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**TRANSVERSE and LONGITUDINAL MICROWAVE
INSTABILITY THRESHOLDS**

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ALTERNATING GRADIENT SYNCHROTRON DEPARTMENT

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Transverse and Longitudinal Microwave Instability Thresholds

Abstract

Taking the first orthogonal polynomials in the conventional radial mode expansion in the eigenvalue type perturbation approach, the usual Keil-Schnell criteria for the microwave instabilities can be obtained. In this way, a close relationship between the two approaches is established. The existing results are reviewed, and some comments and modifications are made.

1 Introduction

A brief review of beam instability analyses shows that its developments either belong to a Vlasov-equation-evolved perturbation approach, or belong to a Keil-Schnell-criterion type approach. In the first approach, see [1,2] and the references therein, both azimuthal and radial expansions are used to explore the particle distribution evolutions in an instability mechanism. Current directions of the development is to include the potential well distortion, see for example [3], and to include the effect of Landau damping, see for example [4]. The development is unlikely to give rise to analytical solutions that can be easily used. On the other hand, the second approach uses crude beam profile (with an exception of the longitudinal coasting beams) to estimate the instability threshold for both bunched and unbunched beams. General results can be found in [5] and the references therein. These results have been proved very useful and often provide guidance to the development and improvement of accelerators. The crude beam profile, however, has certainly posed limitations in the application.

The transverse and longitudinal perturbation formalism shown in [1,2] is based on the eigenvalue problem representing the beam instability mechanism. Corresponding to each eigenvector, there is a radial mode. In [1,2], a conventional expansion of these radial modes by a set of orthogonal polynomials has been used. In this report, we show that the use of the first orthogonal polynomials can give rise to identical results obtained by the Keil-Schnell type criteria. This is owing to the fact that, in general, the first orthogonal polynomial represents approximately the most prominent radial mode. In this way, a close relationship between the two approaches is established. Therefore, some comments can be made regarding to the limitation and the possible error in the applications of the simplified criteria.

In this article, the particle distribution is assumed to be Gaussian. The beam dynamic equations for the bunched beams will be presented using the results in [1,2]. Then, the corresponding microwave instability criteria for bunched beams will be shown by using only the first orthogonal polynomials in the radial mode expansion. Taking the perturbation as delta functions and using some equivalence will lead to the results for coasting beams. The corresponding results in [5] and other widely used criteria presented in literature will be compared. Some modifications will then be developed, if necessary.

2 Transverse

Using the first orthogonal polynomial, the bunched beam dynamic equation shown in [1] becomes a scale equation,

$$\omega - \omega_\beta - m\omega_S = \frac{jeI_0}{2Rm_0\gamma\omega_\beta} \sum_{p=-\infty}^{\infty} Z_T(p) \left(\Lambda_0^{(m)}(p') \right)^2 \quad (2-1)$$

where ω_β and ω_S are the betatron and synchrotron frequencies, respectively, m is the azimuthal mode, R is the machine radius, and I_0 is the beam current defined by,

$$I_0 = \frac{Ne\omega_0}{2\pi} \quad (2-2)$$

with N being the number of particle per bunch, and ω_0 the angular revolution frequency. Also $Z_T(p)$ is the transverse impedance, and $\Lambda_0^{(m)}(p)$ is the Hankel spectrum of the first orthogonal polynomial $f_0^{(m)}(r)$, defined for a transverse weight function $W_T(r)$ by,

$$\Lambda_k^{(m)}(p) = \int_0^\infty W_T(r) f_k^{(m)}(r) J_m(pr) r dr \quad (2-3)$$

where p represents the sampling in frequency domain, r represents the radial position, and $J_m(pr)$ is the m th order Bessel function. The notation p' denotes the sampling regarding to the chromatic effect [1]. The equivalence $\sum_{p=-\infty}^{\infty} Z_T(p) \left(\Lambda_0^{(m)}(p') \right)^2 = \sum_{p=-\infty}^{\infty} Z_T(p'') \left(\Lambda_0^{(m)}(p) \right)^2$ is used in this article, where p'' denotes the frequency shift equals p' but in the opposite direction.

Consider a normalized Gaussian distribution in phase space,

$$\psi_0(r) = \frac{2}{\pi r_\ell^2} e^{-2r^2/r_\ell^2} \quad (2-4)$$

where r_ℓ is the *half bunch length* in rad, which is twice the standard deviation, or the *rms* bunch length, σ . The transverse weight function is defined as,

$$W_T(r) = \psi_0(r) \quad (2-5)$$

and the orthogonal polynomials are defined as,

$$f_k^{(m)}(r) = \left(\frac{\sqrt{2}r}{r_\ell} \right)^m \left(\frac{2\pi k!}{(m+k)!} \right)^{1/2} L_k^{(m)} \left(\frac{2r^2}{r_\ell^2} \right) \quad (2-6)$$

where $L_k^{(m)}(x)$ is the generalized Laguerre polynomial

$$L_k^{(m)}(x) = \sum_{i=0}^k (-1)^i \binom{m+k}{k-i} \frac{x^i}{i!} \quad (2-7)$$

For transverse instabilities, we consider only $m = 0$ mode. The power spectra of the first orthogonal polynomials for the half bunch length $r_\ell = \pi/n$, $n = 2, 4, 6, 8$, are shown in Fig.1 with a zero chromaticity. We note that the peaks of the power spectra are the same at $p = 0$

for different bunches. This is because that the normalized Gaussian distribution is used. In fact for the distributions with different bunch lengths, we have,

$$\left(\Lambda_0^{(0)}(p)\right)^2 \leq \left(\Lambda_0^{(0)}(0)\right)^2 = \frac{1}{2\pi} \quad (2-8)$$

This equation can be shown as follows. We note that $f_0^{(0)}(r) = \sqrt{2\pi}$ and $J_0(0) = 1$, therefore using (2-3) and (2-5), we have,

$$\Lambda_0^{(0)}(0) = \int_0^\infty W_T(r) f_0^{(0)}(r) J_0(0) r dr = \int_0^\infty \frac{2}{\pi r_t^2} e^{-2r^2/r_t^2} \sqrt{2\pi} r dr = \frac{1}{\sqrt{2\pi}} \quad (2-9)$$

where the identity

$$\int_0^\infty e^{-x} dx = 1 \quad (2-10)$$

is used.

Fig.1. Power spectra of $m = 0$ mode

In the following, the instability threshold will be obtained by the widely used rule of thumb, which is,

$$|\Delta\Omega| < \Delta\omega \quad (2-11)$$

where $\Delta\Omega = \omega - \omega_\beta - m\omega_S$ and $\Delta\omega$ is the rms or the half width of half maximum frequency spread. There are double implications in the equation (2-11). One is that if the growth

rate is larger than the frequency spread, then the Landau damping cannot overcome the instability. Another implication is that if the coherent frequency shift is larger than the frequency spread, then the Landau damping becomes ineffective and any small excitation can induce an instability. The corrections of (2-11) for different particle distributions will not be pursued in this article. Note that in Chao [5], correction factors are added to $\Delta\omega$, which is written as $\Delta\omega_{1/2}$ there.

2.1 Bunched beam

A rough estimate of the bunched beam instability threshold is ready to present, which is obtained using (2-1), (2-8) and (2-11),

$$\left| \sum_{p=-\infty}^{\infty} Z_T(p) \right| < \frac{4\pi R m_0 \gamma \omega_\beta}{e I_0} \Delta\omega \quad (2-12)$$

The criterion given in the equation (5.62) by Chao [5] can be written as,

$$\left| \sum_{p=-\infty}^{\infty} Z_T(p) \right| < \frac{2\omega_\beta \gamma T_0^2}{N r_0 c} \Delta\omega \quad (2-13)$$

Using $r_0 = e^2/m_0 c^2$, $T_0 = 2\pi/\omega_0$, $\omega_0 = c/R$ and (2-2), the equation (2-13) becomes identical to (2-12). We note that in (2-13), the unit of the impedance is Ω . If the unit of impedance is s/cm of the *cgs* system, then one has to use $r_0 = 30c \times e^2/m_0 c^2$.

An examination of Fig.1 shows that for a long bunch with a narrow spectrum, the error of using (2-12) can be large, mainly owing to the use of (2-8). Also the chromatic effect, which generates frequency shift of the beam spectrum, can introduce large uncertainties in the situation.

For an improved estimate, therefore, a complete form of $\Lambda_0^{(0)}(p)$ is needed, which is,

$$\Lambda_0^{(0)}(p) = 2\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2r^2/r_\ell^2} J_0\left(pr_\ell \times \frac{r}{r_\ell}\right) \left(\frac{r}{r_\ell}\right) d\left(\frac{r}{r_\ell}\right) = \frac{1}{\sqrt{2\pi}} e^{-p^2 r_\ell^2/8} \quad (2-14)$$

where the following equation in [6, (11.4.29)] is used,

$$\int_0^\infty e^{-a^2 x^2} x^{\nu+1} J_\nu(bx) dx = \frac{b^\nu}{(2a^2)^{\nu+1}} e^{-b^2/4a^2} \quad (2-15)$$

with $\nu = 0$, $a^2 = 2$, $x = r/r_\ell$, and $b = pr_\ell$. Substituting (2-14) into (2-1) for $m = 0$, and considering the chromatic effect, we get,

$$\left| \sum_{p=-\infty}^{\infty} Z_T(p'') e^{-p^2 r_\ell^2/4} \right| < \frac{4\pi R m_0 \gamma \omega_\beta}{e I_0} \Delta\omega \quad (2-16)$$

To compare, we note the following results.

- A criterion is given in the equation (10) by Schnell [7], which can be written as,

$$|\bar{Z}_T(p'')| < \frac{2\omega_\beta \gamma E_0 z_L}{ec^2 I_0} \Delta\omega \quad (2-17)$$

where z_L is the *full bunch length* in meter, and $\bar{Z}_T(p'')$ is the averaged impedance over the width of the bunch spectrum. The width of the bunch spectrum is related to the bunch length by,

$$\omega_{BW} = \frac{4\pi c}{z_L} \quad (2-18)$$

The summation on the right side of (2-1) can be approximately seen as,

$$\sum_{p=-\infty}^{\infty} Z_T(p) \left(\Lambda_0^{(0)}(p') \right)^2 \approx \bar{Z}_T(p'') \frac{1}{2\pi} \frac{\omega_{BW}}{\omega_0} = \bar{Z}_T(p'') \frac{1}{2\pi} \frac{4\pi c R}{z_L c} = \bar{Z}_T(p'') \frac{2R}{z_L} \quad (2-19)$$

Substituting (2-19), using $m_0 = E_0/c^2$ and (2-11), the equation (2-1) becomes

$$|\bar{Z}_T(p'')| < \frac{\omega_{\beta} \gamma E_0 z_L}{e c^2 I_0} \Delta\omega \quad (2-20)$$

Comparing with this equation, the equation (2-17) may need to be corrected by a factor of 2.

- A better formalism is presented in the equation (18) by Sacherer [8]. With $m = 0$, it can be written as,

$$\left| \frac{\sum_{p=-\infty}^{\infty} Z_T(p) h_0(p')}{\sum_{p=-\infty}^{\infty} h_0(p')} \right| < \frac{2\omega_{\beta} \gamma m_0 z_L}{e I_0} \Delta\omega \quad (2-21)$$

where the left side is called the *effective impedance*, which is interpreted in [7] as the averaged impedance $\bar{Z}_T(p'')$, and $h_0(p')$ is the power spectrum of the bunch. If only the first orthogonal polynomial is used, we have $h_0(p') = \left(\Lambda_0^{(0)}(p') \right)^2$. By an examination of the equation (2-21), one may find some redundancy in applying the effective impedance. This is shown as the follows. Using $r_L = z_L/2R$, we can write,

$$\sum_{p=-\infty}^{\infty} h_0(p) = \sum_{p=-\infty}^{\infty} \left(\Lambda_0^{(0)}(p) \right)^2 \approx 2 \int_0^{\infty} \frac{1}{2\pi} e^{-p^2 r_L^2/4} dp = \frac{1}{\sqrt{\pi} r_L} = \frac{2R}{\sqrt{\pi} z_L} \quad (2-22)$$

where the equation [6]

$$\int_0^{\infty} e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2a} \quad (2-23)$$

is used with $a = r_L/2$ and $x = p$. The equation (2-22) shows that if we remove the denominator in the effective impedance to the right side of the equation (2-21), then the bunch length z_L is cancelled. Meanwhile the information of the bunch length has already been represented by the bunch spectrum $h_0(p')$ in the numerator of the effective impedance. Therefore, this triple representation of the bunch length can be seen as some redundancy. In comparison, the use of the total effective impedance shown in the left side of (2-16) seems to be more straightforward. Substituting (2-22) into (2-21), we get,

$$\left| \sum_{p=-\infty}^{\infty} Z_T(p'') e^{-p^2 r_L^2/4} \right| < \frac{8\sqrt{\pi} R m_0 \gamma \omega_{\beta}}{e I_0} \Delta\omega \quad (2-24)$$

This is only different from (2-16), which is tighter, by a factor of 1.13. This difference comes from that in [8] a parabolic distribution is used, and also the original denominator

in the effective impedance was $\sum_{p=-\infty}^{\infty} p h_0(p)$, to get the effective impedance shown in (2-21), the factor p was removed out of the summation by an approximation, using a sinusoidal mode, see part II in [9].

Finally, we note that the sum of the power spectra in (2-22) can be compared with the rough estimate in (2-19), which is larger than the one in (2-22) by a factor of $\sqrt{\pi}$.

2.2 Coasting beam

For a coasting beam, the perturbation used to derive the beam dynamic equation (2-1) in [1] lost its identity in the longitudinal phase space. The power spectrum of the perturbation can be treated as a delta function at a frequency denoted by the sampling p_1 with an amplitude $1/2\pi$. The equation (2-1), therefore, is modified as,

$$\omega - \omega_\beta = \frac{jeI_0}{2Rm_0\gamma\omega_\beta} \sum_{p=-\infty}^{\infty} Z_T(p) \frac{\delta(p - p_1)}{2\pi} \quad (2-25)$$

then the instability threshold can be estimated as,

$$|Z_T(p_1)| < \frac{4\pi Rm_0\gamma\omega_\beta}{eI_0} \Delta\omega \quad (2-26)$$

This equation shows that for the coasting beam, an instability may be excited at the frequency of $p_1\omega_0 + \omega_\beta$ if the impedance there has violated the condition of (2-26). Usually the wave corresponding to $p_1 > 0$ is called the fast wave, and the one of $p_1 < 0$ is the slow wave. Since the fast wave is stable, only the slow wave requires the Landau damping. The criterion given in the equation (5.91) by Chao [5] can read,

$$|Z_T(p)| < \frac{2\omega_\beta\gamma T_0^2}{N\tau_0 c} \Delta\omega \quad (2-27)$$

which is again the identical to the equation (2-26).

To compare, we note the following results.

- The equation (4) by Schnell [7] can be written,

$$|Z_T(p)| < \frac{8FE_0\gamma\omega_\beta}{eI_0R\omega_0^2} \Delta\omega \quad (2-28)$$

where F is the form factor, which is identified to be not much different from unity. Using $E_0 = m_0c^2$ and $\omega_0 = c/R$, the equation becomes,

$$|Z_T(p)| < \frac{8FRm_0\gamma\omega_\beta}{eI_0} \Delta\omega \quad (2-29)$$

Taking $F = 1$, this equation differs from (2-26), which is less tight, by a factor of $\pi/2$.

- The equation (41) by Zotter [10] can be written the same as the equation (2-28), and also the form factor is believed to be about a unity.

One often uses the coasting beam criterion (2-26) for the bunched beam transverse microwave instability estimate, simply replacing the beam current I_0 by the peak current I_p , i.e.

$$|Z_T(p_1)| < \frac{4\pi R m_0 \gamma \omega_\beta}{e I_p} \Delta\omega \quad (2-30)$$

For a Gaussian, the peak current is related to I_0 by,

$$I_p = \frac{2\sqrt{2\pi}}{r_\ell} I_0 \quad (2-31)$$

On the other hand, if the impedance is taken out of the summation, the bunched beam criterion (2-16) becomes,

$$|Z_T(p)| \left| e^{-p^2 r_\ell^2 / 4} \right| = |Z_T(p)| \frac{2\sqrt{\pi}}{r_\ell} < \frac{4\pi R m_0 \gamma \omega_\beta}{e I_0} \Delta\omega \quad (2-32)$$

where the relation (2-23) is used. Using (2-31), it is shown that this equation differs from (2-30) by a factor of $\sqrt{2}$.

3 Longitudinal

Using the first orthogonal polynomial, the longitudinal beam dynamic equation in [2] becomes,

$$\omega - m\omega_S = \frac{j2\pi m\omega_S I_0}{V \cos \phi_S} \sum_{p=-\infty}^{\infty} \frac{Z_L(p)}{p} \left(\Lambda_0^{(m)}(p) \right)^2 \quad (3-1)$$

where ω_S is the synchrotron frequency, and ϕ_S is the synchronous phase, V is the RF gap voltage per ring, and $Z_L(p)$ is the longitudinal impedance. For a Gaussian distribution with the half bunch length r_ℓ , the longitudinal weight function is different from the transverse counterpart, which is,

$$W_L(r) = -\frac{\partial \psi_0(r)}{\partial r} \frac{1}{r} = \frac{8}{\pi r_\ell^4} e^{-2r^2/r_\ell^2} \quad (3-2)$$

and the orthogonal polynomials becomes,

$$f_k^{(m)}(r) = \left(\frac{\sqrt{2}r}{r_\ell} \right)^m \left(\frac{\pi r^2 k!}{2(m+k)!} \right)^{1/2} L_k^{(m)} \left(\frac{2r^2}{r_\ell^2} \right) \quad (3-3)$$

Another difference from the transverse motion is that instead of $m = 0$, the $m = 1$ mode is dominant in the longitudinal instability. The power spectra of the first orthogonal polynomials for the half bunch length $r_\ell = \pi/n$, $n = 2, 4, 6, 8$, are shown in Fig.2.

Note that we have $f_0^{(1)}(r) = \sqrt{\pi}r$. If we take

$$J_1(pr) \approx \frac{pr}{2} \quad (3-4)$$

Fig.2. Power spectra of $m = 1$ mode

then we have

$$\Lambda_0^{(1)}(p) = \int_0^\infty W_L(r) f_0^{(1)}(r) J_1(pr) r dr \approx \int_0^\infty \frac{8}{\pi r_i^4} e^{-2r^2/r_i^2} \sqrt{\pi r} \frac{pr}{2} r dr = \frac{p}{2\sqrt{\pi}} \quad (3-5)$$

where the identity

$$\int_0^\infty x e^{-x} dx = 1 \quad (3-6)$$

is used.

3.1 Bunched beam

Using the equations (3-1) and (3-5), letting $m = 1$, the bunched beam instability threshold is written as,

$$\left| \sum_{p=-\infty}^{\infty} p Z_L(p) \right| < \frac{2V |\cos \phi_S|}{\omega_S I_0} \Delta\omega \quad (3-7)$$

where $\Delta\omega$ in this case is the synchrotron frequency spread. The corresponding criterion by Chao [5] is the equation (5.69), which is,

$$\left| \sum_{p=-\infty}^{\infty} p Z_L(p) \right| < \frac{2\omega_S \gamma (2\pi R)^2}{N r_0 |\eta| c^2 \omega_0} \Delta\omega \quad (3-8)$$

Using $r_0 = e^2/m_0c^2$, $I_0 = Ne\omega_0/2\pi$, and

$$\omega_S^2 = \frac{-\omega_0^2 e\eta V \cos \phi_S}{2\pi E} \quad (3-9)$$

the equation (3-8) is shown to be the same as (3-7). Note that the potential well distortion effect is not included in the equations (3-7) and (3-8).

A comparison of the beam power spectra shown in Fig.2 and the one in (3-5) shows that this criterion is indeed very crude. This is mainly owing to the use of (3-4), which can only be used in a small range $pr < 1$. Also note that in (3-7) $pZ_L(p)$, rather than the usual $Z_L(p)/p$, is shown up in the summation. This is because of the use of (3-5).

An improvement to this criterion, therefore, requires a more accurate spectrum $\Lambda_0^{(1)}(p)$, which can be obtained by writing,

$$\Lambda_0^{(1)}(p) = \frac{8}{\sqrt{\pi} r_\ell} \int_0^\infty e^{-2r^2/r_\ell^2} J_1\left(pr_\ell \times \frac{r}{r_\ell}\right) \left(\frac{r}{r_\ell}\right)^2 d\left(\frac{r}{r_\ell}\right) = \frac{p}{2\sqrt{\pi}} e^{-p^2 r_\ell^2/8} \quad (3-10)$$

where the equation (2-15) is used, with $\nu = 1$, $a^2 = 2$, $x = r/r_\ell$, and $b = pr_\ell$. Substituting (3-10) into (3-1), we get,

$$\left| \sum_{p=-\infty}^{\infty} pZ_L(p) e^{-p^2 r_\ell^2/4} \right| < \frac{2V |\cos \phi_S| \Delta\omega}{\omega_S I_0} \quad (3-11)$$

To compare, we note the following results.

- Consider the result in the equation (5) by Sacherer [9], which can be written, for $m = 1$ and $h = 1$, as,

$$\left| \frac{\sum_{p=-\infty}^{\infty} (Z_L(p)/p) h_1(p)}{\sum_{p=-\infty}^{\infty} h_1(p)} \right| < \frac{6B^3 V |\cos \phi_S| \Delta\omega}{\omega_S I_0} \quad (3-12)$$

where $h_1(p) = (\Lambda_0^{(1)}(p))^2$ is the power spectrum of the $m = 1$ mode, and $B = r_\ell/\pi$ is the bunching factor. Note that we have,

$$\sum_{p=-\infty}^{\infty} h_1(p) \approx 2 \int_0^\infty (\Lambda_0^{(1)}(p))^2 dp = 2 \int_0^\infty \frac{p^2}{4\pi} e^{-p^2 r_\ell^2/4} dp = \frac{1}{\sqrt{\pi} r_\ell^3} = \frac{1}{\pi^{7/2} B^3} \quad (3-13)$$

where in obtaining this result the following equation in [11, (3.461)] is used,

$$\int_0^\infty x^{2a} e^{-bx^2} dx = \frac{(2a-1)!!}{2(2b)^a} \sqrt{\frac{\pi}{b}} \quad (3-14)$$

with $a = 1$, $x = p$, and $b = r_\ell^2/4$. Substituting (3-13) into (3-12), using (3-10), we get,

$$\left| \sum_{p=-\infty}^{\infty} pZ_L(p) e^{-p^2 r_\ell^2/4} \right| < \frac{1.37V |\cos \phi_S| \Delta\omega}{\omega_S I_0} \quad (3-15)$$

which differs from (3-11) by a factor of 1.46, caused by the same reason mentioned in the transverse bunched beam case. Again we consider that the use of the equation (3-15) is more straightforward than (3-12) with the effective impedance. For a non-Gaussian distribution, the calculation needs to be modified, but this makes no essential difference.

- The results using the effective impedance in the equation (6.142) by Chao [5] can be written for $m = 1$ mode as,

$$\left| \frac{\sum_{p=-\infty}^{\infty} (Z_L(p)/p) h_1(p)}{\sum_{p=-\infty}^{\infty} h_1(p)} \right| < \frac{\pi \gamma T_0 \omega_S \omega_0 R^3 r_\ell^3}{\Gamma(1.5) N r_0 |\eta| c^3} \Delta \omega \quad (3-16)$$

where $\Gamma(x)$ is the gamma function of x . Using $r_0 = e^2/m_0 c^2$, $I_0 = Ne\omega_0/2\pi$, and the equations (3-9), (3-10), and (3-13), the equation (3-16) becomes exactly the same as the equation (3-11).

3.2 Coasting beam

The Landau damping in the longitudinal coasting beam is probably the most explored one compared with the others. Detailed particle distribution is often used. Together with the dispersion relation, the stability diagram can be plotted on the real and imaginary impedance plane. Also in this case the focusing force, which for longitudinal bunched beam is the RF focusing and for the transverse case is the betatron focusing, is lost. This leads to an expectation that this case should be completely separated from the others.

The successful application of the coasting beam instability criterion to the bunched beams, i.e. the Boussard criterion, has opened the door to think that at least for the long bunches the effect of the synchrotron focusing is not irreplaceable. It is found that using an equivalence

$$\omega_S = \Delta \omega \quad (3-17)$$

and the local current, the bunched beam criteria is closely related with the coasting beam criteria. Using the relations,

$$\left(\frac{\Delta p}{p} \right)_{rms} = \frac{\omega_S r_\ell}{2 |\eta| \omega_0} \quad (3-18)$$

which is valid if the bunch is smooth, and the usual

$$\left(\frac{\Delta p}{p} \right)_{rms} = \frac{1}{|\eta|} \frac{\Delta \omega}{\omega_0} \quad (3-19)$$

we find that the equation (3-17) implies that,

$$r_\ell = 2 \quad (3-20)$$

In the transverse case, we have used the delta function with an amplitude of $1/2\pi$ for the power spectrum of the $m = 0$ mode. For the longitudinal case with $m = 1$ mode, the beam power spectrum is still a delta function, but the amplitude is no longer constant. We know that the amplitude of a delta function equals the area of the function, i.e. $\sum_{p=-\infty}^{\infty} \left(\Lambda_0^{(1)}(p) \right)^2$, which we have already calculated in the equation (3-13). Thus, removing the impedance $Z_L(p)/p$ out of the summation on the right side of (3-1), using equations (3-9), (3-13), and (3-17), substituting I_0 by I_p , we get from (3-1),

$$\left| \frac{Z_L(p)}{p} \right| < \frac{Er_\ell^4}{2\sqrt{2}e|\eta|\omega_0^2 I_0} (\Delta \omega)^2 \quad (3-21)$$

Substituting (3-20), the equation (3-21) becomes,

$$\left| \frac{Z_L(p)}{p} \right| < \frac{5.66E}{e|\eta|\omega_0^2 I_0} (\Delta\omega)^2 \quad (3-22)$$

The Keil-Schnell criterion shown in the equation (5.131) in [5] can read,

$$\left| \frac{Z_L(p)}{p} \right| < \frac{0.68\gamma T_0^3}{2\pi N r_0 |\eta|} (\Delta\omega)^2 \quad (3-23)$$

where the tri-elliptical spectrum is used. Using $r_0 = e^2/m_0 c^2$, $T_0 = 2\pi/\omega_0$, $I_0 = Ne\omega_0/2\pi$, and $E = m_0 c^2 \gamma$, the equation (3-23) can be written the same as the equation (3-22), except that the factor 5.66 becomes 4.27, i.e., (3-23) is tighter.

- To compare, the equation (1) by Schnell [7] can be written,

$$\left| \frac{Z_L(p)}{p} \right| < \frac{F\gamma E_0 |\eta|}{e I_0} \left(\frac{\Delta p}{p} \right)^2 \quad (3-24)$$

where the form factor F is approximately a unity. Since $\Delta p/p$ is the full momentum spread at half height, using

$$\frac{\Delta p}{p} = 2 \left(\frac{\Delta p}{p} \right)_{rms} \quad (3-25)$$

the equation (3-24) can be written the same as (3-22), with the factor 5.66 in (3-22) becomes 4, which means that the criterion (3-24) is tighter than the one in (3-22).

- The equation (2) by Hofmann [12] can read,

$$\left| \frac{Z_L(p)}{p} \right| < \frac{FE|\eta|}{e I_0} \left(\frac{\Delta E}{E} \right)_{rms}^2 \quad (3-26)$$

where the factor F is believed to be in an order of 6. Therefore, if this equation is written in the same form as (3-22), then the factor 5.66 becomes 6, i.e. this criterion is the least tight one.

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