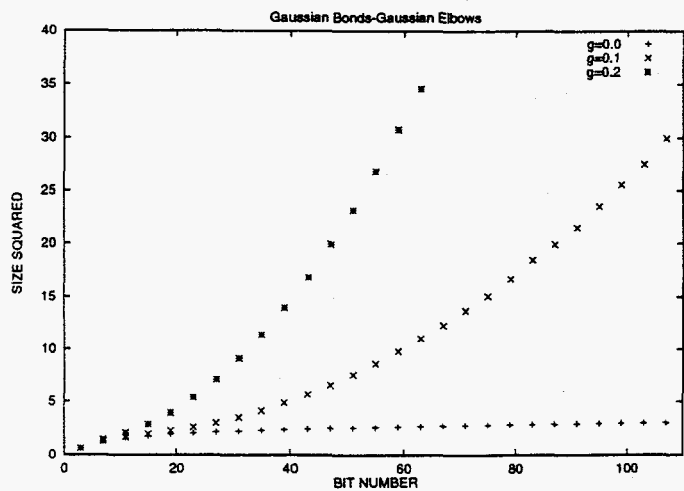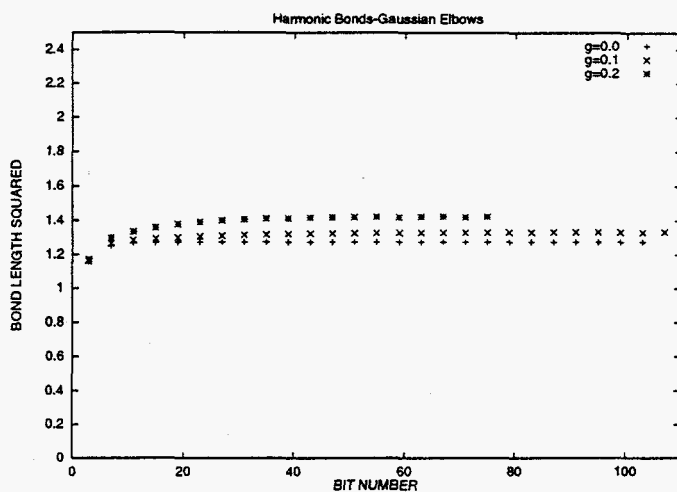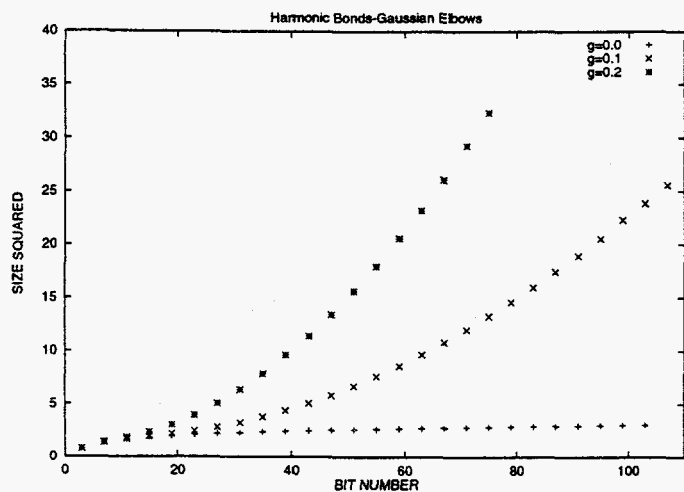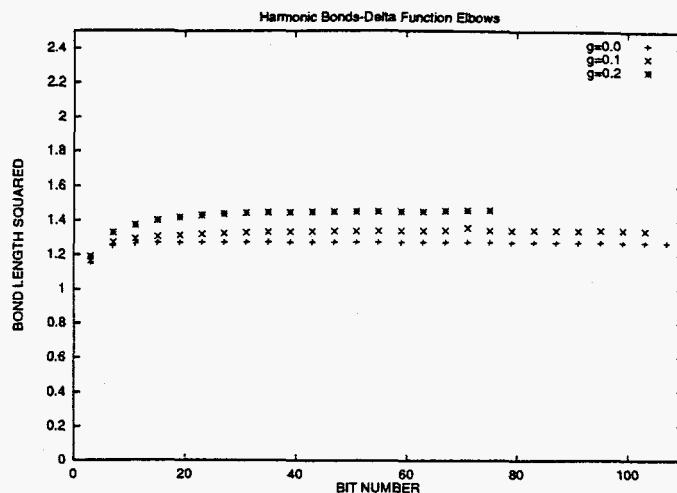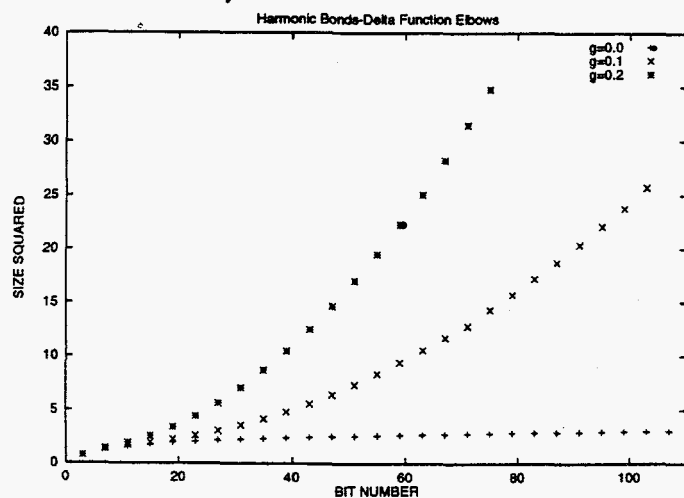
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Figure 1: Results of variational calculation of size and bond length for various nearest-neighbor (bond) potentials and non-nearest-neighbor repulsions (elbows).
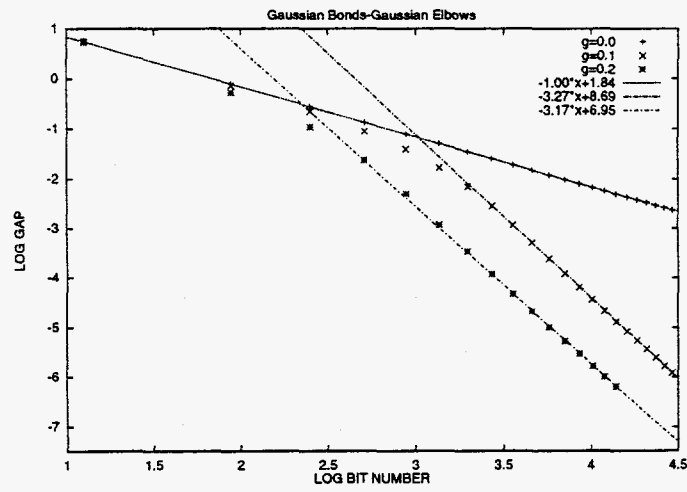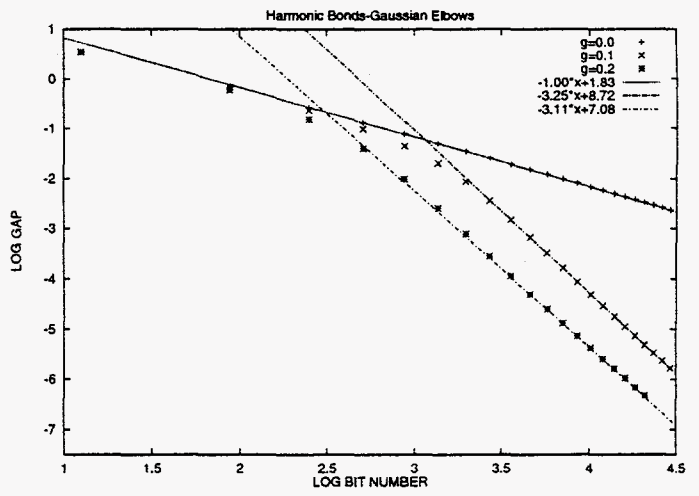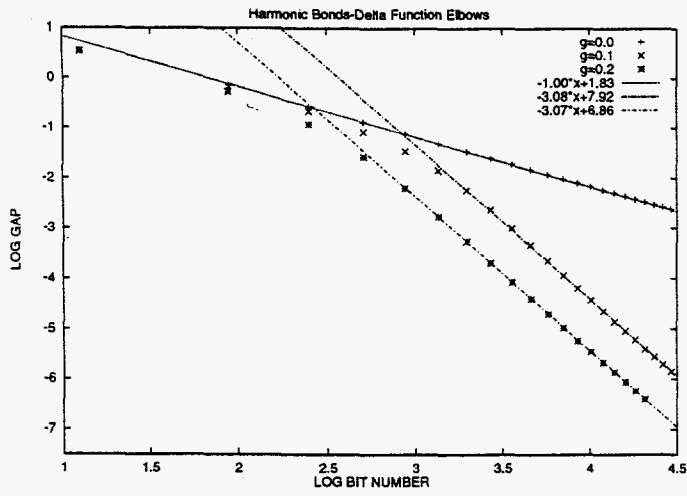
Figure 2: Gap dependence on bit number.

6   Discussion

We have found that our analog string-bit model, simulating the effects of string interactions on the size of a polymer of string-bits, seems to predict a growth with bit number that is much too fast for relativistic string. However, there are several circumstances that indicate that the analog model overstates the self-avoiding aspects of an interacting string. First of all, the interactions of the original bit model are much richer in structure than those of the analog model. Our estimate of the net repulsive character of the interactions (controlled in the analog model by the value of g) involved the competition between huge numbers (of order M^2) of repulsive and attractive contributions. A rough counting gave a net excess of repulsive over attractive contributions of $O(1)$. For larger and larger M the repulsive and attractive contributions are *relatively* more nearly in balance. Moreover, every contribution is not really the same either quantitatively or qualitatively, so it is a bit of a leap to conclude that the net result of the competition is a pure repulsive interaction independent of M. It is conceivable that the net repulsive interaction decreases with M. This would correspond in our analog model to an effective g depending in some way on M.

Furthermore, our analog model has vastly oversimplified the complicated bit rearrangement nature of the interactions, replacing them with interactions that conserve the integrity of each bit. Since a rearrangement term includes a factor of the overlap of different wave functions, it is bound to be smaller than a corresponding "direct" term. This criticism of our analog model is made even sharper by the circumstance that, in fact, *all* of the "direct" contributions exactly cancel out, leaving only contributions which involve at least some bit rearrangement. Thus treating the bit rearrangement properly would certainly reduce the growth rate compared to our analog model. However, it is not at all clear whether the reduction would amount to multiplying the size by a factor smaller than unity but independent of M, or by one that actually decreases with M.

Finally, the models studied in this paper are generic bosonic bit models. There is no opportunity in these models for the further more subtle cancelations that would be present in a supersymmetric model. We have tried to include at least one implication of supersymmetry by starting with a manifestly *positive* Hamiltonian. Indeed it was that positivity that was responsible for the net repulsive character of the interactions in the first place. But imposing positivity without supersymmetry is perhaps a bit heavy-handed, yet another reason to suspect that our analog model overstates the growth rate of string. Thus there is some hope that the real string-bit model could predict a linear growth with bit number.

All of these issues need to be addressed. One could certainly try to apply the variational method in the original bit model by varying within the space of single bare polymer states. But handling bit rearrangement in a tractable way remains a major challenge. As long as bit rearrangement is taken into account, it is probably also a good idea to try to include at least some admixture of bare multi-polymer states in the trial wave function. Clearly, this would require methods beyond those used in this article. To study the role of supersymmetry, we must first find a supersymmetric string-bit model. So far the only candidate we have is the model in $d = 1$ spatial dimension found in [14]. In view of the discussion at the end of section 2 however, the $d = 1$ case is not likely to shed light on the size issue. It is probably most urgent to construct a supersymmetric bit model in realistic dimensions ($d \geq 2$).

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