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Emittance Growth of an Electron Beam in a Periodic Focusing Channel Due to Transfer of Longitudinal Energy to Transverse Energy

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Abstract

Most discussions about emittance growth and halo production for an intense electron beam in a periodic focusing channel assume that the total transverse energy is constant (or, in other words, that the transverse and longitudinal Hamiltonians are separable). Previous analyses that include variations in the total transverse energy are typically based on a transverse-longitudinal coupling that is either from two-dimensional space-charge modes or particle-particle Coulomb collisions. With the space-charge modes, the energy exchange between the transverse and longitudinal directions is periodic, and of constant magnitude. The total energy transfer for the case of the Coulomb collisions is negligible. This limited increase of energy in the transverse direction from these other effects will limit the amount of transverse emittance growth possible. In this paper, we investigate a mechanism in which there is a continual transfer of energy from the longitudinal direction to the transverse direction, leading to essentially unlimited potential transverse emittance growth. This mechanism is caused by an asymmetry of the beam’s betatron motion within the periodic focusing elements. This analysis is based on thermodynamic principles. This mechanism exists for both solenoids and quadrupole focusing, although only solenoid focusing is studied here.
I. Introduction

The nonlinear free-energy concept [1] is a powerful tool to explain emittance growth in a periodic focusing channel due to either an initial mismatch of the beam or nonuniform beam density. This concept is thermodynamic in nature, and the basic idea is that stationary states exist for the transverse beam distribution when the beam is in a periodic focusing channel. These stationary states have certain rms values (including some certain beam emittance), and some average energy per particle. If the initial beam is mismatched, or has a density nonuniformity, the average energy per particle is greater than that of the "equivalent rms beam" stationary state with the same rms values, including the initial beam emittance. The initial distribution will eventually relax to a stationary state with the same average transverse energy per particle, but with larger final rms values, including the beam emittance. This thermodynamic approach has been well supported by both simulations [2,3] and experiments [3,4]. In this analysis, we rely heavily upon the thermodynamic concept, but with the added feature that the average transverse energy per particle can increase due to the beam's interaction with the focusing elements.

For relativistic beams, the average longitudinal energy of an electron beam is extremely larger than the average transverse energy. Just a little transfer of longitudinal energy to the transverse direction can greatly increase the beam's emittance. For example, the average longitudinal energy of a 10 MeV electron beam is about \(2 \times 10^{-12}\) Joule, but the average transverse energy (with an emittance of \(1 \times 10^{-4}\) m and a betatron period of about 10 m) is only about \(5 \times 10^{-18}\) Joules. Thus, not much longitudinal energy needs to be transferred for a large emittance growth, and the longitudinal energy can be essentially thought of as an infinite reservoir of potential energy.

Some previous studies have included longitudinal/transverse couplings. In particular, two-dimensional breathing modes have been studied in [5]. In this model, there is a continuous transfer of energy back and forth between the transverse and longitudinal directions. However, the maximum amount of energy transferred into the transverse direction is limited, as is the emittance growth. Also, slow and fast
longitudinal space-charge waves have been described [6], which travel along the beam, and couple a finite amount of longitudinal energy to the beam's potential energy and thus to the transverse direction. This energy transfer mechanism is also limited in magnitude.

Since magnetic focusing elements (quadrupoles and solenoids) cannot change a particle's total energy, they transfer longitudinal energy to the transverse direction, and back again. This effect is well known, and, in general, it is assumed that the energy exchange, averaged over the entire beam distribution, vanishes. In this analysis, we carefully consider the energy transfer, and show that if the beam space charge is large enough there will be a nonzero-average, continual, leaking of longitudinal energy into the transverse direction.

We will use some thermodynamic assumptions in this analysis. First, we will assume that the initial beam is perfectly matched. We will assume that the average transverse energy growth is sufficiently slow that the beam distribution can adiabatically adjust to it, and that the beam stays matched even as the transverse energy increases. Also, we will make liberal use of the "equivalent rms beam," which is constructed from the uniform density beam with equal rms values as the actual beam distribution. With the equivalent rms beam, we can calculate beam ensemble averages without knowing the exact beam distribution. Finally, we will also be assuming that the beam envelope is very close to the "smooth approximation" envelope, in which the beam encounters a uniform axial magnetic field instead of period focusing elements. In the smooth approximation for a matched distribution, all electrons perform perfect sinusoidal betatron motion, and the beam envelope radius is axially constant. We assume that the a particle's orbit from this perfect betatron motion is very small. Note that a purely axial magnetic field will not transfer any energy from the longitudinal direction to the transverse direction, and that this deviation from the smooth approximation orbits leads to the continual leaking of longitudinal energy.

We will be considering the energy transfer for a periodic channel consisting of solenoids. Much the same physics occurs for a channel consisting of quadrupole magnetics; in fact, the effect is larger for quadrupoles, but the analysis is simpler for solenoids, which is why is it presented here.
Consider the "ideal" solenoid pictured in Fig. 1, where the axial magnetic field is uniform over a central region (and of magnitude $B_0$), and linearly changes in the transition regions. The central region is of no interest to us, because an axial field cannot do any work in the longitudinal direction. But now consider the transition regions, where the magnetic field is given by

$$
\begin{align*}
B_z &= B_0 \frac{z}{z_0} \\
B_r &= B_0 \frac{r}{2z_0} \\
B_z &= B_0 \frac{z_0 - z}{z_0} \\
B_r &= -B_0 \frac{r}{2z_0}
\end{align*}
$$

Solenoid front edge

Solenoid back edge

where here $z$ is taken from the start of the transition region, with a total transition region length of $z_0$. Note that these fields satisfy both the divergence and curl Maxwell Equations. Work is done in the longitudinal direction by these fields, which we now calculate. Because the axial field is radially uniform, the azimuthal velocity of a particle at radius $r$ is given by

$$
\nu_\theta = -\frac{ezB_0}{2m\gamma z_0}
$$

where now $e$ is the electronic charge, $m$ is the electronic mass, and $\gamma$ is the particle’s relativistic mass factor. The force on that particle in the longitudinal direction while the particle is in the transition region is given by

$$
F_z = ev_\theta B_r = \frac{e^2r^2z}{4m\gamma z_0^2}B_0^2.
$$
where the minus sign is used for the initial transition region and the plus sign for the final transition region. Integrating the force over both transition regions gives the net work done on the particle in the longitudinal direction by the solenoid,

\[
W_z = -\frac{e^2}{8m} B_0^2 \left( \frac{r_1^2 - r_2^2}{\gamma_1 - \gamma_2} \right),
\]

where we are now assuming that transitions are short enough that the particle radius remains \( r_1 \) in the first transition and \( r_2 \) in the second transitions, with respective relativistic mass factors of \( \gamma_1 \) and \( \gamma_2 \). The change in the mass factors can only occur due to the potential depression of the beam, resulting from space-charge forces. Note that if the particle is going from a large amplitude radius to a small amplitude radius in the solenoid (as would be the case if the solenoid is focusing an initially parallel beam), energy is taken out of the longitudinal direction and put into the transverse direction, which is clearly true. If the particle is smaller in the first transition than in the second transition, the transverse direction gives up energy to the longitudinal direction.

For a uniform density beam, the relativistic mass factor scales as

\[
\gamma = \gamma_a + \frac{Ir^2}{I_A \rho_{\text{max}}^2}
\]

where \( \gamma_a \) is the mass factor on axis, \( I \) is the beam current, \( I_A \) is very nearly 17 kA, and \( \rho_{\text{max}} \) is the beam radius. The relativistic mass factor variation in (4) is thus second order, and can be ignored. In the rest of this analysis, we will assume that the total particle kinetic energy is a constant.

If the betatron wavelength is \( \lambda_b \), and there are \( N \) solenoids per betatron wavelength, the focal length of each individual solenoid is \( f = N\lambda_b \) [7], where the focal length for a solenoid of length \( l \) is given by
Thus for a matched distribution in a periodic channel, the average increase in a particle’s energy in the transverse direction from a single solenoid is given by

\[ f = \frac{4\gamma^2 m^2 v_z^2}{le^2 B_0^2} \]

where the brackets indicate an average over the particle distribution and \( \beta \) is the axial velocity normalized to the speed of light. This is the energy exchange from a single lens. After the averaging over the particle distribution, this can be either positive or negative. However, we will show in the following analysis that, for a matched beam in a periodic channel, the transverse energy will always increase. From a kinematic point of view, it is hard to quantify the emittance growth due to this energy exchange. However, we can use the thermodynamic concepts to relate an average transverse energy increase to an emittance increase. To use this thermodynamic approach, the beam must be in a stationary state of the periodic channel, with a slow, adiabatic growth in the emittance. In that case, the emittance growth results because the beam continually tries to reach an equilibrium state. The excess transverse energy relaxes to a stationary state with larger rms values through nonlinear space-charge and collisions, among other possible mechanisms. For the case the beam is matched, the average axial rate of increase in energy becomes

\[ \frac{dE_t}{dz} = \frac{NW_t}{\lambda_b} = \frac{\gamma^2 mc^2 \beta^2}{2l^2 \lambda_b^2 \gamma} \left( r_1^2 - r_2^2 \right) \]

In the following sections we will evaluate this expression to quantify the actual energy increase, and we will relate this to the emittance growth itself. In the next section, we will relate the transverse energy to the transverse emittance, for a matched distribution. This will lead to a differential equation relating the emittance growth rate to
the transverse energy growth rate. In the following section, we will show that the transverse energy growth rate as described in Eqn. (8) will vanish if the space-charge forces are sufficiently small. Physically, this occurs because a particle’s motion within a solenoid is symmetric relative to its ideal betatron orbit in the smooth approximation model. In this section, we will also evaluate the transverse energy growth for asymmetric motion within a solenoid, characterized by an offset $\delta$. In Section IV, we will explicitly calculate the offset for the case with appreciable space charge. This offset arises because the particles’ orbits between the solenoids is a hyperbolic sine or hyperbolic cosine function, which is exponential and not symmetric like a sine or cosine function. In the section following that, we assemble the previous parts and derive the differential equation for the emittance growth. The emittance growth is linear if the beam is not being accelerated. The final section contains a discussion about phenomena related to this effect.

II. Relation between RMS Emittance and Transverse Emittance

In this section, we will be using the smooth approximation and additionally assume the beam is perfectly matched into a smooth channel, in order to derive a relation between the beam emittance and average transverse energy.

Consider the phase space distribution shown in Fig. 2, with a maximum transverse displacement of $\rho_{\text{max}}$ and a maximum divergence of $\kappa \rho_{\text{max}}$, which corresponds to the differential equation

$$x'' + \kappa^2 x = 0,$$

where the betatron period is now given by $\lambda_b = 2\pi / \kappa$.

The total energy for a given particle in one transverse direction is given by

$$(mv_z^2 / 2)(x^2 \kappa^2 + x'^2).$$

At this point, we assume that the distribution is uniform in phase space, in which case the average energy is given by
The rms emittance is related to the betatron wavenumber for a uniform density beam by

\[ E_t = \frac{\int_0^{\rho_{\text{max}}} \left( \frac{m v_z^2}{2} \rho^2 \kappa^2 \right) \rho \, d\rho}{\int_0^{\rho_{\text{max}}} \rho \, d\rho} = \frac{m v_z^2}{4} \rho_{\text{max}}^2 \kappa^2. \quad (10) \]

and so the average energy per particle is given in terms of the rms emittance by

\[ \varepsilon_{\text{rms}} = \frac{\rho_{\text{max}}^2 \kappa}{4}, \quad (11) \]

and the corresponding differential equation for the growth of the rms emittance is given by

\[ \kappa m v_z^2 \frac{d\varepsilon_{\text{rms}}}{dz} = \frac{dE_t}{dz}. \quad (13) \]

Once we quantify the average transverse energy growth, we can use this equation to solve for the rate of the emittance growth. The next two sections will derive the necessary tools to quantify the average transverse energy growth. It is worth emphasizing that Eqn. (13) is based on a matched beam, with only adiabatic changes. Thus changes in the transverse energy will lead to changes in the emittance and betatron motion which conform to a new distribution, which is also matched.
III. Transverse Energy Growth for No or Little Space Charge

If space-charge can be neglected, a particle’s motion within the solenoids can be thought of as symmetric parabolic motion, followed by straight-line motion between the solenoids. If the space-charge forces are sufficiently weak, the straight-line motion between the solenoids becomes parabolic, but the motion within the solenoids is still very nearly symmetric (see Figs. 3a and 3b).

Symmetric Motion in Solenoids

Let us assume the differential equation for a particle’s motion within the solenoid is given by Eqn. (9), with solution \( x = a \cos(\kappa z) + b \sin(\kappa z) \). At the center of the solenoid, the particle orbit is tangential to the ideal betatron motion of the smooth approximation, defined by \( x_b = \rho \cos \theta \) and \( x'_b = -\rho \chi_b \sin \theta \), where now \( \chi_b \) is the betatron wavenumber, and as before we will assume particles are uniformly distributed in phase space (for \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \rho \leq \rho_{\text{max}} \) (all that is really necessary is to assume that the particles are uniformly distributed along all betatron phases). Defining the center of the solenoid to be at \( z = 0 \), we see that a particle’s radius within the solenoid is given by

\[
r = \rho \cos \theta \cos(\kappa z) - \frac{\rho \chi_b}{\kappa} \sin \theta \sin(\kappa z) .
\]  

At this point, we assume that the rms equivalent beam (with all second moments equal to those of the actual beam distribution, but with a uniform density distribution in both real space and phase space) can be used to evaluate the ensemble averages needed to determine the average energy transfer per particle. The energy exchange of a single particle from this solenoid (with edges at \( z = \pm l/2 \)) is then given by
\[ W_i = \frac{e^2}{2m\gamma^2} B_0^2 \left( \rho \cos \theta \cos(x_i) + \frac{\rho x_i b}{\kappa} \sin \theta \sin(x_i) \right)^2 \]

\[ - \left( \rho \cos \theta \cos(x_i) - \frac{\rho x_i b}{\kappa} \sin \theta \sin(x_i) \right)^2 \] (15)

\[ = \frac{e^2}{2m\gamma^2} B_0^2 \rho^2 \cos \theta \sin \frac{x_i b}{\kappa} \cos(x_i) \sin(x_i) \]

where we are using \( \bar{t} = t/2 \) for shorthand. Note that the work averaged over all particles vanishes as soon as the averaging is done over all possible azimuthal angles \( \theta \). Even if the distribution is not uniformly distributed along all betatron phases, note that the energy exchange of a single particle will also average to zero over an entire betatron period (assuming that there are many solenoids per betatron period and that the betatron period is not an integral number of solenoid-to-solenoid spacings).

Thus, as long as the particles’ motion is symmetric in the solenoids (defined by the fact that the actual orbits are tangential to the ideal, smooth-approximation betatron orbits at the center of the solenoid), there is no net energy transfer, and the beam emittance will not increase. However, the particle’s motion between the solenoids actually obeys hyperbolic trigonometric functions with space charge, which leads to an appreciable asymmetric motion within the solenoids if the magnitude of the space charge is large enough.

Asymmetric Motion in Solenoids

For the case that the particle’s orbits are asymmetric in the solenoids (Fig. 4), we characterize the asymmetry as a nonzero offset \( \delta \) between the center of the solenoid and the location where the particles’ orbits are tangential to the smooth-approximation betatron orbits. As before, the particles’ orbits in the solenoid are given by Eqn. (9), but now with \( z = 0 \) defined at the offset location and with \( r_1 \) and \( r_2 \) at locations \( z = -1/2 - \delta \) and \( z = 1/2 - \delta \), respectively.

Doing the algebra, we find
\[ r_1^2 - r_2^2 = \left( \rho \cos \theta \cos(\kappa(\tilde{l} + \delta)) + \frac{\rho \chi_b}{\kappa} \sin \theta \sin(\kappa(\tilde{l} + \delta)) \right)^2 \]

\[ - \left( \rho \cos \theta \cos(\kappa(\tilde{l} - \delta)) - \frac{\rho \chi_b}{\kappa} \sin \theta \sin(\kappa(\tilde{l} - \delta)) \right)^2 \]

\[ = 4 \kappa \delta \rho^2 \cos(\kappa \tilde{l}) \sin(\kappa \tilde{l}) \left( \frac{\chi_b^2}{\kappa^2} \sin^2 \theta - \cos^2 \theta \right) + 4 \frac{\chi_b}{\kappa} \rho^2 \cos(\kappa \tilde{l}) \sin(\kappa \tilde{l}) \cos \theta \sin \theta , \]

where again we are using \( \tilde{l} \) to denote \( l/2 \). As in the previous section, the term independent of \( \delta \) vanishes during the \( \theta \) integration. However, the first term does not vanish, and, after the averaging integration leads to

\[ \langle r_1^2 - r_2^2 \rangle = \kappa \delta \rho_{\text{max}}^2 \cos(\kappa \tilde{l}) \sin(\kappa \tilde{l}) \left( \frac{\chi_b^2}{\kappa^2} - 1 \right) . \]  

(17)

The rate of increase in energy now becomes (after equipartitioning into horizontal and vertical directions)

\[ \frac{dE_t}{dz} = \frac{\gamma m c^2 \beta^2}{8 l \lambda_b^2} \kappa \delta \rho_{\text{max}}^2 \cos(\kappa \tilde{l}) \sin(\kappa \tilde{l}) \left( \frac{\chi_b^2}{\kappa^2} - 1 \right) \]  

(18)

and the rate of increase in the transverse emittance becomes

\[ \frac{d\varepsilon_{\text{rms}}}{dz} = \frac{\gamma}{8 l \lambda_b^2} \delta \rho_{\text{max}}^2 \cos(\kappa \tilde{l}) \sin(\kappa \tilde{l}) \left( \frac{\chi_b^2}{\kappa^2} - 1 \right) . \]  

(19)

Note that for small \( \tilde{l} \), the \( \tilde{l} \) terms cancel. Also note that \( \chi_b / \kappa \) is significantly smaller than unity (for short, widely dispersed solenoids). We will find \( \delta \) in the next section, and evaluate the emittance growth in Section V.

IV. Asymmetric Motion in the Solenoids Due to Large Space Charge
In Fig. 5 we show a particle’s motion within two solenoids, and the drift in between, along with the ideal betatron orbit. Note at positions 1 and 4, the particle’s orbit is tangential to the betatron orbit. Positions 2 and 3 refer to the interfaces between the solenoids and the drift. Regions I and III are within the solenoids, and the particle trajectory obeys the differential equation \( x'' + \kappa^2 x = 0 \). In Region II, the particle trajectory obeys \( x'' - \kappa_{sc}^2 x = 0 \). The betatron orbit is defined by \( x_b^2 + \chi_b^2 x_b = 0 \).

Positions 1 and 4 are located a distance of \( \ell/2 + \delta \) after the entrance of the solenoid. In this section, we will find an expression for \( \delta \) which scales linearly with the beam current.

To find the displacement \( \delta \), we explicitly calculate the particle orbit shown in Fig. 5 (for simplicity, we will use the special case the betatron orbit of the particle is at a maximum in the first solenoid). Then we will set this orbit position and velocity equal to the ideal betatron motion at the position \( \ell/2 + \delta \) in the second solenoid. We will find the lowest order dependence of \( \delta \) on the space-charge wavenumber, by taking the derivative of these two equations with respect to \( \kappa_{sc}^2 \). The idea will be to keep the betatron wavenumber \( \chi_b \) the same, by varying \( \kappa \) as the space-charge wavenumber \( \kappa_{sc} \) is varied. We do not need to actually calculate the solenoid wavenumber in the absence of space charge, \( \kappa_0 \); rather we only need to calculate the offset \( \delta \) as a function of \( \kappa_{sc} \).

For a particle with maximum betatron amplitude \( \rho \), we assume the particle’s orbit at position 1 is defined by

\[
\begin{align*}
x_1 &= \rho \\
x_1' &= 0 .
\end{align*}
\]  

At position 2, the particle’s orbit is given by

\[
\begin{align*}
x_2 &= \rho \cos(\kappa(\ell - \delta)) \\
x_2' &= -\rho \kappa \sin(\kappa(\ell - \delta)) .
\end{align*}
\]
Using this to define the boundary conditions for region II, we find that the particle’s orbit at position 3 is given by

\[
x_3 = \rho \cos(\kappa(t - \delta)) \cosh(\kappa_{sc} d) - \frac{DK}{\kappa_{sc}} \sin(\kappa(t - \delta)) \sinh(\kappa_{sc} d) \\
x_3' = \rho \kappa_{sc} \cos(\kappa(t - \delta)) \sinh(\kappa_{sc} d) - \rho \kappa \sin(\kappa(t - \delta)) \cosh(\kappa_{sc} d)
\]

Transferring these boundary conditions to position 4 where this orbit must match the position and angle of the betatron orbit, we find these equations that must be satisfied:

\[
\frac{x_4}{\rho} = \cos((d + \ell) \chi_b) \\
= \left[ \cos(\kappa(t - \delta)) \cosh(\kappa_{sc} d) - \frac{\kappa}{\kappa_{sc}} \sin(\kappa(t - \delta)) \sinh(\kappa_{sc} d) \right] \cos(\kappa(t + \delta)) \\
+ \left[ \frac{\kappa_{sc}}{\kappa} \cos(\kappa(t - \delta)) \sinh(\kappa_{sc} d) - \sin(\kappa(t - \delta)) \cosh(\kappa_{sc} d) \right] \sin(\kappa(t + \delta))
\]

\[
\frac{x_4'}{\rho} = -\chi_b \sin((d + \ell) \chi_b) \\
= -\left[ \cos(\kappa(t - \delta)) \cosh(\kappa_{sc} d) - \frac{\kappa}{\kappa_{sc}} \sin(\kappa(t - \delta)) \sinh(\kappa_{sc} d) \right] \kappa \sin(\kappa(t + \delta)) \\
+ \left[ \frac{\kappa_{sc}}{\kappa} \cos(\kappa(t - \delta)) \sinh(\kappa_{sc} d) - \sin(\kappa(t - \delta)) \cosh(\kappa_{sc} d) \right] \kappa \cos(\kappa(t + \delta)).
\]

At this point, we will take the derivative of these equations with respect to \( \kappa_{sc} d^2 \). We will assume that \( \kappa \) will be adjusted such that the betatron period remains constant. We will expand both \( \kappa \) and \( \delta \) about their values with no space-charge (\( \kappa_0 \) and \( \delta_0 = 0 \)), and will use \( \kappa' \) and \( \delta' \) to refer to their derivatives with respect to \( \kappa_{sc} d^2 \), at \( \kappa_{sc} = 0 \).

We do not take the derivatives with respect to \( \kappa_{sc} d \), because the first order expansions of \( \kappa \) and \( \delta \) vanish (this is easy to show). Thus the derivatives \( \kappa' \) and \( \delta' \) will lead to the lowest order expansion for \( \delta \) in terms of the beam current.
Note these simplifying derivatives, taken at \( \kappa_{sc} = 0 \):

\[
\frac{d}{d(\kappa_{sc}^2d^2)} \cosh(\kappa_{sc}d) = \frac{1}{2} \\
\frac{d}{d(\kappa_{sc}^2d^2)} \left( \frac{\sinh(\kappa_{sc}d)}{\kappa_{sc}d} \right) = \frac{1}{6} \\
\frac{d}{d(\kappa_{sc}^2d^2)} \left( \kappa_{sc}d \sinh(\kappa_{sc}d) \right) = 1 .
\]

(24)

With these expressions, the derivative of the position equation becomes

\[
0 = 0 = \delta' \kappa_0^2 d + \kappa'(-4d \sin(\kappa_0 i) \cos(\kappa_0 i) + \kappa_0 \frac{d^2}{2} \left( \kappa_0^2 d i - \cos^2(\kappa_0 i) - d \sin(\kappa_0 i) \cos(\kappa_0 i) \right) + 2 \kappa_0 d \sin^2(\kappa_0 i) - \kappa_0 i \left( 2 \cos^2(\kappa_0 i) - \sin^2(\kappa_0 i) \right)) \\
+ \frac{\kappa_0^2 i^2 \sin^2(\kappa_0 i)}{d} + \frac{\sin^2(\kappa_0 i) \kappa_0^2 d}{6} - \kappa_0 \sin(\kappa_0 i) \cos(\kappa_0 i) .
\]

(25)

and the derivative of the angle equation becomes

\[
0 = \kappa' \left( 2 \sin(\kappa_0 i) \cos(\kappa_0 i) \left( \kappa_0^2 d i - 1 \right) + 2 \kappa_0 d \sin^2(\kappa_0 i) - \kappa_0 i \left( 2 \cos^2(\kappa_0 i) - \sin^2(\kappa_0 i) \right) \right) \\
+ \frac{\kappa_0^2 i^2 \sin^2(\kappa_0 i)}{d} + \frac{\sin^2(\kappa_0 i) \kappa_0^2 d}{6} - \kappa_0 \sin(\kappa_0 i) \cos(\kappa_0 i) .
\]

(26)

Implicit with the smooth approximation assumption is that \( \kappa_0 i \) is small, or \( \cos \kappa_0 i = 1 \) and \( \sin \kappa_0 i = \kappa_0 i \). With these additional approximations, we find to lowest order (in terms of the space-charge wavenumber) that

\[
\delta' = -\frac{4}{3 + \frac{2i}{\kappa_0^2 d}} ,
\]

(27)

and that the asymmetry offset is given by
\[
\delta = \left( \frac{3}{4} d + 2\bar{I} \right) \frac{\kappa_{\text{sc}}^2}{\kappa_0^2}. \tag{28}
\]

V. Emittance Growth Calculation with Space-Charge

At this point, we have enough material to assemble together to evaluate the emittance growth rate. Using Eqns. (19) and (28), we have

\[
\frac{d\varepsilon_{\text{rms}}}{dz} = -\frac{\gamma \kappa_{\text{sc}}^2}{8I\lambda_0^2\kappa_0^2} \left( \frac{3}{4} d + 2\bar{I} \right) \rho_{\text{max}}^2 \cos(kd) \sin(kd) \left( \frac{\chi_b^2}{\kappa^2} - 1 \right). \tag{29}
\]

Now

\[
\kappa_{\text{sc}}^2 = \frac{2\bar{I}}{I_A (\beta \gamma)^3 \rho_{\text{max}}^2}
\]

so Eqn. (29) becomes

\[
\frac{d\varepsilon_{\text{rms}}}{dz} = \frac{I}{4I_A \lambda_0^2 \kappa_0^2 \gamma^2 \beta^3} \left( \frac{3}{4} d + 2\bar{I} \right), \tag{31}
\]

which leads to a linear normalized emittance growth of the form

\[
\varepsilon_{\text{rms}} = \varepsilon_0 + z \frac{I}{4I_A \lambda_0^2 \kappa_0 \gamma \beta^2} \left( \frac{3}{4} d + 2\bar{I} \right). \tag{32}
\]

Note if there is acceleration, the normalized emittance will grow logarithmically instead of linearly.

We can reduce this expression further by noting that \( \lambda_b = 1 / N\kappa_0^2 \bar{I} \) and defining a filling factor \( \eta = N\bar{I} / \lambda_b \), which leads to
Note that in both the limits the filling factor vanishes and the number of solenoids per betatron period becomes infinite the emittance growth vanishes, as it must. Also note that the emittance always increases for any combination of solenoid width and separation.

VI. Discussion

In this final section we will describe some of the characteristics of this effect. The most important observation is that this analysis only makes sense for a matched beam in equilibrium, with slow, adiabatic changes. This analysis is not applicable to the case of a few, dissimilar solenoids using for matching or focusing the beam, in which the excess transverse energy generated may or may not be transformed into an emittance growth.

Also note first that this effect is not reversible. The effects of a kinematic description of the beam is reversible if the problem is reversed. However, if the matched beam used in this example is reversed after a distance \( z \) has been traveled, the emittance does not decrease but rather increases further. This nonreversibility is due to the second law of thermodynamic, that the entropy will always increase from a statistical point of view. This is equivalent to letting air out of a balloon for a period of time and observing the final state, and then trying to set up a new initial state for a gas cloud from this final state (molecules with same positions but with opposite velocities or some such arrangement) and expecting this new cloud to reenter the balloon as time goes on. If the final state was sufficiently thermalized, this just will not happen, as thermodynamics is statistical in nature. The gas cloud in the new state will just continue to grow.

Although the estimate presented in this paper is for periodic focusing with solenoids, we can make some qualitative statements for periodic focusing involving FODO quadrupole arrays. The same mechanism exists, but with the longitudinal work now being done by the cross product of the transverse velocity with the transverse magnetic field. If there is no space charge, this effect also vanishes with averaging, but
with space charge, an asymmetry again appears. Because of the larger betatron wavenumber in a quadrupole (as opposed to a solenoid), we expect that this effect would be of a larger order, about $\kappa_q / \sqrt{N\lambda p}$ times as big.

We can estimate the emittance growth for "typical" beams, to see if this effect is meaningful. First, consider the case of the Integrated Test Stand, a 6-MeV, 4-kA induction linac at Los Alamos [8]. With reasonable values of $\eta = 1/2$ and $N = 10$, we see that the normalized, rms emittance can grow as much as 250 mm mrad per meter of drift, which is significant for this type of machine.

For a second example, consider the electron accelerator for the Linac Coherent Light Source [9], a 4-kA, 15-GeV machine, designed for a next-generation light source. With reasonable values of $\eta = 1/10$ and $N = 25$, we see that the normalized, rms emittance can grow as much as 0.02 mm mrad per meter of drift. Although this sounds like a small number, the accelerator design emittance is only about 1 mm mrad, and we should expect about an order of magnitude more growth with quadrupoles, and the total lengths of drifts after bunch compression is about 65 meters long.

A particularly exciting feature of this phenomena is that under the right conditions, one could imagine that the transverse energy leaks into the longitudinal direction, damping the transverse emittance as the beam travels down the periodic channel. This feature does not necessarily contradict the second law of thermodynamics, but requires an offset in the solenoids of the opposite sign found for a periodic array. Control of the offset might be possible if the focal lengths of the solenoids can be individually controlled, or if a specific prescription of quadrupole lenses is used.

Acknowledgments

The author would like to thank Professor Martin Reiser for bringing this effect to his attention, and for pointing out that the proper analysis has to be based on thermodynamic principles. The author would also like to acknowledge helpful and insightful discussions with Dr. Richard Sheffield and Dr. Stephen Gierman.
References


Figure Captions

1. Ideal solenoid, showing magnetic field lines and transition regions at the front and back of the solenoid. The axial field is radially constant everywhere. The field components in the transition regions are described in Eqn. (1).

2. Matched transverse phase space distribution for a matched beam, using the smooth approximation, with betatron wavenumber $\kappa_b$.

3. Symmetric particle orbit in the solenoids (a) no space charge and (b) small space charge. The dashed line is the ideal betatron orbit using the smooth approximation.

4. Asymmetric particle orbit in the solenoid. The dashed line is the ideal betatron orbit using the smooth approximation, and the offset from the center of the solenoid is $\delta$. The dashed line is the ideal betatron orbit using the smooth approximation.

5. Asymmetric particle orbit in two solenoids. This figure is used to calculate the offset $\delta$, using the betatron wavenumbers in the solenoid $\kappa$ and between the solenoids $\kappa_{sc}$. The dashed line is the ideal betatron orbit using the smooth approximation, which is tangential to the actual orbit at the offsets. Regions I, II, and III, and position 1-4 are described in the text.
Center of Solenoid  
Point Tangential to the Betatron Orbit

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