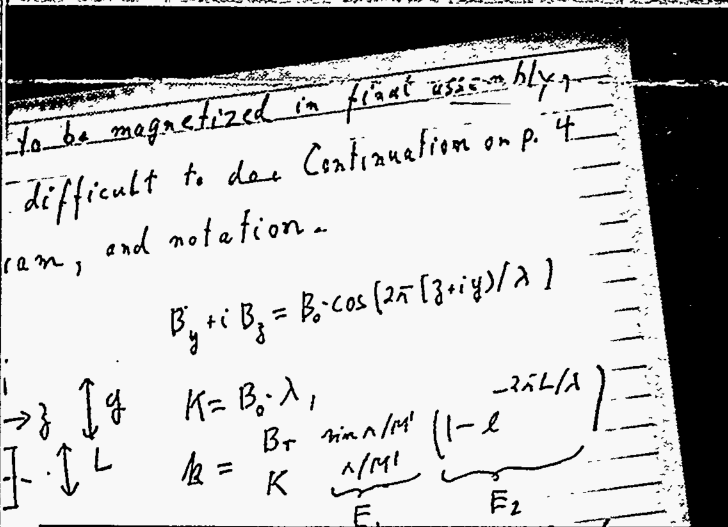
Klaus Halbach, "Matrix Representation of Gaussian Optics," American Journal of Physics, vol. 32, no. 2, pp. 90-108 (February 1964). • Klaus Halbach, "Berechnung linearer, realisierbarer Netzwerke zur Erzielung optimaler Signal/Rauschverhältnisse," Helvetica Physica Acta, vol. 26, no. 1, pp. 65–74 (January 1953). • Klaus Halbach, "Modulation-Effect Corrections for Moments of Magnetic Resonance Line Shapes," Physical Review, vol. 119, no. 4 (August 1960). • Klaus Halbach and William R. Baker, "Plasma Gun Aspects of an E x B System," The Physics of Fluids Supplement (1964).

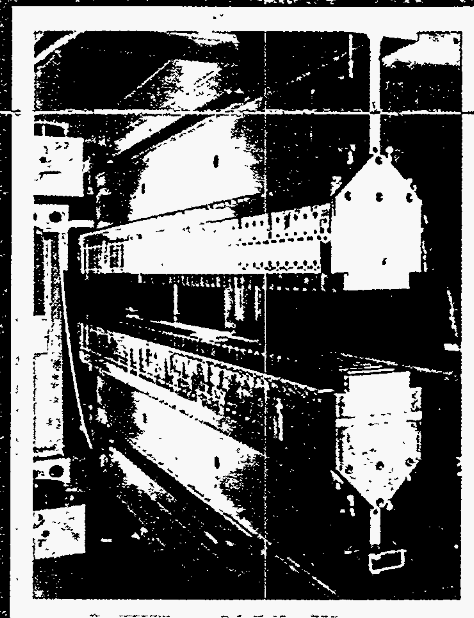
LBL PUB-755

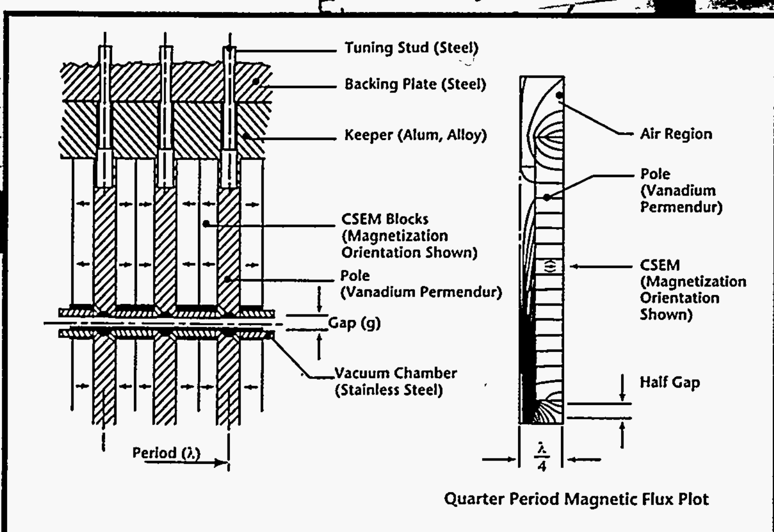
VOLUME 2

The Art and Science of Magnet Design

Selected Notes of Klaus Halbach







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February 1995
Lawrence Berkeley Laboratory
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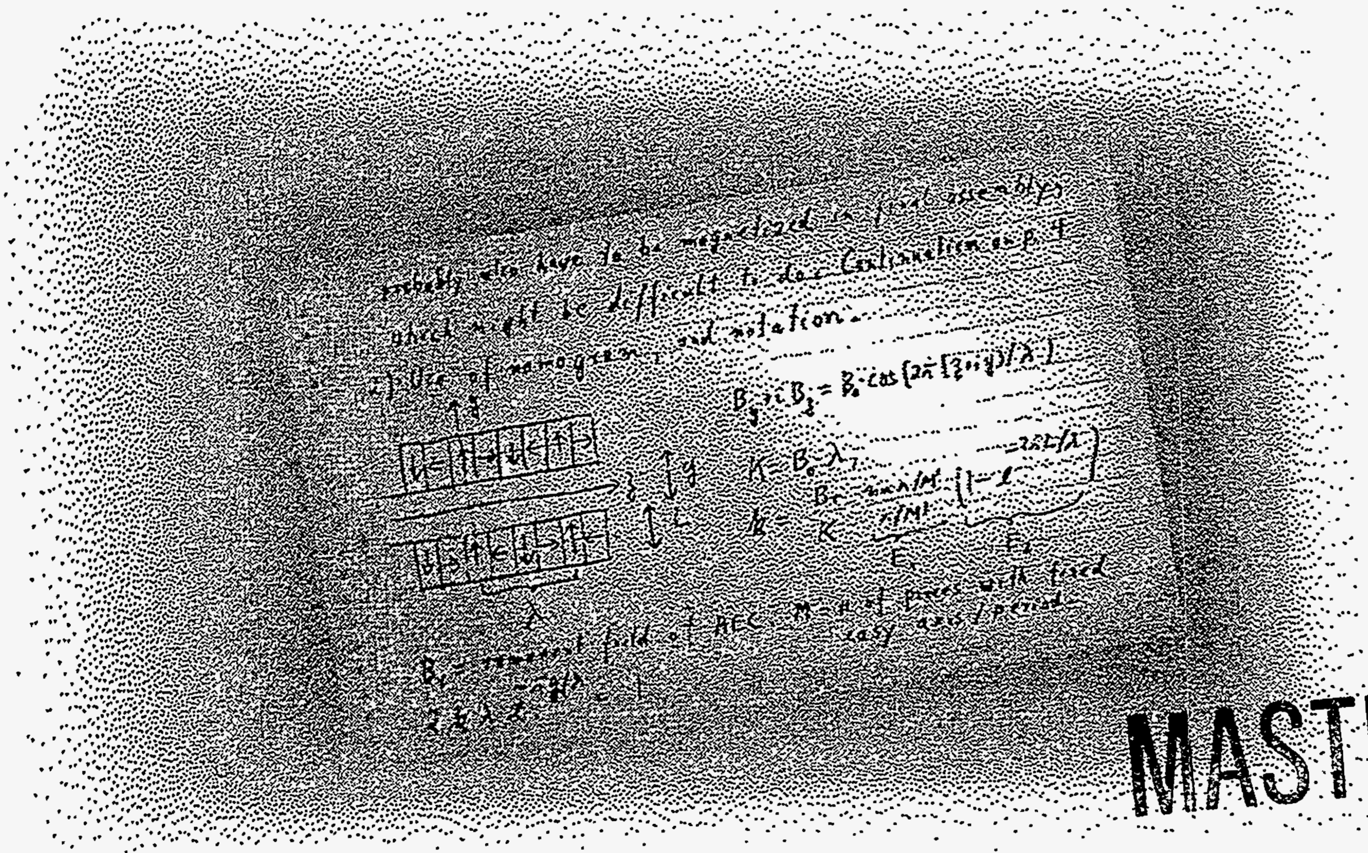
Berkeley, CA 94720

This is the second of two volumes published in conjunction with the Halbach Symposium on Magnet Technology held at Lawrence Berkeley Laboratory on February 3, 1995, in honor of Klaus Halbach's 70th birthday.
Volume 1 (LBL PUB 754), A Festschrift in Honor of Klaus Halbach, contains technical papers and personal remembrances written by Dr. Halbach's colleagues expressly for the Halbach Symposium and dedicated to him.

VOLUME 2

The Art and Science of Magnet Design

Selected Notes of Klaus Halbach



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Preface

This volume contains a compilation of 57 notes written by Dr. Klaus Halbach selected from his collection of over 1650 such documents. It provides an historic snapshot of the evolution of magnet technology and related fields as the notes range from as early as 1965 to the present, and is intended to show the breadth of Dr. Halbach's interest and ability that have long been an inspiration to his many friends and colleagues.

As Halbach is an experimental physicist whose scientific interests span many areas, and who does his most innovative work with pencil and paper rather than at the workbench or with a computer, the vast majority of the notes in this volume were handwritten and their content varies greatly—some reflect original work or work for a specific project, while others are mere clarifications of mathematical calculations or design specifications. As we converted the notes to electronic form, some were superficially edited and corrected, while others were extensively re-written to reflect current knowledge and notation.

The notes are organized under five categories which reflect their primary content: Beam Position Monitors (bpm), Current Sheet Electron Magnets (csem), Magnet Theory (thry), Undulators and Wigglers (u-w), and Miscellaneous (misc). Within the category, they are presented chronologically starting from the most recent note and working backwards in time. The note number, listed in the Table of Contents and at the bottom of each note's first page, comes from a database we have created which includes the titles of the entire collection of notes, and a recently added sixth category, Conformal Transformations (ctr). The appendixes contain a table of all the notes in the database and a list of Dr. Halbach's publications.

The extensive use of hand-written notes by Dr. Halbach leads us to believe that there may be many that were sent to colleagues which were not retained in Dr. Halbach's files, and thus are missing from the database. If you happen to have a note of scientific interest from Dr. Halbach and believe it to be an original, we would appreciate receiving a copy.

Brian M. Kincaid — *November 1994*
Simonetta Turek

Table of Contents

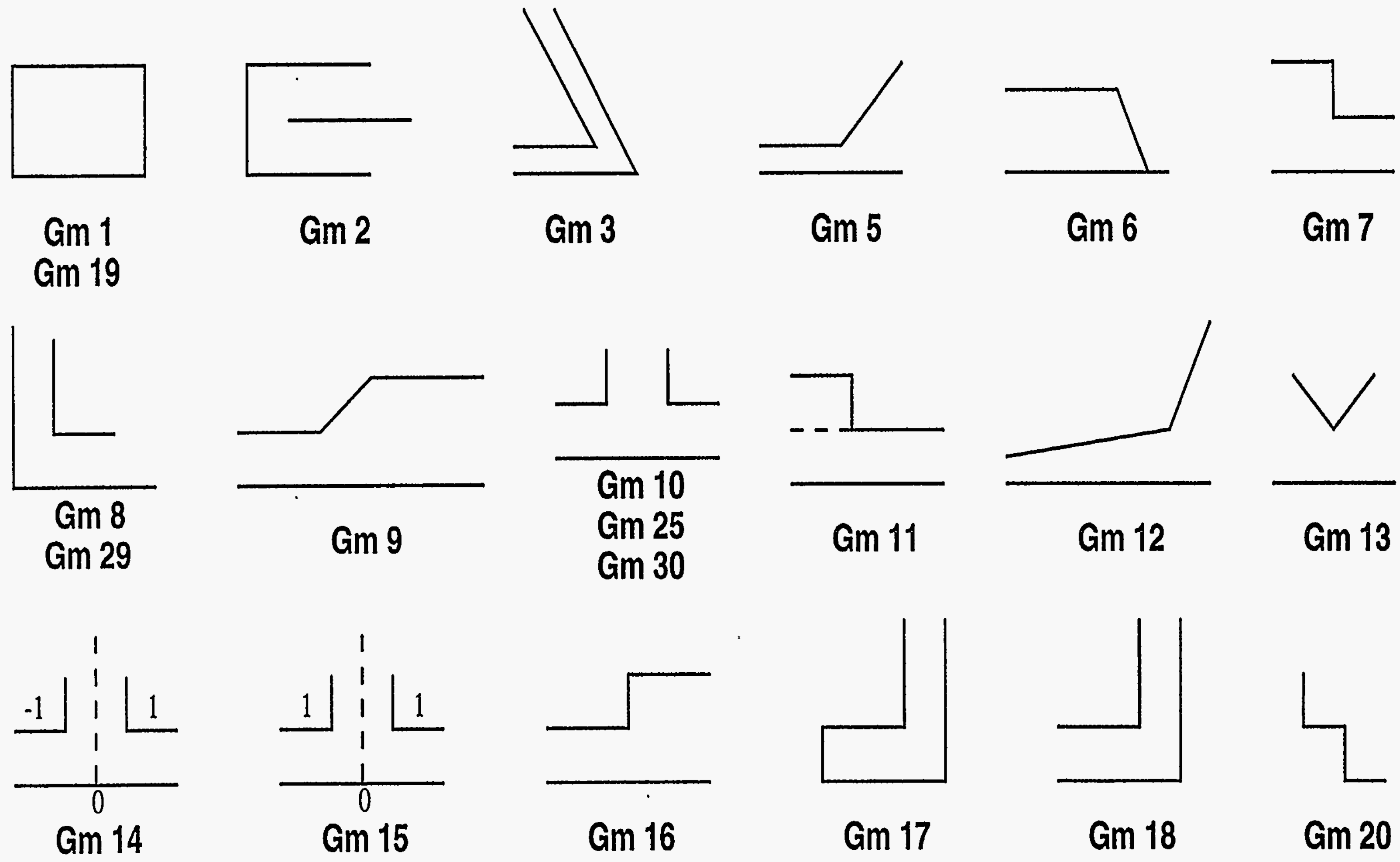
Theory

Undulator-Wiggler

Halbach Geometries

The Halbach Geometries, referred to in the notes as Gm, are a collection of simple geometric shapes, simple function representations, and 2-dimensional electromagnetic geometries for which conformal mapping calculations have been done to compute basic features such as capacitance, excess flux, etc. For examples of calculations of excess flux, see documents 0336csem (p. 5), 0332csem (p. 11), 0183csem (p. 23), and 0131u-w (p. 175).

The following two pages summarize Dr. Halbach's representations and shorthand notations of his "Geometries." The reader is encouraged to refer back to them when encountering such abbreviations as Gm3 or Gm21 while reading the notes. (Note: Not all the Halbach Geometries are referenced in this collection.)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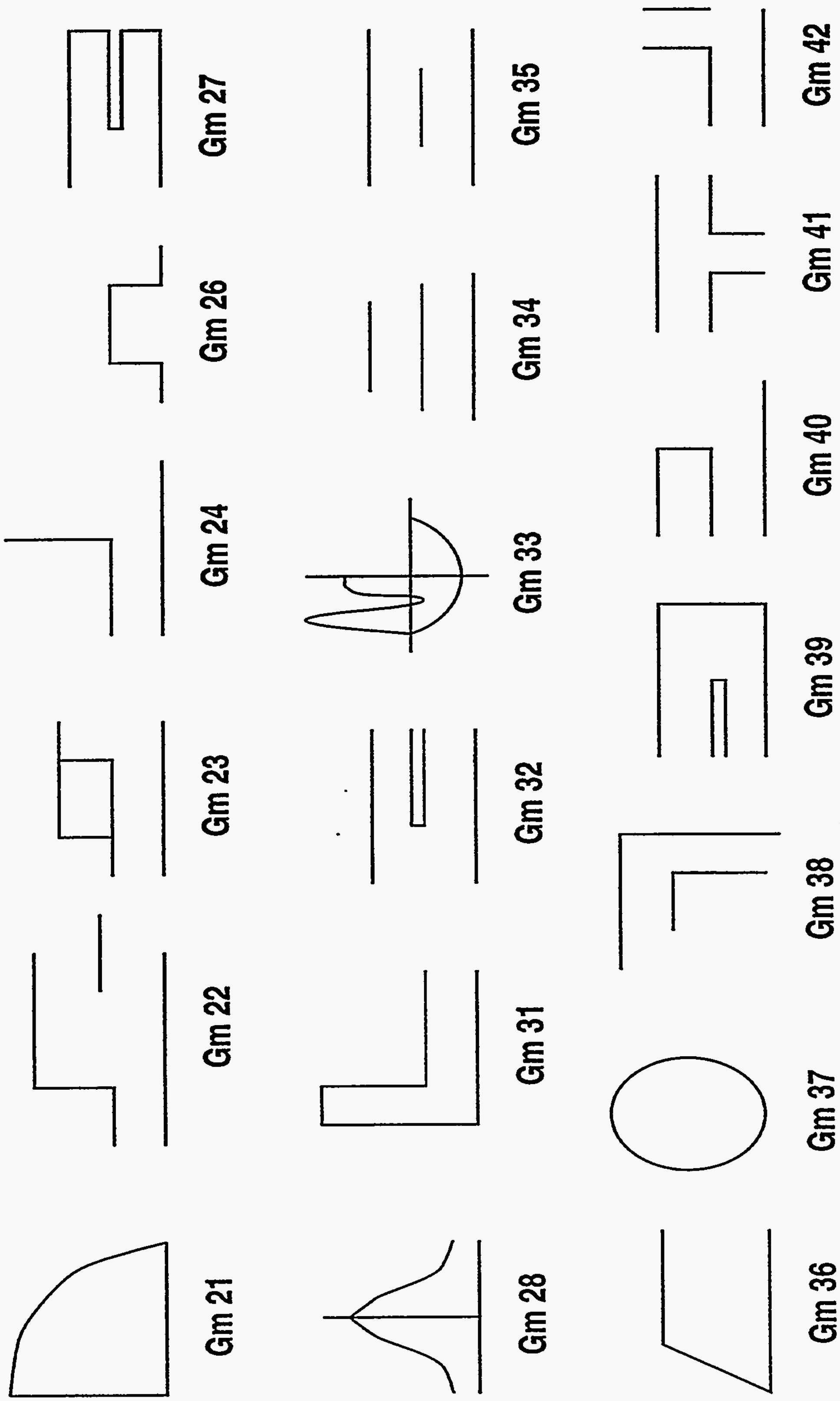
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Gm 37
Gm 38
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Gm 41
Gm 42

Exact, Complete Proofs of Reciprocity Theorems for Electrostatic and Magnetostatic Beam Monitors

The following is an excercise in Maxwell's equations in a region that is bounded by perfect metal walls and contains nothing but moving electric charges.

$$\nabla \times \mathbf{H} = \mathbf{j} + \dot{\mathbf{D}} \quad \text{and} \quad \nabla \cdot \mathbf{H} = 0, \tag{1.1),(1.2}$$

where $\mathbf{j}$ comes from the moving charges represented by charge density $\varrho(x,y,z,t)$ in the beam, and

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \quad \text{and} \quad \nabla \cdot \mathbf{D} = \varrho, \tag{2.1),(2.2}$$

We integrate all equations over time, starting before the front-end wake fields begin, and ending after the end wake fields and RF are gone.

$$\int \dot{\mathbf{D}} dt = 0 \quad \text{and} \quad \int \dot{\mathbf{B}} dt = 0,$$

$$\nabla \times \mathcal{H} = \mathbf{J} \quad \text{and} \quad \nabla \cdot \mathcal{H} = 0, \tag{3,1),(3.2}$$

$$\nabla \times \mathcal{V} = 0 \quad \text{and} \quad \nabla \cdot \mathcal{V} = R. \tag{4,1),(4.2}$$

The new symbols stand for integrals over time at every x, y, z.

Electrostatic Pick-up.

The beam with R produces $V_1(x,y,z)$, with $V_1 = 0$ on wall everywhere, including on the electrode. An electrode on V_{20} produces $V_2(x,y,z)$, with $V_2 = 0$ on wall except on the electrode*.

Using (4.1) and (4.2) we notice that V_2 is the actual potential and V_1 is the integrated potential over time. With $\mathcal{H} = -\Delta V$,

$$\begin{aligned} U &= \int \nabla \cdot (V_1 \nabla V_2 - V_2 \nabla V_1)\, dv \\ &= \int \left(V_1 \underbrace{\nabla \cdot \nabla V_2}_{0} - V_2 \underbrace{\nabla \cdot \nabla V_1}_{-R} \right) dv \\ &= \int V_2 R dv. \end{aligned}$$

December, 1986. Note 0022bpm.

* For subscript 2, the quantities are not averaged over time but are time independent, as are all time integrated quantities.

But it is also true that, for the charge q_e,

$$U = \int (V_1 \nabla V_2 - V_2 \nabla V_1) \cdot d\mathbf{a} = V_{20} \cdot Q \quad \text{with} \quad Q = \int q_e dt,$$

induced by electrons on electrode.

The total charge in the bunch, Q_B, is related to $\varrho(x, y, z, t)$ through

$$\int \varrho dt da = Q_B / v,$$

$$\int R v da = \int \varrho dt \, v da = \int Q'_B da,$$

$$Q \cdot V_{20} = \frac{1}{v} \int V_2 Q'_B dv = \boxed{\frac{1}{v} \int Q_B V_2 dz,}$$

where Q'_B is independent of z, and is the charge going through area da, divided by da. The units for $[Q_B] = A$ sec, and $[Q] = A \sec^2$.

Magnetostatic Pick-up.

We use (3.1) and (3.2). The beam with $\mathbf{J}$ produces $\mathbf{A}_1(x, y, z)$, and $\mathcal{H}_1(x, y, z)$. In the coil, the flux from $\mathbf{J}$ integrated over t is

$$\Phi_2 = \int \mu_0 \mathcal{H}_1 \cdot d\mathbf{a} = \int \nabla \times \mathbf{A}_1 \cdot d\mathbf{a} = \mu_0 \oint \mathbf{A}_1 \cdot d\mathbf{s}.$$

In addition, we use a coil with a current, I_2, that produces $\mathbf{A}_2(x, y, z)$, $\mathcal{H}_2(x, y, z)$. We now use, equivalently to the electrostatic case,

$$U = \int \nabla \cdot (\mathbf{A}_1 \times \mathcal{H}_2 - \mathbf{A}_2 \times \mathcal{H}_1) \, dv,$$

where $\mathbf{A}_1$ is the integrated vector potential, and $\mathbf{A}_2$ is the actual potential associated with I_2.

$$\nabla \cdot \mathbf{A}_1 \times \mathcal{H}_2 = \mathcal{H}_2 \cdot \nabla \times \mathbf{A}_1 - \mathbf{A}_1 \cdot \nabla \times \mathcal{H}_2 = \mathcal{H}_2 \cdot \mathcal{H}_1 - \mathbf{A}_1 \cdot \mathbf{J}_2,$$

thus,

$$U = \int (\mathbf{A}_2 \cdot \mathbf{J}_1 - \mathbf{A}_1 \cdot \mathbf{J}_2) \, dv.$$

With

$$\mathbf{J}_1 = J_1 \mathbf{e_z} \quad \text{and} \quad \int J_1 da = \int j_1 da dt = Q_B,$$

we get

$$\int \mathbf{A}_2 \cdot \mathbf{J}_1 dv = Q_B \cdot \int A_{2z} dz,$$

$$\int \mathbf{A}_1 \cdot \mathbf{J}_2 dv = I_2 \int \mathbf{A}_1 \cdot ds = I_2 \Phi_2 / \mu_0,$$

and get

$$U = Q_B \cdot \int A_{2z} dz - I_2 \Phi_2 / \mu_0 = \int (\mathbf{A}_1 \times \mathcal{H}_2 - \mathbf{A}_2 \times \mathcal{H}_1) \cdot d\mathbf{a}$$

with the last integral taken over the "superconducting" wall.
In the vicinity of the wall we use $\mathcal{H} = -\nabla V$. Thus,

$$U = \int (\mathbf{A}_2 \times \nabla V_1 - \mathbf{A}_1 \times \nabla V_2) \cdot d\mathbf{a}.$$

In general, $\nabla \times (V_1 \mathbf{A}_2) = V_1 \nabla \times \mathbf{A}_2 - \mathbf{A}_2 \times \nabla V_1$, thus

$$U = \int (V_1 \nabla \times \mathbf{A}_2 - V_2 \nabla \times \mathbf{A}_1) \cdot d\mathbf{a} = \int (V_1 \mathcal{H}_2 - V_2 \mathcal{H}_1) \cdot d\mathbf{a} = 0.$$

The last integral vanishes because on the superconducting wall the component of $\mathcal{H}$ perpendicular to the wall (i.e. parallel to $d\mathbf{a}$) is zero. We therefore get

$$\Phi_2 = \mu_0 Q_B \int A_{2z} dz / I_2.$$

The units are $[\Phi_2] = \mu_0 A$ m sec, $[\mathbf{A}_2] = A$, $[\mu_0 Q_B \int A_{2z} dz / I_2] = \mu_0 A$ m sec. It is important to notice that Φ_2 is the integrated flux, and the flux is the integrated induced voltage.

Integral for Excess Flux Calculation

$$J = \int_{t_1}^{t_2} \underbrace{\frac{f(t)}{(t-t_1)^{n_1}(t_3-t)^{n_3}}}_{G} dt$$

We have shown in an earlier note that for $n_1 = n_3 = 1/2$,

$$J = 3\int_{-1}^{1} \frac{f(t)}{\sqrt{4-x^2}} dx, \quad t = \frac{1}{4}(2(t_2+t_1) + (t_2-t_1)x(3-x^2)).$$

For $n_1, n_3 \neq 1/2$, the approach that gave the above equation becomes very complicated, especially if one wants to have generally valid and simple integration. For the general case, we use (arbitrarily, for simplicity)

$$t_2 = 1/2(t_3+t_1) \quad \text{and} \quad J = J_1 + J_3$$

where

$$J_1 = \int_{t_1}^{t_2} G(t)dt \quad \text{and} \quad J_3 = \int_{t_2}^{t_3} G(t)dt\,,$$

We solve for J_1:

$$A\,dx = (t-t_1)^{-n_1}, \quad Ax = \frac{(t-t_1)^{m_1}}{m_1} \quad \text{and} \quad A = \frac{(t_2-t_1)^{m_1}}{m_1} \quad \text{when} \quad x(t_2) = 1\,,$$

with

$$m_1 = 1 - n_1 \quad p_1 = \frac{1}{m_1}$$

$$t = t_1 + (t_2-t_1)x^{p_1}, \qquad t_3 - t = (t_2-t_1)(2-x^{p_1})\,.$$

Thus,

$$J_1 = \frac{(t_2-t_1)^{m_1}}{m1(t_2-t_1)^{n_3}} \int_0^1 \frac{f(t)}{(2-x^{p_1})^{n_3}} dx$$

$$= \boxed{\frac{(t_2-t_1)^{1-n_1-n_3}}{1-n_1} \int_0^1 \frac{f(t)}{(2-x^{p_1})^{n_3}} dx\,.}$$

June, 1993. Note 0336csem.

Equivalently, solving for J_3:

$$-B\,dx = (t_3 - t)^{n_3} dt, \quad Bx = \frac{(t_3 - t)^{m_3}}{m_3} \quad \text{and} \quad B = \frac{(t_2 - t_1)^{m_3}}{m_3},$$

$$t = t_3 - (t_2 - t_1)x^{p_3} \qquad t - t_1 = (t_2 - t_1)(2 - x^{p_3})\,.$$

Thus,

$$\boxed{J_3 = \frac{(t_2 - t_1)^{1-n_1-n_3}}{1 - n_3} \int_0^1 \frac{f(t)dx}{(2 - x^{p_3})^{n_1}}\,.}$$

We may now now conclude that

$$J = \int_{t_1}^{t_3} \frac{f(t)}{(t - t_1)^{n_1}(t_3 - t)^{n_3}} dt$$

$$= \boxed{(t_2 - t_1)^{1-n_1-n_3} \left\{ \int_0^1 \frac{f(t_1 + \Delta t x^{\frac{1}{1-n_1}})\,dx}{\left(2 - x^{\frac{1}{1-n_1}}\right)^{n_3}(1 - n_1)} + \int_0^1 \frac{f(t_3 - \Delta t x^{\frac{1}{1-n_3}})\,dx}{\left(2 - x^{\frac{1}{1-n_3}}\right)^{n_1}(1 - n_3)} \right\}.}$$

We examine a specific case of excess flux in the pole in the geometry of Figure 1,

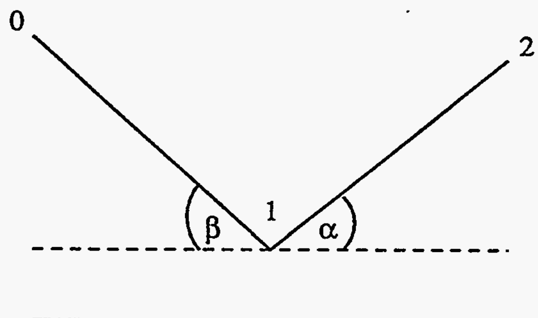


Figure 1.

where,

$$\pi E_{12} = \frac{1}{n_1} \ln\big((n_1 + n_2)I_1\big)\,,$$

$$n_1 = \frac{\alpha}{\pi} \quad \text{and} \quad n_2 = \frac{\beta}{\pi},$$

$$I_1 = \int_0^1 \frac{dt}{t^{n_1}(1-t)^{1-(n_1+n_2)}}$$

$$= \left(\frac{1}{2}\right)^{n_2} \left\{ \frac{1}{1-n_1} \int_0^1 \frac{dx}{\left(2 - x^{\frac{1}{1-n_1}}\right)^{1-(n_1+n_2)}} + \frac{1}{n_1+n_2} \int_0^1 \frac{dx}{\left(2 - x^{\frac{1}{n_1+n_2}}\right)^{n_1}} \right\}.$$

We may conclude that $I_2 = (I_1)_{n_1 \Leftrightarrow n_2}$. Further, the expression $I_1 \frac{\sin\alpha}{\alpha} = I_2 \frac{\sin\beta}{\beta}$ should be true. This is a non-trivial assertion and comes from a derivation of the expression for E_{12} in an earlier note.

H^* at End of CSEM Block

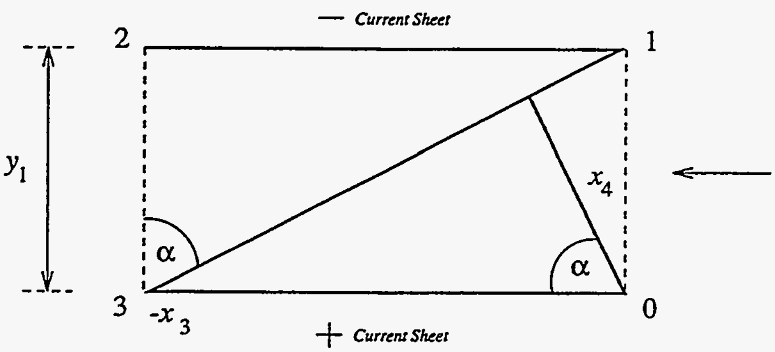


Figure 1.

$$H^*(z) = \frac{I}{2\pi i} \cdot \frac{1}{z - z_0} \longrightarrow -\frac{I'}{2\pi i} \ln \frac{z}{z + x_3} \cdot \frac{z - z_2}{z - z_1}.$$

In the vicinity of $z = 0$,

$$H^* = -\frac{H_c}{2\pi i} \ln \frac{z\, z_2}{z_1 x_3},$$

where

$$\frac{z_1 x_3}{z_2} = \frac{i y_1 x_3}{i y_1 - x_3} = \frac{y_1 x_3}{y_1 + i x_3} = \frac{x_3}{1 + i x_3 / y_1} = x_4 e^{-i\alpha},$$

$$x_4 = \frac{x_3}{\sqrt{1 + x_3^2 / y_1^2}} = x_3 \cos\alpha \quad \text{and} \quad z = r e^{i\varphi}.$$

Thus,

$$\boxed{H^* = -\frac{H_c}{2\pi i} \ln \frac{r e^{i(\varphi + \alpha)}}{x_4} = -\frac{H_c}{2\pi i} \left(\varphi + \alpha + i \ln \frac{x_4}{r}\right).}$$

Field "blows up" at $r = 0$. Thus, for scaling purposes, at location where $\ln \frac{x_4}{r} = 2\pi$, $r = x_4 e^{-2\pi} = x_4 \cdot 1.9 \times 10^{-3}$.

There is a strong local field perpendicular to the "current sheet side", which is not problematic when easy axis is parallel to the "current sheet side". It is easier to see with charge sheet, and it leads to the same answer.

Interesting damage results for block not magnetized in either a perpendicular or parallel direction to the sides.

May, 1993. Note 0335csem.

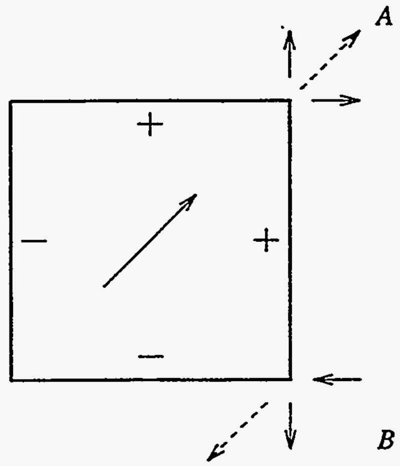


Figure 2.

No damage will result in corner A, but there is a potential of demagnetization at corner B, and at symmetrically located corners.

Summary of Excess Flux Formulae for Gm3, Gm18 and Gm40

Unless otherwise noted, the following definitions hold for all geometries in this Note

$$F = \pi \frac{\Delta A}{V_0}, \quad n = \frac{\alpha}{\pi}, \quad \text{and} \quad a = \frac{h_2}{h_1}.$$

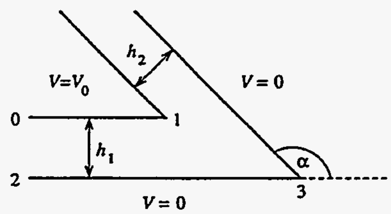


Figure 1.

$$F_{01} = (1+b)\int_0^1 \frac{1-x^n}{(1-x)(b+x)}dx\,, \tag{1}$$

$$F_{23} = F_{01} + \ln b\,, \quad \text{with} \quad b = a^{1/n}\,. \tag{2}$$

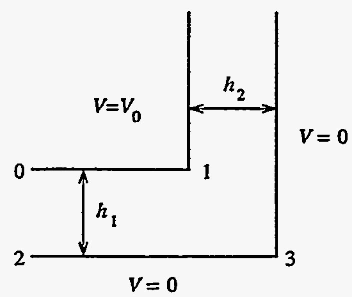


Figure 2.

For Figures 2, 3, and 4 (Gm18 and Gm40) $\alpha = \pi/2$.

$$F_{01} = \ln \frac{1+1/a^2}{4} + 2a \arctan \frac{1}{a}\,, \tag{3}$$

$$F_{23} = \ln \frac{1+a^2}{4} + 2a \arctan \frac{1}{a}\,, \tag{4}$$

April, 1989. Note 0332csem.

We summarize here the the sum of excess fluxes for (1) and (3). For (1), we get

$$(F_{01}(a) + F_{01}(1/a)) = (1+b)^2 \int_0^1 \frac{(1-x^n)(1+x)}{(1-x)(b+x)(1+xb)} dx\,. \tag{1D}$$

And for (3), we have

$$(F_{01}(a) + F_{01}(1/a)) = 2\ln\frac{a+1/a}{4} + 2a\arctan(1/a) + 2(1/a)\arctan a\,. \tag{3D}$$

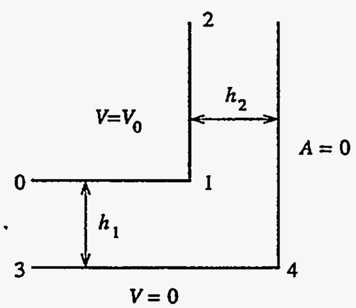


Figure 3.

$$F_{34} = \ln(1+a^2) + 2a\arctan(1/a)\,, \tag{5}$$

$$F_{12} = 2\ln\left(a + \sqrt{1+a^2}\right)\,, \tag{6}$$

$$F_{01} = F_{34} - F_{12}\,. \tag{7}$$

Continued on following page.

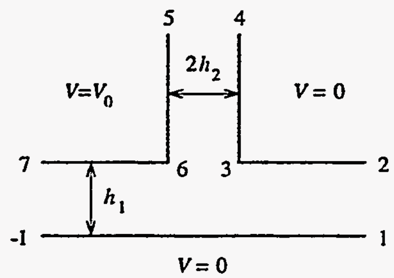


Figure 4.

$$F_{-11} = \ln(1 + a^2) + 2a \arctan(1/a)\,, \tag{8}$$

$$F_{67} = \ln \frac{1 + a^2}{2a\left(a + \sqrt{1 + a^2}\right)} + 2a \arctan \frac{1}{a}\,, \tag{9}$$

$$F_{56} = \ln \frac{\sqrt{1 + a^2}\left(a + \sqrt{1 + a^2}\right)}{2} + \frac{1}{a} \arctan a\,, \tag{10}$$

$$F_{567} = \ln \frac{\left(\sqrt{1 + a^2}\right)^3}{4a} + 2a \arctan \frac{1}{a} + \frac{1}{a} \arctan a\,, \tag{11}$$

$$F_{234} = \ln \frac{\sqrt{1 + 1/a^2}}{4} + \frac{1}{a} \arctan a\,. \tag{12}$$

Anti-Symmetric Undulator to Make Vertically Polarized or Circularly Polarized Light

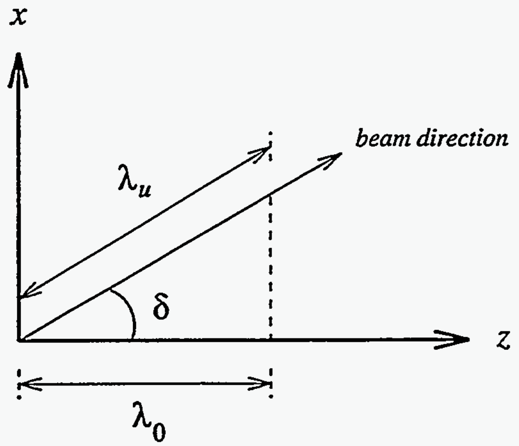


Figure 1.

We have

$$-V = \frac{B_0}{k_0} \sin k_0 z \cosh k_0 y, \tag{1}$$

with fields anti-symmetric to the midplane $y = 0$.

$$B_z = B_0 \cosh k_0 y \cos k_0 z \quad \text{and} \quad B_y = B_0 \sinh k_0 y \sin k_0 z \tag{2}$$

in direction δ relative to the z-axis.

$$\lambda_u = \lambda_0 / \cos \delta, \tag{3.1}$$

$$B_\perp = \underbrace{B_0 \cosh k_0 y \sin \delta}_{B_2} \cos k_0 z, \tag{3.2}$$

$$B_y = \underbrace{B_0 \sinh k_0 y}_{B_1} \sin k_0 z. \tag{3.3}$$

Linearly Polarized Light.

Let $y = 0$, $B_y = 0$,

$$B_2 = \sin \delta B_0 = \sin \delta B_0 \left(\frac{g}{\lambda_u} \cdot \frac{1}{\cos \delta} \right). \tag{4}$$

By the above B_0 is indicated the achievable B_0 as a function of g/λ_0, where λ_0 is the period in the z-direction, and g is the magnet gap.

October, 1984. Note 0208csem.

To have a better understanding, we look at the pure CSEM undulator:

$$B_0 = B_3 e^{-\pi g/\lambda_o},$$

with B_3 equal to the product of $2B_r$, the segmentation factor and the finite height factor.

$$B_2 = B_3 \sin\delta e^{-a/\cos\delta}, \quad \text{with} \quad a = \pi g/\lambda_u. \tag{5.1}$$

We optimize with δ for given g/λ_u. With $B_2' = 0$,

$$\cos\delta - a\tan^2\delta = 0.$$

Instead of solving for given a, we make a table of a *vs.* δ.

$$a = \frac{\cos^3\delta}{\sin^2\delta}, \tag{5.2}$$

$$B_2 = B_3 \sin\delta e^{-\cot^2\delta} = B_3 y. \tag{5.3}$$

Compared to a "normal" undulator:

$$B_{2n} = B_3 e^{-a} = B_3 e^{-\cos\delta/\tan^2\delta} = B_3 y, \tag{5.4}$$

$$\frac{B_2}{B_{2n}} = \sin\delta e^{-\cot^2\delta} e^{\cos\delta\cot^2\delta} = \frac{y}{y_o}, \tag{5.5}$$

δ	g/λ_u	y	y/y_0
30.00	0.83	0.02	0.33
35.00	0.53	0.07	0.40
40.00	0.35	0.16	0.46
45.00	0.23	0.26	0.53
50.00	0.14	0.38	0.60
55.00	0.09	0.50	0.66
60.00	0.05	0.62	0.73

Table 1.

Circularly Polarized Light.

We set $y = y_1$, and $k_0 y_1 = \beta$. From (3.2) and (3.3), we see we need to satisfy $\cosh\beta \sin\delta = \sinh\beta$ for the helical undulator action, thus

$$\sin\delta = \tanh\beta, \tag{6.1}$$

or

$$\cos\delta = \sqrt{1-\tanh^2\beta} = 1/\cosh\beta, \tag{6.2}$$

or

$$\tan\delta = \sinh\beta, \tag{6.3}$$

$$B_u = B_0 \sinh\beta = B_0 \tan\delta = B_0 \left(\frac{g/\lambda_u}{\cos\delta}\right) \tan\delta. \tag{7}$$

We assess the reasonableness and feasability of the above analysis.

Clearly,

$$\varepsilon = \frac{2y_1}{g} \tag{8.1}$$

is an important parameter.

$$\beta = \frac{2\pi y_1}{\lambda_0} = \frac{2y_1}{g} \cdot \pi \cdot \frac{g}{\lambda_u \cos\delta},$$

and for $p = g/\lambda_u$,

$$\beta = \varepsilon p \frac{\pi}{\cos\delta} = \varepsilon \frac{a}{\cos\delta}, \tag{8.2}$$

$$\varepsilon = \frac{\beta\cos\delta}{a}. \tag{8.3}$$

The indicated procedure is as follows. Given $p = g/\lambda_u$ and $B_0(g/\lambda_0) = B_0(p/\cos\delta)$, we optimize B_u with δ and get ε from (8.3).

For a pure REC undulator,

$$B_0 = B_3 e^{-a/\cos\delta} \quad \text{with} \quad a = \pi g/\lambda_u = \pi p,$$

$$B_u = B_3 \tan\delta \cdot e^{-a/\cos\delta}.$$

With $B_u' = 0$,

$$\frac{1}{\cos^2\delta} - a\tan\delta \cdot \frac{\sin\delta}{\cos^2\delta} = 0,$$

$$a\frac{\sin^2\delta}{\cos\delta} = a\frac{1-\cos^2\delta}{\cos\delta} = 1,$$

$$\boxed{\cos\delta = -\frac{1}{2a} + \sqrt{\frac{1}{4a^2}+1}.} \tag{9.1}$$

For $\cosh\beta = 1/\cos\delta$,

$$\boxed{\beta = \ln\left(\sqrt{\cosh^2\beta - 1} + \cosh\beta\right).} \tag{9.2}$$

$$\boxed{\varepsilon = \beta\cos\frac{\delta}{a}.} \tag{9.3}$$

$$\boxed{\frac{B_u}{B_3} = \tan\delta\cdot e^{-a/\cos\delta}.} \tag{9.4}$$

For extreme "legal" $\varepsilon = 1$:

$$\beta\cos\frac{\delta}{a} = \sin^2\delta\cdot\beta = \sin^2\delta\cdot\ln\left(\sqrt{1+\tan^2\delta}+\tan\delta\right) = \sin^2\delta\cdot\ln\frac{1+\sin\delta}{\cos\delta},$$

$$\beta = 60.27^\circ,\quad g/\lambda_u = .21,\quad \text{and}\quad B_u/B_3 = .46.$$

Hybrid Undulator with Superimposed Quadrupole Field

With the electron beam in the z-direction, and the midplane the xz-plane, the normal undulator fields can be described by

$$B_z = iB_y = B_1^* = iF_1' = iB_1 \cos k(z+iy).$$

For the complex potential $F_1 = A_1 + V_1$, with A_1 and V_1 the vector and scalar potentials respectively, it follows that

$$F_1 = \frac{B_1}{k}\sin k(z+iy) \quad \text{and} \quad V_1 = \frac{B_1}{k}\cos kz \sinh ky.$$

The desired normal quadrupole fields are given by

$$B_0^* = iF_0' = iB_0'z \quad F_0 = \frac{1}{2}B_0'z^2, \quad \text{and} \quad V_0 = B_0'xy.$$

For the scalar potential and the combined undulator and quadrupole fields, we therefore have

$$V = V_1 + V_0 = \frac{B_1}{k}\cos kz \sinh ky + B_0'xy.$$

Setting this equal to a constant gives the associated surface of a pole made with infinitely permeable material. With y_0 the half-gap of the pole at $z = x = 0$,

$$0 = \cos kz \sinh ky - \sinh ky_0 + \frac{B_0'}{kB_1}kxky.$$

With the following substitutions:

$$a = \cos kz, \quad \mathcal{E} = \frac{B_0'}{kB_1}, \quad u = kx, \quad \text{and} \quad v = ky,$$

we arrive at the following equation for the ideal 3D pole:

$$\boxed{a \sinh v - \sinh v_0 + \mathcal{E}uv = G = 0.}$$

To understand what this means, we look at some derived quantities:

$$\boxed{y' = \frac{dy}{dx} = -\frac{G_x'}{G_y'} = -\frac{G_u'}{G_v'} = -\frac{\mathcal{E}v}{a\cosh v + \mathcal{E}u}}$$

January, 1984. Note 0187csem.

is the slope of the pole in the xy-plane. For $x = z = 0$ it is reduced to

$$\boxed{y'_0 = -\frac{B'_0 y_0/B_1}{\cosh v_0} = -\frac{\mathcal{E} v_0}{\cosh v_0}.}$$

Looking at the slope just above the axis of the system, i.e. for $x = 0$,

$$\boxed{a \sinh v_1 = \sinh v_0,}$$

and

$$y' = -\frac{\mathcal{E} v_1}{a \cosh v_1},$$

where the subscript 1 refers to the case of $x = 0$ and z equal to anything. For $z = 0$ this reduces to

$$y' = -\frac{B'_0 y_0/B_1}{\cosh v_0} = -\frac{\mathcal{E} v_0}{\cosh v_0}.$$

Eliminating $a = \cos kz$ gives

$$\boxed{y' = -\mathcal{E} v_1 \frac{\tanh v_1}{\sinh v_0}.}$$

For the curvature of the pole in the xy-plane we need

$$\begin{aligned} y'' &= \frac{\partial y'}{\partial x} + \frac{dy}{dx} \cdot \frac{\partial y'}{\partial y} \\ &= k \left(\frac{\partial y'}{\partial u} + y' \frac{\partial y'}{\partial v} \right) \\ &= k \left(\frac{\mathcal{E}^2 v}{(a \cosh v + \mathcal{E} u)^2} + \frac{\mathcal{E}^2 v}{(a \cosh v + \mathcal{E} u)} \cdot \frac{a \cosh v + \mathcal{E} u - a v \sinh v}{(a \cosh v + \mathcal{E} u)^2} \right). \end{aligned}$$

For $u = 0$, this reduces to

$$\begin{aligned} y'' &= \frac{k \mathcal{E}^2 v_1 a}{a^3 \cosh^3 v_1} (2 \cosh v_1 - v_1 \sinh v_1) \\ &= \mathcal{E}^2 k \frac{v_1 \tanh^2 v_1}{\sinh^2 v_0} (2 - v_1 \tanh v_1). \end{aligned}$$

We let $|y'| \ll 1$, and therefore $\sqrt{1 + (y')^2}^3 \approx 1$. With radius of curvature R, we get

$$\frac{1}{k \mathcal{E}^2} \cdot \frac{\sinh^2 v_0}{v_1 \tanh^2 v_1 (2 - v_1 \tanh v_1)} = \boxed{R = \frac{y_0}{\mathcal{E}^2} \cdot \frac{\sinh^2 v_0}{v_0 v_1 \tanh^2 v_1 (2 - v_1 \tanh v_1)},}$$

For $v_1 = v_0$, and $y_0' = -\mathcal{E}v_0/\cosh v_0$:

$$\boxed{R_0 = \frac{y_0}{\mathcal{E}^2} \cdot \frac{\cosh^2 v_0}{v_0^2(2 - v_0 \tanh v_0)} = \frac{y_0}{(y_0')^2} \cdot \frac{1}{v_0^2(2 - v_0 \tanh v_0)}}.$$

We make the following assignments:

$$\frac{1}{k\mathcal{E}^2} = \frac{kB_1^2}{(B_0')^2} = y_0 \frac{B_1^2}{y_0^2(B_0')^2} v_0 \quad \text{and} \quad b = \frac{B_1^2}{y_0^2(B_0')^2},$$

and re-write

$$\boxed{R = y_0 b \cdot \frac{v_0}{v_1} \cdot \frac{\sinh^2 v_0}{\tanh^2 v_1 (2 - v_1 \tanh v_1)}},$$

where for $2 - v \tanh v = 0, v = \dfrac{2\pi y}{\lambda} = 2.0653.$

Excess Flux into Gm13

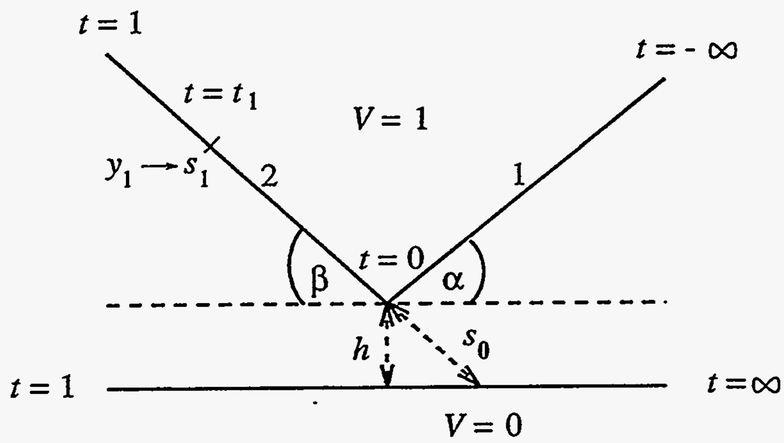


Figure 1.

For the above graph,

$$\frac{\alpha}{\pi} = n_1\,, \quad \frac{\beta}{\pi} = n_2\,, \quad \text{and} \quad n_1 + n_2 = n_3\,.$$

The conformal map is described by

$$\dot{z} = a\frac{t^{n_3}}{(t-1)^{1+n_2}}, \quad a \in \Re\,.$$

From

$$\pi\dot{F} = \frac{1}{t-1} \quad \text{follows} \quad \pi F = \ln(t-1)\,.$$

We describe the flux into surface 2 of Figure 1 as

$$\Phi_2 = -\frac{1}{\pi}\ln(1-t_1) = \int_{s_0}^{s_1} \frac{ds}{\beta s} + \Delta A_2\,.$$

Thus

$$\begin{aligned}\beta\Delta A_2 &= -n_2\ln(1-t_1) - \ln\left(1 + \frac{s_1 - s_0}{s_0}\right)\\ &= \ln\frac{(1-t_1)^{-n_2}}{\left(1 + \dfrac{a}{s_0}\displaystyle\int_0^{t_1}\frac{t^{n_3}\,dt}{(1-t)^{1+n_2}}\right)}\end{aligned}$$

September, 1983. Note 0183csem.

and by l'Hôpital's Rule,

$$= \ln \frac{n_2(1-t_1)^{-n_2-1}}{\left(\frac{a}{s_0} t_1^{n_3}(1-t_1)^{-n_2-1}\right)}$$
$$= \ln \frac{n_2 s_0}{a},$$

and therefore

$$\boxed{\Delta A_2 = \frac{1}{\beta} \ln \left(\frac{h}{\pi} \frac{\beta}{a \sin \beta} \right) .} \tag{1}$$

We need to calculate h/a, and we begin with

$$y_1 = h + a \sin \beta \int_0^{t_1} \frac{t^{n_3} dt}{(1-t)^{n_2+1}} . \tag{2}$$

We now calculate y_1 by going around the singularity at $t = 1$ on circle with $\varrho = \varrho_1 = 1 - t_1$, that is

$$t = 1 + \varrho_1 e^{i\varphi} \quad \text{and} \quad dt = i \varrho_1 e^{i\varphi} d\varphi ,$$

and thus

$$y_1 = \Im i \varrho_1 a \int_0^{\pi} \frac{(1 + \varrho_1 e^{i\varphi})^{n_3}}{\varrho_1^{1+n_2} e^{i\varphi(1+n_2)}} e^{i\varphi} d\varphi$$
$$= \Im \frac{ia}{\varrho_1^{n_2}} \int_0^{\pi} (1 + \varrho_1 e^{i\varphi})^{n_3} e^{-in_2\varphi} d\varphi .$$

For ϱ_1 sufficiently small,

$$y_1 = \Im \frac{ia}{\varrho_1^{n_2}} \frac{e^{-in_2\pi} - 1}{-in_2} = \frac{a \sin \beta}{n_2 \varrho_1^{n_2}} .$$

Re-writing (2) with $t = 1 - \varrho$

$$y_1 = h + a \sin \beta \int_{\varrho_1}^{1} \frac{(1-\varrho)^{n_3}}{\varrho^{n_2+1}} d\varrho = \frac{a \sin \beta}{n_2 \varrho_1^{n_2}} .$$

Thus

$$\frac{h}{a \sin \beta} = \frac{1}{n_2 \varrho_1^{n_2}} - \int_{\varrho_1}^{1} \frac{(1-\varrho)^{n_3}}{\varrho^{n_2+1}} d\varrho$$
$$= \frac{1}{n_2 \varrho_1^{n_2}} + \left. \frac{(1-\varrho)^{n_3}}{n_2 \varrho^{n_2}} \right|_{\varrho_1}^{1} + \frac{n_3}{n_2} \int_{\varrho_1}^{1} \frac{(1-\varrho)^{n_3-1}}{\varrho^{n_2}} d\varrho ,$$

and

$$\frac{hn_2}{a\sin\beta} = \frac{h\beta}{\pi a\sin\beta} = n_3 I_2$$

where

$$\boxed{I_2 = \int_0^1 \frac{dt}{t^{n_2}(1-t)^{1-n_3}} \quad \text{and} \quad I_1 = \int_0^1 \frac{dt}{t^{n_1}(1-t)^{1-n_3}}\,.}$$

Therefore we may now summarize from (1) that

$$\boxed{\Delta A_2 = \frac{1}{\beta}\ln(n_3 I_2)\,,}$$

and equivalently

$$\boxed{\Delta A_1 = \frac{1}{\alpha}\ln(n_3 I_1)\,.}$$

Further, since

$$\frac{h}{\pi a} = n_3 I_2 \frac{\sin\beta}{\beta} = n_3 I_1 \frac{\sin\alpha}{\alpha}\,,$$

$$\boxed{n_3 I_1 = n_3 I_2 \frac{\sin\beta/\beta}{\sin\alpha/\alpha}\,.}$$

A different way to look at what was done earlier:

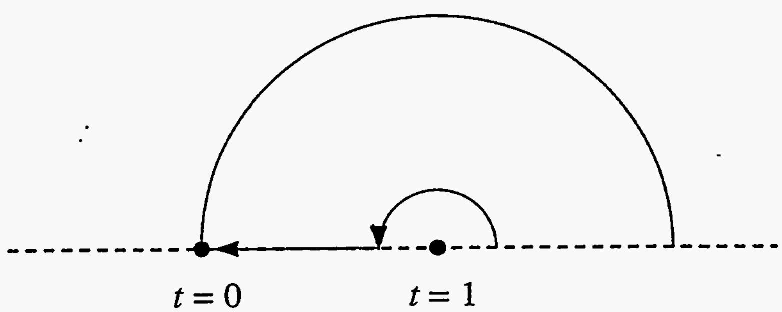


Figure 2.

Since, clearly, $h = \Im\int_{1+}^{0} \dot{z}dt$, one may take a path from any point on the real t-axis to the right of $t = 1$ to $t = 0$.

In this note the path followed a $\varrho_1 \Rightarrow 0$ half-circle around $t = 1$, and then on axis to $t = 0$.

Flux Distribution Symmetry Theorem

Even though this case is the same as the electrostatic case, it is formulated for magnetic fields because of the application to CSEM in hybrid systems.

Theorem.

There are N bodies with $\mu = \infty$. The matrix M, which describes the relationship between potentials V_n on fluxes F_n leaving body n, is symmetrical, i.e.: $F = MV$, $M^t = M$. In this notation, the subscript 0 indicates the reference body on potential $V = 0$. Thus, the theorem states

$$M_{nm} = M_{mn}.$$

Analysis.

Stored energy in the field is unique: it depends only on the initial state (we assume $V_n = 0$) and the end state. By going from the initial to the end state by bringing bodies in any sequence from $V_n = 0$ to the final V_n, and doing so by moving magnetic charges from infinity, we get

$$\mathcal{E} = \int V^t dF = \int V^t M dV.$$

$\mathcal{E}$ must be independent of sequence in which this is done: the right hand side must be a complete differential leading to $M_{nm} = M_{mn}$. We show this explicitly for V_1, V_2 and all other $V_n = 0$:

$$\begin{aligned}\mathcal{E} &= \int (\, V_1 \quad V_2 \,) \begin{pmatrix} M_{11} dV_1 - M_{12} dV_2 \\ -M_{21} dV_1 + M_{22} dV_2 \end{pmatrix} \\ &= \int M_{11} V_1 dV_1 + \int M_{22} V_2 dV_2 - \underbrace{\int (M_{12} V_1 dV_2 + M_{21} V_2 dV_1)}_{G}.\end{aligned}$$

We simplify G by making the following substitutions:

$$M_{12} = S + D, \quad M_{21} = S - D, \quad \text{and} \quad S = \frac{1}{2}(M_{12} + M_{21}), \quad D = \frac{1}{2}(M_{12} - M_{21}),$$

$$G = S \int \underbrace{(V_1 dV_2 + V_2 dV_1)}_{(a)} + D \underbrace{\int (V_1 dV_2 - V_2 dV_1)}_{(b)},$$

where (a) is $d(\, V_1 \quad V_2 \,)$ and is therefore independent of the sequence, and (b) would

November, 1986. Note 0143csem.

be dependent on our sequence. Since G must be independent of the sequence, it follows that $D = 0$.

In a CSEM circuit, $\mathbf{F}$ equals the vector of charges deposited by the CSEM on the surfaces (with all $V = 0$). Therefore, a hybrid system can be represented by magnetic capacitors and sources that deposit DC charges. If one finds this more convenient, one may also do this analysis with capacitors and AC current sources, or with resistors and DC currents.

The theorems known for these circuits apply, such as Kirchhoff's reciprocity theorem, i.e. the nodal equations, etc. One can also use 2×2 matrix methods for systems like ladder networks, and apply them directly to hybrid wigglers. One can use concepts like characteristic impedance of networks and quadrupole theory, i.e. all the tools that have been developed for analog network analysis.

Stored Energy in CSEM

We define the energy density as

$$\mathcal{E} = \int \mathbf{H} \cdot d\mathbf{B}.$$

We can look at the easy-axis direction and the direction perpendicular to the easy-axis separately. The lower integral limit is arbitrary, but must be fixed.
With $B_{\|} = B_r + \mu_0 \mu_{\|} H_{\|}$ and $B_{\perp} = \mu_0 \mu_{\perp} H_{\perp}$:

$$\mathcal{E}_{\|} = \int_{H_{\|}=0}^{H_{\|}} H_{\|} dB_{\|} = \int_{H_{\|}=0}^{H_{\|}} H_{\|} \frac{dB_{\|}}{dH_{\|}} dH_{\|} = \mu_0 \mu_{\|} \int_{H_{\|}=0}^{H_{\|}} H_{\|} dH_{\|}.$$

Thus,

$$\boxed{\mathcal{E}_{\|} = \frac{1}{2} \mu_0 \mu_{\|} H_{\|}^2,}$$

and similarly,

$$\boxed{\mathcal{E}_{\perp} = \frac{1}{2} \mu_0 \mu_{\perp} H_{\perp}^2,}$$

with

$$\boxed{\mathcal{E}_{\text{tot}} = \mathcal{E}_{\|} + \mathcal{E}_{\perp}.}$$

This obviously gives $H_{\|}$ a unique role.

November, 1982. Note 0142csem.

Earnshaw's Theorem for Non-Permeable Material

Problem: There is an assembly of "frozen" magnetic charges $\varrho(\mathbf{r})^{\dagger}$ in an external magnetic field (produced by solenoid or other REC assembly) without any permeable material in the system.

Question: What is the restoring force for small displacements?

Analysis: The force components in the (x_1, x_2, x_3) coordinate system are

$$F_1 = -\int \varrho V_1' dv, \quad F_2 = -\int \varrho V_2' dv, \quad \text{and} \quad F_3 = -\int \varrho V_3' dv.$$

We displace the system by $\Delta x_1, \Delta x_2, \Delta x_3$, which is the same as displacing the external fields by $-\Delta x_1, -\Delta x_2, -\Delta x_3$ without changing ϱ, and get

$$\begin{aligned}\Delta F_n &= \int \sum \Delta x_m \frac{\partial}{\partial x_m} \varrho \frac{\partial V}{\partial x_n} dv \\ &= \sum M_{nm} \Delta x_m \qquad \text{with} \qquad M_{nm} = \int \varrho \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_m} V \, dv,\end{aligned}$$

$$\Delta \mathbf{F} = M \cdot \Delta \mathrm{x}.$$

In general, M will not be a diagonal matrix. We assume that it can be made diagonal (by going to a new coordinate system) with matrix C, where

$$C \Delta \mathbf{F} = \Delta \mathbf{F}_d = CMC^{-1} \cdot C \Delta \mathrm{x} = CMC^{-1} \cdot \Delta \mathrm{x}_d,$$

$$CMC^{-1} = M_d = \int \varrho \begin{pmatrix} V''_{xx} & 0 & 0 \\ 0 & V''_{yy} & 0 \\ 0 & 0 & V''_{zz} \end{pmatrix} dv,$$

where x, y, z are the new coordinates. From this, it follows that

$$\boxed{\frac{\Delta F_x}{\Delta x} + \frac{\Delta F_y}{\Delta y} + \frac{\Delta F_z}{\Delta z} = \int \varrho \underbrace{\left(V''_{xx} + V''_{yy} + V''_{zz} \right)}_{= \nabla^2 V = 0} dv = 0.}$$

Since a stable system requires a negative restoring force in each of the three coordinate directions, any such system will be unstable.

June, 1981. Note 0076csem.

† REC can be considered this way if one assumes differential permeability $\equiv 1$.

In applications, it will often be clear *a priori* in which coordinate system the matrix M will be diagonal, and the above equation can then be used directly in its final form. This means that for systems with cylindrical symmetry about the z-axis, that because

$$\frac{\Delta F_y}{\Delta y} = \frac{\Delta F_x}{\Delta x} \Rightarrow \frac{\Delta F_r}{\Delta r} \quad \text{and} \quad 2\frac{\Delta F_r}{\Delta r} + \frac{\Delta F_z}{\Delta z} = 0,$$

only one of the stiffnesses needs to be calculated from basics.

It is interesting to note that Earnshaw's Theorem is often stated in an overly broad fashion. For instance, stable support *is* possible if one allows forces not derived from a potential satisfying $\nabla^2 V = 0$. The forces between contacting solid materials, for example, such as mechanical bearings, are due to the quantum nature of solids, and hence do not obey Earnshaw's Theorem. It is also clear without any mathematics that a permanent magnet is stably supported in a superconducting bowl. This is similarly true for an extreme diamagnetic bowl.

Harmonics Produced by Rectangular REC Block

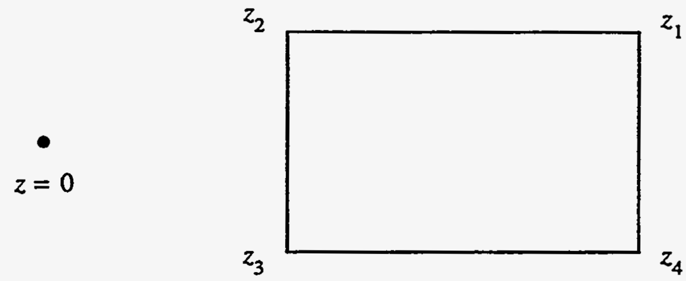


Figure 1.

In this document we refer to the above diagram and (4b) and (16b) of the *Nuclear Instruments and Methods* article[†]

$$\underline{B}^*(z_0) = \sum_{n=1} b_n z_0^{n-1},$$

$$b_n = \frac{\underline{B}_r}{2\pi i} \oint \frac{dx}{z^n}.$$

For $n = 1$,

$$b_1 = \frac{\underline{B}_r}{2\pi i} \ln \frac{z_2}{z_1} \cdot \frac{z_4}{z_3} = \frac{\underline{B}_r}{2\pi i} \ln \frac{z_2/z_1}{z_3/z_4}$$

$$= \boxed{\frac{\underline{B}_r}{\pi} \Im \ln \frac{z_2}{z_1}.}$$

For $n \geq 2$,

$$b_n = -\frac{\underline{B}_r}{2\pi i} \cdot \frac{1}{n-1} \left(\frac{1}{z_2^{n-1}} - \frac{1}{z_1^{n-1}} + \frac{1}{z_4^{n-1}} - \frac{1}{z_3^{n-1}} \right)$$

$$= \boxed{-\frac{\underline{B}_r}{\pi} \cdot \frac{1}{n-1} \Im \left(\frac{1}{z_2^{n-1}} - \frac{1}{z_1^{n-1}} \right).}$$

October, 1980. Note 0059csem.

† Halbach, H., Nuclear Instruments and Methods **169**, 1 (1980).

A Possible REC Undulator for SSRL

I. Reason for REC.

It may be possibe to use some very specific ferrite. All other materials, like the Alnicos, are not only significantly inferior in performance, but would probably also have to be magnetized in final assembly which may be difficult to do.

A potential future advantage is that the permanent magnet undulator can be scaled down in physical size without difficulty. One can therefore envision the following scheme: design a REC undulator for very small gap and λ and have it inside vacuum. Move the two arrays apart to have the large gap necessary during beam formation. When beam is established, move the 2 arrays together to form the design gap that the beam allows. Clearly, this would be nearly impossible with either a conventional, or even a superconducting, undulator.

II. Use of Nomogram and Notation.

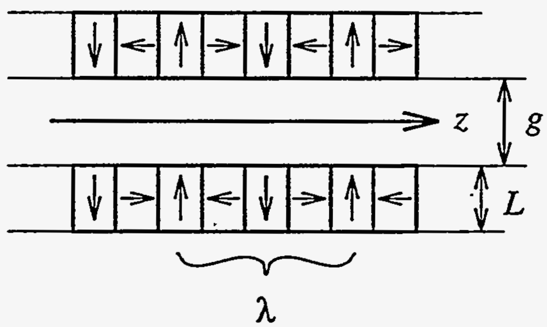


Figure 1.

$$B_y + iB_z = B_0 \cos \frac{2\pi(z+iy)}{\lambda},$$

$$K = B_0 \lambda,$$

$$k = \frac{B_r}{K} \underbrace{\frac{\sin \pi/M'}{\pi/M'}}_{E_1} \underbrace{\left(1 - e^{-2\pi L/\lambda}\right)}_{E_2},$$

where B_r is the remanent field of REC, M' is the number of pieces with fixed easy-axis

April, 1979. Note 0038csem.

per period.

$$2k\lambda e^{-\pi g/\lambda} = 1.$$

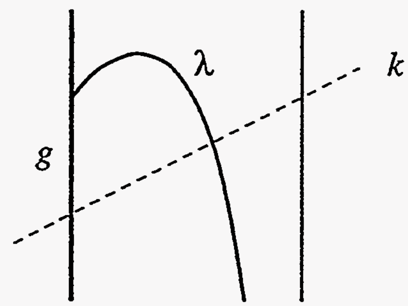


Figure 2.

Because E_1, E_2 are close to 1, and one usually chooses $K \approx 1$ T(cm), k (cm^{-1}) is numerically close to B_r.

In this document, all lengths will be in cm, and B's in Tesla.

III. Design Considerations.

• End Section. Normalize center to $z = 0$ and that piece has easy-axis parallel to the y axis. The last pieces at both ends must have the same easy-axis as piece at $z = 0$, but should have only half of normal length in the z-direction. One may want to use coils to fine-tune the end sections, but it would not be surprising if this were unnecessary.

In order to reduce the effects from finite length in x-direction (or to get away with shorter length in that direction) and to avoid 3D fringe effects at the ends in z-direction by cutting end fields off rapidly, one should back-up REC with a soft steel plate with reasonable overhang in z and x directions. This will not affect the amplitude of the $\cos(2\pi(z + iy)/\lambda)$ term, but will introduce a very weak, unnoticeable in the midplane, third harmonic (for $M' = 4$).

• Length of REC in x-direction. The present estimate is that it should be approximately the sum of the width of the beam and $2g$. The 3D effects discussed in the previous section are easily analyzed computationally and should be done before ordering materials!

• Choice of M' and L. It is recommended , at least for the first undulator, to use $M' = 4$ (giving $E_1 = .9$) and $L = \lambda/2$ (giving $E_2 = .96$) or $L = \lambda/4$ (giving $E_2 = .79$). With these choices, the undulator can be assembled from identical REC pieces with square cross-section and the easy-axis parallel to a side. The exception would be with the end pieces which could be obtained by cutting or grinding the normal pieces.

IV. Specific Calculations.

For a realistic undulator with $g = 2.8\text{cm}$, $B_r = .95\text{T}$, $K = 1\text{Tcm}$ and $M' = 4$:

L (cm)	k (cm^{-1})	λ (cm)
$\lambda/4 = 1.18$	.68	4.73
$\lambda/2 = 2.22$	.82	4.44

Table 1.

Since $\lambda/4$ uses only half the REC of the $\lambda/2$ case and λ is only less than 10% larger, this is the preferable design. The volume for $\lambda/4$ is $V = 3540$ cm^3, and for $\lambda/2$ is $V = 6660$ cm^3.

The REC price would probably be approximately \$1-2/cm^{-3}.

A Simple Derivation of the Lorentz Transformation Without Talking About Light

Postulate: Physics is independent of location, time and uniform motion of the system in which the experiment is performed.

We look at two systems that move with velocity, v, relative to each other. We establish clocks and space (x) markers in each system.

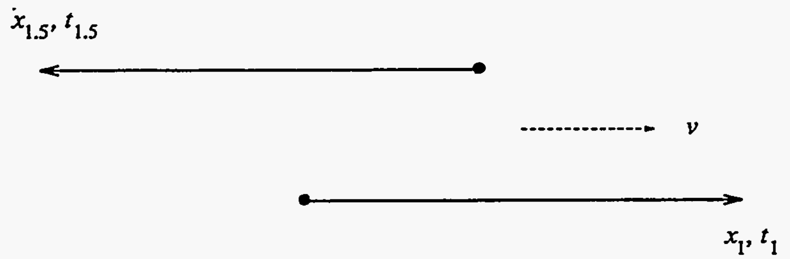


Figure 1.

We locate the origins and synchronize the clocks so that $x_1 = 0$, $t_1 = 0$, $x_{1.5} = 0$, and $t_{1.5} = 0$. Also notice that the "1.5" system has x increasing in the opposite direction from the "1" system.

We want to know $(x_{1.5}, t_{1.5})$ as a function of (x_1, t_1).

We know that $\Delta x_{1.5}/\Delta x_1$, $\Delta x_{1.5}/\Delta t_1$, $\Delta t_{1.5}/\Delta x_1$, and $\Delta t_{1.5}/\Delta t_1$ can not depend on x_1, t_1 because of our postulate. This means that the relationship between the two systems is linear, and can be expressed as a 2 by 2 matrix.

$$\left.\begin{aligned} x_{1.5} &= a_{11}x_1 + a_{12}t_1 \\ t_{1.5} &= a_{21}x_1 + a_{22}t_1 \end{aligned}\right\} \implies r_{1.5} = \begin{pmatrix} x_{1.5} \\ t_{1.5} \end{pmatrix} = A \cdot r_1$$

The velocity of a particle in system "1.5" (e.g. at $x_{1.5} = 0$) as seen from system "1" is $v = -a_{12}/a_{11}$. Thus

$$\boxed{a_{12} = -a_{11}v}$$

with $a_{11} \neq 0$ always true.

The choice of the relative sign of x in the two systems means that the observer in each system sees the other system move in the positive x–direction with velocity v.

July 30, 1992. Note 0287misc.

Therefore,

$$r_1 = A \cdot r_{1.5} = A \cdot A \cdot r_1,$$

and also,

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

must be satisfied. By multiplication and substitution,

$$A^2 = \begin{pmatrix} a_{11}(a_{11} - a_{21})v & -a_{11}v(a_{11} + a_{22}) \\ a_{21}(a_{11} + a_{22}) & a_{22}^2 - a_{21}a_{11}v \end{pmatrix}.$$

Therefore,

$$\boxed{a_{11} = -a_{22}}$$

and

$$\boxed{a_{21} = (1 - a_{22}^2)/(a_{22}v).}$$

By further substitution,

$$A = \begin{pmatrix} -a_{22} & a_{22}v \\ (1/a_{22} - a_{22})/v & a_{22} \end{pmatrix}.$$

We introduce $x_2 = -x_{1.5}, t_2 = t_{1.5}$ and

$$r_2 = B \cdot r_1, \quad B = \begin{pmatrix} a_{22} & -a_{22}v \\ (1/a_{22} - a_{22})/v & a_{22} \end{pmatrix}.$$

We further define

$$\gamma = a_{22}.$$

$$g = 1/a_{22}^2 - 1 = 1/\gamma^2 - 1,$$

and therefore

$$B = \gamma \begin{pmatrix} 1 & -v \\ g/v & 1 \end{pmatrix}.$$

It is important to notice that the diagonal elements are identical.

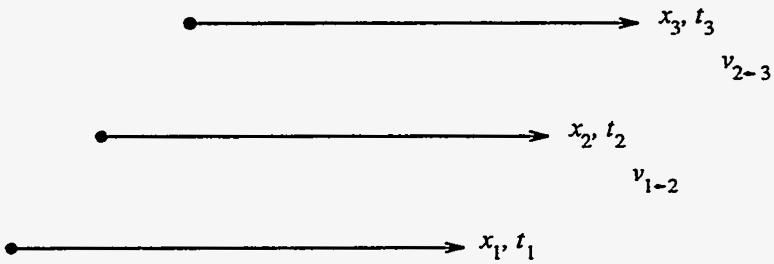


Figure 2.

We define

$$\begin{aligned} r_2 &= B_{1\to 2} \cdot r_1 \\ r_3 &= B_{2\to 3} \cdot r_2 \\ &= B_{2\to 3} \cdot B_{1\to 2} \cdot r_1 \end{aligned}$$

and thus

$$B_{1\to 3} = \gamma_{1\to 2}\gamma_{2\to 3} \begin{pmatrix} 1 & -v_{2\to 3} \\ g_{2\to 3}/v_{2\to 3} & 1 \end{pmatrix} \begin{pmatrix} 1 & -v_{1\to 2} \\ g_{1\to 2}/v_{1\to 2} & 1 \end{pmatrix}$$

and further,

$$B_{1\to 3} = \gamma_{1\to 2}\gamma_{2\to 3} \begin{pmatrix} 1 - v_{2\to 3}g_{1\to 2}/v_{1\to 2} & -(v_{1\to 2} + v_{2\to 3}) \\ g_{2\to 3}/v_{2\to 3} + g_{1\to 2}/v_{1\to 2} & 1 - v_{1\to 2}g_{2\to 3}/v_{2\to 3} \end{pmatrix}.$$

By the identical diagonal elements we have:

$$\frac{v_{2\to 3}g_{1\to 2}}{v_{1\to 2}} = \frac{v_{1\to 2}g_{2\to 3}}{v_{2\to 3}} \implies \frac{g_{1\to 2}}{v_{1\to 2}^2} = \frac{g_{2\to 3}}{v_{2\to 3}^2}.$$

Thus, we may generalize our equation and we have

$$g/v^2 = \frac{(1/\gamma^2 - 1)}{v^2} = k = \text{constant of nature.}$$

Here

$$\boxed{\gamma = \frac{1}{\sqrt{1 + kv^2}}.}$$

We have to verify that other relationships are also satisfied (e.g. relation between elements [11] and [12], etc.). We have shown that if the postulate is true, the relationship between x and t of the systems moving with velocity v relative to each other must be

$$\begin{pmatrix} x_2 \\ t_2 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ kv & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ t_1 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1 + kv^2}}.$$

We have not shown that $k \neq 0$, but the value of k can be obtained from "any" experiment, e.g. lifetime of meson, etc., and experiments do not have to use light.

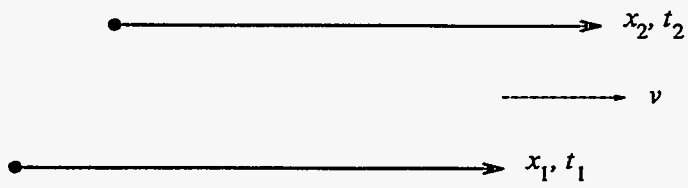


Figure 3.

$$\begin{pmatrix} x_2 \\ t_2 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v/c^2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ t_1 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ t_1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v/c^2 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ t_2 \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

1) The lifetime of particle at rest at $x_1 = 0$ in system "1" is τ_1. What is it in system "2"?

$$x_1 = 0, \quad t_1 = \tau_1 \quad \Longrightarrow \quad \boxed{t_2 = \tau_2 = \gamma\tau_1}$$

2) The distance x_1 covered by observer at rest at $x_2 = 0$ in time t_2:

$$x_2 = 0, \quad x_1 = \gamma v t_2, \quad \boxed{x_1/t_2 = \gamma v}$$

Note that

$$v\gamma = c \text{ for v/c} = 1/\sqrt{2}.$$

Dimensional Analysis of the Trajectory of Non-Relativistic Charged Particles in Stationary Electric and Magnetic Fields (MKS units, with and without space charge)

Motivation.

To explain the structure of trajectory equations to engineers working on cyclotron-mass spectrometer.

We use linear scaling length D_0, and represent $\mathbf{B}$ and $\mathbf{E}$ fields by the scaling quantities B_0 and $E_0 = V_0 D_0$ times the appropriate dimensionless functions of x/D_0, y/D_0, and z/D_0. We must be able to represent the trajectory $x(t)$ ($t = 0$ at start of trajectory) as the product of D_0 and a function of dimensionless products P_n. The list of parameters to form P's has, in addition to $D_0, B_0, V_0(E_0), t$, the quantity m/e due to the equation of motion. Thus, the complete list consists of $m/e, D_0, B_0, V_0(E_0), t$.

We construct P's by first finding the appropriate physics relationship, then re-writing them in product form with parameters from the above list, and finally by solving for P, i.e.:

$$m\ddot{\mathbf{x}} = e\mathbf{E} \quad \rightarrow \quad m\frac{D_0}{t^2} = e\frac{V_0}{D_0} \quad \rightarrow \quad \boxed{P_1 = t^2 \frac{e}{m}\frac{V_0}{D_0^2}.}$$

For construction of next P, consider the parameter list without e/m:

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} \quad \rightarrow \quad E_0 = B_0 \frac{D_0}{t} \quad \rightarrow \quad P_2 = \frac{B_0}{E_0}\frac{D_0}{t} = \frac{B_0 D_0^2}{V_0 t}.$$

We use P_1 to form P_3 without t, and use P_3 instead of P_2:

$$P_3 = P_2^2 P_1 = \frac{B_0^2}{E_0^2}\frac{e}{m}V_0 \quad \rightarrow \quad \boxed{P_3 = \frac{e}{m}\frac{B_0^2 D_0^2}{V_0}.}$$

We now remove B_0 from the parameter list, leaving only $D_0, V_0(E_0), t$, and we see that no additional P's are possible. Thus, we have

$$\boxed{x(t) = D_0 F_x(P_1, P_3) = D_0 F_x\left(t^2 \frac{e}{m}\frac{V_0}{D_0^2}, \frac{e}{m}\frac{B_0^2 D_0^2}{V_0}\right),}$$

and this is equivalently true for $y(t)$ and $z(t)$.

April, 1992. Note 0278misc.

These expressions show the available options for changing the values of parameters if one of these has to be changed in a particular way and if one does not want to change the trajectory. If one does not care how long it takes for the particle to traverse its trajectory, then P_3 is the only P that is to be kept constant. P_1 can be considered to be an expression for the time to traverse the system.

If one wants to include space charge effects, one must include I, e, and ε_0 to the remaining list of parameters D_0, V_0, t. If the magnetic fields produced by the moving charges are important, one must add μ_0 as well.

When writing the P's, we shall use the fact that space charge and magnetic fields for charged particles go to 0 as $1/\varepsilon_0$ and μ_0 go to 0. The space charge effects come from

$$\nabla \cdot \varepsilon_0 \mathbf{E} = \varrho \quad \rightarrow \quad \varepsilon_0 \frac{E_0}{D_0} = I \frac{t}{D_0^3} \quad \rightarrow \quad P_4 = \frac{I}{\varepsilon_0} \frac{t}{V_0 D_0}.$$

We remove t with P_2 to get

$$P_5 = P_4 P_2 = \frac{I}{\varepsilon_0} \frac{1}{V_0} \frac{B_0}{E_0} \quad \rightarrow \quad P_5 = \frac{I}{\varepsilon_0} \frac{B_0 D_0}{V_0^2}.$$

We remove B_0 with P_3 to get

$$P_6 = \frac{P_5^2}{P_3} = \left(\frac{I}{\varepsilon_0}\right)^2 \frac{D_0^2}{V_0^4} \frac{m}{e} \frac{V_0}{D_0^2} \quad \rightarrow \quad \boxed{P_6 = \left(\frac{I}{\varepsilon_0}\right)^2 \frac{m/e}{V_0^3}}$$

with P_4, P_5 discarded.

We remove I from our parameter list and are left with $e, \varepsilon_0, D_0, t, V_0$ for

$$\nabla \cdot \varepsilon_0 \mathbf{E} = \varrho, \quad \varepsilon_0 E_0 = \frac{e}{D_0^2} \quad \rightarrow \quad \boxed{P_7 = \frac{e}{\varepsilon_0 D_0 V_0}.}$$

By removing ε_0 we see that no more P's are possible with just e, D_0, t, V_0. Thus, we have

$$\boxed{\begin{aligned} x(t) &= D_0 F_x(P_1, P_3, P_6, P_7) \\ &= D_0 F_x\left(\frac{e}{m}\frac{V_0 t^2}{D_0}, \frac{e}{m}\frac{B_0^2 D_0^2}{V_0}, \frac{I^2}{\varepsilon_0^2}\frac{m/e}{V_0^3}, \frac{e}{\varepsilon_0 D_0 V_0}\right), \end{aligned}}$$

and this is equivalently true for $y(t)$ and $z(t)$.

We expect that some P's are not significant if ε_0 is large enough so that P_6 and/or P_7 are small enough. For instance, for $D_0 = 10^{-2}$m, $V_0 = 10^3$V, and $e =$ charge of electron, we have $P_7 = 1.8 \times 10^{-9} \ll 1$, thus P_7 is probably not important.

We now add μ_0 to the parameter list, and for e, D_0, t, V_0, μ_0, we have

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \quad \rightarrow \quad \frac{E_0}{D_0} = \frac{V_0}{D_0^2} = \frac{\mu_0 (e/t)/D_0}{t} = \frac{\mu_0 e}{t^2 D_0},$$

$$P_8 = \frac{\mu_0 e D_0}{t^2 V_0}.$$

Using other P's, we get

$$\boxed{P_9 = \frac{\mu_0 I}{D_0 B_0}.}$$

We remove μ_0 from the parameter list, and see that no more P's are possible, and we have

$$\boxed{x(t) = D_0 F_x(P_1, P_3, P_6, P_7, P_9),}$$

and this is equivalently true for $y(t)$ and $z(t)$.

Application to System with Fixed D_0, B_0 (Cyclotrino).

We ignore P_1 since it determines traversal time. Without space charge and current-field effects, we must keep $V_0 m/e$ constant to get same behavior when the particle is changed, i.e. $V_0 \sim e/m$ is necessary. To see how space charge limitation affects "permissible" current, one must look at P_6:

$$\frac{I^2 m/e}{V_0^3} = \frac{I^2 (m/e)^4}{(V_0 m/e)^3} = \text{constant}$$

and this implies that $(m/e)^2 \cdot I$ should be a constant or small enough. As stated earlier, P_7 will be small enough to cause no problems, and the same will be true for P_9.

Further Comments.

While this theory was formulated with scale factors in mind, the P's also have local meaning. That is, if the "local" V † is interpreted as potential energy (divided by e), it becomes clear that P_6 and P_7 (with the local V and D) cannot be sufficiently small to be ignored everywhere since the particles start somewhere with $eV = 0$. But if the ion source is considered as a separate entity the ignorability argument will hold. It is also clear that looking at the P's with subscripted V, V_0 applies not only to applied potentials within the structure, but also to the energy of the incoming beam.‡

† Without the subscript 0 that identifies the "global" scale.

‡ This study was motivated by Tony Young's question about how V_0 has to be changed when B_0 differs from its original design value. Using P_3 we must have B_0^2/V_0 = constant.

Some Practical Numbers.

Use γ sufficiently smaller than 1 to make a P ignorable. Then P_7, $\sqrt{P_6}$ can be ignored if

$$P_7 = \frac{e}{\varepsilon_0 D_0 V_0} < \gamma \quad \rightarrow \quad \boxed{D_0 V_0 = \frac{e}{\varepsilon_0}\frac{1}{\gamma} > \frac{1.8 \times 10^{-8}}{\gamma},}$$

$$\sqrt{P_6} = \frac{I}{\varepsilon_0}\frac{\sqrt{m/e}}{V_0^{3/2}} < \gamma \quad \rightarrow \quad I < V_0^{3/2}\gamma\varepsilon_0\sqrt{e/m}.$$

With e_c = electron charge, and m_p = proton mass,

$$I < V_0^{3/2}\gamma\sqrt{\frac{e/e_c}{m/m_p}}\varepsilon_0\sqrt{\frac{e_c}{m_p}} = V_0^{3/2}\gamma\sqrt{\frac{e/e_c}{m/m_p}}8.7 \times 10^{-8}$$

$$\boxed{I < V_0^{3/2}\gamma\sqrt{\frac{e/e_c}{m/m_p}}8.7 \times 10^{-8}.}$$

Analog Integrator Dynamics

Contrary to conventional analysis, which expresses the output signal in terms of the input signal, the quantity one wants (time integral over input integral) is expressed in terms of output signal (in digital form or as a scope trace), with all dynamic effects taken into account. In addition to dynamic terms being caused by the frequency response of the operational amplifier, the first order sensitivity is also affected by its dynamic behavior.

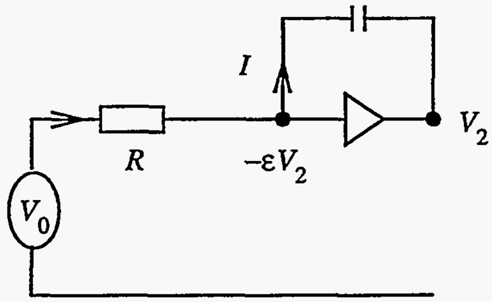


Figure 1.

For p the Laplace transform variable, and $RC = \tau_1$,

$$\frac{V_0 + \varepsilon V_2}{R} = -V_2(1+\varepsilon)pC,$$

where $\mu = 1/\varepsilon \gg 1$ is the open loop gain of the operational amplifier, and

$$V_0 = -V_2(p\tau_1(1+\varepsilon)+\varepsilon) = -V_2 \cdot F.$$

We use the following rough numbers:

$$\varepsilon = 0, \quad \int V_0 dt = 10^{-6}\text{Vsec}, \quad \tau_1 = 10^{-3}\text{sec}, \quad \text{and} \quad -V_2 = \frac{\int V_0 dt}{\tau_1} = 10^{-3}\text{V}.$$

The frequency response of the operational amplifier is

$$\mu_1(p) = \frac{\mu_0}{1+p\tau_0}.$$

It actually behaves in this fashion until the open loop gain is much less than 1. If the operational amplifier were not to behave this way, it would be useless for many

January, 1989. Note 0267misc.

applications. We characterize the frequency response either by the time constant τ_0, or the frequency (times 2π) where the amplitude gain is reduced to 1:

$$1 = \frac{\mu_0}{\omega_2 \tau_0} = \frac{1}{\omega_2 \varepsilon_0 \tau_0} = \frac{1}{\omega_2 \tau_2}$$

with

$$\tau_2 = \frac{1}{\omega_2} \approx 10^{-7}\text{sec}, \quad \mu_0 = 1/\varepsilon_0 \approx 10^6, \quad \text{and} \quad \varepsilon = \varepsilon_0(1 + p\tau_0) = \varepsilon_0 + p\tau_2,$$

and for $\tau_1^* = (\tau_1 + \tau_2)$:

$$\begin{aligned} F &= p\tau_1(1 + p\tau_2) + \varepsilon_0 + p\tau_2 \\ &= p\tau_1^* + \varepsilon_0 + p^2\tau_1\tau_2 \\ &= p\tau_1^* + \varepsilon_0 + p^2\tau_2(\tau_1^* - \tau_2), \end{aligned}$$

$$\boxed{-\frac{V_0}{p\tau_1^*} = V_2\left(1 + \frac{1}{p\mu_0\tau_1^*} + \frac{p\tau_2\tau_1}{\tau_1^*}\right).}$$

In the time domain, the quantity of interest, $\int V_0(\tau)d\tau$, is expressed in terms of the measured quantity $V_2(t)$ by

$$\int V_0(\tau)d\tau = V_2(t) \cdot \tau_1^* + \frac{\int V_2(\tau)d\tau}{\mu_0} + \dot{V}_2(t)\tau_2\tau_1.$$

One has to choose the time constants and open loop gain such that the second and third terms are small compared to the first term so that they can effectively be ignored or corrections can easily be made. It should be noted that the frequency response of the operational amplifier can make a small, but noticeable, correction to the effective time constant τ_1^* through τ_2.

Local Interpolation with Continuous Function and its First N Derivatives

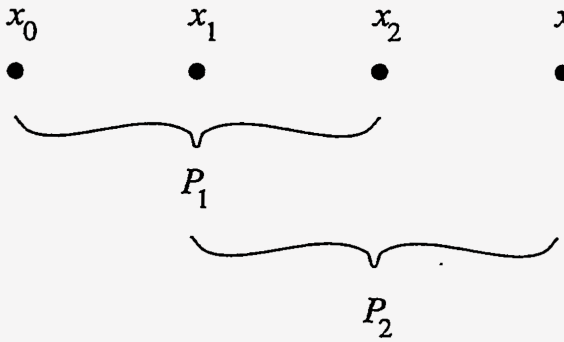


Figure 1.

1. Real function $y_0(x)$ must have known values at $x = x_0, x_1, \ldots, x_n$.

2. Establish interpolation functions $P_{1,\ldots,n-1}(x)$, that have properties appropriate to model $y_0(x)$ in small regions. This necessitates continuous functions, and continuous and meaningful first N derivatives. $P_j(x)$ must reproduce $y_0(x)$ exactly for $x = x_{j-1}$, $x = x_j$ and $x = x_{j+1}$, for $1 \leq j \leq n-1$.

3. Calculate the approximate function $y(x)$ from $y(x) = P_1(x)W_1(x) + P_2(x)W_2(x)$ in interval $x_1 \leq x \leq x_2$, and similarly in other intervals. Make the choices, to some degree arbitrary, for the weight functions $W_{1,\ldots,n-1}(x)$ so that the desired goal is obtained in a reasonable fashion.

4. If, P_1 and P_2 are the same as $y_0(x)$, we do not want the interpolation scheme to destroy the relationship $y(x) = y_0(x)$. Therefore, we must have that

Condition 1: $\quad W_1(x) + W_2(x) = 1\,.$

And if the above is satisfied, it is also true that

Condition 2: $\quad W_1^{(n)}(x) + W_2^{(n)}(x) = 0\,.$

5. We examine $y^{(n)}(x)$:

$$
\begin{aligned}
y^{(n)}(x) = {} & P_1^{(n)}W_1 + nP_1^{(n-1)}W_1^{(1)} + \ldots + nP_1^{(1)}W_1^{(n-1)} + P_1W_1^{(n)} \\
& + P_2^{(n)}W_2 + nP_2^{(n-1)}W_2^{(1)} + \ldots + nP_2^{(1)}W_2^{(n-1)} + P_2W_2^{(n)}\,,
\end{aligned}
$$

May, 1981. Note 0177misc.

for $n = 1, 2, \ldots, N$. We choose $W^{(1)}$ so that all needed derivatives exist. At $x = x_1$ or $x = x_2$,

$$P_1 W_1^{(n)} + P_2 W_2^{(n)} = P_1(W_1^{(n)} + W_2^{(n)}) = 0$$

because P_1, P_2 fit $y_0(x)$ exactly at $x = x_1$, $x = x_2$, and due to Condition 2. We now choose the weight functions such that at $x = x_1$, $y^{(n)} = P_1^{(n)}$, and at $x = x_2$, $y^{(n)} = P_2^{(n)}$. We do this by requiring that weight functions fulfill

Condition 3: $\quad W_1(x_1) = 1, \quad W_1(x_2) = 0, \quad$ and
$\quad W_2(x_1) = 0, \quad W_2(x_2) = 1,$

and fulfill

Condition 4: $\quad W_1^n(x_1) = W_1^n(x_2) = 0 \quad$ and
$\quad W_2^n(x_1) = W_2^n(x_2) = 0 \quad$ for $\quad n = 1, 2, \ldots, N-1$.

With the above choices, y and its first N derivatives at $x = x_n$ depend only on P_n, independently of whether we get to x_n from an upper or lower interval, i.e., y and its derivatives are continuous everywhere.

6. The construction of the weight functions that satisfy Conditions 1 (and therefore Condition 2), 3, and 4, is not unique. We introduce

$$u(x) = \frac{x - (x_1 + x_2)/2}{(x_2 - x_1)/2} = \frac{2x - (x_1 + x_2)}{(x_2 - x_1)}.$$

This gives us

$$\begin{aligned} u(x_1) &= -1, \\ u((x_1 + x_2)/2) &= 0, \\ u(x_2) &= 1. \end{aligned}$$

We now have that

$$\boxed{W_2(x) = \frac{1}{2}(1 + g_N(x)),}$$

$$\boxed{W_1(x) = \frac{1}{2}(1 - g_N(x)),}$$

$$\boxed{\begin{aligned} y(x) &= \frac{1}{2}(P_2(x) + P_1(x) + (P_2(x) - P_1(x)) \cdot g_N(x)) \\ &= P_1(x)W_1(x) + P_2(x)W_2(x), \end{aligned}}$$

where,

$$\boxed{g_N(x) = a_N \int\limits_0^{u(x)} (1-v^2)^{N-1} dv \quad \text{and} \quad \frac{1}{a_N} = \int\limits_0^1 (1-v^2)^{N-1} dv\,.}$$

We may now conclude that, clearly, Conditions 1 and 3 are satisfied, and from

$$\begin{aligned} W_2^{(1)}(x) &= \frac{g_N(x)}{x_2 - x_1} \\ &= \frac{a_N}{x_2 - x_1}(1-u^2)^{N-1} \\ &= \frac{a_N}{x_2 - x_1}(1-u)^{N-1}(1+u)^{N-1}\,, \end{aligned}$$

it follows that Condition 4 is satisfied as well.

We introduce here some further details. Given

$$\frac{1}{a_N} = \int\limits_0^1 (1-v^2)^{N-1} dv = b_N\,,$$

we have that

$$\begin{aligned} b_N &= \int\limits_0^1 (1-v^2)^{N-2}(1-v^2) dv \\ &= b_{N-1} - \int\limits_0^1 (1-v^2)^{N-2} v^2 dv\,. \end{aligned}$$

For

$$du = -1(1-v^2)^{N-2} v dv \quad \text{and} \quad u = \frac{(1-v^2)^{N-1}}{2(N-1)}\,,$$

$$r = v \quad \text{and} \quad dr = dv\,,$$

we have that

$$b_N = \frac{b_{N-1}}{1 + \dfrac{1}{2(N-1)}}\,,$$

$$\boxed{a_N = a_{N-1}\left(1 + \frac{1}{2(N-1)}\right)\,.}$$

Thus, for $a_1 = 1$

$$\boxed{a_N = a_{N-1}\left(1 + \frac{1}{2(N-1)}\right) = \prod_{n=1}^{N-1}\left(1 + \frac{1}{2(N-1)}\right).}$$

And further,

$$a_N = \prod_{1}^{N-1}\frac{2n+1}{2n} = \frac{3\cdot 5\cdots(2N-1)}{2^{N-1}(N-1)!} = \frac{(2N-1)!}{2^{N-1}2^{N-1}(N-1)!^2} = \frac{(2N-1)!}{4^{N-1}(N-1)!^2}.$$

Summary.

P_1 fits y_0 exactly at $x = x_0, x_1, x_2$.
P_2 fits y_0 exactly at $x = x_1, x_2, x_3$.

$$\begin{aligned}
y(x) &= P_1(x)W_1(x) + P_2(x)W_2(x),\\
W_2(x) &= \frac{1}{2}(1 + g_N(x)),\\
W_1(x) &= \frac{1}{2}(1 - g_N(x)),\\
u(x) &= \frac{2x - (x_1 + x_2)}{(x_2 - x_1)},\\
\frac{1}{a_N} &= \int_0^1 (1 - v^2)^{N-1}dv,\\
g_N(x) &= a_N \int_0^{u(x)} (1 - v^2)^{N-1}dv.
\end{aligned}$$

Special Cases.

$N = 1$: $\boxed{g_1 = u.}$

$N = 2$: $g_2 = a_2 \int_0^u (1 - v^2)dv = a_2 u(1 - u^2/3), \quad 1/a_2 = 2/3$

$$\boxed{g_2 = \frac{1}{2}u(3 - u^2).}$$

$N = 3$: $g_3 = a_3 \int_0^u (1 - 2v^2 + v^4) = a_3 u\left(1 - \frac{2}{3}u^2 + \frac{u^4}{5}\right), \quad \frac{1}{a^3} = \frac{1}{3} + \frac{1}{5} = \frac{8}{15},$

$$\boxed{g_3 = \frac{1}{8}u(15 - 10u^2 + 3u^4).}$$

Linear Least Squares with Erroneous Matrix

When one is dealing with a system in which the relationships between parameter changes, Δp, and the system performance changes, Δs, are in good approximation represented by the linear relationship

$$\Delta s = M \Delta p$$

achieving a desired performance change is simply accomplished by parameter changes

$$\Delta p = M^{-1} \Delta s$$

as long as one has as many parameters as system performance characteristics.

When the desired change, Δs, has more components than Δp, it is often adequate to minimize the weighted sum of the deviations from the desired performance, i.e. one minimizes

$$S = \Delta s^t W \Delta s,$$

where W is a diagonal square matrix with appropriate weights on the diagonal. S is minimized in the first iteration if the parameter vector is changed by

$$\boxed{\Delta p_1 = A \Delta s_1} \quad \text{where} \quad A = (M^t W M)^{-1} M^t W.$$

If the matrix M used for this operation deviates by $\Delta M = M_{\Re} - M$ from the real matrix $M_{\Re}$, the desired change Δs_2 with the new parameters is given by

$$\boxed{\Delta s_2 = \Delta s_1 - (M + \Delta M)\Delta p_1 = (I - MA - \Delta M A)\Delta s_1}$$

If the effort to determine $M_{\Re}$ (often by elaborate measurements) is too large one can iterate the procedure, and it would be of interest to estimate the asymptotic Δs_∞. To obtain this, we introduce

$$\boxed{I - MA = B} \quad \text{and} \quad \boxed{-\Delta M A = D}$$

Thus,

$$\boxed{\Delta s_n = (B + D)^{n-1} \Delta s_1} \quad \text{and} \quad \boxed{\Delta p_n = A(B + D)^{n-1} \Delta s_1}$$

Notice that $AM = I$, $AB = 0$, $DB = 0$ and $B^2 = I - 2MA + MAMA = B$.

November, 1971. Note 0038misc.

Further,

$$(B+D)^2 = B(I+D) + D^2$$

$$(B+D)^3 = B(I+D+D^2) + D^3$$

and so forth such that

$$\boxed{(B+D)^n = B(I-D)^{-1}(1-D^n) + D^n.}$$

Therefore,

$$\boxed{\Delta s_n = (B(I-D)^{-1}(1-D^{n-1}) + D^{n-1})\Delta s_1}$$

$$\boxed{\Delta p_n = AD^{n-1}\Delta s_1}$$

as it must be, because for $\Delta M = 0$ and $n \geq 2$, $\Delta p_n = 0$ and $\Delta s_n = \Delta s_2$.

If ΔM is small enough, the absolute values of the eigenvalues of D will be less than 1, resulting in the following for large enough n:

$$\boxed{\Delta s_\infty = B(I-D)^{-1}\Delta s_1 = (I-MA)(1+\Delta MA)^{-1}\Delta s_1.}$$

Judging whether one is close to this value is possible by observing the decrease in Δp_n with increasing n.

Matrix Describing Second Order Effects to Second Order in One Dimension

No Momentum Errors.

The normalized equation of motion is

$$y'' = y + by^2.$$

Expand y in terms of initial conditions y_0, y_0' up to 2^{nd} order:

$$y = a_{11}y_0 + a_{12}y_0' + a_{13}y_0^2 + a_{14}y_0y_0' + a_{15}{y_0'}^2.$$

Initial conditions for $a(x)$

$$a_{11}(0) = a_{12}'(0) = 1,$$

all others are 0. The equation for $a(x)$ is:

$$a_{11}'' = a_{11}, \quad \text{and} \quad a_{12}'' = a_{12} \quad \Longrightarrow \quad \begin{cases} a_{11} = \cosh x, & a_{12} = \sinh x, \\ a_{21} = \sinh x, & a_{22} = \cosh x. \end{cases}$$

$$a_{13}'' = a_{13} + ba_{11}^2, \quad a_{14}'' = a_{14} + 2ba_{11}a_{12}, \quad \text{and} \quad a_{15}'' = a_{15} + ba_{12}^2.$$

Because in all three cases $a(0) = a'(0) = 0$: $\mathcal{L}(a'') = p^2\mathcal{L}(a)$.

For a_{13},

$$a_{11}^2 = \frac{1}{4}\left(e^{2x} + e^{-2x} + 2\right) \rightleftharpoons \frac{1}{4}\left(\frac{1}{p-2} + \frac{1}{p+2} + \frac{2}{p}\right).$$

In general,

$$\frac{1}{(p-1)(p+1)(p+c)} \rightleftharpoons \frac{e^x}{2(1+c)} + \frac{e^{-x}}{2(1-c)} + \frac{e^{-cx}}{c^2-1},$$

thus,

$$\begin{aligned} \frac{4a_{11}^2}{p^2-1} &\rightleftharpoons \frac{4}{3}\cdot\frac{e^x}{2} + \frac{4}{3}\cdot\frac{e^{-x}}{2} + \frac{e^{2x}+e^{-2x}}{3} - 2 \\ &= \frac{4}{3}\cosh x + \frac{2}{3}\cosh 2x - 2. \end{aligned}$$

Therefore,

$$\boxed{a_{13} = \frac{b}{6}\left(2\cosh x + \cosh 2x - 3\right)} \quad \text{and} \quad \boxed{a_{13}' = a_{23} = \frac{b}{3}\left(\sinh x + \sinh 2x\right).}$$

February, 1966. Note 0006misc.

For a_{14},

$$a_{11}a_{12} = \frac{1}{4}\left(e^{2x} - e^{-2x}\right) \risingdotseq \frac{1}{4}\left(\frac{1}{p-2} - \frac{1}{p+2}\right),$$

and thus,

$$\begin{aligned}\frac{4a_{11}a_{12}}{p^2-1} &\risingdotseq \frac{-4}{3}\cdot\frac{e^{x}}{2} + \frac{4}{3}\cdot\frac{e^{-x}}{2} + \frac{e^{2x}-e^{-2x}}{3} \\ &= -\frac{4}{3}\sinh x + \frac{2}{3}\sinh 2x.\end{aligned}$$

Therefore,

$$\boxed{a_{14} = \frac{b}{3}\left(\sinh 2x - 2\sinh x\right)} \quad \text{and} \quad \boxed{a'_{14} = a_{24} = \frac{2b}{3}\left(\cosh 2x - \cosh x\right).}$$

Similarly, for a_{15},

$$a_{12}^2 = \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right) \risingdotseq \frac{1}{4}\left(\frac{1}{p-2} + \frac{1}{p+2} - \frac{2}{p}\right),$$

thus,

$$\begin{aligned}\frac{4a_{12}^2}{p^2-1} &\risingdotseq \frac{-8}{3}\cdot\frac{e^{x}}{2} + \frac{-8}{3}\cdot\frac{e^{-x}}{2} + \frac{e^{2x}+e^{-2x}}{3} + 2 \\ &= -\frac{8}{3}\cosh x + \frac{2}{3}\cosh 2x + 2.\end{aligned}$$

Therefore,

$$\boxed{a_{15} = \frac{b}{6}\left(\cosh 2x - 4\cosh x + 3\right)} \quad \text{and} \quad \boxed{a'_{15} = a_{25} = \frac{b}{3}\left(\sinh 2x - 2\sinh x\right).}$$

Inclusion of Momentum Error α.

The normalized equation of motion is

$$y'' = y + \alpha + by^2.$$

First, add the term linear in α to the expansion in y_0, y_0': add $a_{16}\alpha$. The initial conditions are

$$a_{16}(0) = a_{16}'(0) = 0, \quad a_{16}'' = a_{16} + 1 \quad \Longrightarrow \quad \boxed{a_{16} = \cosh x - 1.}$$

Second, take the terms α^2, αy_0, $\alpha y_0'$ into account, where the procedure is the same as in the calculation of a_{13}, a_{14}, and a_{15}.

Third, do not add any terms, but introduce $z = y + \beta$ (β is a constant) in the differential equation. Thus,

$$z'' = z - \beta + \alpha + b(z^2 - 2z\beta + \beta^2),$$

$$\beta^2 - \frac{\beta}{b} + \frac{\alpha}{b} = 0 \quad \Longrightarrow \quad \beta = \frac{1}{2b}\left(1 - \sqrt{1 - 4\alpha\beta}\right),$$

$$z(1 - 2b\beta) + bz^2 = \boxed{z'' = z\sqrt{1 - 4\alpha\beta} + bz^2.}$$

This procedure requires the calculation of a new matrix for every α of interest, but this will give more insight in return.

General Procedure.

Description of higher order effects with power expansion and the consequences for stability.

We describe deviations from the closed orbit by the column vector

$$\mathbf{v} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}$$

where the components of y_1 are y and y', and components of $y_2, y_3, \ldots, y_k$ are, respectively, the second, third, $\ldots$, k^{th} order contributions of y and y'. Then,

$$\mathbf{v}_2 = M\mathbf{v}_1, \quad \text{with} \quad M = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \ldots & A_{1k} \\ 0 & A_{22} & A_{23} & \ldots & A_{2k} \\ 0 & 0 & A_{33} & \ldots & A_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & A_{kk} \end{pmatrix}.$$

In M, A_{11} describes the first order effects, A_{12} the second order effects, etc. The other matrices reproduce the higher than first order components of $\mathbf{v}$. The diagonal elements A_{22}, $A_{33}, \ldots, A_{kk}$ depend only on the matrix element of A_{11}. The eigenvalues of M do not depend on $A_{12}, \ldots, A_{1k}$. Thus, the stability of the system does not depend, in this approximation, on the non-linear effects described by these elements. Since stability obviously can depend on non-linear effects, this implies that the power expansion for many passes through the system has a progressively shrinking radius of convergence. One can thus conclude that although this method is worthless to evaluate the effect of non-linearities on stability, it might still yield valuable information provided the system does not become unstable because of the non-linearities.

We show rough numbers for

$$F_3 = a_{13}/b, \quad F_4 = a_{14}/b, \quad F_5 = a_{15}/b,$$

$$F_3' = a_{13}'/b = a_{23}/b, \quad F_4' = a_{14}'/b = a_{24}/b, \quad F_5' = a_{15}'/b = a_{25}/b.$$

x	$\pi/4$	$\pi/2$	π
e^x	2.2	4.8	23
e^{-x}	0.46	0.21	0
$\cosh x$	1.33	2.5	11.5
$\sinh x$	0.87	2.3	11.5

x	$\pi/4$	$\pi/2$
$F_3(x)$	0.77	2.25
$F_4(x)$	0.18	2.3
$F_5(x)$	0.03	0.75

x	$\pi/4$	$\pi/2$
$F_3'(x)$	1.06	4.6
$F_4'(x)$	0.78	6
$F_5'(x)$	0.18	2.3

Curvature of 2D Magnetic Field Lines and Scalar Potential Lines

I. Preparation and Background.

Magnetic fields. 2D magnetic fields can be derived from a scalar potential V or a vector potential A, or the complex potential $F(z) = A + iV$, an analytic function of the complex variable $z = x + iy$, according to

$$B_x - iB_y = B^* = i\frac{dF}{dz} = iF'. \tag{1}$$

Field lines and scalar potential lines in the z-plane are the $z(F)$ maps of straight lines parallel to either the imaginary or real axis of the F-plane.

Modification of the curvature by a conformal map. If a curve in the z-plane has a local tangent in the direction $e^{i\alpha_z}$, the conformal map $w(z)$ of that region has a local tangent in the direction

$$e^{i\alpha_z} = e^{i\alpha_z}\frac{w'}{|w'|}. \tag{2}$$

This equation shows that the angle of intersection of any two curves in the z-plane is preserved under the transformation $w(z)$, hence the name conformal transformation. If the curve at that location in the z-plane is k_z, then the curvature of the map of that point can be shown to be

$$k_w = \frac{\left(k_z + \Im\left(\frac{w''}{w'}e^{i\alpha_z}\right)\right)}{|w'|}. \tag{3}$$

The sign convention used for this formula is such that a positive curvature means that if one proceeds in the direction of the tangent, the curve turns to the left, i.e. the conventional mathematically positive direction.

II. Application of (2).

Fundamental relationships. There are several ways to apply (2) to this problem. The most natural way to do so seems to be, at least at first, to assign quantities w and z in (2) to the the variables F and z of our problem, since we are looking at the map of a region of the F-plane to the z-plane. For most applications, this is not very practical since one then gets the curvature of the maps of constant potential lines as a function of A and V, when in fact one wants the curvature as a function of x and

January, 1994. Note 0611thry.

y. We therefore proceed in the following manner: we assign z and w to z and F, and look in

$$k_F = \frac{\left(k_z + \Im\left(\frac{F''}{F'}e^{i\alpha_z}\right)\right)}{|F'|}, \tag{4}$$

for $k_F = 0$, i.e. the curvature of maps of straight lines in the F-plane is given by

$$k_z = -\Im\left(\frac{F''}{F'}e^{i\alpha_z}\right). \tag{5}$$

To get a more practical formula, we express $e^{i\alpha_z}$ with the help of (2) through

$$e^{i\alpha_z} = e^{i\alpha_F}\frac{|F'|}{F'}, \tag{6}$$

and the derivatives of F through the fields as given in (1), yielding

$$k_z = -\Re\left(\frac{B^{*'}}{B^{*2}}|B|e^{i\alpha_F}\right). \tag{7}$$

For some expressions of the fields, it is more convenient to write this as

$$k_z = +\Re\left(\left(\frac{1}{B^*}\right)'|B|e^{i\alpha_F}\right). \tag{8}$$

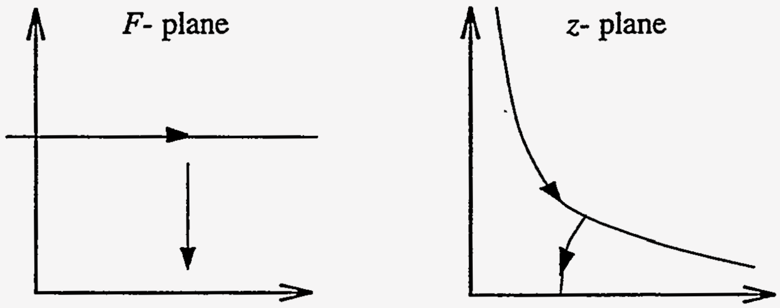
In both (7) and (8), $e^{i\alpha_F}$ has the absolute value 1 and is real if one is looking at a scalar potential line, and is purely imaginary if one looks at a field line.

Comments. It is worth noting that in order to calculate the curvatures of interest, one needs only the expressions for the complex field, not the complex potential. Under most circumstances, the expression for the complex potential is not more complicated than the expression for the complex field. There are, however, exceptions. For instance, the field of a modified sextupole is given by

$$B^* = iz^2e^{az^2}. \tag{9}$$

Integrating this to get the complex potential, (1), leads to the error function in the complex plane.

III. Applications.



Figures 1(a,b).

(i) *The regular multipole.* For a multipole of order n with the field perpendicular to the midplane, the field is given by

$$B^* = iz^{n-1}. \tag{10}$$

Substituting in (8) gives directly

$$k_z = (n-1)\left|z^{n-1}\right| \Re\left(iz^{-n}e^{i\alpha_w}\right). \tag{11}$$

Using, for $e^{i\alpha_F}$, the phases corresponding to the arrows in Figures 1(a) and 1(b), and using $z = re^{i\varphi}$, gives, for the curvature of the field line and the scalar equipotential:

$$k_z = (n-1)\cos\frac{n\varphi}{r}, \tag{12A}$$

$$k_z = (n-1)\sin\frac{n\varphi}{r}. \tag{12V}$$

(ii) *The modified sextupole.* This particular implementation of a modified sextupole has the field in the midplane perpendicular to the midplane, and behaves like a good sextupole close to the origin, but has a stronger modified field, proportional to $x^2e^{ax^2}$, $a \in \Re$, as one moves away from the origin of the coordinate system. The complex field is therefore given by

$$B^* = ize^{ax^2}.$$

Fringe Field Model Function for Dipoles

For a number of beam optics tasks, it is important to have an analytical function that describes the field in the fringe field region of a dipole[†]. We restrict ourselves to the simple case of a dipole that has a straght effective field boundary, making this a very simple problem of describing two dimensional fields. Putting the x-axis into the midplane of a dipole whose half gap is normalized to be equal to 1, with large $x > 0$ describing the outside of the magnet, and the far negative end of the x-axis the deep inside region of the magnet, the field in the region of interest can be described by

$$\frac{B_y(x,y) + iB_x(x,y)}{B_0} = G(z) = D_1(z) + D_2(z) + D_3(z), \tag{0.0}$$

$$z = x + iy, \tag{0.1}$$

and the functions D_1, D_2, D_3 chosen such that the asymptotic behavior of $G(z)$ reflects the properties of the fields in the regions deep inside and far outside the magnet. In addition, $G(z)$ should not have any singularities for the space within $-1 < y < 1$. The following functions satisfy these conditions:

$$D_1(z) = \frac{1 + nAe^{\pi z/2}}{\left(1 + Ae^{\pi z/2}\right)^n}, \tag{1}$$

$$D_2(z) = C_1 e^{-C_2(z-x_2)^2}, \tag{2}$$

$$D_3(z) = K_1 \cdot \frac{\left(1 + K_3 e^{-\pi z/2}\right)^m}{\left(1 + K_2(z - x_3)^2\right)^{3/2}}, \tag{3}$$

with all coefficients real, $n \geq 2$, $K_2 > 1$, and $A, C_2, K_3, m > 0$. The fields deep inside the magnet are dominated by the "longest surviving" term $e^{\pi z}$ from $D_1(z)$, while far outside the magnet the field is dominated, as desired, by the "longest surviving" term proportional to $1/z^3$ from D_3, with clearly no singularities for $-1 < y < 1$. $D_2(z)$ has been added (and one could add more such terms) to allow a good fit of $G(z)$ to measured or computed data in the transition region. While this suggested model function $G(z)$ has enough free parameters to fit data, the quality of such a fit has not been tested on a real problem, but the $G(z)$ given here should contain a sufficient number of suggestions that this approach to the Enge function promises to be successful.

December, 1993. Note 0610thry.

† See document 0609thry, *Comments about RAYTRACE.*

Comments about RAYTRACE

Introduction.

Several years ago I was asked at a workshop to comment on the representation of magnetic fields in the RAYTRACE code, a computer program that was developed by H. A. Enge and his students in the 1960's†. Since my comments contained not only some academically interesting points, but also suggestions for improvement of this enormously successful code, several people asked me to put my thoughts on this subject on paper. After describing the specific aspects of the code that I want to discuss, I will elaborate on what I would characterize as shortcomings, together with suggestions for eliminating them, and a description of some mathematical detail at the end.

Fields in RAYTRACE.

Even though it is not a major effort to generalize my comments, I restrict the discussion to the case of the fringe field region of a dipole magnet that has a straight effective field boundary in the region of interest. This means that we are dealing with two dimensional fields, with all the associated simplifications that make it possible to address the core of the problem without unnecessary distractions.

Using the midplane of the magnet as the x-axis of the xy coordinate system, with large positive x representing the region far outside the dipole, and the other extreme the region deep inside the magnet, the field is characterized by the following function, commonly called the Enge function :

$$B_y(x,0) = \frac{B_0}{1+e^{P(x)}}, \tag{1a}$$

where

$$P(x) = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n, \tag{1b}$$

with n an odd integer and $C_n > 0$.

The coefficients are obtained by fitting measured or computed field values in the midplane to (1), and fields off the midplane are obtained by using a Taylor series expansion, with the derivatives obtained from (1).

Comments and Suggestions.

I have problems with three tightly linked aspects of this procedure:

December, 1993. Note 0609thry.

† Spencer, J. E. and Enge, H. A., Nuclear Instruments and Methods 49, 181 (1967).

(A) It is true, in general, that if one fits parameters of a function so that the fields on the surface of a volume are well represented by that function, the quality of the fields computed with that function inside the volume is at least as good as (but usually better than) the original data on the surface. It is, of course, assumed that the function and field calculation algorithm satisfy all the relevant vacuum field equations. Conversely, calculating fields in the volume from a function whose free parameters were determined on a line inside the volume gives fields that are not nearly as accurate as the original data. These facts are qualitatively clear if one thinks of the fringe fields in the midplane of the dipole: significantly different pole contours produce very similar fields in the midplane. That means that if one calculates fields off the midplane accurately from the fields in the midplane, small differences in the function there will give significantly different fields far away from the midplane.

(B) Calculating fields off the midplane with a Taylor series expansion makes no sense in this case for the following reasons: since $B_x - iB_y$ or, more conveniently in this case, $B_y + iB_x$, is an analytical function of the complex variable $z = x + iy$, the field at location (x, y) can be obtained directly, without any approximation, by evaluating (1) for complex argument:

$$B_y(x,y) + iB_x(x,y) = B_y(x+iy, 0) = \frac{B_0}{1 + e^{P(x+iy)}}. \tag{2}$$

This very simple evaluation of fields from a midplane model function makes it obviously easy to fit the parameters of the model, no matter the nature of that function, to fields off the midplane, thus eliminating the objection raised in (A).

(C) It seems to me that the form of the Enge function is not well suited to this problem for two reasons: 1) the function does not have ther appropriate asymptotic behavior far away from either end of the magnet; and 2) unless one makes a careful study of the Enge function, it may have one or more singularities in the "business" region. Avoiding that kind of disaster by evaluating the field only approximately is clearly not a satisfactory answer to this problem. While it is fairly easy with the help of (2) to make the singularity check (see Appendix), it might be simpler to "design" a function that can not have that kind of singularity, in addition to having the proper asymptotic behavior. I have some very promising candidates but have not made the effort to test them on some real problems.

Appendix.

For the Enge function to have no damaging singularity it is necessary and sufficient that the equation

$$P(z) = im\pi \quad \text{with} \quad m = \text{odd integer} \neq 0 \tag{3}$$

has no solution for z between the midplane and a line parallel to the midplane one half gap, h, away from the midplane. This test is most easily carried out with the argument principle that states, in this case, that the number of zeroes of $w(z)$ within a region of the z-plane equals the number of times $w(z)$ goes around $w = 0$ when z traces the boundary of the region. Since, in this case,

$$w(z) = P(z) + im\pi, \tag{4}$$

with all C_n in $P(z)$ real, it is only necessary to find the locations where the map of the straight line parallel to the midplane at distance h intersects the imaginary axis of P(z), i.e. one has to find $\Im P(z)$ at the locations where $\Re P(x + ih) = 0$. Since $\Re P(x+ih) = 0$ means nothing more than finding the real roots (in x) of a polynomial of order n, this a very simple exercise for a computer. Having these points, it is trivial to see whether $w(z) = 0$ is possible for any odd m. I have carried out that test for the example given by Spencer and Enge, and for four cases given to me by S. Kowalski. I am happy to report that while none of these cases had singularities within one half gap of the midplane, there were some singularities just outside the end of the dipole only a little more than a half gap away from the midplane.

Stored Energy in H-Magnet for $\mu = \infty$

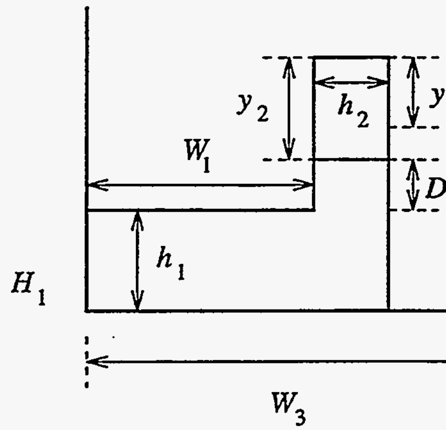


Figure 1.

$$2\mathcal{E} = \int \mathbf{B} \cdot \mathbf{H} dv = \int \mathbf{H} \cdot \nabla \times \mathbf{A} dv.$$

From

$$\nabla \cdot (\mathbf{A} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{H}),$$

we have that, with $\mathbf{j} = j\mathbf{e}_z$,

$$\begin{aligned} 2\mathcal{E} &= \int \mathbf{A} \cdot \mathbf{j} dv + \int (\mathbf{A} \times \mathbf{H}) \cdot d\mathbf{a} \\ &= \int \mathbf{A} \cdot \mathbf{j} dv + \int \mathbf{A} \cdot (\mathbf{H} \times d\mathbf{a}), \end{aligned}$$

where $\mathbf{H} \times d\mathbf{a} = 0$ on $\mu = \infty$ surface.

In case of a long magnet, $\int j \, da \equiv 0$ which means that we can add any constant to A without changing anything. We make $A = 0$ along the y-axis. We now use $\mathbf{A} = \mu_0 A \mathbf{e}_z$, so that the total energy per unit length is

$$\boxed{\mathcal{E}' = \frac{1}{2}\mu_0 \, j \int A dx dy}$$

where the integral is evaluated over the coil in the first quadrant.

January, 1993. Note 0607thry.

To get

$$J_1 = \int A(x,y)dxdy$$

we look at

$$J_2(y) = \int_0^{y<y_2} \int_0^{h_2} H_x dxdy = \int\int \frac{\partial A}{\partial y} dydx$$
$$= \int \left(A(x,y) - A_t\right) dx = \int A(x,y)dx - A_t h_2,$$

where A_t is A at top of the coil slot.

We integrate the original expression for J_2 over x first, and by Ampère's Law,

$$J_2(y) = \int_0^y \frac{I}{y_2} y dy = \frac{Iy^2}{2y_2}.$$

Therefore,

$$\int A(x,y)dx = \frac{Iy^2}{2y_2} + A_t h_2,$$

$$\boxed{J_1 = \int_0^{y_2} \int A(x,y)dxdy = \frac{Iy_2^2}{6} + A_t h_2 y_2.}$$

From *H-Magnet With Minimal Yoke Flux Density*† we know that

$$A_t = I\left(\frac{W_1}{h_1} + \frac{D + y_2/2}{h_2} + E_1\right),$$

and thus we have that

$$J_1 = Ih_2 y_2 \left(\frac{W_1}{h_1} + \frac{D + y_2/2}{h_2} + E_1 + \frac{y_2}{6h_2}\right).$$

$$\boxed{jJ_1 = I^2\left(\frac{W_1}{h_1} + \frac{2y_2}{3h_2} + \frac{D}{h_2} + E_1\left(\frac{h_1}{h_2}\right)\right).}$$

$$\boxed{\mathcal{E}' = \frac{1}{2}\mu_0(jJ_1),}$$

† Document 0606thry.

where

$$I = H_1 h_1 = j h_2 y_2,$$

$$E_1(a) = a + \frac{2}{\pi}\left(\ln\frac{a+1/a}{4} + \left(\frac{1}{a} - a\right)\arctan a\right) \quad \text{with} \quad a = h_2/h_1.$$

H-Magnet With Minimal Yoke Flux Density

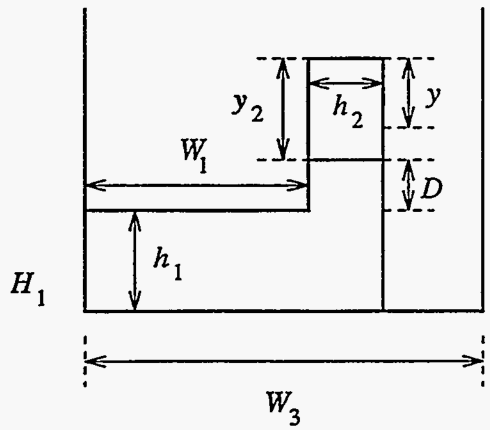


Figure 1.

W_1, H_1, W_3, D, and h_1 are given. We want to minimize $\overline{B}_{\text{yoke,max}}$ for $\mu = \infty$ by chosing the proper h_2.

Procedure.

Calculate flux for 0-thickness coil at top of coil slot using excess flux coefficient E_1 for corner. Subtract "window frame flux" from combination of real coil and 0-thickness coil.

For $V = H_1 h_1 = j h_2 y_2$ we chose h_2 and j. Thus,

$$A = V \left(\frac{W_1}{h_1} + \frac{D + y_2}{h_2} + E_1 \left(\frac{h_1}{h_2} \right) - \frac{y_2}{2h_2} \right) \mu_0$$

$$= \boxed{\mu_0 V \left(\frac{W_1}{h_1} + \frac{D + y_2/2}{h_2} + E_1 \left(\frac{h_1}{h_2} \right) \right)}$$

$$\boxed{\overline{B}_{\text{yoke,max}} = \frac{A}{W_3 - W_1 - h_2}.}$$

We determine the minimum value of $\overline{B}_{\text{yoke,max}}$ by varying h_2, and we define

$$E_1(a) = a + \frac{2}{\pi} \left(\ln \frac{a + 1/a}{4} + \left(\frac{1}{a} - a \right) \arctan a \right) \quad \text{with} \quad a = h_2/h_1.$$

January, 1993. Note 0606thry.

Dipole with Small Gap Bypass

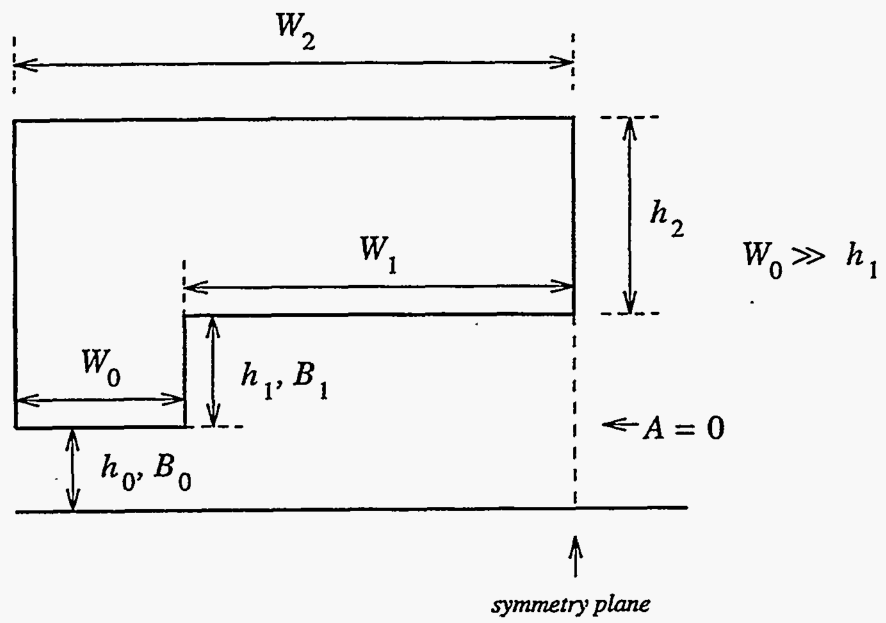


Figure 1.

$$\boxed{B_0 h_0 + h_1 \mu_0 H(B_0) = B_1(h_1 + h_0)} \tag{1}$$

For $W_0/W_2 = \varepsilon_0$,

$$B_2 W_2 = B_0 W_0 + B_1 W_1 \quad \text{and thus} \quad B_2 = B_0 \varepsilon_0 + B_1(1 - \varepsilon_0)\,,$$

$$V_{00} = \frac{B_1}{\mu_0}(h_0 + h_1) + h_2 H(B_0 \varepsilon_0 + B_1(1 - \varepsilon_0))$$

The exact equation

$$\boxed{\mu_0 V_{00} = B_1(h_0 + h_1) + \mu_0 h_2 H(B_0 \varepsilon_0 + B_1(1 - \varepsilon_0))} \tag{2}$$

has the following implementations

$$B_1 \to B_0 \quad \text{and then} \quad B_1, B_0 \to V_{00}.$$

We will now examine three special cases.

June, 1993. Note 0591thry.

In **Case 1**, V_{00} is so small that $\mu_0 H(B) = \gamma B$,

$$B_0(h_0 + \gamma h_1) = B_1(h_0 + h_1),$$

$$\boxed{\mu_0 V_{00} = B_1 \underbrace{\left(h_0 + h_1 + \gamma h_2 \left(\varepsilon_0 \frac{h_0 + h_1}{h_0 + \gamma h_1} + 1 - \varepsilon_0\right)\right)}_{K},}$$

where

$$\frac{h_0 + h_1}{h_0 + \gamma h_1} = 1 + \frac{(1-\gamma)h_1}{h_0 + \gamma h1} \quad \text{and} \quad K'_{h_0} = 1 - \frac{\gamma h_1 \varepsilon_0 h_2 (1-\gamma)}{(h_0 + \gamma h_1)^2},$$

$$B_1 = \frac{\mu_0 V_{00}}{K}, \quad B_1' = \frac{-\mu_0 V_{00}}{K^2} K' = \frac{\mu_0 V_{00}}{K^2} \left(\frac{\gamma h_1 \varepsilon_0 h_2 (1-\gamma)}{(h_0 + \gamma h_1)^2} - 1\right).$$

For $h_0 \ll \gamma h_1$:

$$B_1' = \frac{\mu_0 V_{00}}{K^2} \left(\frac{\varepsilon_0 h_2 (1-\gamma)}{\gamma h_1} - 1\right) > 0.$$

$K' = 0$ for $h_0 + \gamma h_1 = \sqrt{\varepsilon_0 \gamma h_1 h_2 (1-\gamma)}$ such that

$$h_0 = \gamma h_1 \left(\underbrace{\sqrt{\frac{\varepsilon_0 h_2 (1-\gamma)}{\gamma h_1}}}_{\gg 1} - 1\right) \approx \sqrt{\varepsilon_0 h_1 h_2 (1-\gamma)\gamma}.$$

For $h_1 = 1$, $h_2 = 5$, $\varepsilon_0 = 1/2$, $\gamma = 10^{-3}$, we have

$$h_0 = \sqrt{1/2 \cdot 1 \cdot 5 \cdot 10^{-3}} = \boxed{\frac{1}{20}\,\text{cm}.}$$

In **Case 2**, we need V_{00} large enough so that $B_0 \approx B_s$, but small enough so that for (2) $\mu_0 H(B) = \gamma B$, thus

$$\mu_0 V_{00} = B_1(h_0 + h_1) + B_1 h_2 \gamma (1 - \varepsilon_0) + B_s h_2 \gamma \varepsilon_0,$$

where

$$\boxed{B_1 = \frac{\mu_0 V_{00} - B_s \gamma h_2 \varepsilon_0}{h_0 + h_1 + h_2 \gamma (1 - \varepsilon_0)}.}$$

In a third, simple case, **Case 3**, for a still higher V_{00},

$$B_s \varepsilon_0 + B_1(1 - \varepsilon_0) \to B_s,$$

i.e. it is independent of V_{00}, and thus

$$\boxed{B_1 = B_s.}$$

Boundary Condition at Iron-Air Interface for AC and Application to 2-Dimensional Cylinder

Interface at $y = 0$,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0 \quad \text{and} \quad \frac{\partial}{\partial y} = {}'.$$

$$\mathbf{H} = H\mathbf{e}_x, \quad \mathbf{j} = \mathbf{e}_z \sigma E \quad \text{and} \quad \mathbf{E} = \mathbf{e}_z E.$$

$$(\nabla \times \mathbf{H})_z = -H' = \sigma E \quad \text{and} \quad E = -\varrho H'.$$

$$(\nabla \times \mathbf{E})_x = E' = -\mu_0 \mu p H = -\varrho H'' \quad \text{and} \quad H'' - k^2 H = 0$$

where, depending on the application, p is either the Laplace transform variable or, for sinusoidal excitation, $i\omega$.

$$\boxed{k = \sqrt{\sigma \mu_0 \mu p} = \frac{1}{D_1}}$$

$$H = H_0 e^{-ky} \quad \text{and} \quad \Phi = \mu_0 \mu \int_0^\infty H dy = \frac{H_0 \mu_0 \mu}{k} = \mu_0 H_0 \mu D_1.$$

Therefore, with

$$\boxed{\mu D_1 = D_2}$$

$$\frac{d\Phi}{dx}\Delta x = \mu_0 D_2 \frac{\partial H_0}{\partial x}\Delta x = \mu_0 H_y \Delta x,$$

$$H_y = D_2 \frac{\partial H_0}{\partial x} \quad \Longrightarrow \quad \boxed{H_\perp = D_2 \frac{\partial H_\|}{\partial s_\|}}.$$

Given an iron cylinder with D_2, of radius 1, in far-field $\mathbf{H} = H_\infty \mathbf{e}_x$, we try to solve for the complex potential F. Ansatz: the superposition of the macro and micro dipoles, with normalized units.

$$F = -iH_1(z + 1/z) - iH_2(z - 1/z).$$

with $z = x + iy$, normalized with radius r_0 of the cylinder.

May, 1988. Note 0492thry.

On $|z| = 1$

$$F = A + iV = 2H_2 \sin\varphi - i2H_1 \cos\varphi,$$

$$H^* = iF' = H_x - iH_y \quad \text{and} \quad \mathcal{H} = H_r + iH_\varphi = He^{-i\varphi},$$

$$\mathcal{H}^* = H^* e^{i\varphi}.$$

and the boundary condition is, with

$$H_\| = H_\varphi \quad \text{and} \quad H_\perp = -H_r,$$

$$\boxed{H_r = -D_2 \frac{\partial H_\varphi}{\partial \varphi}}$$

$$H^* = H_1(1 - 1/z^2) + H_2(1 + 1/z^2) = e^{-i\varphi}\mathcal{H}^*.$$

On the surface,

$$\mathcal{H}^* = H_r - iH_\varphi = 2iH_1 \sin\varphi + 2H_2 \cos\varphi,$$

$$\boxed{H_r = 2H_2 \cos\varphi} \quad \text{and} \quad \boxed{H_\varphi = 2H_1 \sin\varphi}$$

and the boundary condition is, with

$$2H_2 \cos\varphi = D_2 2H_1 \sin\varphi + 2H_2 \cos\varphi \quad \text{and} \quad H_2 = D_2 H_1,$$

$$H_\infty = H_1 + H_2 = H_1(1 + D_2),$$

$$H_1 = \frac{H_\infty}{1 + D_2}, \quad H_2 = \frac{H_\infty D_2}{1 + D_2},$$

$$\boxed{H_r = 2H_\infty \frac{D_2}{1 + D_2} \cos\varphi} \quad \text{and} \quad \boxed{H_\varphi = -2H_\infty \frac{\sin\varphi}{1 + D_2}}$$

normalized with radius r_0 of the cylinder.

Using SI units, we choose $\sigma\mu_0 = 10$, $\mu = 10^4$ and $\omega = 2\pi \cdot 60\text{Hz}$, and therefore we have

$$|D_2| = \sqrt{\frac{\mu}{\sigma\mu_0\omega}} = \frac{10^2}{\sqrt{10^4(.12\pi)}},$$

$$\boxed{|D_2| = 1.6\text{m}}$$

$$\boxed{|D_1| = .16\text{mm}} \quad \text{and} \quad |D_1|\sqrt{2} = .23\text{mm}.$$

For sinusoidal excitation,

$$D_2 = |D_2|\frac{(1-i)}{\sqrt{2}},$$

$$\boxed{|1 + D_2| = \sqrt{\left(1 + \frac{|D_2|}{\sqrt{2}}\right)^2 + \frac{|D_2|^2}{2}} = \sqrt{1 + |D_2|^2 + |D_2|\sqrt{2}}\,.}$$

Normalized, where r_0 is the radius of the cylinder:

$$D_2 = \frac{D_2(\text{m})}{\text{r}_0(\text{m})}\,.$$

That is, for same material and frequency, $|D_2|$ is large for a small cylinder and $|D_2|$ is small for a large cylinder.

Unfortunately, if $H_\perp = D_2 \dfrac{\partial H_\parallel}{\partial s_\parallel}$ is valid in z-geometry, it is not satisfied in conformally mapped w-geometry, i.e. dealing with this problem in mapped geometry is not practical.

Flux Into A Rectangular Box

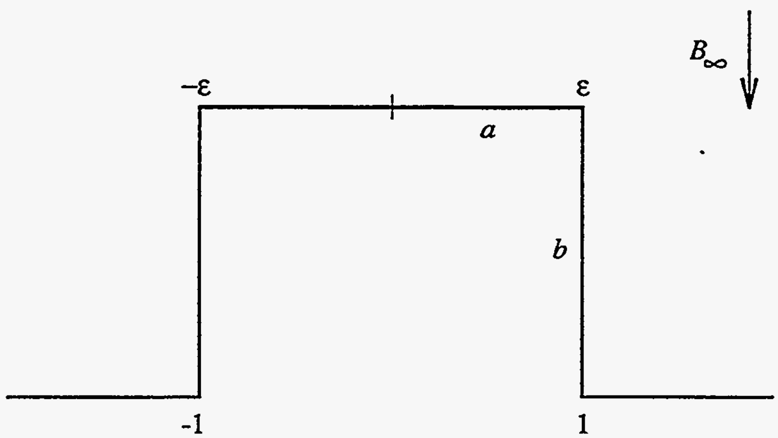


Figure 1.

$$\dot{z} = c\frac{\sqrt{t^2-\varepsilon^2}}{\sqrt{t^2-1}}$$

For

$$t = \varepsilon\sin\varphi, \quad dt = \varepsilon\cos\varphi\, d\varphi \quad \sqrt{1-\varepsilon^2} = \varepsilon_1\,,$$

we have

$$\frac{a}{c} = \int_0^{\varepsilon} \frac{\sqrt{\varepsilon^2-t^2}}{\sqrt{1-t^2}} = \varepsilon^2 \int_0^{\pi/2} \frac{\cos^2\varphi\, d\varphi}{\sqrt{1-\varepsilon^2\sin^2\varphi}} = \int_0^{\pi/2} \frac{(1-\varepsilon^2\sin^2\varphi)+(\varepsilon^2-1)}{\sqrt{1-\varepsilon^2\sin^2\varphi}}\, d\varphi$$

$$\frac{a}{c} = \int_0^{\pi/2} \sqrt{1-\varepsilon^2\sin^2\varphi}\, d\varphi - \varepsilon_1^2 \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\varepsilon^2\sin^2\varphi}}$$

$$\boxed{\frac{a}{c} = E(\varepsilon^2) - \varepsilon_1^2 K(\varepsilon^2)\,.}$$

For

$$t = \cos\varphi, \quad dt = -\sin\varphi\, d\varphi, \quad \varepsilon = \cos\alpha, \quad \varepsilon_1 = \sin\alpha,$$

May, 1988. Note 0491thry.

$$\sin\varphi = \varepsilon_1 \sin\psi, \quad d\varphi = \frac{\varepsilon_1 \cos\psi \, d\psi}{\sqrt{1-\varepsilon_1^2\sin^2\psi}},$$

we have

$$\frac{b}{c} = \int_\varepsilon^1 \frac{\sqrt{t^2-\varepsilon^2}}{\sqrt{1-t^2}}\, dt = \int_0^\alpha \sqrt{\varepsilon_1^2 - \sin^2\varphi}\, d\varphi = \varepsilon_1^2 \int_0^{\pi/2} \frac{\cos^2\psi}{\sqrt{1-\varepsilon_1^2\sin^2\psi}}\, d\psi$$

$$\boxed{\frac{b}{c} = E(\varepsilon_1^2) - \varepsilon^2 K(\varepsilon_1^2)\,.}$$

Thus

$$\boxed{\frac{a}{b} = \frac{E(\varepsilon^2) - \varepsilon_1^2 K(\varepsilon^2)}{E(\varepsilon_1^2) - \varepsilon^2 K(\varepsilon_1^2)}\,.}$$

For

$$F(t) = cB_\infty t, \quad F' = B_\infty \frac{\sqrt{t^2-1}}{\sqrt{t^2-\varepsilon^2}}, \quad \boxed{B_0 = \frac{B_\infty}{\varepsilon}\,.}$$

and therefore

$$\boxed{F(\varepsilon) = \frac{aB_\infty \varepsilon}{E(\varepsilon^2) - \varepsilon_1^2 K(\varepsilon^2)}\,,}$$

$$\boxed{F(1) = \frac{aB_\infty}{E(\varepsilon^2) - \varepsilon_1^2 K(\varepsilon^2)}\,.}$$

Given a square box, with dimensions $\varepsilon^2 = 1/2$, $E(1/2) = 1.3506$, $K(1/2) = 1.8541$,

$$F(\varepsilon) = F(\sqrt{1/2}) = 1.67aB_\infty, \quad F(1) = 2.361aB_\infty, \quad B_0 = 1.41B_\infty.$$

Propagation of Fast Perturbation in Dipole

We describe the boundary condition as

$$H_y(h) = D_2 \frac{\partial H_x}{\partial x} \quad \text{with} \quad D_2 = \mu D_1 = \frac{\mu}{\sqrt{i\omega\sigma\mu_0\mu}} .$$

Ansatz:

$$\boxed{H_y(x,y) = \sum a_n \cos k_n y \, e^{-k_n x}}$$

where we look to satisfy the $H_y(-y) = H_y(y)$ symmetry only.
$\nabla^2 H_y = 0$ is obviously satisfied.

$$\frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x} = -\sum a_n k_n \cos k_n y \, e^{-k_n x}$$

$$\boxed{H_x = -\sum a_n \sin k_n y \, e^{-k_n x} .}$$

At the boundary we have

$$\sum a_n \cos k_n h \, e^{-k_n x} = \sum a_n D_2 k_n \sin k_n h \, e^{-k_n x} .$$

For

$$D_2 k_n \tan k_n h = 1 \quad \text{and} \quad \boxed{\alpha_n = k_n h ,}$$

we therefore have

$$\boxed{\alpha_n \tan \alpha_n = \frac{h}{D_2}}$$

where

$$\frac{1}{D_2} = \sqrt{\frac{i\omega\sigma\mu_0}{\mu}} .$$

November, 1987. Note 0489thry.

Case 1: "normal" case, $\mu \to \infty$ $\alpha_n = n\pi$.
Case 2: superconducting case, $\sigma \to \infty$ $\alpha_n = (n + 1/2)\pi$.
Case 3: using iron with $\omega = 2\pi 60\text{Hz}$, and given that $D_1 = \dfrac{1}{\sqrt{i\omega\sigma\mu_0\mu}}$,

$$|D_1| = \frac{1}{\sqrt{2\pi \cdot 60 \cdot 10^{1+3}}} = 5.2 \cdot 10^{-4}\text{m} = .52\text{mm}$$

$$|D_2| = \mu|D_1| = 52\text{cm}\,.$$

We introduce

$$\alpha_0 \tan\alpha_0 = \frac{h}{D_2} = \varepsilon = h\sqrt{\frac{i\omega\sigma\mu_0}{\mu}}$$

and for $|\varepsilon| << 1$, $\boxed{\alpha_0 \approx \sqrt{\varepsilon}\,.}$ Thus, for $\alpha_n = n\pi + \delta_n$,

$$(n\pi + \delta_n)\tan\delta_n = \varepsilon \quad\Longrightarrow\quad \boxed{\delta_n \approx \frac{\varepsilon}{n\pi}\,.}$$

For a better notation of α_0 we have that, for

$$\alpha_0^2 + \alpha_0^4/3 = \varepsilon \quad\Longrightarrow\quad \alpha_0^2 = -3/2 + \sqrt{9/4 + 3\varepsilon}\,.$$

and it follows that

$$\boxed{\alpha_0^2 = \frac{3\varepsilon}{\frac{3}{2} + \sqrt{\frac{9}{4} + 3\varepsilon}} = \frac{2\varepsilon}{1 + \sqrt{1 + \frac{4\varepsilon}{3}}}\,,}$$

$$\boxed{\alpha_0^2 = \frac{2\varepsilon}{2 + \frac{2\varepsilon}{3}} = \frac{\varepsilon}{1 + \frac{\varepsilon}{3}}\,.}$$

To determine a_n from $H_y(y)$ at $x = 0$ we try

$$\int_0^h H_y(y)\cos k_m y\,dy = \sum a_n \int_0^h \cos k_n y\,\cos k_m y\,dy\,.$$

Since $2\cos k_n y\ \cos k_m y = \cos(k_n + k_m)y + \cos(k_n - k_m)y$,

$$\begin{aligned}
\frac{2}{h}\int_0^h \cos k_n y\ \cos k_m y\, dy &= \frac{\sin(\alpha_n + \alpha_m)}{\alpha_n + \alpha_m} + \frac{\sin(\alpha_n - \alpha_m)}{\alpha_n - \alpha_m} \\
&= \frac{\alpha_n(\sin(\alpha_n + \alpha_m) + \sin(\alpha_n - \alpha_m))}{\alpha_n^2 - \alpha_m^2} \\
&\qquad - \frac{\alpha_m(\sin(\alpha_n + \alpha_m) - \sin(\alpha_n - \alpha_m))}{\alpha_n^2 - \alpha_m^2} \\
&= \frac{2(\alpha_n \sin\alpha_n \cos\alpha_m - \alpha_m \cos\alpha_n \sin\alpha_m)}{\alpha_n^2 - \alpha_m^2} \\
&= \frac{2\cos\alpha_n \cos\alpha_m(\alpha_n \tan\alpha_n - \alpha_m \tan\alpha_m)}{\alpha_n^2 - \alpha_m^2} \\
&= 0 \quad \text{for} \quad n \neq m.
\end{aligned}$$

Note: this orthogonality condition is not satisfied for $\int_0^h \sin k_n y \sin k_m y\, dy$. So, for instance, $V(0, y)$ would not work "directly". One would have to first calculate $H_y(0, y)$.

Description of the Properties of an Ellipse

For many problems, one needs integrals over the circumference of an ellipse, whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

One may describe the ellipse by the parametric representation

$$z = a\cos\varphi + ib\sin\varphi,$$

and use φ as the integration variable.

However, in many cases, it is mathematically more convenient to use z on the circumference as the integrations variable. If one can represent all quantities of interest *on the circumference* as analytic functions of z, one can then use the Cauchy Theorem to execute the integration.

In using the parametric representation, one usually does something similar by introducing $e^{i\varphi}$ as the new integration variable. While this oftens works very well, it can lead to difficulties: for example, when e^{kz} appears in the function to be integrated.

In general, problems are much simpler for circles, where $a = b$. When $b \neq a$ it often becomes so difficult to execute the integral that it is most convenient to expand in a quantity that is equivalent to $a - b$ and thus the formulas will be easily written and therefore the expansion will be similarly easy.

Thus,

$$z = a\cos\varphi + ib\sin\varphi = e^{i\varphi}\frac{a+b}{2} + e^{-i\varphi}\frac{a-b}{2}, \tag{1}$$

$$\begin{aligned} 0 &= e^{i\varphi} - 2\frac{z}{a+b} + e^{-i\varphi}\frac{a-b}{a+b} = 0 \\ &= e^{2i\varphi}(a+b)^2 - 2ze^{i\varphi}(a+b) + \varepsilon, \end{aligned}$$

$$\boxed{\varepsilon = a^2 - b^2} \quad \text{and} \quad \boxed{W_1 = \sqrt{1 - \varepsilon/z^2}.} \tag{2}$$

$$e^{i\varphi} = \frac{z + \sqrt{z^2 - \varepsilon}}{a+b} = z\frac{1+W_1}{a+b}, \tag{3.1}$$

January, 1988. Note 0476thry.

$$e^{-i\varphi} = 2\frac{z}{a-b} - e^{i\varphi}\frac{a+b}{a-b}$$

$$= \frac{z - \sqrt{z^2-\varepsilon}}{a-b} = z\frac{1-W_1}{a-b} \tag{3.2}$$

$$= \frac{(a+b)/z}{1+W_1}. \tag{3.3}$$

For $\cos\varphi$, $\sin\varphi$:

$$\frac{1}{a+b} + \frac{1}{a-b} = \frac{2a}{\varepsilon} \quad \text{and} \quad \frac{1}{a+b} - \frac{1}{a-b} = \frac{-2b}{\varepsilon},$$

$$\cos\varphi = \frac{az - b\sqrt{z^2-\varepsilon}}{\varepsilon} = \frac{z^2+b^2}{az + b\sqrt{z^2-\varepsilon}} = \frac{z^2+b^2}{z(a+bW_1)}, \tag{4.1}$$

$$i\sin\varphi = \frac{-bz + a\sqrt{z^2-\varepsilon}}{\varepsilon} = \frac{z^2-a^2}{bz + a\sqrt{z^2-\varepsilon}} = \frac{z^2-a^2}{z(b+aW_1)}. \tag{4.2}$$

$$ds = \sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}\, d\varphi. \tag{5}$$

From (3.1) we have

$$ie^{i\varphi}d\varphi = \frac{\left(1 + (z/\sqrt{z^2-\varepsilon})\right)dz}{a+b} = \frac{e^{i\varphi}dz}{zW_1},$$

$$d\varphi = \frac{1}{i}\frac{dz}{zW_1}. \tag{6.0}$$

Thus,

$$\begin{aligned}
\varepsilon^2 G &= \varepsilon^2(a^2\sin^2\varphi + b^2\cos^2\varphi) \\
&= z^2(b^2(a-bW_1)^2 - a^2(b-aW_1)^2) \\
&= z^2(W_1^2(b^4-a^4) + 2abW_1(a^2-b^2)) \\
&= z^2\left(\frac{\varepsilon(a^2+b^2)}{z^2} - (a-b)^2 - 2ab\frac{1-W_1^2}{1+W_1}\right),
\end{aligned}$$

$$G = a^2 + b^2 - z^2\frac{\varepsilon}{(a+b)^2} - \frac{2ab}{1+W_1},$$

where

$$\frac{2}{1+W_1} = 1 + \frac{1-W_1}{1+W_1}$$

and thus

$$
\begin{aligned}
G &= a^2 + b^2 - ab - z^2\frac{\varepsilon}{(a+b)^2} - ab\frac{1-W_1}{1+W_1} \\
&= (a-b)^2 + ab - z^2\frac{\varepsilon}{(a+b)^2} - ab\frac{1-W_1}{1+W_1} \\
&= \frac{\varepsilon^2}{(a+b)^2} + ab - z^2\frac{\varepsilon}{(a+b)^2} - ab\frac{1-W_1}{1+W_1} \\
&= \boxed{ab - \varepsilon\frac{z^2-\varepsilon}{(a+b)^2} - ab\frac{1-W_1}{1+W_1}.}
\end{aligned}
\tag{7.1}
$$

To expand an expression like

$$\frac{1-W_1}{1+W_1} \quad \text{with} \quad W_1 = \sqrt{1-\varepsilon/z^2},$$

in ε, it is often convenient to break it up into an even and odd part in ε:

$$2F(\varepsilon) = F(\varepsilon) + F(-\varepsilon) + F(\varepsilon) - F(-\varepsilon) \quad \text{with} \quad W_2 = \sqrt{1+\varepsilon/z^2},$$

$$2\frac{1-W_1}{1+W_1} = 2H = \frac{1-W_1}{1+W_1} + \frac{1-W_2}{1+W_2} + \frac{1-W_1}{1+W_1} - \frac{1-W_2}{1+W_2},$$

$$
\begin{aligned}
2H(1 + W_1W_2 + W_1 + W_2) &= (1-W_1)(1+W_2) + (1+W_1)(1-W_2) \\
&\quad + (1-W_1)(1+W_2) - (1+W_1)(1-W_2) \\
&= 2(1 - W_1W_2 + W_2 - W_1),
\end{aligned}
$$

$$\frac{1-W_1}{1+W_1} = \frac{1-\sqrt{1-\varepsilon^2/z^4}+\sqrt{1+\varepsilon/z^2}-\sqrt{1-\varepsilon/z^2}}{1+\sqrt{1-\varepsilon^2/z^4}+\sqrt{1+\varepsilon/z^2}+\sqrt{1-\varepsilon/z^2}}.$$

To second order in ε:

$$\boxed{\frac{1-W_1}{1+W_1} = \frac{\varepsilon}{z^2}\frac{1+\varepsilon/2z^2}{4}.} \tag{7.2}$$

A comment about the expansion in ε and subsequent integration: the expansion has to be valid and good for z on the ellipse. If, to carry out the integration, one later modifies the integration path (in particular, to a very small circle around $z = 0$), this will *not* invalidate the original expansion.

Addendum. A different way to derive G.

For

$$W_0 = \sqrt{z^2 - \varepsilon},$$

$$\varepsilon = a^2 - b^2 \quad \text{and} \quad s = a^2 + b^2,$$

$$s^2 - \varepsilon^2 = 4a^2b^2 \quad \text{and} \quad 2ab = \sqrt{s^2 - \varepsilon^2},$$

we have

$$\begin{aligned}
\varepsilon^2 G &= (b\varepsilon\cos\varphi + ia\varepsilon\sin\varphi)(b\varepsilon\cos\varphi - ia\varepsilon\sin\varphi) \\
&= W_0(2abz - W_0(a^2+b^2)) \\
&= sW_0\left(z\sqrt{1-\varepsilon^2/s^2} - W_0\right) \\
&= sW_0\frac{z^2(1-\varepsilon^2/s^2) - z^2 + \varepsilon}{z\sqrt{1-\varepsilon^2/s^2} + W_0},
\end{aligned}$$

$$G = sW_0\frac{1-\varepsilon z^2/s^2}{z\sqrt{1-\varepsilon^2/s^2}+W_0} = \boxed{sW_1\frac{1-\varepsilon z^2/s^2}{\sqrt{1-\varepsilon^2/s^2}+W_1}.}$$

To first order in ε:

$$G = s\frac{1}{2}\left(1-\frac{\varepsilon}{2z^2}\right)\left(1-\frac{\varepsilon z^2}{s^2}\right)\left(1+\frac{\varepsilon}{4z^2}\right) = \frac{s}{2}\left(1-\varepsilon\left(\frac{z^2}{s^2}+\frac{1}{4z^2}\right)\right),$$

for $s = 2a^2$ and $s^2 = 4a^4$,

$$\boxed{\sqrt{G} = a\left(1-\frac{\varepsilon}{2}\left(\frac{z^2}{s^2}+\frac{1}{4z^2}\right)\right),}$$

$$\boxed{\sqrt{G} = a\left(1-\frac{\varepsilon}{8a^2}\left(\frac{z^2}{a^2}+\frac{a^2}{z^2}\right)\right).}$$

Characterization of Dipole Fringe Fields with Field Integrals

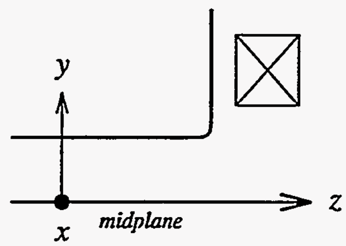


Figure 1.

Background and Introduction.

The quantity $\int B_y(x,y,z)dz$ was measured as a function of y for a fixed x, with integration region beginning in the homogenous field region inside the dipole magnet and reaching into the essentially field-free region outside. This resulted in the approximate plotted curve of Figure 1 below.

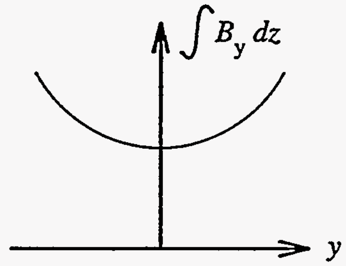


Figure 2.

The conclusion reached pointed to the coil being too close to or too far from the midplane. For didactic purposes this is a very interesting problem for two reasons.

(1) The coil position is only indirectly responsible. The fact that $\int B_y dz$ depends on y indicates that this is a 3D problem: namely, $\int B_y(x,0,z)dz$ will have a curvature of opposite polarity (i.e. effective field boundary is curved). This is due either to a curvature of the pole ends (when projected into the xz-plane) or to the finite width in the x-direction. If the latter is the cause, the problem is magnified by the absence (or incorrect design) of the field clamp *and* by a coil that is too far from the midplane.

(2) The characterization of the fringe field by measuring $\int B_y(x,0,z)dz$ gives, in case of midplane symmetry, more information than $\int B_y(0,y,z)dz$ alone.

May, 1986. Note 0438thry.

Analysis.

We assume midplane symmetry. Violation of midplane symmetry should be detected and/or measured, preferably with null method.

In vacuum, full 3D, the following hold:

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0, \tag{1.1}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0. \tag{1.2}$$

We now investigate the properties of

$$\int_{z_1}^{z_2} B_x(x,y,z)dz/L = b_x(x,y) \quad \text{and} \quad \int_{z_1}^{z_2} B_y(x,y,z)dz/L = b_y(x,y) \tag{2a, b}$$

where z_1, z_2 are constants, i.e. they are not considered variables; L is a convenient length that is used only for normalization purposes. Integration is performed over (1.1). Integration and differentiation can be interchanged and thus

$$\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} = 0. \tag{3.1}$$

Integration is performed over (1.2) and

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = (B_z(x,y,z_1) - B_z(x,y,z_2))/L. \tag{3.2}$$

If, independently of x and y, B_z at the two end-points is the same†, we have

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = 0. \tag{3.3}$$

(3.1) and (3.2) mean that b_x and $-b_y$ satisfy the Cauchy-Riemann conditions of real and imaginary parts of the analytic function of $Z = x + iy$:

$$b_x - ib_y = b^*(Z). \tag{4}$$

We use the Taylor series to represent $b^*(Z)$:

$$b^*(Z) = -i\sum_0^n a_n Z^n,$$

where n = multipole order -1, i.e. $n = 0 \Longrightarrow$ dipole, $n = 1 \Longrightarrow$ quadrupole, etc. Because of midplane symmetry all a_n must be real. Notice that for $y = 0$,

† In the case under discussion here, $B_z = 0$ at both ends even though $B_z \neq 0$ in the fringe field region.

$b_y(x,0) = \sum a_n x^n$, i.e. all harmonics contribute; while for $x = 0$, only a_n with even n contributes to $b_y(0,y)$. If one measures for $x = 0$ both b_y and b_x, then one gets information about all harmonics. There is an advantage to measuring both $b_y(0,y)$ and $b_x(0,y)$ since one gives only the odd harmonics and the other only the even, while they are all mixed when calculating $b_y(x,0)$.

Looking at (5) it is obvious that if $b_y(0,y)$ is not constant, but depends on y, then $b_y(x,0)$ must depend on x. That is, one is in fact dealing with 3D fields which must be due to curved (in projection on xy-plane) poles or poles of insufficient width in the $x-$direction, and failure to use a field clamp. It is also qualitatively clear that these 3D effects get more pronounced with increasing distance of the coils from the midplane.

Specifically, for a known value of $b_y(0,y)$, what is $b_y(x,0)$?

$$b_y(0,y) = \sum_0^m a_{2m} y^{2m} (-1)^m, \qquad (6.1)$$

$$b_y(x,0) = \sum_0^m a_{2m} x^{2m} + \sum_0^m a_{2m+1}, \qquad (6.2)$$

where in (6.2) a_{2m+1} is not obtainable from $b_y(0,y)$, but can be obtained from $b_x(0,y)$. If one measures $b_x(0,y)$, one gets

$$b_x(0,y) = \sum_0^m a_{2m+1} y^{2m+1} (-1)^m. \qquad (6.3)$$

For simple analysis, one should plot $b_y(0,y)$ *vs* y^2, and $b_x(0,y)/y$ *vs* y^2. One has to be careful to make the measurements in such a way that they really mean something. The flux loop and integrator method is perhaps best because it can practically always be done in such a way that one makes a null measurment.

Penetration of Solenoidal Field through Conducting Shell

Preliminaries: Solenoid, shield infinitely long. Thin shell, circular cross-section: treat eddy currents in it in plane geometry with proper boundary values. Only one shell: matrix formulation not needed.

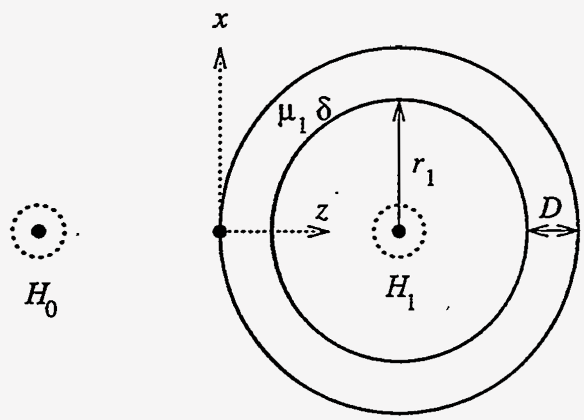


Figure 1.

At $z = 0$,

$$H_y = H_0(p) = \text{given solenoid current.}$$

At $z = D$,

$$E_x = \frac{\mu_0 p H_1 \pi r_1^2}{2\pi r_1} = \frac{r_1}{2}\mu_0 p H_1.$$

In shell,

$$\mathbf{H} = \mathbf{e}_y H, \quad \mathbf{E} = \mathbf{e}_x E, \quad \text{and} \quad \sigma = \frac{\partial}{\partial z}.$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} \quad \Longrightarrow \quad -H' = \sigma E,$$

$$\nabla \times \mathbf{E} = -\mu_0 \mu p \mathbf{H} \quad \Longrightarrow \quad E' = -\mu_0 \mu p H.$$

For $\mu_0 \mu \sigma p = k^2$,

$$H'' = -\sigma E' = \mu_0 \mu \sigma p H = k^2 H.$$

$$H = H_0 \cosh kz + b \sinh kz,$$

and for $\gamma = kD$,

$$H_1 = H_0 \cosh\gamma + b \sinh\gamma, \quad \text{and thus} \quad b = \frac{H_1 - H_0 \cosh\gamma}{\sinh\gamma},$$

May, 1986. Note 0437thry.

$$E_1 = \frac{r_1}{2}\mu_0 p H_1 = -\frac{H_1'}{\sigma} = -\frac{k}{\sigma}(H_0 \sinh\gamma + b\cosh\gamma),$$

$$H_1 = \frac{\mu_0\sigma p r_1}{2k} = -\left(H_0 \sinh\gamma + \frac{\cosh\gamma(H_1 - H_0\cosh\gamma)}{\sinh\gamma}\right),$$

where, for

$$\frac{\mu_0\sigma p r_1}{2k} = \frac{k^2 r_1/\mu}{2k} = \frac{r_1}{2}\frac{k}{\mu} = \gamma\varepsilon, \quad \text{and thus} \quad \boxed{\varepsilon = \frac{r_1}{2\mu D},}$$

where diffusion occurs for $\mu \gg 1$. Thus, for

$$H_1(\varepsilon\gamma\sinh\gamma + \cosh\gamma) = H_0,$$

$$\boxed{H_1(p) = \frac{H_0}{\cosh\gamma + \varepsilon\gamma\sinh\gamma}.}$$

The zeroes of $\cosh\gamma + \varepsilon\gamma\sinh\gamma$ are given, with

$$\gamma_n = D\sqrt{\mu_0\mu\sigma p_n} = i\alpha_n.$$

And

$$\cosh\gamma + \varepsilon\gamma\sinh\gamma = \cos\alpha_n - \varepsilon\alpha_n \sin\alpha_n = 0,$$

$$\boxed{\frac{1}{\varepsilon} = \alpha_n \tan\alpha_n,}$$

$$\boxed{p_n = \frac{-\alpha_n^2}{\mu_0\mu\sigma D^2}.}$$

One may expect difficulties for some calculations because $\gamma = D\sqrt{\mu_0\mu\sigma p}$, but this is not so, for

$$\begin{aligned}\cosh\gamma + \varepsilon\gamma\sinh\gamma &= 1 + \gamma^2\left(\frac{1}{2} + \varepsilon\right) + \gamma^4\left(\frac{1}{4!} + \frac{\varepsilon}{3!}\right) + \gamma^6\left(\frac{1}{6!} + \frac{\varepsilon}{5!}\right) + \cdots \\ &= 1 + \sum_{n=1}^{\infty}\gamma^{2n}\left(\frac{1}{(2n)!} + \frac{\varepsilon}{(2n-1)!}\right),\end{aligned}$$

and $\cosh\gamma + \varepsilon\gamma\sinh\gamma$ is a simple function of γ^2, not a complicated function of γ. For

$$\varepsilon \gg 1 \quad\Longrightarrow\quad \alpha_0^2 \approx \frac{1}{\varepsilon} = \mu_0\mu\sigma p_0 D^2 = \frac{2\mu D}{r_1} \quad\Longrightarrow\quad \boxed{\mu_0\sigma D r_1 p_0 = 2,}$$

with σD implying non-diffusion.

The roots, in quadrants 1 and 3, are given by,

$$\tan\alpha_n = \frac{1}{\alpha_n\varepsilon}.$$

Where, for $\alpha_n\varepsilon \gg 1$, $\alpha_n = n\pi + \sigma_n$, and thus

$$\boxed{\sigma = \frac{1}{\varepsilon(n\pi+\sigma_n)} \approx \frac{1}{\varepsilon n\pi}.}$$

The residue contribution from $\cosh\gamma + \varepsilon\gamma\sinh\gamma$ is given by

$$\frac{1}{\cosh\gamma+\varepsilon\gamma\sinh\gamma} \quad\Longrightarrow\quad R = \frac{1}{\dfrac{d\gamma}{dp}(\sinh\gamma+\varepsilon\sinh\gamma+\varepsilon\gamma\cosh\gamma)};$$

where $d\gamma/dp = \gamma/2p$ and $\varepsilon\gamma = -\cosh\gamma/\sinh\gamma$. Thus,

$$R = \frac{2p}{\gamma}\frac{\sinh\gamma}{\varepsilon\sinh^2\gamma-1} = \frac{2p}{\gamma^2}\frac{\gamma\sinh\gamma}{\varepsilon\sinh^2\gamma-1} = \frac{2p}{\gamma^2}\frac{\alpha_n\sin\alpha_n}{\varepsilon\sin^2\alpha_n+1}.$$

For $\cot\alpha_n = \alpha_n\varepsilon$, and $\sin\alpha_n = 1/\sqrt{1+\cot^2\alpha^2} = 1/\sqrt{1+\alpha_n^2\varepsilon^2}$,

$$\boxed{R_n = \frac{2}{\mu_0\mu D^2\sigma}\frac{\alpha_n}{\sqrt{1+\alpha_n^2\varepsilon^2}\left(1+\dfrac{\varepsilon}{1+\alpha_n^2\varepsilon^2}\right)} = \frac{2}{\mu_0\mu D^2\sigma}\frac{\alpha_n\sqrt{1+\alpha_n^2\varepsilon^2}}{1+\alpha_n^2\varepsilon^2+\varepsilon}.}$$

For $\alpha_0^2 = 1/\varepsilon$,

$$R_0 = \frac{2}{\mu_0\mu\sigma D^2}\frac{1}{\sqrt{\varepsilon}}\frac{\sqrt{1+\varepsilon}}{1+2\varepsilon} = \frac{1}{\mu_0\mu\sigma D^2\varepsilon}\frac{\sqrt{1+1/\varepsilon}}{1+1/2\varepsilon}.$$

Rogowski Dipole

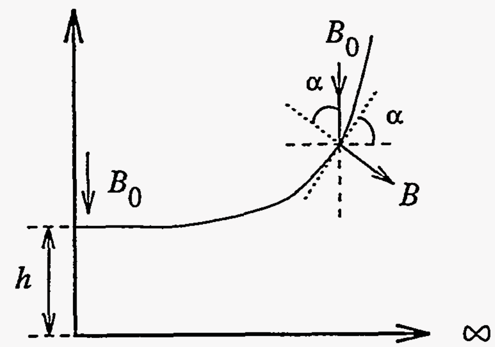
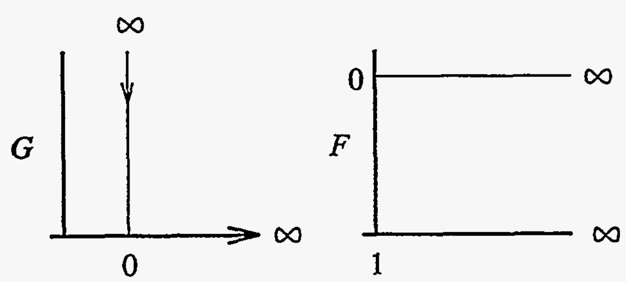


Figure 1.

$$B = B_0 \cos\alpha \cdot \frac{e^{i\alpha}}{i} \quad \text{and} \quad B^* = iB_0 \cos\alpha \cdot e^{-i\alpha},$$

$$\frac{iB_0}{B^*} = \boxed{G = \frac{e^{i\alpha}}{\cos\alpha} = 1 + i\tan\alpha} \cdot \text{with} \quad G = \frac{1}{F'} = \frac{\dot{z}}{\dot{F}}\,.$$

On pole: $\Re G = 0$. On $0, 1, \infty : \Im G = 0$.



Figures 2(a,b).

$$\dot{G} = \frac{a/2}{\sqrt{t}} \quad \text{and} \quad \boxed{G = 1 + a\sqrt{t},}$$

$$\dot{F} = \frac{b/2}{\sqrt{t}\sqrt{t-1}} \quad \text{and} \quad \boxed{F = b\ln\left(\sqrt{t} + \sqrt{t-1}\right).}$$

Thus

$$\dot{z} = \dot{F}G = \frac{b}{2}\left(\frac{1}{\sqrt{t}\sqrt{t-1}} + \frac{a}{\sqrt{t-1}}\right) \text{ and } \boxed{z = b\left(\ln\left(\sqrt{t} + \sqrt{t-1}\right) + a\sqrt{t-1}\right).}$$

February, 1981. Note 0397thry.

Since $ih = b(\ln i + ia) = ib(\pi/2 + a)$, we let

$$\boxed{\pi/2 + a = C} \quad \text{and} \quad \boxed{h = bC.}$$

On pole: $t = -\tau < 0$,

$$x + iy = b(i\pi/2 + \ln(\sqrt{\tau} + \sqrt{\tau + 1}) + ia\sqrt{\tau + 1}),$$

with

$$\boxed{\tau = \sinh^2 \alpha,} \quad \boxed{x = b\alpha,}$$

$$b\left(\frac{\pi}{2} + a\cosh\alpha\right) = \frac{h}{C}\left(C + a\left(\cosh\frac{x}{b} - 1\right)\right) = \boxed{y = h\left(1 + \frac{a}{c}\left(\cosh C\frac{x}{h} - 1\right)\right),}$$

and

$$G(1) = 1 + a = \frac{B(z = ih)}{B(z = 0)}.$$

Rogowski Quadrupole: Formulation of Problem

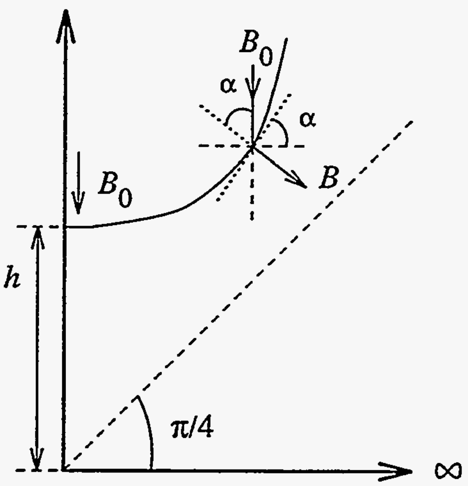


Figure 1.

$$\mu = \infty, \quad \text{and} \quad B^* = +iB_0.$$

$$B = B_0 \cos\alpha \cdot \frac{e^{i\alpha}}{i}, \quad \text{and} \quad B^* = iB_0 \cos\alpha \cdot e^{-i\alpha}.$$

Thus,

$$\frac{iB_0}{B^*} = G = \frac{e^{i\alpha}}{\cos\alpha} = 1 + i\tan\alpha,$$

on the pole, with $\Re G = \text{constant} = 1$, and

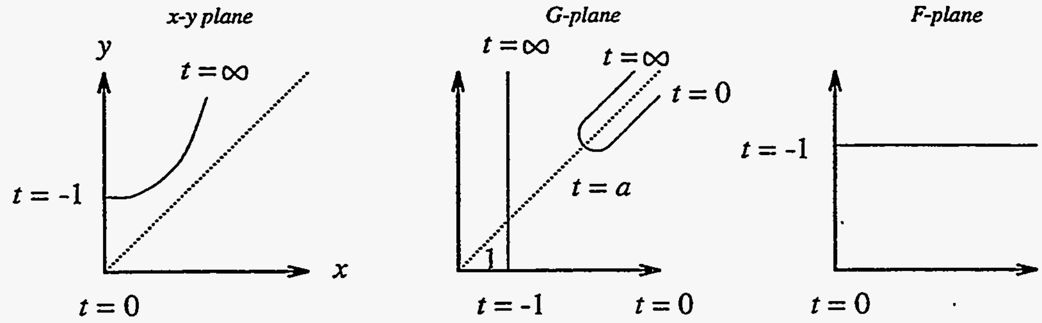
$$|G| = \frac{B_0}{|B|}.$$

On the 45° line,

$$B \approx e^{i\pi/4}/i, \quad B^* \approx ie^{i\pi/4}, \quad \text{and} \quad G \approx \epsilon^{i\pi/4}.$$

February, 1981. Note 0326thry.

Observe Figures 2(a,b,c):



Figures 2(a,b,c).

$$\boxed{\frac{dG}{dt} = b\frac{t-a}{t^{5/4}\sqrt{t+1}}e^{i\pi/4}.}$$

$$B^* = iB_0\frac{dF}{dz}, \quad \frac{dz}{dF} = \frac{\dot{z}}{\dot{F}} = G \quad \text{and} \quad \boxed{\dot{z} = G(t)\cdot\dot{F}.}$$

In the F-plane, for $-1 \leq t \leq 0$:

$$\boxed{\pi\dot{F} = \frac{c:}{\sqrt{t}\sqrt{t+1}}.}$$

The mathematical difficulty arises in the integration of $\dot{G}$. For

$$t = w^4, \quad dt = 4w^3 dw, \quad \frac{dt}{t^{5/4}} = \frac{4w^3 dw}{w^5} = \frac{4dw}{w^2},$$

we solve the elliptic integral

$$\boxed{G = b\int \frac{w^4-a}{w^2\sqrt{1+w^4}}dw.}$$

The integration of $\dot{z} = (\dot{GF}) - \dot{G}F$ leads to

$$\boxed{z = GF - be^{i\pi/4}\int \frac{t-1}{t^{5/4}\sqrt{t+1}}F dt}$$

and therefore

$$\boxed{F = 2C\ln\left(\sqrt{t}+\sqrt{t+1}\right).}$$

Eddy Currents for Fast Permanent Magnet Magnetization

Sometimes permanent magnets are magnetized by "hitting" them for a short time with high H. It is of interest to know how the magnetization front propagates through the material. Since this is a highly non-linear problem, a strongly simplified model is used in this document, but one which has all the essential features of the real process.

At the left edge of the material, assume a step function excitation starting at $t = 0$, with amplitude H_{00}. Our model is a 1-dimensional block of material with the left edge at $x = 0$, and

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0, \quad \mathbf{H} = \mathbf{e}_y H, \quad \mathbf{E} = \mathbf{e}_z E, \quad \text{and} \quad \mathbf{j} = \sigma \mathbf{E}.$$

For the times of interest, $H \geq 0$ everywhere. We use a strongly simplified $B(H)$ curve.

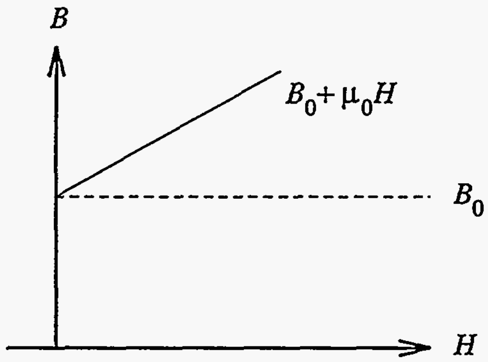


Figure 1.

Figure 1 shows that in the beginning, the material "sees" no H and $B = 0$. As soon as it "sees" $H > 0$, it becomes magnetized according to the above curve.

To reach an initial understanding of the problem, only the propagation in a medium that is, at least at first, unlimited to the right is treated.

$$\nabla \times \mathbf{H} = \mathbf{j} \quad \Longrightarrow \quad H' = \sigma E, \quad \text{with} \quad H' = \frac{\partial}{\partial x}.$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \quad \Longrightarrow \quad +E' = +\dot{B}.$$

August, 1977. Note 0264thry.

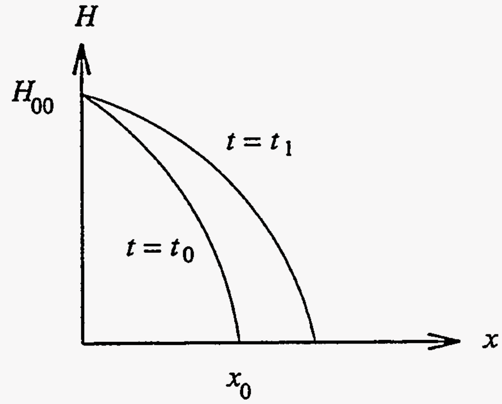


Figure 2.

The "location" of the front is designated by $x_0(t)$. We integrate $E' = \dot{B}$ over x across $x_0(t)$:

$$E(x_0 + \varepsilon) - E(x_0 - \varepsilon) = -E(x_0) = \int \dot{B} dx = B_0 \dot{x}_0,$$

giving the equation of the front for both **Case I** and **Case II** below.

Case I: $B(H) = B_0$.

For $x < x_0$:

$$E' = 0, \quad \text{and} \quad E = -B_0 \dot{x}_0 = \varrho H'.$$

$$H(x) = H_{00} - \sigma B_0 x \dot{x}_0,$$

meaning that for this $B(H)$ model, the $H(x)$ curves of Figure 2 are straight lines.

$$H(x_0) = 0 = H_{00} - \sigma B_0 x_0 \dot{x}_0.$$

Integrating over t gives $H_{00}t - \sigma B_0 x_0^2/2$, giving the following result for the propagation of the front:

$$\frac{2tH_{00}}{B_0 \sigma} = \frac{B_{00}}{B_0} \cdot \frac{2t}{\mu_0 \sigma} = \boxed{x_0^2 = r \frac{2t}{\mu_0 \sigma},} \quad \text{with} \quad r = \frac{\mu_0 H_{00}}{B_0} = \frac{B_{00}}{B_0}.$$

Case II: $B = B_0 + \mu_0 H$.

For $x < x_0$: $\quad H' = \sigma E, \quad$ and $\quad E' = \mu_0 \dot{H}$ †.

For $x = x_{0_+}$: $\quad H = 0$.

† The right side of this last equation is now non-zero in contrast to Case I where $\dot{B} = 0$ in the magnetized part of the material; i.e. $\dot{B} \neq 0$ only at the propagating front.

For $x = x_{0-}$: $B = B_0$, $E = -B_0\dot{x}_0 = \varrho H'$.

For $x = 0$: $H = H_{00}$.

The differential equation is $\boxed{H'' = \mu_0\sigma\dot{H}.}$

We introduce

$$\sqrt{\frac{t}{\mu_0\sigma}} = \tau \quad \text{and thus} \quad t = \tau^2\mu_0\sigma,$$

and get

$$\dot{H} = \frac{\partial H}{\partial \tau}\cdot\frac{d\tau}{dt} = \frac{\partial H}{\partial \tau}\cdot\frac{1}{2\tau\mu_0\sigma}, \quad \text{and} \quad \frac{\partial^2 H}{\partial x^2} = \frac{\partial H}{\partial \tau}\cdot\frac{1}{2\tau}.$$

We use the dimensional analysis argument that x and τ are the only dimensional quantities entering the problem. This means that H must be a function of x/τ. We let $u = x/2\tau$. For $H = F(u)$:

$$\frac{\partial^2 H}{\partial x^2} = \frac{F''}{4\tau^2} = \frac{1}{2\tau}\cdot F'\cdot\frac{-u}{\tau} = \frac{-F'u}{2\tau^2}, \quad \text{and thus} \quad F'' + 2uF' = 0,$$

$$\ln\frac{F'}{F_0'} + u^2 = 0, \quad \text{and} \quad \boxed{F' = -ae^{-u^2},}$$

with $F_0' = -a$ because F' has to be less than 0.

The boundary conditions, with fixed $\tau > 0$, are

$$\boxed{F = H(u) = H_{00} - a\int_0^u e^{-u^2}du.}$$

$u_0 = x_0/2\tau$ and x_0 refer to the location of front.

$$H(u_0) = H_{00} - a\int_0^{u_0} e^{-u^2}du = 0, \quad \text{with} \quad a = \frac{H_{00}}{\int_0^{u_0} e^{-u^2}du}.$$

Just as in **Case I**, the location x_0 of the front is proportional to $\sqrt{t}$. Thus,

$$\frac{\partial H(u_0)}{\partial x} = \frac{-ae^{-u_0^2}}{2\tau} = \sigma E = -B_0\sigma\dot{x}_0 = -B_0\sigma\cdot\frac{dx_0}{d\tau}\cdot\frac{1}{2\tau\mu_0\sigma}$$

$$\boxed{a = e^{u_0^2}\cdot\frac{B_0}{\mu_0}\cdot\frac{dx_0}{d\tau}} \quad \text{and} \quad \boxed{\frac{B_{00}}{B_0} = r = \frac{dx_0}{d\tau}\cdot e^{-u_0^2}\cdot\int_0^{u_0} e^{-u^2}du.}$$

Since $u_0 = x_0/2\tau$, this is a first order differential equation that looks difficult at first.

However, it is obvious that from dimensional considerations the solution must be

$$\boxed{x_0 = 2\tau g(r),}$$

where the factor of 2 is for neatness. That is, $u_0 = g(r)$, and then $g(r)$ is determined by

$$\boxed{r = 2ge^{g^2}\int_0^g e^{-u^2}\,du.}$$

For small g:

$$r = 2g^2 \quad \Longrightarrow \quad g = \sqrt{r/2}, \quad \text{and} \quad x_0 = \tau\sqrt{2r} = \sqrt{r\cdot\frac{2t}{\mu_0\sigma}},$$

where x_0 has the same solution as the case of $B = B_0$, as it has to be.

Evaluation with a $TI59$ gives the following results:

g	0.01	0.1	0.2	0.3	0.4	0.6	0.8	1.0
r	2.0×10^{-4}	2.01×10^{-2}	8.22×10^{-2}	.191	.356	.920	2.00	4.06

g	1.1	1.2	1.3	1.4	1.5	1.6	1.8	2.0
r	5.76	8.17	11.7	16.8	24.4	35.8	80.6	193

If r is of order 4, then

$$g \approx 1 \quad \Longrightarrow \quad \boxed{x_0 = 2\tau = 2\sqrt{\frac{t}{\mu_0\sigma}}.}$$

Case III: H after complete penetration of the slab.

When the front has reached $x_0 = x_1$, where $2x_1$ is the slab thickness, the boundary conditions change. In contrast to our earlier analysis, a given length x_1 enters the problem implicitly, and thus the dimensional analysis argument that H must be equal to $F(x/2\tau)$ is no longer valid. If time is counted anew, with $t = 0$ when $x_0 = x_1$, we are dealing with a linear system with boundary condition

$$H(x_1 + \Delta x) = H(x_1 - \Delta x) \quad \text{for} \quad t \geq 0,$$

with known and given $H(x)$ for $t = 0$ and $0 \leq x \leq 2x$.

If $H(x) - H_{00}$ is defined to be an odd function of x with respect to $x = 0$, and an even function with respect to $x = \pm x_1$, and this function is expanded into a Fourier series, then the period is $4x_1$, and

$$H(x) - H_{00} = \sum a_n(t) \sin\left(n \frac{2\pi}{4x_1} x\right).$$

To satisfy these symmetry conditions n must be odd, and we get

$$H(x) = H_{00} + \sum a_{2m+1}(t) \sin\left((2m+1)\frac{\pi x}{2x_1}\right)$$

and $a_{2m+1}(0)$ from known $H(x)$ at $t = 0$, the time of complete penetration.
Recalling the differential equation, $H'' = \mu_0 \sigma \dot{H}$, with $n = 2m+1$ we have

$$-a_n \left(\frac{n\pi}{2x_1}\right)^2 = \mu_0 \sigma \dot{a}_n,$$

$$\boxed{a_n(t) = a_n(0) \cdot e^{-\left(\frac{n\pi}{2x_1}\right)^2 \frac{t}{\mu_0\sigma}},}$$

$$\boxed{a_{2m+1}(t) = a_{2m+1}(0) \cdot e^{-(2m+1)^2 \frac{\pi^2 t}{4x_1^2 \mu_0 \sigma}}.}$$

At $t = 0$:

$$H(x) = H_{00} - H_{00} \frac{\int_0^u e^{-u^2} du}{\int_0^{u_0} e^{-u^2} du} = H_{00}\left(1 - \frac{\int_0^{g\frac{x}{x_1}} e^{-u^2} du}{\int_0^g e^{-u^2} du}\right),$$

$$\boxed{-H_{00} \frac{\int_0^{g\frac{x}{x_1}} e^{-u^2} du}{\int_0^g e^{-u^2} du} = \sum a_{2m+1}(0) \sin\left((2m+1)\frac{\pi x}{2x_1}\right).}$$

To determine $a_{2m+1}(0)$ we let

$$a_n(0) \cdot \int_0^1 \sin^2\left(n\frac{\pi}{2}v\right) dv = -\frac{H_{00}}{\int_0^g e^{-u^2} du} \cdot \int_0^1 \underbrace{\left(\int_0^{gv} e^{-u^2} du\right)}_{\zeta} \underbrace{\sin\left(n\frac{\pi}{2}v\right) dv}_{d\eta},$$

with

$$\int_0^1 \sin^2\left(n\frac{\pi}{2}v\right) dv = \frac{1}{2}\int_0^1 (1 - \cos(n\pi v)) dv = \frac{1}{2},$$

$$\zeta = \int_0^{gv} e^{-u^2} du, \quad \text{and} \quad d\zeta = g e^{-g^2 v^2},$$

$$d\eta = \sin\left(n\frac{\pi}{2}v\right) dv, \quad \text{and} \quad \eta = -\frac{\cos\left(n\frac{\pi}{2}v\right)}{n\frac{\pi}{2}v},$$

$$\left(\int_0^1 \zeta \, d\eta\right)_{n=\text{odd}} = \frac{2g}{n\pi}\int_0^1 e^{-g^2v^2} \cos\left(n\frac{\pi}{2}v\right) dv.$$

Thus,

$$\boxed{a_{2m+1} = -H_{00} \cdot \frac{4}{(2m+1)\pi} \cdot \frac{\displaystyle\int_0^1 e^{-g^2v^2} \cos\left((2m+1)\frac{\pi}{2}v\right) dv}{\displaystyle\frac{1}{g}\int_0^g e^{-u^2} du}.}$$

One would let

$$\frac{1}{g}\int_0^g e^{-u^2} du = \int_0^1 e^{-g^2v^2} dv,$$

if one wants to evaluate it with the integral of the numerator. Otherwise, one may return to the definition of g and use

$$\frac{1}{g}\int_0^g e^{-u^2} du = \frac{re^{-g^2}}{2g}.$$

The total time required to get good magnetization is determined as follows:

(1) Time to reach $x = x_1$: $\quad t_1 = \mu_0\sigma(x_1^2/4)$.

(2) Time for a_1 to decay by a factor of e : $\quad t_2 = \mu_0\sigma x_1^2(4/\pi^2)$.

Thus,

$$\mu_0\sigma x_1^2\left(\frac{1}{4} + \frac{4}{\pi^2}\right) = \boxed{t_{\text{total}} \approx .65\mu_0\sigma x_1^2.}$$

(3) If time for a_1 to decay by e^{π} is used, $t_2 = \mu_0\sigma x_1^2(4/\pi)$, and

$$\mu_0\sigma x_1^2(0.25 + 1.27) = \boxed{t_{\text{total}} \approx 1.5\mu_0\sigma x_1^2.}$$

Change of Determinant for Small Changes of One Element of the Matrix that Describes a System that Is Least Squares Optimized with Restraints and Has Least Squares Limitations on Parameters

We let

$$\boxed{A = \begin{pmatrix} M^tWM + V & N^t \\ N & 0 \end{pmatrix}}$$

and consider only those terms linear in ΔM_{nm} or ΔN_{nm}.

1)

$$\Delta M_{ik} = \Delta M_{nm}\delta(i-n)\delta(k-m) \quad \text{with} \quad \Delta M_{nm} = a.$$

Here and below we sum over indices appearing more than once.

$$\begin{aligned}(M^tWM)_{ik} &= M_{li}W_{ll}M_{lk} \\ &\rightarrow \quad (M_{li} + a\delta(l-n)\delta(i-m))\,W_{ll}\,(M_{lk} + a\delta(l-n)\delta(k-m)) \\ &= M_{li}W_{ll}M_{lk} + aW_{nn}\,(M_{ni}\delta(k-m) + M_{nk}\delta(i-m)) + a^2.\end{aligned}$$

To get $\|A + \Delta A\|$ to first order in a, one must differentiate $\|A + \Delta A\|$ with respect to a and then evaluate, knowing that A^{-1} is Hermitian.

$$\| A + \Delta A \| = \| A \| + aW_{nn} \cdot 2M_{nk}K_{mk},$$

where K is the co-factor, and summation over k is done only over values consistent with the number of rows in M.

$$\boxed{\frac{\| A + \Delta A \|}{\| A \|} = 1 + \frac{\Delta M_{nm}}{M_{nm}} \cdot 2W_{nn}M_{nm} \cdot \sum_k M_{nk}A^{-1}_{km}}$$

where $W_{nn}\sum_k M_{nk}A^{-1}_{km} = (A^{-1}M^tW)_{mn}$.

2)

$$\Delta N_{ik} = \Delta N_{nm}\delta(i-n)\delta(k-m),$$

$$\| A + \Delta A \| = \| A \| + 2\Delta N_{nm}K_{nm},$$

$$\boxed{\frac{\| A + \Delta A \|}{\| A \|} = 1 + \frac{\Delta N_{nm}}{N_{nm}} \cdot 2N_{nm}A^{-1}_{mn}.}$$

November, 1968. Note 0072thry.

Sensitivity of Solution of Linear Equations to Change of an Individual Matrix Element

We let

$$MP = S, \quad M^{-1} = N \quad \text{and} \quad P = NS,$$

$$(M + \Delta M)(P + \Delta P) = M(I + A)(P + \Delta P) = S \quad \text{where} \quad A = N\Delta M,$$

$$P + \Delta P = (I + A)^{-1}P \quad \text{and} \quad \Delta P = \left((I + A)^{-1} - I\right) P.$$

Further

$$\Delta M_{nm} = a\delta(n - n_0)\delta(m - m_0) \quad \text{where} \quad a = \Delta M_{n_0m_0},$$

$$A_{km} = N_{kn}\Delta M_{nm} = aN_{kn}\delta(n - n_0)\delta(m - m_0) = aN_{kn_0}\delta(m - m_0),$$

$$A^2_{km} = A_{ki}A_{im} = a^2N_{kn_0}\delta(i - m_0)N_{in_0}\delta(m - m_0),$$

where $A^2 = aN_{m_0n_0}A$, and we let $\alpha = aN_{m_0n_0}$.

$$(I + A)^{-1} = I + \gamma A \quad \text{and} \quad (I + A)(I + \gamma A) = I + A(I + \gamma + \gamma\alpha) = I,$$

thus,

$$\boxed{\gamma = -\frac{1}{1 + \Delta M_{n_0m_0}N_{m_0n_0}}} \quad \text{with} \quad \boxed{\Delta M_{n_0m_0}N_{m_0n_0} \neq -1}$$

$$\boxed{\Delta P = \gamma AP}$$

$$\Delta P_k = \gamma A_{km}P_m = \gamma aN_{kn_0}\delta(m - m_0)P_m,$$

$$\boxed{\Delta P_k = -\frac{\Delta M_{n_0m_0}P_{m_0}}{1 + \Delta M_{n_0m_0}N_{m_0n_0}} \cdot N_{kn_0}}$$

That the matrix M becomes exactly singular for $\Delta M_{n_0m_0} = -1/N_{m_0n_0}$ is easily shown with *Cramer's Rule*. Let K_{nm} be the co-factor to the nm element, and $\|M + \Delta M\| = \|M\| + \Delta M_{n_0m_0}K_{m_0n_0}$:

$$\Delta M_{n_0m_0} \cdot \frac{K_{n_0m_0}}{\|M\|} = \Delta M_{n_0m_0}N_{m_0n_0} = -1$$

which is the necessary condition for a singular matrix.

November, 1968. Note 0071thry.

This condition can easily be used to judge whether a matrix is "close" to being singular. One would test

$$\frac{M_{n_0 m_0}}{\Delta M_{n_0 m_0}} = -M_{n_0 m_0} N_{m_0 n_0}$$

and when the result is large compared to the inverse of the relative error of $M_{n_0 m_0}$, one is likely to be in trouble. This is of particular importance when the matrix elements are experimentally determined.

Fourier Analysis of Numerical Data

We assume that the spacing between data points is uniform, $2\pi/N$. Representing $F(\varphi)$ by a Fourier series with unknown coefficients and making the coefficients such that

$$\sum_{\varphi_n}\left(F(\varphi_n)-\sum_m a_m e^{im\varphi_n}\right)^2 = \min$$

gives the same coefficients that one obtains by evaluating the integral

$$\int F(\varphi)e^{-im\varphi}d\varphi$$

with trapezoidal rule applied to the whole integrand:

$$a_m^* = \frac{1}{2\pi}\int F(\varphi)e^{im\varphi}d\varphi \quad \Longrightarrow \quad \frac{\Delta\varphi}{2\pi}\sum F(\varphi_n)e^{im\varphi_n} = \frac{1}{N}\sum F(\varphi_n)e^{im\varphi_n}.$$

A better way to integrate would be to assume that not the whole integrand changes linearly over an individual interval, but that only $F(\varphi)$ changes linearly over the interval.

For one interval,

$$\int F(\varphi)e^{i(m\varphi+\alpha)}d\varphi = \int (a+b\varphi)e^{i(m\varphi+\alpha)}d\varphi = (a+b\varphi)\frac{e^{i(m\varphi+\alpha)}}{im} + \frac{be^{i(m\varphi+\alpha)}}{m^2}.$$

When summing over the whole range of φ, the first term contributions cancel. With $b=(F_2-F_1)/\Delta\varphi$, we get

$$\begin{aligned} I &= \int\limits_{\text{interval}} F(\varphi)e^{i(m\varphi+\alpha)}d\varphi \\ &= \frac{F_2-F_1}{m^2\Delta\varphi}\left(e^{i(m\varphi_2+\alpha)} - e^{i(m\varphi_1+\alpha)}\right) \\ &= \frac{F_2e^{i(m\varphi_2+\alpha)}\left(1-e^{-im\Delta\varphi}\right) - F_1e^{i(m\varphi_1+\alpha)}\left(e^{im\Delta\varphi}-1\right)}{m^2\Delta\varphi}. \end{aligned}$$

August, 1968. Note 0059thry.

When summing over the whole circle, we get:

$$\int_0^{2\pi} F(\varphi)e^{i(m\varphi+\alpha)}d\varphi = \sum \frac{F(\varphi_n)\left(1 - e^{-im\Delta\varphi} - e^{im\Delta\varphi} + 1\right)}{m^2\Delta\varphi} e^{i(m\varphi_n+\alpha)}$$

$$= \sum \frac{F(\varphi_n) 4\sin^2\varepsilon}{m^2\Delta\varphi} e^{i(m\varphi_n+\alpha)} \quad \text{with} \quad \varepsilon = \frac{m\Delta\varphi}{2}$$

$$= \boxed{\left(\frac{\sin\varepsilon}{\varepsilon}\right)^2 \Delta\varphi \sum F(\varphi_n) e^{i(m\varphi_n+\alpha)}}$$

with

$$\frac{m\Delta\varphi}{2} = \boxed{\varepsilon = m\frac{\pi}{N}.}$$

A parabolic approximation for F over two intervals, $-\Delta\varphi \leq \varphi \leq \Delta\varphi$, without $e^{i\alpha}$, gives after some calculation:

$$\boxed{I_2 = \left(\frac{\sin^4\varepsilon}{\varepsilon^2} + \cos\varepsilon\left(\frac{\sin\varepsilon}{\varepsilon}\right)^3\right) \cdot \Delta\varphi \cdot \sum_n F(\varphi_n) e^{i(m\varphi_n+\alpha)}}$$

with $\varepsilon = m\pi/N$, and

$$\left(\frac{\sin^4\varepsilon}{\varepsilon^2} + \cos\varepsilon\left(\frac{\sin\varepsilon}{\varepsilon}\right)^3\right) = t = \left(\frac{\sin\varepsilon}{\varepsilon}\right)^2 \left(\sin^2\varepsilon + \cos\varepsilon\frac{\sin\varepsilon}{\varepsilon}\right).$$

Program to Calculate t, and Results.

```
5  CLS
10 FOR N=1 TO 18
15 E=N*3.14159265/36
20 PRINT ((SIN(E)/E)^2*((SIN(E))^2+COS(E)*SIN(E)/E)
30 NEXT N
```

For ε = N*5:

ε	5°	10°	15°	20°	25°	30°	35°	40°	45°
t	.9999	.9998	.9988	.9962	.9911	.9821	.9682	.9482	.9213

ε	50°	55°	60°	65°	70°	75°	80°	85°	90°
t	.8870	.8450	.7957	.7397	.6780	.6120	.5434	.4739	.4053

Program to Calculate K_1/ε^2 and K_2/ε^2, and Results.

```
5 CLS
10 FOR N=1 TO 18
20 E=N*3.14159265/36
30 PRINT N*5, (3+COS(4*E)-SIN(4*E)/E)/(4*E^2), (SIN(2*E)/(2*E)-COS(2*E))/(E^2)
40 NEXT N
```

ε	K_1/ε^2	K_2/ε^2
5°	.67069299	1.3292762
10°	.68235368	1.3171576
15°	.70042915	1.2971353
20°	.72300072	1.2694693
25°	.74761149	1.2345172
30°	.77147163	1.1927287
35°	.79169091	1.1446375
40°	.80551841	1.0908528
45°	.81056947	1.0320491
50°	.8050208	.96895504
55°	.78775796	.90234147
60°	.75866353	.83300908
65°	.71763969	.76177546
70°	.6665645	.68946226
75°	.60718709	.61688241
80°	.54197173	.54482766
85°	.47370526	.47405682
90°	.40528474	.40528474

Curvature of Field Lines in a Quadrupole

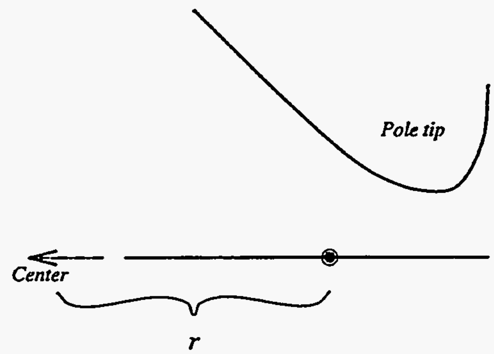


Figure 1.

$$F(z) = (z+r)^2 \quad \text{with} \quad r = 1/K.$$

The field line is described by $\Re(z+r)^2 = (x+r)^2 - y^2 =$ constant, thus

$$(x+r)^2 - y^2 = (x_0+r)^2 \quad \longrightarrow \quad x = -r + \sqrt{y^2 + (x_0+r)^2}.$$

Further, $1/R = x''/\left(1+(x')^2\right)^{3/2}$, with

$$x' = \frac{y}{\sqrt{y^2+(x_0+r)^2}} \quad \text{and} \quad x'' = \frac{(x_0+r)^2}{(y^2+(x_0+r)^2)^{3/2}}$$

$$1+(x')^2 = \frac{2y^2+(x_0+r)^2}{\sqrt{y^2+(x_0+r)^2}}, \quad \text{and} \quad \left(1+(x')^2\right)^{3/2} = \frac{\left(2y^2+(x_0+r)^2\right)^{3/2}}{(y^2+(x_0+r)^2)^{3/2}}.$$

Thus,

$$\frac{1}{R} = \frac{(x_0+r)^2}{(2y^2+(x_0+r)^2)^{3/2}} \quad \text{and} \quad \boxed{R = (x_0+r)\left(1+\frac{2y^2}{(x_0+r)^2}\right)^{3/2}}$$

for field line starting at x_0.

May, 1986. Note 0009thry.

The field line at x, y is described by

$$\frac{\left((x+r)^2+y^2\right)^{3/2}}{(x+r)^2-y^2} = \boxed{R=(x+r)\frac{\left(1+\left(\frac{y}{x+r}\right)^2\right)^{3/2}}{1-\left(\frac{y}{x+r}\right)^2}.}$$

We make the following substitutions:

$$\frac{y}{x+r}=\tan\alpha, \quad 1+\tan^2\alpha=\frac{1}{\cos^2\alpha}, \quad \text{and} \quad 1-\tan^2\alpha=\frac{\cos 2\alpha}{\cos^2\alpha},$$

and thus,

$$\boxed{R=\frac{(x+r)}{\cos\alpha\cos 2\alpha}.}$$

Also,

$$\sqrt{(x+r)^2+y^2}\left(\frac{1+\tan^2\alpha}{1-\tan^2\alpha}\right)=\boxed{R=\frac{\sqrt{(x+r)^2+y^2}}{\cos 2\alpha}.}$$

Skin Effect in Fe

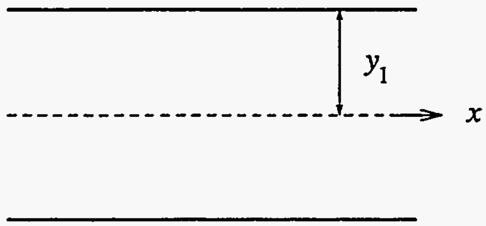


Figure 1.

We introduce initial conditions and definitions:

$$\mathbf{E} = \mathbf{e}_x E, \quad \text{and} \quad \mathbf{B} = \mathbf{e}_z B \quad \text{with} \quad B = \mu_0 \mu_{\text{rel}} H = \mu H,$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} = 0, \quad \text{and} \quad \frac{\partial}{\partial y} \neq 0,$$

$$\nabla \times \mathbf{H} = H' = \sigma E = j, \quad \text{and} \quad \nabla \times \mathbf{E} = -E' = -i\omega\mu H,$$

$$\sigma E' = j' = i\omega\mu\sigma H.$$

We let $i\omega\mu\sigma = k^2$, and

$$H'' - k^2 H = 0, \quad \boxed{H = H_0 \cosh ky,} \quad \text{and} \quad \boxed{j = H_0 k \sinh ky.}$$

The average field in the sheet, $\overline{H}$, compared to the field outside, H_1, is given by

$$\overline{H} = \frac{1}{2y_1} \int_{-y_1}^{y_1} H dy = H_0 \frac{\sin ky_1}{ky_1}, \quad H_1 = H_0 \cosh ky_1, \quad \text{and} \quad H_0 = \frac{H_1}{\cosh ky_1},$$

$$\boxed{\frac{\overline{H}}{H_1} = \frac{\tanh ky_1}{ky_1}.} \tag{1}$$

In (1), we let $x = ky_1$, and solve

February, 1966. Note 0007thry.

$$\frac{\overline{H}}{H_1} = \frac{\tanh x}{x} \approx \frac{1 + x^2/6 + x^4/120}{1 + x^2/2 + x^4/24}$$

$$\approx \left(1 + \frac{x^2}{6} + \frac{x^4}{120}\right)\left(1 - \frac{x^2}{2} + \frac{5x^4}{24} + \cdots\right)$$

$$\approx \boxed{1 - \frac{1x^2}{3} + \frac{2x^4}{15}.} \tag{2}$$

The power dissipation per cubic meter is given by

$$P = \frac{\varrho}{2}|j|^2 = \frac{1}{2}\varrho H_0^2 |k|^2 |\sinh^2 ky|.$$

We let $ky = \alpha + i\alpha$ where $\alpha = |k|/\sqrt{2}$ thus

$$\sinh(\alpha + i\alpha) = \sinh\alpha\cos\alpha + i\cosh\alpha\sin\alpha,$$

$$\begin{aligned} |\sinh(\alpha + i\alpha)|^2 &= \sinh^2\alpha\cos^2\alpha + \cosh^2\alpha\sin^2\alpha \\ &= \sinh^2\alpha \cdot (1 - \sin^2\alpha) + (1 - \sinh^2\alpha) \cdot \sin^2\alpha \\ &= \sinh^2\alpha + \sin^2\alpha \\ &= \frac{1}{2}(1 - \cos 2\alpha + \cosh 2\alpha - 1) \\ &= \frac{1}{2}(\cosh 2\alpha - \cos 2\alpha), \end{aligned}$$

and

$$\begin{aligned} \overline{|\sinh(\alpha + i\alpha)|^2} &= \frac{1}{2y_1}\int_0^{y_1}(\cosh 2\alpha - \cos 2\alpha)dy \\ &= \frac{1}{2\alpha_1}\int_0^{\alpha_1}(\cosh 2\alpha - \cos 2\alpha)d\alpha \\ &= \frac{1}{4\alpha_1}(\sinh 2\alpha_1 - \sin 2\alpha_1). \end{aligned}$$

Thus,

$$\boxed{\overline{P} = \frac{H_0^2\mu}{2} \cdot \omega \cdot \frac{\sinh 2\alpha_1 - \sin 2\alpha_1}{4\alpha_1}} \quad \text{with} \quad \alpha_1 = y_1\sqrt{\frac{\omega\mu\sigma}{2}}.$$

For

$$\frac{\sinh x - \sin x}{x} \approx \frac{2 \cdot \left(\frac{x^3}{3!} + \frac{x^7}{7!}\right)}{x} = \frac{x^2}{3}\left(1 + \frac{x^4}{7!/3!}\right) = \frac{x^2}{3}\left(1 + \frac{x^4}{840}\right) \approx \frac{x^2}{3},$$

and thus,

$$\frac{\sinh 2\alpha_1 - \sin 2\alpha_1}{4\alpha_1} = \frac{(2\alpha_1)^2}{6} = \frac{(\sqrt{2}\gamma_1)^2}{6} = \frac{\gamma_1^2}{3}, \quad \text{with} \quad \gamma_1 = y_1\sqrt{\omega\mu\sigma} = \sqrt{2}\alpha_1.$$

Therefore,

$$\boxed{\overline{P} = \frac{H_0^2\mu}{2} \cdot \omega \cdot \frac{y_1^2\omega\mu\sigma}{3}.}$$

For $H_0\mu = B_0$,

$$\boxed{\overline{P} = \frac{B_0^2}{2} \cdot \frac{y_1^2\omega^2\sigma}{3} = \frac{B_0^2}{2} \cdot \frac{(2y_1)^2\omega^2\sigma}{12}.}$$

Resulting, thermally, in a trivial geometry:

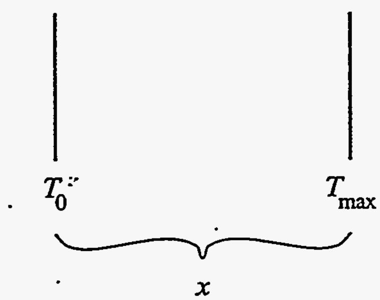


Figure 2.

For heat conductivity, $S = \lambda T'$ in power/m^2,

$$\Delta S(x) = \overline{P}\Delta x, \quad \text{and thus} \quad \overline{P} = S' = \lambda T'',$$

and thus,

$$\boxed{T_{\text{max}} - T_0 = \frac{x^2\overline{P}}{2\lambda}.}$$

Typical Numbers for Dynamo Steel.

We let

$$\varrho = 46\mu\Omega\,\text{cm} = 4.6 \times 10^{-7}\Omega\,\text{m},$$

$$\mu(14\text{kG}) = 2100, \quad \text{and} \quad \mu(18\text{kG}) = 125,$$

$$\lambda = 27\text{–}36\ \frac{\text{BTU/h}}{\text{ft}\,^\circ F} \quad \text{with} \quad 1\ \frac{\text{BTU/h}}{\text{ft}\,^\circ F} = \frac{0.293}{1.84}\ \frac{\text{Watts}}{\text{m}\,^\circ C}$$
$$= 47.5\text{–}63.5\ \text{Watts/m}\,^\circ C.$$

(1) For $|x^2|/3 = 1/10$ we have

$$\frac{|x^2|}{3} = \frac{\omega\mu}{3\varrho} \cdot y_1^2 = \frac{1}{10}, \quad \text{and} \quad y_1 = \sqrt{\frac{0.3\varrho}{\omega\mu}} = 10^{-3}\sqrt{\frac{2.3}{16.8}} = 0.37 \times 10^{-3}\text{m} = 0.37\text{mm},$$

$$\boxed{2y_1 = 0.74\text{mm.}}$$

(2) For $B_0 = 14$ kG $=1.4$T and the above $2y_1$,

$$\boxed{\overline{P} = \frac{60}{4.6} \times 10^3 = 13 \times 10^3 \text{Watts/m}^3 = 13 \times 10^{-3}\text{Watts/cm}^3.}$$

(3) For $x = 0.4$m and the above $\overline{P}$,

$$\Delta T = \frac{(.16)(13 \times 10^3)}{(2)(50)} = (13)(.16)^\circ C \approx 21^\circ C.$$

If the field is a sinusoidal function between $B = 0$T and 14kG, one has to use $B_0 =$ 7kG.

A More Detailed Expression for $\overline{H}/H_1$.

With $2y_1 = D$, we let

$$x = \frac{kD}{2} = \frac{D}{2}\sqrt{i\omega\mu\sigma} = \frac{D}{2}\sqrt{2}\sqrt{i\frac{\omega\mu\sigma}{2}}.$$

With $\lambda = \sqrt{2/\omega\mu\sigma}$,

$$x = \sqrt{i}\frac{D}{\sqrt{2}\lambda} = \sqrt{i}\varepsilon, \quad \text{where} \quad \varepsilon = \frac{D}{\sqrt{2}\lambda}.$$

Therefore,

$$\frac{\overline{H}}{H_1} = 1 - i\frac{\varepsilon^2}{3} - \frac{2\varepsilon^4}{15} \quad \text{and} \quad \boxed{\tan\varphi \approx \frac{\varepsilon^2}{3} = \frac{(D/\lambda)^2}{6},}$$

$$\left|\frac{\overline{H}}{H_1}\right|^2 = 1 - \frac{7\varepsilon^4}{45}, \quad \text{and} \quad \left|\frac{\overline{H}}{H_1}\right| = 1 - \frac{7\varepsilon^4}{90},$$

and thus,

$$\boxed{\left|\frac{\overline{H}}{H_1}\right| = 1 - \frac{7}{(4)(90)}\frac{D^4}{\lambda}.}$$

Results for Al, Cu and Fe at 60Hz.

For Al and Cu, we let

$$\lambda = \sqrt{\frac{2\varrho}{\sigma\mu_0}} = \sqrt{\frac{10^4\varrho}{2.4}}.$$

$$\varrho_{\mathrm{Al}} = 2.8 \times 10^{-8} \quad \text{and} \quad \lambda_{\mathrm{Al}} = 1.08\text{cm},$$

$$\varrho_{\mathrm{Cu}} = 1.7 \times 10^{-8} \quad \text{and} \quad \lambda_{\mathrm{Cu}} = 0.84\text{cm},$$

and $D = (1/4)\text{in} = 0.635\text{cm}$:

	D/λ	$(D/\lambda)^2$	$(D/\lambda)^2/6$	$0.7\ (D^2/6\lambda^2)^2$
Cu	0.755	0.570	0.095	0.00635
Al	0.587	0.345	0.0575	.0023

For Fe with $\mu \approx 2000$,

$$\varrho_{\mathrm{Fe}} = 4.6 \times 10^{-7} \quad \text{and} \quad \lambda_{\mathrm{Fe}} \approx 1\text{mm},$$

and $D = 14\text{mm}$:

	D/λ	$(D/\lambda)^2$	$(D/\lambda)^2/6$	$0.7\ (D^2/6\lambda^2)^2$
Al	0.350	0.1225	0.0205	0.00029

Magnetic Field Energy Calculations

$$E = \frac{1}{2}\int(\mathbf{B}\cdot\mathbf{H})d\tau = \frac{1}{2}\int(\mathbf{B}\cdot\nabla V)d\tau = \frac{1}{2}\int\mathbf{H}\cdot(\nabla\times\mathbf{A})d\tau,$$

with

$$\begin{aligned}\mathbf{B}\cdot\nabla V &= \nabla\cdot(V\mathbf{B}) - V\nabla\cdot\mathbf{B} = \nabla\cdot(V\mathbf{B}),\\ \mathbf{H}\cdot(\nabla\times\mathbf{A}) &= \mathbf{A}\cdot(\nabla\times\mathbf{H}) = \mathbf{A}\cdot\mathbf{j}.\end{aligned}$$

Field Energy in the Airspace of a Long, Symmetrical Bending Magnet.

The airspace is bounded by the midplane, an equipotential and two field lines (lines starting at two locations on the midplane).

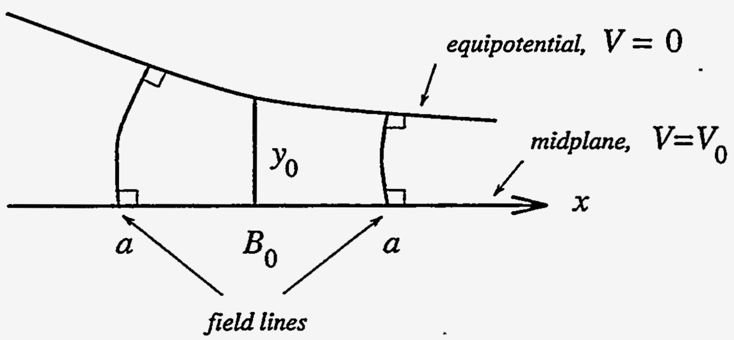


Figure 1.

Derive B from the potential: $\mathbf{B} = \nabla V$,

$$2\mu_0 E = \int(\mathbf{B}\cdot\nabla V)d\tau = \int\nabla\cdot(V\mathbf{B})d\tau = \int V\mathbf{B}\cdot d\boldsymbol{\sigma}.$$

Normalize $V = 0$ on equipotential, then contribution on equipotential is 0, as well as being 0 along the field lines:

$$2\mu_0 E = LV_0\int B_y dx,$$

where L is the length of the magnet.

February, 1966. Note 0006thry.

For $B_y = B_0(1 + Kx)$:

$$\int_{-a}^{a} B_y dx = 2aB_0, \quad \text{and} \quad V_y'|_{x=0} = B_y = B_0 \quad \Longrightarrow \quad V_0 = yB_0.$$

Thus,

$$2\mu_0 E = LB_0 y_0 2aB_0 \quad \Longrightarrow \quad \boxed{\frac{E}{L} = \frac{B_0^2}{2\mu_0} 2ay_0.}$$

y_0 equipotential is the hyperbola tangent to an ellipse with half-axis a:

$$y_0 = \frac{c}{r_0} = \frac{b}{r_0 a}\sqrt{(x + r_0)^3 x} = b\frac{r_0}{a}\sqrt{\frac{x}{r_0}\left(\frac{x}{r_0} + 1\right)^3}.$$

For $a/r_0 = \varepsilon$,

$$s = \frac{x}{r_0} = \frac{2\varepsilon^2}{1 + \sqrt{1 + 8\varepsilon^2}} = \frac{1}{4}\left(\sqrt{1 + 8\varepsilon^2} - 1\right),$$

we redefine $y_0 = bF(\varepsilon)$, where

$$\boxed{F(\varepsilon) = \sqrt{\frac{s(s+1)^3}{\varepsilon^2}} = \sqrt{\frac{2}{1 + \sqrt{1 + 8\varepsilon^2}}\left(\frac{3 + \sqrt{1 + 8\varepsilon^2}}{4}\right)^3}.}$$

For $8\varepsilon^2 \ll 1$:

$$\sqrt{1 + 8\varepsilon^2} = 1 + 4\varepsilon^2.$$

$$F^2(\varepsilon) = \frac{2}{2 + 4\varepsilon^2}\left(\frac{4 + 4\varepsilon^2}{4}\right)^3 = (1 - 2\varepsilon^2)(1 + 3\varepsilon^2),$$

$$\boxed{F(\varepsilon) = 1 + \varepsilon^2/2.}$$

For $\varepsilon = 1/2$, $F(1/2) = 1.2$, while if $\varepsilon = 1$, $F(1) = 1.3$.

Magnetic Energy of 2D Vacuum Field Inside Arbitrary Boundary.

Represent $\mathbf{B}$ by scalar potential: $\mathbf{B} = \nabla V$,

$$2\mu_0 E = \int (\mathbf{B} \cdot \nabla V) d\tau = \int \nabla \cdot (V\mathbf{B}) d\tau = \int V (\mathbf{B} \cdot d\boldsymbol{\sigma}) .$$

The expression for scalar product of two vectors in 2-dimensional space, when vectors are expressed by the complex numbers $a = a_x + ia_y$ and $b = b_x + ib_y$, is

$$\mathrm{a} \cdot \mathrm{b} = a_x b_x + a_y b_y = \Re ab^* = \Re a^* b.$$

Thus, for $d\sigma = iLdz$:

$$\boxed{2\mu_0 E/L = \Re \int ViB^* dz.}$$

For $V = v$, and $iB^* = F'$:

$$\begin{aligned} 2\mu_0 E/L &= \Re \int vF'dz = \Re \int vdF = \Re \int v(du + idv) \\ &= \boxed{\int vdu = \int v(u'_x + u'_y y')dx.} \end{aligned}$$

Special Case.

The energy of field derived from $F = (B_0 K/2)(z + r_0)^2$, with $r_0 = 1/K$, inside the ellipse described by $(x/a)^2 + (y/b)^2 = 1$, is given by

$$F = \frac{1}{2} B_0 K \left((x + r_0)^2 - y^2 + 2iy(x + r_0) \right) .$$

With

$$u'_x = B_0 K(x + r_0), \quad u'_y = -B_0 K y, \quad v = B_0 K y(x + r_0), \quad y' = -\frac{b^2}{a^2}\frac{x}{y},$$

and

$$x = a \sin\varphi, \quad dx = a\cos\varphi d\varphi, \quad y = b\sqrt{1 - (x/a)^2} = b\cos\varphi,$$

and $a/r_0 = \varepsilon$,

$$\begin{aligned}
2\mu_0 E/L &= B_0^2 K^2 \int y(x+r_0)\left(x + r_0 + y\frac{b^2 x}{a^2 y}\right) dx \\
&= B_0^2 K^2 \int (x+r_0)\left(r_0 + x\left(1+\frac{b^2}{a^2}\right)\right) y dx \\
&= B_0^2 ab \int_0^{2\pi} (1+\varepsilon\sin\varphi)\left(1+\varepsilon\sin\varphi\left(1+\frac{b^2}{a^2}\right)\right)\cos^2\varphi d\varphi \\
&= B_0^2 ab \int_0^{2\pi} \left(1+\varepsilon\sin\varphi\left(2+\frac{b^2}{a^2}\right)+\varepsilon^2\sin^2\varphi\left(1+\frac{b^2}{a^2}\right)\right)\cos^2\varphi d\varphi \\
&= B_0^2 ab\left(\pi + 0 + \varepsilon^2\frac{\pi}{4}\left(1+\frac{b^2}{a^2}\right)\right) \\
&= \boxed{B_0^2 ab\pi\left(1+\frac{a^2+b^2}{4r_0^2}\right).}
\end{aligned}$$

Scalar Potential for 3D Fields in "Business Region" of Insertion Device with Finite Width Poles

Task.

In the absence of random errors, we are interested in the formulation of 3-dimensional $V(x,y,z)$ (with $\nabla^2 V = 0$) for $\mathbf{B}$ in the "business region" of an insertion device (hybrid or electro-magnetic) with finite width poles, and containing only a small number of free, easily measured constants.

Notation and Coordinate System.

The beam will be in the direction of the z-axis. The midplane will be in the xz-plane. The field will be in the y-direction in the midplane.

Field Symmetries.

B_y will be the even function of x, y and B_x will be the odd function of x, y.

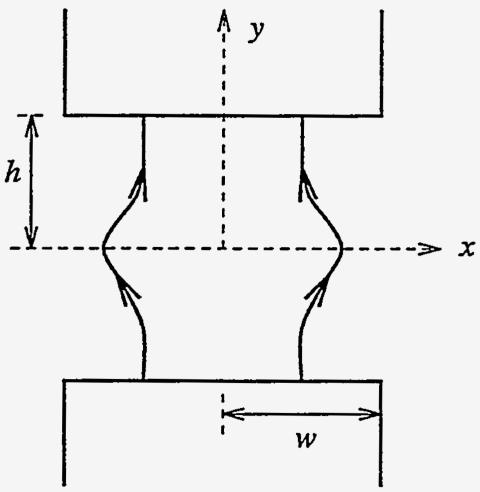


Figure 1.

Representation of $V(x,y,z)$.

We represent $V(x,y,z)$ by a Fourier series in z:

$$V(x,y,z) = \sum_{n=\mathrm{odd}} \cos nk_3 z \cdot G_n(x,y) \quad \text{with} \quad k_3 = 2\pi/\lambda, \tag{1}$$

where

$$\nabla^2 = 0 \quad \Longrightarrow \quad \nabla^2 G_n = n^2 k_3^2 G_n. \tag{2}$$

February, 1992. Note 0144u-w.

For an infinitely wide pole:

$$\partial G_n/\partial x = 0 \quad \Longrightarrow \quad G_n(x,y) = a_n \sinh nk_3 y,$$

which is a standard 2D solution. The effect of the finite width pole is described by

$$G_n = a_n \sinh nk_3 y + g_n(x,y),$$

where g_n is the effect of the finite width pole. Thus

$$\nabla^2 g_n = n^2 k_3^2 g_n. \tag{3}$$

Case 1:

$$\mu_{\mathrm{Fe}} = \infty \quad \Longrightarrow \quad B_x(x, \pm h, u\lambda/2) = 0 \tag{4}$$

where u is an integer.

1.1) We initially assume that

$$G_n = g_n = 0 \quad \text{for} \quad n > 1$$

and we see that the symmetries and (4) give $g'_{1x}(x,0) = g'_{1x}(x,\pm h) = 0$. We expand $g'_{1x}(x,y)$ in a Fourier series in y with a $2h$ period, and see that

$$g'_{1x}(x,y) = b'_m(x) \sin mk_2 y \quad \text{with} \quad k_2 = \pi/h. \tag{5}$$

We now substitute (5) into (3). We have $b'''_m = b'_m(m^2k_2^2 + n^2k_3^2)$, with $n = 1$. A solution which satisfies this equation and the symmetries is given by

$$b_m(x) = c_m \cosh k_{nm} x, \quad \text{where} \quad k_{nm} = (m^2k_2^2 + n^2k_3^2)^{1/2}. \tag{6}$$

A complete solution is given by

$$V(x,y,z) = \cos k_3 z \cdot a_0 \left(\sinh k_3 y + \sum_{m=1} \sin mk_2 y \cosh k_{1m} x \cdot \frac{a_m}{\cosh k_{1m} w} \right). \tag{7}$$

c_m is chosen such that a_m can be expected to be only weakly dependent on w.

1.2) We now assume that

$$G_n = g_n \neq 0 \quad \text{for} \quad n \geq 1.$$

(7) can be generalized, but one must realize that the resulting formula does not have the same force of logic that was inherent in its original derivation. This generalized

formula allows the possibility that contributions from many n could combine to make $B_x(x, \pm h, u\lambda/2) = 0$.

$$
\begin{aligned}
V(x,y,z) &= \sum_{n=\text{odd}} \cos nk_3 z \cdot G_n(x,y) \\
G_n(x,y) &= A_{n0}\left(\sinh nk_3 y + \sum_{m=1} \sin mk_2 y \cosh k_{nm}x \cdot \frac{a_{nm}}{\cosh k_{nm}w}\right) \qquad (8)\\
k_{nm} &= (n^2k_3^2 + m^2k_2^2)^{1/2} \quad \text{with} \quad k_3 = 2\pi/\lambda \quad \text{and} \quad k_2 = \pi/h.
\end{aligned}
$$

Notice that in the above set of equations A_{n0} has units of Tesla-meters and a_{nm} is dimensionless.

We expect that a_{nm} is of the first order, the finite width effects decrease with increasing n and m, and further, that only a few a_{nm} are needed.

Case 2:

$$
\mu_{\text{Fe}} < \infty \quad \Longrightarrow \quad B_x(x, \pm h, u\lambda/2) = -B_x(-x, \pm h, u\lambda/2) \neq 0. \qquad (9)
$$

The contributions from Fe alone are given by the addition of $Q_n(x,y)$:

$$
V = \sum \cos nk_3 z \cdot Q_n(x,y) \quad \text{where} \quad \nabla^2 Q_n = n^2 k_3^2 Q_n.
$$

For the sake of simplification, we shall look at one Q_n, normalize lengths so that $nk_3 = 1$, and denormalize at the end

$$
\nabla^2 Q = Q. \qquad (10)
$$

We follow the logic of **Case 1** as well as also satisfying $Q_y'(x,0) \approx x^2$ for sufficiently small x. Thus, we start with

$$
Q(x,y) = (\cosh \eta x - 1)P_1(y) + P_2(y), \qquad (11)
$$

where η is real and arbitrary. Later we will let $\eta \to 0$. We substitute (11) into (10) and get

$$
\begin{aligned}
\nabla^2 Q &= \eta^2 c P_1 + (c-1)P_1'' + P_2'' \\
&= Q \\
&= (c-1)P_1 + P_2.
\end{aligned}
$$

where $c = \cosh \eta x$ and P_1, P_2 are unknown. Separating into terms with and without

c, we have

$$\eta^2 P_1 + P_1'' = P_1, \quad P_2'' - P_1'' = P_2 - P_1, \tag{12}$$

$$p^2 = 1 - \eta^2, \quad P_1'' - p^2 P_1 = 0 \quad \Longrightarrow \quad P_1 = \sinh py, \tag{13}$$

$$P_2'' - P_2 = P_1'' - P_1 = -\eta^2 P_1 = -\eta^2 \sinh py,$$

$$P_2 = \sinh py - p \sinh y, \tag{14}$$

in which the second term serves to satisfy the condition $P'_{2y}(0) = 0$.

$$Q(x,y) = (\cosh \eta x - 1) \sinh py + \sinh py - p \sinh y, \tag{15}$$

with $\eta \to 0$ and $p \to (1 - \eta^2/2)$.

$$Q(x,y) = \frac{\eta^2}{2}(x^2 \sinh y - y \cosh y + \sinh y). \tag{16}$$

We denormalize (16) and drop $\eta^2/2$ and get

$$Q_n(x,y) = (nk_3x)^2 \sinh nk_3y - nk_3y \cosh nk_3y + \sinh nk_3y, \tag{17}$$

$$V(x,y,z) = \sum_{n=\text{odd}} \cos nk_3z(G_n(x,y) + Q_n(x,y)). \tag{18}$$

Magnetic Measurements.

First, the simplest implementation consists of measuring the Fourier coefficients of the expansion of B_z, B_y, B_x in $\sin nkz$ and $\cos nkz$ and determining the value of the free coefficients in G_n and Q_n that best fit the data. Use a "filter" to remove the random errors from the data sets.

Second, choose x, y very carefully for each of these sets of measurements in order to take advantage of the properties of $G_n(x,y)$ and $Q_n(x,y)$ and its derivatives with respect to x, y. This is particularly important for the contributions originating from $\sin mk_2y$ in $G_n(x,y)$.

Third, investigate suitability of less conventional magnetic measurements, like a Hall probe or flux loop that vibrates in the x-direction, with phase sensitive de-modulation.

Use of Model.

After verification of the validity region of the model is completed, it can be used for trajectory calculations. Furthermore, one can use this model to determine the maximal narrowness of the pole before detrimental effects become intolerable.

In application to existing hardware, one can break up the total field into the ideal 3D field and the random errors.

Magnetic Measurement and Data Reduction to Identify Some Specific Error Field Consequences

Measurement of Steering in Wigglers and Undulators.

Prefer null measurement method, if it can be done.

In "body" of wiggler or undulator: use coil with length equal to the product of period and integer.

In the end-region, from the field-free region to the periodic part: measure using long coil reaching from the outside to the periodic part, together with an attached compensation coil in the periodic part. This gives a signal that depends only on the steering integral, and is independent of position in the periodic part. It is an important tool for correcting the ends.

Normalized sensitivity of system, for $\varphi = kz = 2\pi z/\lambda$ is given by

$$S(\varphi) = S_0(\varphi) + S_1(\varphi),$$

where S_0 refers to the main coil, and S_1 to the compensation coil. With $\varphi = 0$ referring to the end of the main coil, and $\varphi = -\alpha$ to the center of the correction coil (of length $2\varphi_1$) relative to the end of the main coil, we have, in the coil coordinate system,

$$\begin{aligned}
S_0(\varphi) &= 1 \text{ at } -\infty \leq \varphi \leq 0,\\
S_0(\varphi) &= 0 \text{ at } 0 \leq \varphi,\\
S_1(\varphi) &= \varepsilon \text{ at } -\alpha - \varphi_1 \leq \varphi \leq -\alpha + \varphi_1,\\
S_1(\varphi) &= 0 \text{ at } \varphi \text{ outside the above region.}
\end{aligned}$$

For the periodic region, $\varphi > 0$ and

$$B = \sum_{n=\text{odd}} n a_n \cos n\varphi = \Re \sum n a_n e^{in\varphi},$$

with the end of main coil at $\varphi_0 > \pi$ in the field coordinate system, the signal from the main coil is

$$F_0 = \int_{-\infty}^{0} B d\varphi + \int_{0}^{\varphi_0} \Re \sum n a_n e^{in\varphi} d\varphi = \text{Steering} \int + \Re \sum a_n \left(e^{in\varphi_0} - 1\right)/i,$$

and similarly, the signal from the compensating coil is

$$F_1 = \varepsilon \int_{\varphi_0-\alpha-\varphi_1}^{\varphi_0-\alpha+\varphi_1} \sum n a_n e^{in\varphi} d\varphi = \Re \sum a_n e^{in(\varphi_0-\alpha)} 2\varepsilon \sin\varphi_1,$$

September, 1993. Note 0142u-w.

Presented at the ID Measurement Workshop, ANL, September, 1993.

$$F_0 - F_1 = \text{Steering} \int + \Re \sum a_n e^{in\varphi_0} \left(\frac{1}{i} - 2\varepsilon e^{-in\alpha} \sin n\varphi_1 \right).$$

We want $F_0 - F_1$ independent of φ_0, thus

$$2\varepsilon \sin n\alpha \sin n\varphi_1 = 1, \quad \text{and} \quad 2\varepsilon \cos n\alpha \cos n\varphi_1 = 0.$$

When harmonics are weak (undulator), we need to satisfy these conditions only for $n = 1$, but when strong harmonics are present (wiggler), we need to satisfy them for all odd n: to get

$$\cos n\alpha = 0 \qquad \text{choose} \quad \boxed{\alpha = \pi/2,}$$

$$\sin n\alpha = (-1)^{(n-1)/2} \qquad \text{choose} \quad \boxed{\varphi_1 = \pi/2,} \qquad \varepsilon = 1/2.$$

$\varphi_1 = \pi/2$ needs to be done by hardware, $\alpha = \pi/2$, $\varepsilon = 1/2$, can be done by "tuning" if one provides for it. Other solutions should be obvious.

This scheme can also be implemented with simple coil (or Hall probes) and software. However, software implementation is *not* a null method and therefore suffers much more from equipment imperfections.

Phase Shifts of Emitted Light Due to Error Fields.

This is one of a number of ways to develop more insight into why or how synchrotron light properties deteriorate because of error fields.

We make the following definitions:

$$x'' = \frac{g}{\gamma} B, \quad \text{with} \quad g = \frac{e}{m_0 c},$$

$$' = \frac{\partial}{\partial z}, \quad \varphi = kz, \quad \text{and} \quad k = \frac{2\pi}{\lambda}.$$

For the reference trajectory:

$$B(\varphi) = B_0 \cos \varphi,$$

$$x_0' = \frac{gB_0}{\gamma k} \sin \varphi = \frac{K}{\gamma} \sin \varphi, \quad \text{with} \quad \frac{gB_0}{k} = K = .934 \cdot B_0(\mathrm{T}) \cdot \lambda(\mathrm{cm}),$$

$$x_0 = -\frac{K}{k\gamma} \cos \varphi,$$

$$\boxed{x_W = \frac{K}{k\gamma}.}$$

We define the trajectory length error as

$$\Delta s = \int_{-\infty} \left(\sqrt{1+(x_0'+\Delta x')^2} - \sqrt{1+{x_0'}^2}\right) dz$$
$$= \int \left(x_0'\Delta x' + \frac{1}{2}\Delta x'^2\right) dz$$
$$= x_0\Delta x' - \int x_0 \Delta x'' dz + \frac{1}{2}\int \Delta x'^2 dz.$$

For $D = \Delta B/B_0$ as a function of φ:

$$\Delta x'' = \frac{gB_0}{\gamma}\frac{\Delta B}{B_0} = \frac{gB_0}{\gamma}D = \frac{Kk}{\gamma}D,$$

$$\Delta x' = \frac{K}{\gamma}\int D d\varphi.$$

Thus,

$$\Delta s = \left(\frac{K}{\gamma}\right)^2 \frac{1}{k}\left(-\cos\varphi \int D d\varphi + \int \cos\varphi D d\varphi + \frac{1}{2}\int \left(\int D d\varphi\right)^2 d\varphi\right),$$

With $\Delta t = \Delta s/c$, $\Delta\Phi = \omega_L \Delta t = \Delta s \omega_L/c = \Delta s k_L$, and

$$\lambda_L = \frac{\lambda}{2\gamma^2}\left(1+\frac{K^2}{2}\right) = \frac{\lambda K^2}{4\gamma^2}\left(1+\frac{2}{K^2}\right),$$

$$\boxed{\Delta\Phi = P\left(\underbrace{-\cos\varphi \int D d\varphi}_{(G_1)} + \underbrace{\int \cos\varphi D d\varphi}_{(G_2)} + \underbrace{\frac{1}{2}\int \left(\int D d\varphi\right)^2 d\varphi}_{(G_3)}\right)},$$

where

$$P = \frac{4}{1+2/K^2}.$$

Notice that, as a function of K, $\Delta\Phi \approx K^2$ for $K^2 \ll 2$, and $\Delta\Phi$ is independent of K^2 for $K^2 \gg 2$. G_1 produces harmonics and reduces the intensity of the fundamental, but only if steering is not 0. The $G_2 \neq 0$ contribution depends on symmetry of D, not on the presence or absence of steering. In G_3 only steering errors contribute and it always gives $\Delta\Phi$ of the same sign. G_3 being of second order, we may expect a significant contribution only for a long undulator. We will see below that G_3 can be surprisingly large even for a short undulator.

We present order of magnitude estimates for $\overline{G_2^2}, \overline{G_3^2}$, for the ensemble.

For G_2 from one primary source:

$$G_2 = \varepsilon_2 D_2 \frac{\lambda}{2} \frac{2\pi}{\lambda} = \varepsilon_2 D_2 \pi.$$

For n_2 sources per period: after $N_1 = z/\lambda$ periods

$$\overline{G_2^2} = \overline{(\varepsilon_2 D_2 \pi)^2} n_2 \frac{z}{\lambda} = \overline{D_2^2} \pi^2 \varepsilon_2^2 n_2 \frac{z}{\lambda}.$$

Without $\cos\varphi$ in the integrand, we may expect from steering:

$$\overline{G_0^2} = \overline{D_0^2} \pi^2 \varepsilon_0^2 n_0 \frac{z}{\lambda} = \overline{D_0^2} \pi^2 \varepsilon_0^2 n_0 N_1.$$

$$\overline{G_3} = \frac{1}{2} \overline{D_0^2} \pi^2 \varepsilon_0^2 n_0 \int_0^{L=N_1\lambda} \frac{z}{\lambda} dz \frac{2\pi}{\lambda} = \frac{1}{2} \overline{D_0^2} \pi^3 \varepsilon_0^2 n_0 N_1^2.$$

$$\varepsilon_4 = \frac{\overline{G_3}}{\sqrt{\overline{G_2^2}}} = \frac{\overline{D_0^2} \pi^3 \varepsilon_0^2 n_0 N_1^2 / 2}{\sqrt{\overline{D_2^2}} \pi \varepsilon_2 \sqrt{n_2} \sqrt{N_1}} = \frac{\varepsilon_0^2 n_0 \overline{D_0^2}}{\sqrt{\varepsilon_2 n_2 \overline{D_2^2}}} \frac{\pi^2}{2} N_1^{3/2}.$$

To get a feeling for the order of magnitude, we assume that the first order term, $\sqrt{\overline{G_2^2}}$, contributes twice the contribution of the second order term, $\overline{G_3}$, i.e. $\varepsilon_4 = 1/2$, and further, $\varepsilon_0 = \varepsilon_2 = 1$ $n_0 = n_2 = 4$, $N_1 = 10^2$ $\overline{D_0^2} = \overline{D_2^2}$, then

$$\sqrt{\overline{D_0^2}} = \frac{1}{20 \times 10^3} = 5 \times 10^{-5}$$

has to be satisfied. And for $\sqrt{\overline{D_0^2}} = 10^{-3}$, $\varepsilon_0 = 1$, $n_0 = 4$, $N_1 = 10^2$,

$$\Delta\Phi = .6 \,\text{radians}.$$

$\Delta\Phi$ over-estimates the damage done to the emitted light because $\Delta\Phi = a + bz$ causes no real damage. We subtract the straight line from the original $\Delta\Phi(z) = f(z)$, and then normalize the length of the undulator to 1:

$$H = \int_0^1 (a + bz - f(z))^2 dz$$

and minimize H with a, b,

$$\int f(z)dz = F_0, \quad \int zf(z)dz = F_1, \quad \text{and} \quad \int f^2(z)dz = F_2.$$

The solution gives

$$\boxed{a = 2(2F_0 - 3F_1),} \quad \text{and} \quad \boxed{b = 6(2F_1 - F_0),}$$

and this gives

$$\boxed{H = F_2 - 4(F_0^2 + 3F_1(F_1 - F_0)).}$$

For a specific function $f(z) = u\sqrt{z} + vz^2$, corresponding to the ensemble model used above, and after optimization (see Appendix A for details),

$$H = \left(\frac{1}{180}\right)\frac{2u^2}{5} - \frac{8uv}{7} + v^2,$$

With $\alpha = v/u$, we get the following improvement factor:

$$\frac{H}{F_2} = \left(\frac{1}{36}\right)\frac{\alpha^2 - 8\alpha/7 + 2/5}{\alpha^2 + 20\alpha/7 + 5/2}.$$

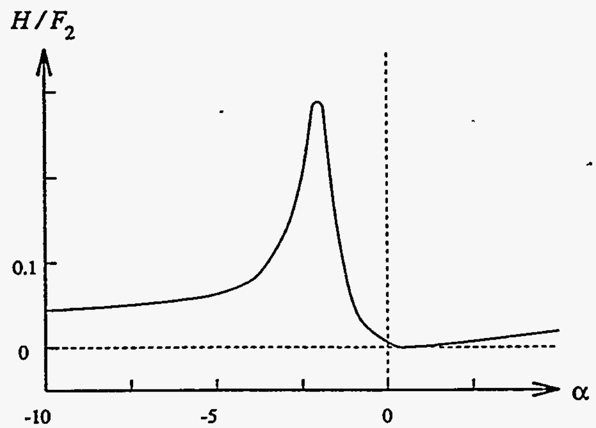


Figure 1.

For Figure 1,

$$(H/F_2)_{\min} = 4.5 \times 10^{-4} \qquad \text{at } \alpha = .605,$$

$$(H/F_2)_{\max} = .274 \qquad\qquad \text{at } \alpha = -1.65,$$

Even if one does not consider the model of $f(z) = u\sqrt{z} + vz^2$ a realistic one, it is quite clear that (a) one should optimize not $\Delta\Phi(z)$, but $\Delta\Phi(z)$ minus "best" straight line, and that (b) gains can be remarkable. In other words, the quality of the light generated may be much better than one would think if one were to only look at $\Delta\Phi(z)$ or $D(z)$. We make a trivial, but interesting observation: since H (before or after subtraction of straight line) is a quadratic function of u, v, an *increase* in the G_2 or G_3 contribution may lead to a *decrease* of H.

For the measurement of $G_2 = \int \cos\varphi D d\varphi$, consider the following 2-coil configuration:

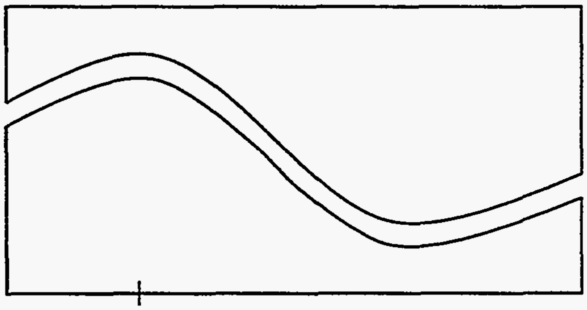


Figure 2.

The above design will measure the integral over $\cos\varphi \cdot B(\varphi)$. But, $\cos\varphi \cdot B_0 \cos\varphi$ gives a large signal, therefore the null-coil system is needed. The proposed system cancels $B_0 \cos\varphi$ but also "sees" steering; thus, it is fine only if steering is small enough or known. Therefore, we give below only the basic design and performance equations for system components, and one system.

Since $\Delta\Phi$ is only relevant for undulator, we ignore the harmonics.

The compensation coil is the same for measurement of steering at ends.

The main coil sensitivity is $S_0 = \cos(\alpha_0 + \varphi)$ at $-\varphi_2 \le \varphi \le \varphi_2$, and $S_0 = 0$ outside this region. At the center of the coil, sensitivity is $\cos\alpha_0$, and $B = \cos\varphi_0 = \Re e^{i\varphi_0} = B$.

$$\begin{aligned} F_0 &= \Re \int_{-\varphi_2}^{\varphi_2} e^{i(\varphi_0+\varphi)} \cos(\alpha_0 + \varphi) d\varphi \\ &= \frac{1}{2} \Re e^{i\varphi_0} \int_{-\varphi_2}^{\varphi_2} \left(e^{i\alpha_0} e^{2i\varphi} + e^{-i\alpha_0}\right) d\varphi \\ &= \boxed{\frac{1}{2} \Re e^{i\varphi_0} \left(e^{i\alpha_0} \sin 2\varphi_2 + e^{-i\alpha_0} 2\varphi_2\right).} \end{aligned}$$

The compensation coil sensitivity is $S_1(\varphi) = \varepsilon_1$ at $-\varphi_1 \le \varphi \le \varphi_1$, and $S_1(\varphi) = 0$ outside this region. At the center of the coil, $B = \cos(\varphi_0 + \beta) = \Re e^{i(\varphi_0+\beta)}$.

$$F_1 = \Re\varepsilon \int_{-\varphi_1}^{\varphi_1} e^{i(\varphi_0+\beta+\alpha)} d\varphi$$
$$= \boxed{2\varepsilon_1 \Re e^{i(\varphi_0+\beta)} - \sin\varphi_1.}$$

With $\varepsilon = 4\varepsilon_1 \sin\varphi_1$ and $2\varphi_2 = \gamma$,

$$F_0 + F_1 = \frac{1}{2}\Re e^{i\varphi_0}\left(e^{i\alpha_0}\sin\gamma + e^{-i\alpha_0}\gamma + \varepsilon e^{i\beta}\right).$$

To get no signal for all φ_0, we must satisfy

$$\cos\alpha_0(\sin\gamma + \gamma) = -\varepsilon\cos\beta, \quad \text{and} \quad \sin\alpha_0(\sin\gamma - \gamma) = -\varepsilon\sin\beta,$$

and since there are four parameters to satisfy two equations there are many possible solutions. We pick one with $\beta = 0$ and $\alpha_0 = 0$ and have

$$\varepsilon = 4\varepsilon_1 \sin\varphi_1 = -(\gamma + \sin\gamma), \quad \boxed{\varepsilon_1 = -\frac{2\varphi_2 + \sin 2\varphi_2}{4\sin\varphi_1}.}$$

If $|\varepsilon_1| > 1$, we can use a combined coil system as follows, with $\varphi_2 = \varphi_1 = 3\pi/2$ and therefore $\varepsilon_1 = 3\pi/4$.

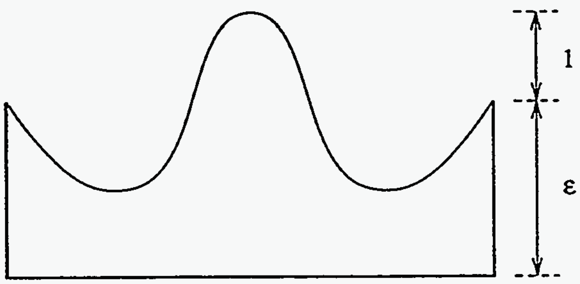


Figure 2.

This is not the ultimate answer, but only a first step, and it may start similar thinking on other issues.

Appendix A.

For the execution of the optimization of H, we let

$$a + b/2 = F_0, \quad 2F_1 - F_0 = b/3, \quad a/2 + b/3 = F_1,$$

$$\boxed{b = 6(2F_1 - F_0)} \quad \boxed{a = 2(2F_0 - 3F_1).}$$

With the above, we have

$$\begin{aligned}
H &= a^2 + b^2/3 + F_2 - 2aF_0 - 2bF_1 + ab \\
&= a\underbrace{(a + b/2)}_{F_0} + b\underbrace{(b/3 + a/2)}_{F_1} + F_2 - 2aF_0 - 2bF_1 \\
&= F_2 - aF_0 - bF_1 \\
&= F_2 - 2F_0(2F_0 - 3F_1) - 6F_1(2F_1 - F_0) \\
&= \boxed{F_2 - 4F_0^2 - 12F_1^2 + 12F_0F_1.}
\end{aligned}$$

For the special case of $f(z) = \sqrt{z} + \alpha z^2$:

$$F_0 = 2/3 + \alpha/3 = (\alpha + 2)/3, \quad \text{and} \quad F_1 = 2/5 + \alpha/4,$$

and for $f^2(z) = z + \alpha^2 z^4 + 2\alpha z^{2.5}$, $F_2 = \frac{}{2} + \frac{\alpha^2}{5} + \frac{4\alpha}{7}$. Therefore,

$$\begin{aligned}
F_2 - H &= \frac{4}{9}(\alpha + 2)^2 + 12\left(\frac{\alpha}{4} + \frac{2}{5}\right)^2 - 4(\alpha + 2)\left(\frac{\alpha}{4} + \frac{2}{5}\right) \\
&= \frac{7\alpha^2}{36} + \frac{26\alpha}{45} + \frac{112}{225},
\end{aligned}$$

$$\begin{aligned}
H &= \alpha^2\left(\frac{1}{5} - \frac{7}{36}\right) + \alpha\left(\frac{4}{7} - \frac{26}{45}\right) + \frac{1}{2} - \frac{112}{225} \\
&= \boxed{\frac{1}{180}\left(\alpha^2 - \frac{8\alpha}{7} + \frac{2}{5}\right)} = \boxed{\frac{1}{180}\left(v^2 - \frac{8uv}{7} + \frac{2u^2}{5}\right).}
\end{aligned}$$

$$\boxed{\frac{H}{F_2} = \frac{1}{36}\frac{\alpha^2 - 8\alpha/7 + 2/5}{\alpha^2 + 20\alpha/7 + 5/2}.}$$

Least Square Fit of $f(z)$ with $a + bz$ in $0 \leq z \leq 1$

Origin and Purpose of Study. If $f(z)$ = phase shift, the difference between $f(z)$ and $a + bz$ is the only damaging property of $f(z)$ since a would be an irrelevant shift of phase reference, and b represents a shift of center of line without any broadening. We define

$$S = \int (a + bz - f(z))^2 dz.$$

For $S'_a = 0$: $\quad a + b/2 = F_0 = \int f(z)dz,$
For $S'_b = 0$: $\quad a/2 + b/3 = F_1 = \int zf(z)dz.$

Therefore,

$$\boxed{b = 12F_1 - 6F_0} \quad \text{and} \quad \boxed{a = 4F_0 - 6F_1.}$$

For $\int f(z)^2 dz = F_2$,

$$\begin{aligned} S &= a^2 + b^2/3 + F_2 + ab - 2aF_0 - 2bF_1 \\ &= a(a + b/2) + b(a/2 + b/3) - 2aF_0 - 2bF_1 + F_2 \\ &= F_2 - aF_0 - bF_1 \\ &= F_2 - F_0(4F_0 - 6F_1) - F_1(12F_1 - 6F_0) \\ &= \boxed{F_2 - 4(F_0^2 + 3F_1^2 - 3F_0F_1).} \end{aligned}$$

For a specific function, $f(z) = \sqrt{z} + \alpha z^2$,

$$f(z)^2 = z + 2\alpha z^{5/2} + \alpha^2 z^4,$$

$$F_0 = \frac{2}{3} + \frac{\alpha}{3}, \quad F_1 = \frac{2}{5} + \frac{\alpha}{4}, \quad \text{and} \quad F_2 = \frac{1}{2} + \frac{4\alpha}{7} + \frac{\alpha^2}{5},$$

thus,

$$\begin{aligned} S &= \frac{\alpha^2}{5} + \frac{4\alpha}{7} + \frac{1}{2} - 4\left(\frac{1}{9}(\alpha^2 + 4\alpha + 4) + 3\left(\frac{\alpha^2}{16} + \frac{\alpha}{5} + \frac{4}{25}\right) - (\alpha + 2)\left(\frac{\alpha}{4} + \frac{2}{5}\right)\right) \\ &= \alpha^2\left(\frac{1}{180}\right) - \alpha\left(\frac{2}{315}\right) + \frac{1}{450} \\ &= \alpha^2 a_2 + \alpha a_1 + a_0 \end{aligned}$$

with

$$a_2 = \frac{1}{180}, \quad a_1 = -\frac{2}{315}, \quad \text{and} \quad a_0 = \frac{1}{450},$$

August, 1993. Note 0141u-w.

therefore,

$$\boxed{S = \frac{1}{180}\left(\alpha^2 - \frac{8\alpha}{7} + \frac{2}{5}\right),}$$

$$\boxed{\frac{S}{F_2} = \frac{1}{36} \cdot \frac{\alpha^2 - \frac{8\alpha}{7} + \frac{2}{5}}{\alpha^2 + \frac{20\alpha}{7} + \frac{5}{2}}.}$$

For $\alpha \gg 1$, $\sqrt{S/F_2}$ is improved by a factor of 6, and for $\alpha \ll 1$, $\sqrt{S/F_2}$ is improved by a factor of 15.

S/F_2 is a strongly peaked function:

$(S/F_2)_{max} \approx .274$ at $\alpha \approx -1.65$,
$S/F_2 \approx .125$ at $\alpha \approx -3$,
$S/F_2 \approx .004$ at $\alpha \approx 0$.
$(S/F_2)_{min} \approx 4.5 \times 10^{-4}$ at $\alpha \approx .605$,

Since $a + bz$ represents the error-free condition, looking at the deviation of phase shift from the straight line may represent the best way to characterize the consequences of the error fields.

$$S_{min} = \frac{1}{180}\left(\frac{2}{5} - \frac{16}{49}\right) = \frac{1}{50 \cdot 49} \quad \text{with} \quad \alpha = \frac{4}{7},$$

and the values for a, b are

$$a = 4F_0 - 6F_1 = \frac{4}{3}(\alpha + 2) - 6\left(\frac{\alpha}{4} + \frac{2}{5}\right) = \boxed{-\frac{\alpha}{6} + \frac{4}{15},}$$

$$b = 12F_1 - 6F_0 = 3\left(\alpha + \frac{8}{5}\right) - 2(\alpha - 2) = \boxed{\alpha + \frac{4}{5}.}$$

Normalizations Factors ε_1 and ε_2 for Comparison of First and Second Order Phase Shifts, with Analytical Model of $b(z)$

At the center

$$\begin{aligned}\Delta B &= b(0)\\ &= \frac{V_1}{h}\frac{1}{k_0(x_2-x_1)}\ln\frac{\cosh k_0x_2+1}{\cosh k_0x_1+1}\\ &= \boxed{\frac{V_1}{h}\cdot\underbrace{\frac{\left(\ln\dfrac{\sinh k_0x_2/2}{\sinh k_0x_1/2}\right)}{\left(\dfrac{k_0(x_2-x_1)}{2}\right)}}_{g_0}},\end{aligned}$$

with $k_0 = \pi/h$ and $k_1 = 2\pi/\lambda$. Further,

$$\begin{aligned}\int b(z)\cos(k_1z)k_1dz &= \frac{V_1}{h}\cdot\frac{2}{\pi}\cdot k_1\cdot\frac{2\pi}{k_0}\cdot\int_{x_1}^{x_2}\frac{\sin k_1x}{\sinh(\pi k_1/k_0)}\cdot\frac{dx}{x_2-x_1}\\ &= \boxed{\frac{V_1}{h}\cdot\underbrace{\frac{4k_1}{k_0}\cdot\frac{\sin(k_1\Delta x/2)/(k_1\Delta x/2)}{\sinh(\pi k_1/k_0)}}_{g_1}},\end{aligned}$$

$$\boxed{\frac{V_1}{h}=\frac{\Delta B}{g_0},}$$

$$\boxed{\int b(z)\cos(k_1z)k_1dz=\frac{\Delta B}{g_0}g_1,}$$

$$\boxed{\int b\,dz=\frac{V_1}{h}\cdot\frac{\lambda}{2}}=\frac{\Delta B\cdot\lambda/2}{g_0}.$$

Thus,

$$\int\frac{\Delta B}{B_0}k_1dz=\varepsilon_2\pi\frac{\Delta B}{B_0}=\frac{\Delta B}{B_0}\,\frac{\lambda/2}{g_0}\,\frac{2\pi}{\lambda},$$

August, 1993. Note 0140u-w.

$$\boxed{\varepsilon_2 = \frac{1}{g_0}.}$$

Similarly,

$$\frac{1}{B_0}\int b\cos(k_1 z)k_1 dz = \frac{\Delta B}{B_0}\frac{g_1}{g_0} = \varepsilon_1 \pi \frac{\Delta B}{B_0},$$

$$\boxed{\varepsilon_1 = \frac{g_1}{\pi g_0}.}$$

Thus,

$$\boxed{\frac{\varepsilon_1}{\varepsilon_2^2} = \frac{g_0 g_1}{\pi},}$$

and

$$\frac{g_0 g_1}{\pi} = \frac{4}{\pi}\cdot \ln\left(\frac{\sinh k_0 x_2/2}{\sinh k_0 x_1/2}\right)\cdot \frac{\sin(k_1 \Delta x/2)}{\sinh(\pi k_1/k_0)}\cdot \frac{(k_1/k_0)^2}{(k_1 \Delta x/2)^2}.$$

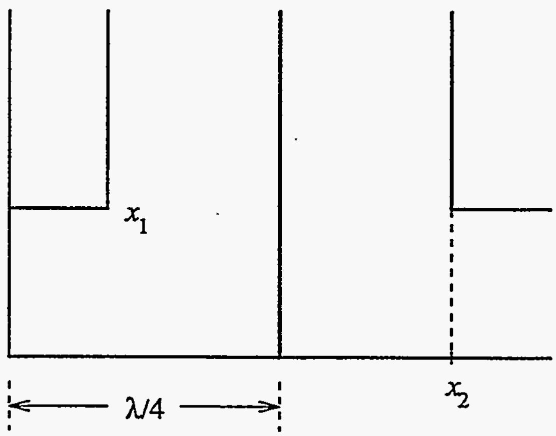


Figure 1.

For the above figure,

$$x_1 + \frac{1}{2}(x_2 - x_1) = \frac{\lambda}{4} = \frac{x_1 + x_2}{2},$$

$$\frac{\lambda/4}{x_1} = 3, \quad \boxed{x_1 = \frac{\lambda}{12},}$$

$$x_2 = \frac{\lambda}{2} - x_1 = \lambda\frac{5}{12}.$$

Further,

$$k_1 \frac{\Delta x}{2} = \frac{2\pi}{\lambda} \cdot \frac{1}{2} \cdot \frac{4\lambda}{12} = \boxed{\frac{\pi}{3},}$$

$$k_0 \frac{x_2}{2} = \frac{\pi}{h} \cdot \frac{5\lambda}{24} = \boxed{\frac{\lambda}{h} \cdot \frac{5\pi}{24},}$$

$$k_0 \frac{x_1}{2} = \frac{\pi}{h} \cdot \frac{\lambda}{24} = \boxed{\frac{\lambda}{h} \cdot \frac{\pi}{24},}$$

$$\frac{k_1}{k_0} = \frac{2h}{\lambda}, \quad \frac{\lambda}{h} = a.$$

Thus,

$$\frac{g_0 g_1}{\pi} = \frac{36 \cdot 4}{\pi^3} \cdot \frac{\sqrt{3}}{2} \cdot \ln\left(\frac{e^{a5\pi/12} - 1}{e^{a\pi/12} - 1}\right) \cdot \frac{1}{a^2} \cdot \frac{1}{\sinh(2\pi/a)},$$

where for $\dfrac{2h}{\lambda} = b = \dfrac{2}{a}$, and $a = \dfrac{2}{b}$,

$$\frac{g_0 g_1}{\pi} = \frac{36\sqrt{3}}{\pi^3} \cdot b^2 \cdot \ln\left(\frac{e^{5\pi/6b} - 1}{e^{\pi/6b} - 1}\right) \cdot \frac{1}{2\sinh(\pi b)}.$$

Comparison of First and Second Order Contributions of Error Fields to Phase Shift

We introduce, for $kz = \varphi$, $kdz = d\varphi$, and $\dfrac{\Delta B}{B_0} = D$,[†]

$$\Delta\Phi = P\left(\underbrace{-\cos\varphi\int D d\varphi}_{(A_1)} + \underbrace{\int \cos\varphi D d\varphi}_{(B_1)} + \underbrace{\frac{1}{2}\int\left(\int D d\varphi\right)^2 d\varphi}_{(B_2)}\right),$$

where

$$P = \frac{4}{1 + 2/K^2}.$$

Notice that $\Delta\Phi \approx K^2$ for $K^2 \ll 2$, and $\Delta\Phi$ is independent of K^2 for $K^2 \gg 2$.

We denote the "typical" case of B_1 as

$$B_1 = \varepsilon_1 D \frac{\lambda}{2}\frac{2\pi}{\lambda} = \varepsilon_1 \pi D.$$

At every error source, B_1 changes by the above "typical" value of B_1[‡]. We assume n contributions per period. After $N = z/\lambda$ periods, the total expectation value is

$$\overline{B_1^2} = n\frac{z}{\lambda}\overline{(\varepsilon_1 \pi D)^2}.$$

When

$$B_1 = \int D d\varphi,$$

we expect

$$\overline{B_1^2} = n\frac{z}{\lambda}\overline{(\varepsilon_2 \pi D)^2}.$$

At the end of insertion device with N periods we expect

$$\langle B_1^2\rangle = \varepsilon_1^2 \pi^2 \overline{D^2} n N,$$

August, 1993. Note 0139u-w.

† See document 0138u-w for the origins of this equation.

‡ See document 0140u-w for derivations of ε_1 and ε_2.

$$
\begin{aligned}
\langle B_2 \rangle &= \frac{1}{2}\varepsilon_2^2 \pi^2 \overline{D^2} n \frac{(N\lambda)^2}{2\lambda}\frac{2\pi}{\lambda} \\
&= \frac{1}{2}\varepsilon_2^2 \pi^3 \overline{D^2} n N^2 \\
&= \varepsilon_3 \sqrt{\langle B_1^2 \rangle} \quad \text{with} \quad \varepsilon_3 < 1.
\end{aligned}
$$

This means that

$$
\frac{1}{2}\varepsilon_2^2 \pi^3 \overline{D^2} n N^2 = \varepsilon_3 \varepsilon_1 \pi \sqrt{\overline{D^2}} \sqrt{n} \sqrt{N},
$$

$$
\boxed{\sqrt{\overline{D^2}} = \frac{2\varepsilon_3\varepsilon_1/\varepsilon_2^2}{\pi^2\sqrt{n}\left(\sqrt{N}\right)^3}.}
$$

We make the following definitions:

$$
\varepsilon_1 = \varepsilon_2 = 1, \quad 2\varepsilon_3 = 1, \quad n = 4 \quad \text{and} \quad N = 81.
$$

Thus,

$$
\sqrt{\overline{D^2}} \approx \frac{1}{20 \times 10^3} \approx 5 \times 10^5.
$$

This means that the second order contributions will dominate. Or, similarly

$$
\varepsilon_3 = \frac{\pi^2}{2}\frac{\varepsilon_2^2}{\varepsilon_1}\sqrt{\overline{D^2}}\sqrt{n}\left(\sqrt{N}\right)^3,
$$

and for $\sqrt{\overline{D^2}} = 10^{-3}$,

$$
\varepsilon_3 \approx 5 \times 2 \times 10^{-3+} \times 10^3 \approx 10
$$

still demonstrating that second order contributions will dominate.

The magnitude of the $\Delta\Phi$ shift along the length of the insertion device with second order contributions is

$$
\begin{aligned}
\Delta\Phi &= 4\frac{1}{2}\varepsilon_2^2 \pi^3 \overline{D^2} n N^2 \\
&\approx 2 \times 30 \times 10^{-6} \times 4 \times 6.5 \times 10^3 \\
&\approx 30 \times 50 \times 10^{-3} \\
&\approx 1.5 \text{ radians.}
\end{aligned}
$$

Thus,

$$
\frac{\varepsilon_1}{\varepsilon_2^2} \approx .5 \rightarrow D
$$

for equal contributions (i.e.: $\varepsilon_3 = 1$), and for

$$
D \approx 5 \times 10^{-5} \rightarrow D \approx 5 \times 10^{-3} \rightarrow \alpha = 100
$$

for representation of phase shift by straight line.

Connection Between Undulator Field Errors and Optical Phase

We begin with the following definitions:

$$x'' = \frac{g}{\gamma}B \quad \text{and} \quad g = \frac{e}{m_0 c}.$$

We introduce the following references:

$$B(z) = B_0 \cos kz,$$

$$x_0' = \frac{gB_0}{\gamma k}\sin kz = \frac{K}{\gamma}\sin kz \quad \text{with} \quad \frac{gB_0}{k} = K,$$

$$x_0 = -\frac{K}{k\gamma}\cos kz,$$

$$\boxed{x_W = \frac{K}{k\gamma}.}$$

We now proceed with the analysis.

$$\Delta s = \int \left(\sqrt{1+(x_0'+\Delta x')^2} - \sqrt{1+(x_0')^2}\right) dz = \int \left(x_0'\Delta x' + \frac{1}{2}\Delta(x')^2\right) dz.$$

By integration by parts, with $du = x_0' dz$, $u = x_0$, $v = \Delta x'$, and $dv = \Delta x'' dz$,

$$\Delta s = x_0 \Delta x' - \int x_0 \Delta x'' dz + \frac{1}{2}\int \Delta(x')^2 dz,$$

with

$$\Delta x' = \frac{gB_0}{\gamma k}\int \frac{\Delta B}{B_0} k dz \quad \text{and} \quad \Delta x'' = \frac{gB_0}{\gamma}\frac{\Delta B}{B_0},$$

where $gB_0/\gamma k = K/\gamma$, and thus

$$\Delta s = \frac{K^2}{\gamma^2 k}\left(-\cos kz \int \frac{\Delta B}{B_0} k dz + \int \cos(kz)\frac{\Delta B}{B_0} k dz + \frac{1}{2}\int \left(\int \frac{\Delta B}{B_0} k dz\right)^2 k dz\right).$$

Furthermore,

$$\Delta t = \frac{\Delta s}{c},$$

August, 1993. Note 0138u-w.

$$\Delta\varphi = \omega\Delta t = \Delta s\frac{\omega}{c} = \Delta s \cdot k_L,$$

where,

$$\lambda_L = \frac{\lambda}{2\gamma^2}\left(1+\frac{K^2}{2}\right),$$

$$k_L = k\frac{4\gamma^2}{K^2\left(1+\dfrac{2}{K^2}\right)},$$

and therefore,

$$\Delta\varphi = P\left(\underbrace{-\cos kz\int\frac{\Delta B}{B_0}kdz}_{(a)} + \underbrace{\int\cos(kz)\frac{\Delta B}{B_0}kdz}_{(b)} + \underbrace{\frac{1}{2}\int\left(\int\frac{\Delta B}{B_0}kdz\right)^2 kdz}_{(c)}\right),$$

with

$$P = \frac{4}{1+2/K^2}.$$

Term (c) is of second order and is important only for a long insertion device. Term (a) gives harmonics and reduces line intensity for steering errors, but produces no effect if there is no steering. Term (b) produces phase shift and line broadening. Whether or not it is equal 0 depends on such elements as symmetry, but not on presence of net steering.

ϱ, A_0/B_1 for Hybrid Insertion Device

This note is a result of ANL lecture notes, and from *Simple Analytical Model For Fields From One Pole Of Hybrid Insertion Device,*† with $k_1 = 2\pi/\lambda$ and $k_n = nk_1$, and

$$\boxed{B(x) = \sum_{n=\text{odd}} B_n \cos k_n x,}$$

from poles with $\pm$ "1", and

$$\boxed{B_n = \frac{2}{\pi} \int_{-\infty}^{\infty} b(x) \cos k_n x \, dk_1 x,}$$

where $b(x)$ is the field from one pole with excitation $+1$.

For V from 0 to V_1 at the edge of pole going from $-x_1$ to x_1,

$$\boxed{B_n = 4\frac{k_1}{k_0} \frac{\sin k_n x_1}{\sinh(\pi k_n/k_0)} \frac{V_1}{h}} \quad \text{with} \quad k_0 = \pi/h.$$

For V going linearly from 0 to V_1 over thickness of CSEM $= (x_2 - x_1)/2$, we have

$$\boxed{A_0 = \int_{-\infty}^{\infty} b(x) dx = \frac{V_1}{h} \frac{\lambda}{2},}$$

$$\boxed{B_n = 4\frac{k_1}{k_0} \frac{V_1}{h} \frac{\sin(k_n(x_2 - x_1)/2)}{(k_n(x_2 - x_1)/2)} \frac{1}{\sinh(\pi k_n/k_0)}.}$$

For $k_1(x_2 - x_1)^2/6 \ll 1$,

$$\boxed{B_1 = 4\frac{k_1}{k_0} \frac{V_1}{h} \frac{1}{\sinh(\pi k_1/k_0)} = \frac{4}{\pi} \frac{V_1}{h} \frac{\alpha}{\sinh \alpha}} \quad \text{with} \quad \alpha = \pi\frac{k_1}{k_0} = 2\pi\frac{h}{\lambda} = \pi\frac{g}{\lambda}.$$

We may now conclude that

$$\boxed{\frac{A_0}{B_1} = \lambda \frac{\pi}{8} \frac{\sinh \alpha}{\alpha}.}$$

August, 1993. Note 0137u-w.

† Document 0136u-w

For

$$\varrho = \frac{A_0}{\int_{-\lambda/4}^{\lambda/4} B_1 \cos k_1 x \, dx} = \frac{B_1 \lambda \frac{\pi}{8} \frac{\sinh \alpha}{\alpha}}{B_1 (2/k_1)} = \frac{k_1 \lambda \pi}{16} \frac{\sinh \alpha}{\alpha} = \frac{\pi^2}{8} \frac{\sinh \alpha}{\alpha},$$

we have

ϱ	1.5	2.0	2.5	3.0	4.0	5.0
g/λ	.35	.57	.70	.79	.94	1.05

Simple Analytical Model for Fields from One Pole of Hybrid Insertion Device

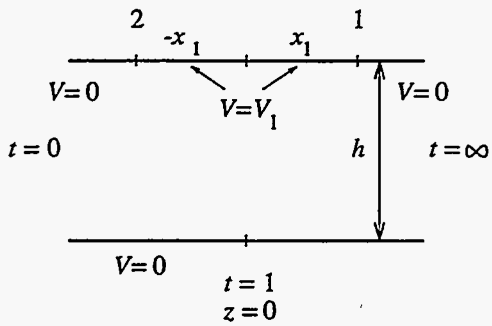


Figure 1.

Model: midplane on $V = 0$, and pole from $-\infty$ to $+\infty$ on $V = 0$, except on $V = V_1$ for $-x_1 \leq x \leq x_1$.

The above geometry is described by the following conformal map

$$\pi z = \frac{h}{t},$$

and the following elements

$$k_0 z = \ln t, \quad t = e^{k_0 z} \quad \text{and} \quad k_0 = \frac{\pi}{h},$$

and where x_1 is the half-width of the pole. Putting $\pm$ current filaments at $x = \pm x_1$,

$$\pi F = V_1 \ln \frac{(t - t_1)}{(t - t_2)},$$

$$\begin{aligned} F' &= \frac{V_1}{h}\left(\frac{t}{t - t_1} - \frac{t}{t - t_2}\right) \\ &= \frac{V_1}{h}\left(\frac{1}{t/t_1 - 1} - \frac{1}{t/t_2 - 1}\right). \end{aligned}$$

where $t = e^{k_0 z}$, $t_1 = e^{k_0(x_1 + ih)} = -e^{k_0 x_1}$. Thus,

August, 1993. Note 0136u-w.

$$
\begin{aligned}
F'\frac{h}{V_1} &= \frac{1}{e^{k_0(z+x_1)}+1} - \frac{1}{e^{k_0(z-x_1)}+1} \\
&= -\frac{e^{k_0 z}\left(e^{k_0 x_1} - e^{-k_0 x_1}\right)}{e^{2k_0 z}+1+e^{k_0 z}\left(e^{k_0 x_1}+e^{-k_0 x_1}\right)} \\
&= \boxed{-\frac{\sinh k_0 x_1}{\cosh k_0 z + \cosh k_0 x_1}} = -b(z). \qquad (1)
\end{aligned}
$$

The odd harmonics of the field are described by

$$
B_N = a \int_{-\infty}^{\infty} b(x)\cos(Nk_1 x)dx, \qquad (2)
$$

where $N = 2n+1$, $k_N = k_1 N = k_1(2n+1)$, $k_1 = 2\pi/\lambda$, and a is a constant, thus

$$
\begin{aligned}
\frac{B_N}{a} &= \sinh k_0 x_1 \int_{-\infty}^{\infty} \frac{\cos(k_N x)dx}{\cosh k_0 x + \cosh k_0 x_1} \\
&= \sinh k_0 x_1 \cdot G_N, \qquad (3)
\end{aligned}
$$

$$
G_N = \Re \int_{-\infty}^{\infty} \frac{e^{ik_N x}}{\cosh k_0 x + \cosh k_0 x_1} dx, \qquad (4)
$$

That is, to evaluate this integral, one can integrate a line integral along the real axis of the complex z-plane, and close it at ∞ in the upper half-plane without changing its value.

There are singularities at $\cosh k_0 z = -\cosh k_0 x_1$ in the upper half-plane, with $k_0 z = \pm k_0 x_1 + i\pi(2m+1)$, for $m = 0, 1, 2, \ldots$. We take the first singularity at $k_0 z = +k_0 x_1 + i\pi M$ and do others later by replacing x_1 by $-x_1$. We integrate over the upper half-plane:

$$
\begin{aligned}
G_{N+} &= \Re \int \frac{e^{ik_N z}\,dz}{\cosh k_0 z + \cosh k_0 x_1} \\
&= \Re \sum_{m=0} \frac{e^{ik_N(x_1 + i\pi M/k_0)}}{k_0 \sinh k_0 z_m} \cdot 2\pi i \\
&= +\frac{2\pi}{k_0} \cdot \frac{\sin k_N x_1}{\sinh k_0 x_1} \cdot \sum_{m=0} e^{-\pi(k_N/k_0)(2m+1)} \\
&= +\frac{2\pi}{k_0} \cdot \frac{\sin k_N x_1}{\sinh k_0 x_1} \cdot \frac{e^{-\pi k_N/k_0}}{1 - e^{-2\pi k_N/k_0}} \\
&= +\frac{2\pi}{k_0} \cdot \frac{\sin k_N x_1}{\sinh k_0 x_1} \cdot \frac{1}{2\sinh(\pi k_N/k_0)} \\
&= \frac{\pi}{k_0} \cdot \frac{\sin k_N x_1}{\sinh k_0 x_1} \cdot \frac{1}{\sinh(\pi k_N/k_0)}.
\end{aligned}
$$

One solves for G_{N-} similarly. Thus, we may re-write (3),

$$
\frac{B_N}{a} = \frac{2\pi}{k_0} \cdot \frac{\sin k_N x_1}{\sinh(\pi k_N/k_0)}, \tag{5}
$$

and further,

$$
\boxed{\frac{B_{(2n+1)}}{B_1} = \frac{\sin(k_1(2n+1)x_1)}{\sinh((\pi k_1/k_0)(2n+1))} \cdot \frac{\sinh(\pi k_1/k_0)}{\sin(k_1 x_1)},} \tag{6}
$$

$$
\text{with} \quad k_0 = \frac{\pi}{h}, \quad k_1 = \frac{2\pi}{\lambda}, \quad \text{and thus} \quad \pi\frac{k_1}{k_0} = 2\pi\frac{h}{\lambda}.
$$

This model of $b(z)$, and the resultant B_N, assume that the potential increases like a step function at the edge of the pole. As a next approximation, to improve this model, one would assume that the potential increases linearly over the size of the CSEM and represent this by the operation

$$
\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} dx_1,
$$

which is easily executed on both $b(z)$ and B_N. For $b(z)$ we have

$$
\int \frac{\sinh k_0 x_1}{\cosh k_0 z + \cosh k_0 x_1} \cdot \frac{dx_1}{(x_2 - x_1)} = \frac{1}{k_0(x_2 - x_1)} \ln \frac{\cosh k_0 z + \cosh k_0 x_2}{\cosh k_0 z + \cosh k_0 x_1}, \tag{7}
$$

and for B_N we have

$$\int \sin(Nk_1x_1)\frac{dx_1}{(x_2-x_1)} = \frac{\cos Nk_1x_1 - \cos Nk_1x_2}{Nk_1(x_2-x_1)}$$

$$= -\sin\left((2n+1)k_1\frac{x_2+x_1}{2}\right)\frac{\sin\left((2n+1)k_1\dfrac{x_2-x_1}{2}\right)}{\dfrac{(2n+1)}{k_1}\cdot\dfrac{x_2-x_1}{2}} \quad (8)\ .$$

where

$$-\sin\left(k_1(2n+1)\left(\frac{x_2+x_1}{2}\right)\right) = (-1)^{n+1},$$

$$\frac{x_2+x_1}{2} = \frac{\lambda}{4} \quad \text{and} \quad (2n+1)k_1\frac{x_2-x_1}{2} = \frac{\pi}{2} + n\pi.$$

The argument of the log function can, and should be, operated on in the same manner, such that for

$$\cosh k_0 z = C_0, \quad \cosh k_0 x_1 = C_1, \quad \text{and} \quad \cosh k_0 x_2 = C_2,$$

$$\frac{C_0+C_2}{C_0+C_1} = \frac{C_0+a+b}{C_0+a-b} = \frac{1+\dfrac{b}{C_0+a}}{1-\dfrac{b}{C_0+a}}$$

where

$$a = \frac{C_2+C_1}{2}, \quad b = \frac{C_2-C_1}{2}, \quad \text{and} \quad \ln\frac{H\varepsilon}{1-\varepsilon} = 2\left(\varepsilon + \frac{\varepsilon^3}{3} + 10\right).$$

Wiggler Parameter K Definitions

For $v = c$ we have

$$m_0 \gamma v^2 x'' = evB = evA',$$

$$x' = \frac{1}{\gamma}\frac{e}{m_0 v} A \quad \text{and} \quad x'_{\text{max}} = \frac{K_1}{\gamma}.$$

Definition 1:

$$\boxed{K_1 = \frac{e}{m_0 c} A_{\text{max}} = \frac{e}{2\pi m_0 c} 2\pi A_{\text{max}}.}$$

For a pure sinusoidal field we have

$$B = B_0 \sin kz \quad \text{and} \quad A = \frac{B_0}{k} \cos kz.$$

Thus

$$A_{\text{max}} = \frac{B_0}{k} \quad \Longrightarrow \quad \boxed{K_1 = \frac{e}{2\pi m_0 c} B_0 \lambda_u.}$$

Definition 2:

The "path length" slippage in λ_u equals λ_{light}. (We shall refer to λ_{light} as λ_{L} for the remainder of this document.)

$$\Delta t = \frac{s}{c\beta} - \frac{\lambda_u}{c} = \frac{\lambda_{\text{L}}}{c} \quad \text{and thus} \quad \lambda_{\text{L}} = \lambda_u \left(\frac{s}{\lambda_u \beta} - 1 \right)$$

where s = path length over one period, and $s' = \sqrt{1 + x'^2}$.

Proceeding from above, we now have that

$$\frac{\lambda_{\text{L}}}{\lambda_u} = \frac{1}{\beta \lambda_u} \int_0^{\lambda_u} \left(1 + \frac{x'^2}{2} \right) dz - 1$$

$$= \frac{1}{\beta} - 1 + \frac{1}{2\lambda_u} \int_0^{\lambda_u} x'^2 \, dz.$$

June, 1993. Note 0135u-w.

By introducing

$$\beta^2 + \frac{1}{\gamma^2} = 1 \quad \text{and thus} \quad \beta^{-1} = \left(1 - \frac{1}{\gamma^2}\right)^{-1/2} = 1 + \frac{1}{2\gamma^2},$$

we further simplify

$$\begin{aligned}\frac{\lambda_L}{\lambda_u} &= \frac{1}{2\gamma^2} + \frac{1}{2\lambda_u}\int_0^{\lambda_u} x'^2 dz \\ &= \frac{1}{2\gamma^2}\left(1 + \underbrace{\left(\frac{e}{m_0 c}\right)^2 \frac{\int_0^{\lambda_u} A^2 dz}{\lambda_u}}_{K_2^2/2}\right).\end{aligned}$$

and we now arrive at our definition

$$\boxed{K_2^2 = \left(\frac{e}{m_0 c}\right)^2 \frac{2}{\lambda_u}\int_0^{\lambda_u} A^2 dz\,.}$$

For $A = \dfrac{B_0}{k}\cos kz$ we have

$$K_2^2 = \left(\frac{e}{m_0 c}\right)^2 \frac{2}{\lambda_u}\frac{B_0^2}{k^2}\frac{\lambda_u}{2} = \left(\frac{e}{2\pi m_0 c}B_0\lambda_u\right)^2$$

where $(e/2\pi m_0 c) = .934 \cdot 10^2$ in SI units.

We define $(2\pi m_0 c/e) = A_e$ and thus $1/A_e = .934 \cdot 10^2$ MKS.
We now reformulate our **Definitions 1** and **2** such that

$$\boxed{K_1 = \frac{2\pi}{A_e}A_{\max}}$$

and

$$\boxed{K_2 = \frac{2\pi}{A_e}\sqrt{\frac{2\int_0^{\lambda_u/4} A^2 dz}{\lambda_u/4}}.}$$

Definition 3:

$$K_3 = \frac{B_0\lambda_u}{A_e} = \frac{V_0}{A_e}\,4\frac{\lambda_u/4}{D_4}.$$

Where D_4 refers to the NPOLE1.BAS program variable which describes the distance factor in the transformation from scalar potential to the field.

NPOLE

A recreation, with "Korea modification," of a program (for HP71B) to design and analyze $\lambda/4$ of hybrid insertion device.

We will begin by establishing some background information for the conformal map and the limits for t_1 and t_2.

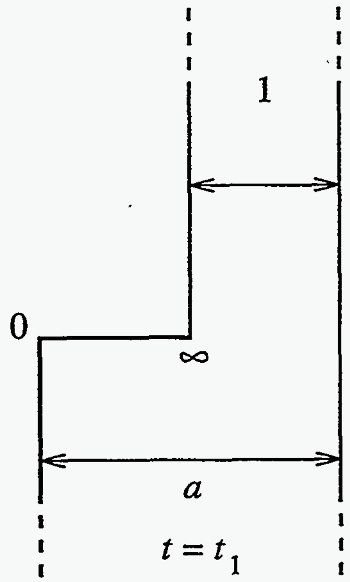


Figure 1.

For the map of Figure 1,

$$\pi \dot{z} = -i \frac{1 - t_1}{\sqrt{t}(t - t_1)(t - 1)}$$

$$a = \frac{1}{\sqrt{t_1}} \quad \text{and thus} \quad t_1 = \frac{1}{a^2}.$$

June, 1993. Note 0134u-w.

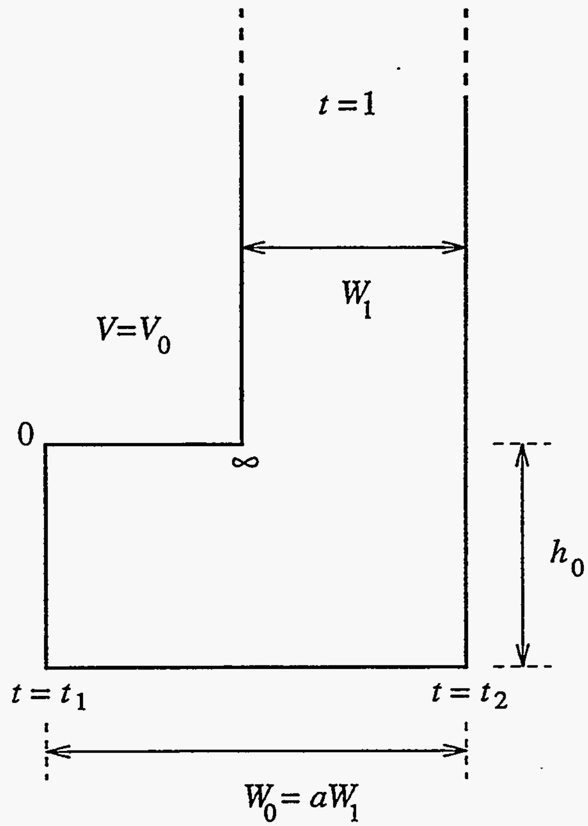


Figure 2.

We proved in Korea in 1987 that

$$0 < t_1 < 1/a^2 < t_2 < 1 \quad \text{and} \quad \frac{1}{a^2} = \left(\frac{W_1}{W_0}\right)^2$$

for the geometry of Figure 2.

Therefore, in program NPOLE1.BAS (included at the end of this document),

$$t_1 = \frac{1}{a^2}\text{RANG(C1)}, \quad t_2 = \frac{1}{a^2} + \left(1 - \frac{1}{a^2}\right)\text{RANG(C2)},$$

with $0 < \text{RANG}(x) < 1$ as used in the first version, and $\text{RANG}(x) = 1/(1+e^x)$ as specifically used now.

The map for geometry with corners at t_1, t_2 is described by

$$\boxed{\dot{z} = -iW_1 \frac{Q_1}{\sqrt{t}\sqrt{t-t_1}\sqrt{t-t_2}(t-1)}} \quad \text{and} \quad \boxed{Q_1 = \frac{\sqrt{1-t_1}\sqrt{1-t_2}}{\pi}.}$$

We determine t_1 and t_2 from

$$\boxed{\frac{h_0}{W_1} = Q_1 \int_0^{t_1} \frac{dt}{\sqrt{t}\sqrt{t_1 - t}\sqrt{t_2 - t}(1-t)}}$$

$$\boxed{\frac{W_0}{W_1} = Q_1 \int_{t_1}^{t_2} \frac{dt}{\sqrt{t}\sqrt{t - t_1}\sqrt{t_2 - t}(1-t)}}.$$

To evaluate the integrals, we use

$$\boxed{\int_{t_1}^{t_2} \frac{f(t)dt}{\sqrt{t-t_1}\sqrt{t_2-t}} = 3\int_{-1}^{1} \frac{f(t)}{\sqrt{4-x^2}}\,dx\,,}$$

$$\boxed{t = \frac{2(t_2+t_1) + (t_2 - t_1)\,x(3-x^2)}{4}}$$

We use Gaussian integration with segmented intervals for testing and accuracy purposes. We use a "2D" secant equation solver to determine t_1 and t_2 from the above integrals.

We describe the complex potentials for fluxes, fields:

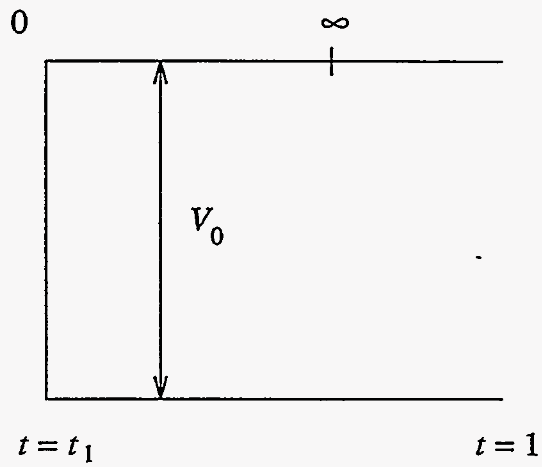


Figure 3.

$$\dot{F} = -\frac{Q_2 V_0}{\sqrt{t}\sqrt{t-t_1}\,(t-1)} \quad \text{and} \quad Q_2 = \frac{\sqrt{1-t_1}}{\pi}.$$

We therefore have

$$F = -Q_2 V_0 \int \frac{dt}{t^2 \sqrt{1 - t_1/t}\,(1 - 1/t)} = -Q_2 \int \frac{2du}{t_1\,(1 - (1 - u^2)/t_1)} V_0$$

where

$$\sqrt{1 - \frac{t_1}{t}} = u, \quad \frac{t_1}{t} = 1 - u^2, \quad \text{and thus} \quad \frac{dt}{t^2} = \frac{2udu}{t_1}$$

$$\boxed{1 - t_1 = t_3^2\,.}$$

Thus,

$$\begin{aligned}
\frac{1}{V_0} F &= -Q_2 \int \frac{2du}{u^2 - t_3^2} \\
&= -Q_2 \int \left(\frac{1}{u - t_3} - \frac{1}{u + t_3} \right) \frac{du}{t_3} \\
&= \frac{Q_2}{t_3} \ln \frac{u + t_3}{u - t_3} \\
&= \boxed{\frac{Q_2}{t_3} \ln \frac{\sqrt{1 - t_1/t} + t_3}{\sqrt{1 - t_1/t} - t_3}\,.}
\end{aligned}$$

Flux into pole / V_0:

$$\begin{aligned}
E_p &= \frac{Q_2}{t_3} \ln \left| \frac{\sqrt{1 + t_1/\tau} + t_3}{\sqrt{1 + t_1/\tau} - t_3} \right| \Bigg|_0^\infty \\
&= \boxed{\frac{Q_2}{t_3} \ln \frac{1 + t_3}{1 - t_3}\,.}
\end{aligned}$$

Flux into midplane (for K):

$$\begin{aligned}
E_M &= \left(F(t_2) - F(t_1) \right) / V_0 \\
&= \boxed{\frac{Q_2}{t_3} \ln \frac{t_3 + \sqrt{1 - t_1/t_2}}{t_3 - \sqrt{1 - t_1/t_2}}\,,}
\end{aligned}$$

with $K_1 = 2\pi V_0 E_M (1/A_e)$ where $(1/A_e) = .934 \cdot 10^2$ MKS.

We calculate the excess flux coefficient for the side of the pole ($V_0 = 1$):

$$\left(A(t) - A(\infty) = \frac{y(t) - y(\infty)}{W_1} + E_s \right)_{t \to 1}$$

$$E_s = \int_1^\infty \left(-\dot{F} + \frac{\dot{z}}{i}\right) dt = \int_1^\infty \frac{1}{\sqrt{t}\sqrt{t-t_1}(t-1)} \underbrace{\left(Q_2 - \frac{Q_1}{\sqrt{t-t_2}}\right)}_{G_1} dt$$

where

$$\begin{aligned} G_1 &= Q_2\left(1 - \frac{\sqrt{1-t_2}}{\sqrt{t-t_2}}\right) \\ &= \frac{Q_2}{\sqrt{t-t_2}}(\sqrt{t-t_2} - \sqrt{1-t_2}) \\ &= \frac{Q_2(t-1)}{\sqrt{t-t_2}\,(\sqrt{t-t_2}+\sqrt{1-t_2})} \end{aligned}$$

and thus

$$\begin{aligned} E_s &= Q_2 \int_1^\infty \frac{dt}{\sqrt{t}\sqrt{t-t_1}\sqrt{t-t_2}\,(\sqrt{t-t_2}+\sqrt{1-t_2})} \\ &= \boxed{Q_2 \int_0^1 \frac{dt}{\sqrt{1-t_1 t}\sqrt{1-t_2 t}\,(\sqrt{t-t_2 t}+\sqrt{t}\sqrt{1-t_2})}.} \end{aligned}$$

The field B_0 at $t = t_1$ is $(\dot{F}/\dot{z})$. With V_0 on pole,

$$B_0 = \frac{V_0}{W_1}\frac{Q_2}{Q_1}\sqrt{t_2 - t_1} = \frac{V_0}{W_1}\frac{\sqrt{t_2-t_1}}{\sqrt{1-t_2}} = \frac{V_0}{D_4}$$

where D_4 is an old notation and

$$\boxed{D_4 = W_1 \frac{\sqrt{1-t_2}}{\sqrt{t_2-t_1}}.}$$

For the second definition of K,

$$K_2 = 2\pi \frac{1}{A_e}\sqrt{\frac{2\int_0^{\lambda_u/4} A^2 dz}{\lambda_u/4}}.$$

We need $\int F^2 dz$, thus,

$$\int A^2 dz = \int_{t_1}^{t_2} F^2 \dot{z}\, dt = G_2 .$$

Therefore, we have that

$$G_2 = \frac{Q_1 Q_2^2}{t_3^2} V_0^2 W_1 \int_{t_1}^{t_2} \left(\ln \frac{t_3 + \sqrt{1-t_1/t}}{t_3 - \sqrt{1-t_1/t}}\right)^2 \frac{dt}{\sqrt{t}\sqrt{t-t_1}\sqrt{t_2-t}(1-t)} .$$

$$\frac{G_2}{\lambda_u/4} = \frac{G_2}{W_0} = V_0^2 \frac{W_1}{W_0} \frac{Q_1 Q_2^2}{t_3^2} J$$

where

$$J = \int_{t_1}^{t_2} \left(\ln \frac{t_3 + \sqrt{1 - t_1/t}}{t_3 - \sqrt{1 - t_1/t}} \right)^2 \frac{dt}{\sqrt{t}\sqrt{t - t_1}\sqrt{t_2 - t}(1 - t)},$$

and thus we may summarize

$$\boxed{K_2 = 2\pi V_0 \frac{1}{A_e} \sqrt{\frac{2 W_1 Q_2^2 Q_1}{t_3^2 W_0} J}\ .}$$

Program NPOLE1.BAS

```
PRINT:PRINT DATE$;"  ";TIME$;"  NPOLE1"
'GOTO BYPASS
PRINT "Determines parameter values and evaluates flux into midplane (Em) and"
PRINT "pole (Ep) of ID, and excess flux coefficient for side of pole (Es)."
PRINT "K1,K2,K3 are obtained by multiplying the printed values by the scalar"
PRINT "potential of pole in Tcm.  K1 is for maximum deflection angle, K2 for"
PRINT "trajectory length effect, and K3 for B0*period.  D4=V0/B0 for V0=1."
REM--List of P1() elements:0>W01^(-2),1>T1,2>T2,3>T3,5>Q1,6>Q2,9>Function ID
BYPASS:
DEFINT J:DEFDBL A-Z
PI=4*ATN(1):A1$="##.###^^^^ ":TAP=0
A2$=" Em         Ep         Es         K1         K2         K3         D4"
A3$="                 H0=##.##   W0=##.##   W1=##.##"
DIM P1(0:9),GX(1:4),GW(1:4)
SHARED PI,GX(),GW(),P1(),A1$
REM--GX,GW=(normalized) abscissas; P1=parameters for Gauss integrator
DATA .1834346425,.3626837834,.5255324099,.3137066459
DATA .7966664774,.2223810345,.9602898565,.1012285363
FOR J1=1 TO 4:READ GX(J1),GW(J1):NEXT J1:REM--Abscissas, weights for Gauss
REM-------------------------------------------------------------------------
'C10=1:C20=1
PRINT:PRINT TAB(TAP);:PRINT A2$

DO
AGAIN:
INPUT;"H0,W0,>W1=",H00,W00,W10:REM--INPUT unnormalized 1/2gap, period/4,
IF H00>0 THEN H0=H00:REM--pole to symmetry line distance, stored temporarily
IF W00>0 THEN W0=W00:REM--in H00,W00,W10, then in H0,W0,W1(=not-normalized).
IF W10>0 THEN W1=W10:REM--H01,W01=normalized with W1, used in program.
IF H00=0 AND W00=0 AND W10=0 THEN END
IF W0<W1 THEN PRINT TAB(20);:PRINT "W0 must be larger than W1!":GOTO AGAIN
PRINT TAB(24);:PRINT USING A3$;H0;W0;W1
H01=H0/W1:W01=W0/W1:P1(0)=1/(W01*W01):DC1=.1:DC2=.1
GOSUB SOLVIT
PRINT TAB(TAP);:PRINT USING A1$;EM;EP;ES;K1;K2;K3;D4
LOOP

SOLVIT:
C11=C10+DC1:C21=C20:C12=C10:C22=C20+DC2
CALL EVAL(C10,C20,S10,S20):S10=S10-H01:S20=S20-W01:S00=ABS(S10)+ABS(S20)
CALL EVAL(C11,C21,S11,S21):S11=S11-H01:S21=S21-W01:S01=ABS(S11)+ABS(S21)
CALL EVAL(C12,C22,S12,S22):S12=S12-H01:S22=S22-W01:S02=ABS(S12)+ABS(S22)
DO
```

```
REM--REARR puts "worst" set into last column, to be discarded later
GOSUB REARR
N1=1/((S11-S10)*(S22-S20)-(S12-S10)*(S21-S20)): REM--Start of 2D secant
D1=N1*(S20*S12-S10*S22):D2=N1*(S10*S21-S20*S11):REM--equation solver
DC1=(C11-C10)*D1+(C12-C10)*D2:DC2=(C21-C20)*D1+(C22-C20)*D2
C12=C11:C22=C21:C11=C10:C21=C20:S12=S11:S22=S21:S11=S10:S21=S20
S02=S01:S01=S00:C10=C10+DC1:C20=C20+DC2:REM--Recommended new parameters
CALL EVAL(C10,C20,S10,S20):S10=S10-H01:S20=S20-W01:S00=ABS(S10)+ABS(S20)
LOOP UNTIL S00<.001
P1(9)=3:CALL SGAUSSINT8(0,1,G2,-.001):Q2=P1(6):T1=P1(1):T2=P1(2):T3=P1(3)
Q1=P1(5):ES=Q2*G2:EP=Q2/T3*LOG((1+T3)/(1-T3)):D4=W1*SQR((1-T2)/(T2-T1))
EM=Q2/T3*LOG(2/(1-SQR(1-T1/T2)/T3)-1):K1=2*PI*EM*.934:K3=4*W0/D4*.934
P1(9)=4:CALL SGAUSSINT8(-1,1,K2,-.001):K2=2*PI*Q2/P1(3)*SQR(2*Q1*3*K2/W01)*.934
RETURN

REARR:
IF S00>S01 THEN GOSUB SW01
IF S01>S02 THEN GOSUB SW12
RETURN

SW01:
SWAP S00,S01:SWAP S10,S11:SWAP S20,S21:SWAP C10,C11:SWAP C20,C21:RETURN
SW12:
SWAP S01,S02:SWAP S11,S12:SWAP S21,S22:SWAP C11,C12:SWAP C21,C22:RETURN

SUB EVAL(C1,C2,S1,S2):REM--Calculates H01,W01 for set of parameters C1,C2>T1,T2
T1=P1(0)*RANG(C1):T2=P1(0)+(1-P1(0))*RANG(C2)
T3=SQR(1-T1):P1(1)=T1:P1(2)=T2:P1(3)=T3
Q2=T3/PI:Q1=Q2*SQR(1-T2):P1(5)=Q1:P1(6)=Q2
P1(9)=1:CALL SGAUSSINT8(-1,1,G2,-.001):S1=3*Q1*G2
P1(9)=2:CALL SGAUSSINT8(-1,1,G2,-.001):S2=3*Q1*G2
END SUB

SUB SGAUSSINT8(X0,X3,G2,DG):REM--Gauss integrator, with interval segmentation
IF DG>0 THEN E1=DG:E2=0 ELSE E1=0:E2=-DG:REM--For DG>/<0,DG=absol./rel.  error
CALL GAUSSINT8(X0,X3,G2):J1=1:J4=16:REM--J4=largest # subdiv.
DO
  G1=G2:G2=0:J1=2*J1:DX=(X3-X0)/J1:REM--G1/G2=last/next computed integral
  IF J1>J4 THEN PRINT "   Not converged":END
  FOR J2=0 TO J1-1
    CALL GAUSSINT8(X0+J2*DX,X0+(J2+1)*DX,G3)
    G2=G2+G3:REM---------------------------------G2=integral
  NEXT J2
LOOP UNTIL ABS(G2-G1)<E1+E2*ABS(G2) OR J1>J4
END SUB
```

```
SUB GAUSSINT8(X1,X2,G2):REM--------------Integrator; G2=value of integral
X0=.5*(X2+X1):X3=X0-X1:G2=0
ON P1(9) GOTO INTEGRAND1,INTEGRAND2,INTEGRAND3,INTEGRAND4
INTEGRAND1:
FOR J1=1 TO 4
 DX=GX(J1)*X3:G2=G2+GW(J1)*(GCT1(X0+DX)+GCT1(X0-DX))
NEXT J1:G2=G2*X3
EXIT SUB
INTEGRAND2:
FOR J1=1 TO 4
 DX=GX(J1)*X3:G2=G2+GW(J1)*(GCT2(X0+DX)+GCT2(X0-DX))
NEXT J1:G2=G2*X3
EXIT SUB
INTEGRAND3:
FOR J1=1 TO 4
 DX=GX(J1)*X3:G2=G2+GW(J1)*(GCT3(X0+DX)+GCT3(X0-DX))
NEXT J1:G2=G2*X3
EXIT SUB
INTEGRAND4:
FOR J1=1 TO 4
 DX=GX(J1)*X3:G2=G2+GW(J1)*(GCT4(X0+DX)+GCT4(X0-DX))
NEXT J1:G2=G2*X3
END SUB

FUNCTION GCT1(X):REM--------------First of functions to be integrated.
TT=P1(1)*(2+X*(3-X*X))/4:GCT1=1/SQR((P1(2)-TT)*(4-X*X))/(1-TT)
END FUNCTION

FUNCTION GCT2(X)
TT=((P1(2)+P1(1))*2+(P1(2)-P1(1))*X*(3-X*X))/4
GCT2=1/SQR(TT*(4-X*X))/(1-TT)
END FUNCTION


FUNCTION GCT3(X)
S1=SQR(1-P1(2)*X):GCT3=1/SQR(1-P1(1)*X)/(S1*(S1+SQR(X*(1-P1(2)))))
END FUNCTION

FUNCTION GCT4(X)
TT=((P1(2)+P1(1))*2+(P1(2)-P1(1))*X*(3-X*X))/4:T4=SQR(1-P1(1)/TT)/P1(3)
GCT4=(LOG((1+T4)/(1-T4)))^2/(1-TT)/SQR(TT*(4-X*X))
END FUNCTION
```

```
FUNCTION RANG(X):REM-----------------------------0<RANG(X)<1
RANG =1/(1+EXP(X))
END FUNCTION

DEF FNPOLE(X)=((X+1)*LOG(X+1)-(X-1)*LOG(X-1))/PI
```

Program Results

```
06-28-1993  10:14:02  NPOLE1
Determines parameter values and evaluates flux into midplane (Em) and
pole (Ep) of ID, and excess flux coefficient for side of pole (Es).
K1,K2,K3 are obtained by multiplying the printed values by the scalar
potential of pole in Tcm.  K1 is for maximum deflection angle, K2 for
trajectory length effect, and K3 for B0*period.  D4=V0/B0 for V0=1.

    Em          Ep          Es          K1          K2          K3          D4
H0, W0,>W1 = .5,1.4                        H0= 0.50    W0= 1.00    W1= 0.40
 1.373E+00  1.515E+00  2.532E-01  8.055E+00  7.890E+00  7.310E+00  5.111E-01
```

Error of Flux Calculation for Finite Pole Width with Excess Flux Coefficient

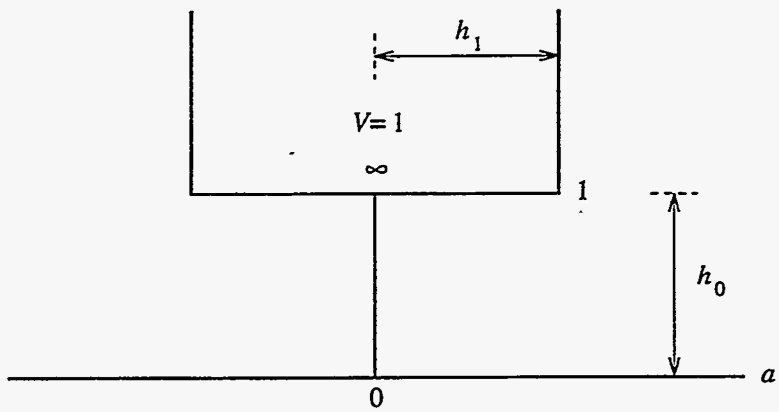


Figure 1.

$$\boxed{\dot{z} = -\frac{\sqrt{t^2-1}}{\sqrt{t^2-a^2}^3}}$$

We have

$$h_0 = \int_0^\infty \frac{\sqrt{1+t^2}}{\sqrt{a^2+t^2}^3}\,dt \quad \text{and} \quad h_1 = \int_1^\infty \frac{\sqrt{1-t^2}}{\sqrt{t^2-a^2}^3}\,dt\,.$$

We introduce

$$\sqrt{a^2+t^2} = \frac{1}{u}, \quad t = \sqrt{\frac{1}{u^2}-a^2} = \frac{\sqrt{1-a^2u^2}}{u}$$

$$dt = -\frac{\dfrac{a^2u^2}{\sqrt{1-a^2u^2}} + \sqrt{1-a^2u^2}}{u^2}\,du = -\frac{1}{u^2\sqrt{1-a^2u^2}}\,du$$

$$1+t^2 = 1-a^2+\frac{1}{u^2} = b^2+\frac{1}{u^2}, \qquad b^2 = 1-a^2$$

we therefore have

$$h_0 = -\frac{1}{a}\int_0^{1/a} \frac{\sqrt{1+b^2u^2}}{\sqrt{1-a^2u^2}}\,du\,.$$

June, 1993. Note 0133u-w.

And given $au = \sin\varphi$, and $du = \dfrac{\cos\varphi\, d\varphi}{a}$,

$$
\begin{aligned}
h_0 &= -\frac{1}{a}\int_0^{\pi/2}\sqrt{1+\frac{b^2}{a^2}\sin^2\varphi}\,d\varphi \\
&= -\frac{1}{a}\int_0^{\pi/2}\sqrt{1+\frac{b^2}{a^2}-\frac{b^2}{a^2}\cos^2\varphi}\,d\varphi \\
&= -\frac{1}{a^2}\int_0^{\pi/2}\sqrt{1-b^2\cos^2\varphi}\,d\varphi \\
&= \boxed{-\frac{1}{a^2}E(b^2)\,.}
\end{aligned}
$$

For $t = \dfrac{1}{u}$, $dt = -\dfrac{du}{u^2}$, and $u = \sin\alpha$

$$
\begin{aligned}
h_1 &= -\int_0^1 \frac{\sqrt{1/u^2-1}}{\sqrt{1/u^2-a^2}^3}\,\frac{du}{u^2} \\
&= -\int_0^1 \frac{\sqrt{1-u^2}}{\sqrt{1-a^2u^2}^3}\,du \\
&= -\int_0^{\pi/2} \frac{\cos^2\alpha\, d\alpha}{\sqrt{1-a^2\sin^2\alpha}^3}
\end{aligned}
$$

From Jahnke and Emde[1]:

$$
\boxed{h_1 = -\frac{K(a^2)-E(a^2)}{a^2}}
$$

and therefore

$$
\boxed{\frac{h_1}{h_0} = \frac{K(a^2)-E(a^2)}{E(1-a^2)}\,.}
$$

1 *Table of Functions with Formulae and Curves,* Dover Publications, 1945: p. 56.

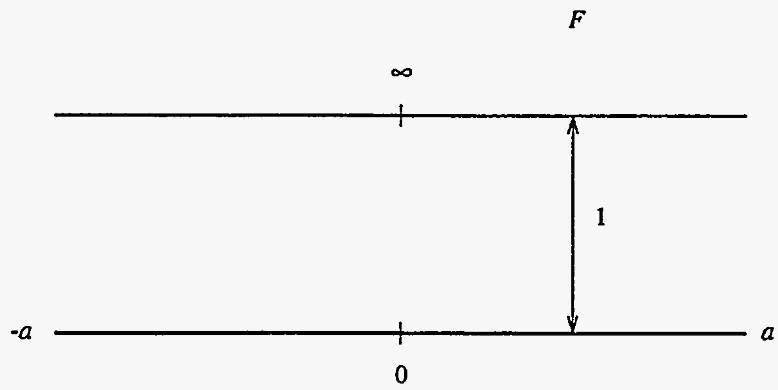


Figure 2.

$$\pi \dot{F} = -\frac{2a}{t^2 - a^2} = \frac{1}{t+a} - \frac{1}{t-a}$$

$$\boxed{\pi F = \ln \frac{t+a}{t-a} = \ln \frac{1+a/t}{1-a/t}}$$

The flux into the poleface is

$$A(1) - A(\infty) = A_{\text{ideal}} = \boxed{\frac{1}{\pi} \ln \frac{1+a}{1-a}\,.}$$

Comparing this flux to the homogeneous flux and the excess flux for the end of a semi-infinite pole with half-gap $= h_0$, we have

$$\begin{aligned} A_{\text{approx}} &= \frac{h_1}{h_0} + \frac{1}{\pi}(2 - \ln 4) \\ &= \boxed{\frac{K(a^2) - E(a^2)}{E(1-a^2)} + .195\,.} \end{aligned}$$

Therefore we have

$$G\left(\frac{h_1}{h_0}\right) = \frac{A_{\text{approx}} - A_{\text{ideal}}}{A_{\text{ideal}}} = \frac{A_{\text{approx}}}{A_{\text{ideal}}} - 1 = \boxed{\frac{\frac{h_1}{h_0} + \frac{1}{\pi}(2 - \ln 4)}{\frac{1}{\pi} \ln \frac{1+a}{1-a}} - 1\,.}$$

Program EXCFLTST.BAS

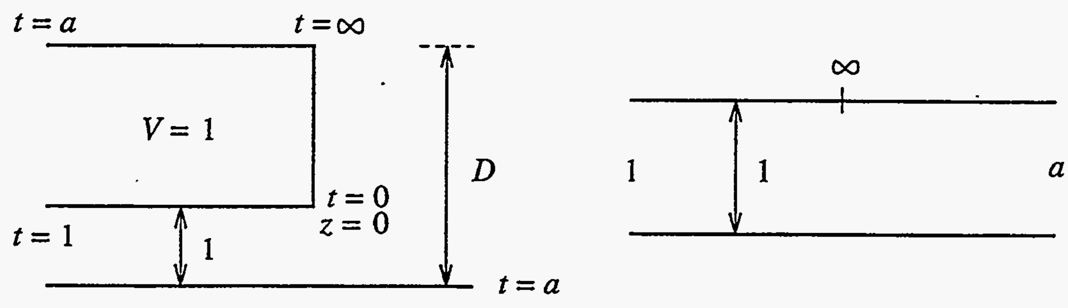
```
CLS
DEFDBL A-Z
PRINT DATE$ ;"  ";TIME$ ;"  EXCFLTST"
REM--Error of flux calculation for finite width pole with excess flux
REM--coefficient.    INPUT parameter = 1/2-width of pole / 1/2-gap.
PI=4*ATN(1):A1$="dA=##.###^^^^  dA/A=##.###^^^^  dX/H1=##.###^^^^"
E1=(2-LOG(4))/PI:X2=1:DY=1E-6
DIM P1(0:2)
DO
INPUT;"H1/H0=",H0
X1=.9*X2:P1(0)=H0:REM--G1=2/(1+EXP(PI*H0+2)):E1=2/PI*(1-G1-LOG(2-G1))
CALL SECANTS(X1,X2,DY,Y2,P1())
A1=1/SQR(1+EXP(X2)):A0=LOG((1+A1)/(1-A1))/PI:AE=H0+E1
PRINT TAB(15);:PRINT USING A1$;AE-A0;AE/A0-1;(AE-A0)/H0
LOOP

SUB SECANTS(X1,X2,DY,Y2,P1())
CALL FCTY(X1,Y1,P1()):CALL FCTY(X2,Y2,P1())
IF ABS(Y1)<ABS(Y2) THEN SWAP Y1,Y2:SWAP X1,X2
J1%=0
DO
DX=Y2*(X1-X2)/(Y2-Y1)
X1=X2:Y1=Y2:X2=X1+DX:J1%=J1%+1
CALL FCTY(X2,Y2,P1())
LOOP UNTIL ABS(Y2)<DY OR J1%=15
IF J1%=15 THEN PRINT "NOT CONVERGED"
END SUB

SUB FCTY(X1,Y1,P1())
A2=1/(1+EXP(X1))
Y1=(ELLK(A2)-ELLE(A2))/ELLE(1-A2)-P1(0)
END SUB

FUNCTION ELLK(X1)
X=1-X1:S1=.01451196212*X+.03742563713:S1=S1*X+.03590092383
S1=S1*X+.09666344259:S1=S1*X+1.38629436112:S2=.00441787012*X+.03328355346
S2=S2*X+.06880248576:S2=S2*X+.12498593597:S2=S2*X+.5
ELLK=S1-S2*LOG(X)
END FUNCTION
```

```
FUNCTION ELLE(X1)
X=1-X1:S1=.01736506451*X+.04757383546:S1=S1*X+.0626060122
S1=S1*X+.44325141463:S2=X*.00526449639+.04069697526
S2=S2*X+.09200180037:S2=S2*X+.2499836831
ELLE=X*S1+1-X*S2*LOG(X)
END FUNCTION
```

Program Results

```
06-16-1993  09:09:24  EXCFLTST
H1/H2=.1      dA= 4.832E-02  dA/A= 1.956E-01  dX/H1= 4.832E-01
H1/H2=.2      dA= 2.240E-02  dA/A= 6.022E-02  dX/H1= 1.123E-01
H1/H2=.3      dA= 1.129E-02  dA/A= 2.331E-02  dX/H1= 3.762E-02
H1/H2=.4      dA= 5.848E-03  dA/A= 9.920E-03  dX/H1= 1.462E-02
H1/H2=.5      dA= 3.074E-03  dA/A= 4.441E-03  dX/H1= 6.149E-03
H1/H2=.6      dA= 1.627E-03  dA/A= 2.050E-03  dX/H1= 2.712E-03
H1/H2=.7      dA= 8.645E-04  dA/A= 9.665E-04  dX/H1= 1.235E-03
H1/H2=.8      dA= 4.602E-04  dA/A= 4.626E-04  dX/H1= 5.752E-04
H1/H2=.9      dA= 2.452E-04  dA/A= 2.239E-04  dX/H1= 2.725E-04
H1/H2=1       dA= 1.307E-04  dA/A= 1.094E-04  dX/H1= 1.307E-04
H1/H2=2       dA= 3.008E-07  dA/A= 1.370E-07  dX/H1= 1.504E-07
```

Excess Flux Into Pole and Flux Into Side of Gm40



z-plane *F-plane*

Figure 1.

The conformal map is described by

$$\pi \dot{z} = \frac{\sqrt{t}(a-1)^2}{(t-1)(t-a)^2}.$$

To determine the value a that produces the desired D, we use $t = a + \tau$, $|\tau| \ll a$. Expanding in τ gives

$$\pi \frac{dz}{d\tau} = \frac{(a-1)^2\sqrt{a}\left(1+\dfrac{\tau}{2a}\right)}{(a-1)\left(1+\dfrac{\tau}{a-1}\right)} \cdot \frac{1}{\tau^2}.$$

Expanding more, and then integrating over the half-circle around $t = a$, we get

$$\begin{aligned} D &= -(a-1)\sqrt{a}\left(\frac{1}{2a} - \frac{1}{a-1}\right) \\ &= \sqrt{a}\cdot\frac{a+1}{2a} \\ &= \boxed{\frac{1}{2}\left(\sqrt{a} + \frac{1}{\sqrt{a}}\right).} \end{aligned}$$

By substitution and integration, we have

$$\frac{\pi z}{(a-1)^2} = \frac{\partial}{\partial a}\int \frac{\sqrt{t}\,dt}{(t-1)(t-a)} = \frac{\partial}{\partial a}J,$$

April, 1993. Note 0131u-w.

where, for $t = W^2$ and $dt = 2WdW$,

$$\begin{aligned}
J &= \int \frac{2tdW}{(t-1)(t-a)} \\
&= \int \frac{2}{a-1}\left(\frac{a}{t-a} - \frac{1}{t-1}\right) dW \\
&= \frac{1}{a-1}\int \left(\sqrt{a}\left(\frac{1}{W-\sqrt{a}} - \frac{1}{W+\sqrt{a}}\right) - \left(\frac{1}{W-1} - \frac{1}{W+1}\right)\right) dW \\
&= \boxed{\frac{1}{a-1}\left(\sqrt{a}\ln\frac{\sqrt{a}-W}{\sqrt{a}+W} + \ln\frac{1+W}{1-W}\right) :}
\end{aligned}$$

Further, we have that

$$\sqrt{a}\,\frac{\partial}{\partial a}\ln\frac{\sqrt{a}-W}{\sqrt{a}+W} = \frac{1}{2}\left(\frac{1}{\sqrt{a}-W} - \frac{1}{\sqrt{a}+W}\right) = \frac{W}{a-t}.$$

$$\left(\frac{1}{\sqrt{a}-1/\sqrt{a}}\right)' = -\frac{1}{2a}\cdot\frac{\sqrt{a}+1/\sqrt{a}}{(\sqrt{a}-1/\sqrt{a})^2} = -\frac{D}{(a-1)^2},$$

$$\left(\frac{1}{a-1}\right)' = -\frac{1}{(a-1)^2}.$$

Thus,

$$J' = -\frac{1}{(a-1)^2}\left(\ln\frac{1+W}{1-W} + D\ln\frac{\sqrt{a}-W}{\sqrt{a}+W}\right) + \left(\frac{1}{a-1}\cdot\frac{W}{a-t}\right),$$

and therefore,

$$\boxed{\pi z = (a-1)\frac{W}{a-t} + D\ln\frac{\sqrt{a}+W}{\sqrt{a}-W} - \ln\frac{1+W}{1-W}.}$$

Further, for

$$\pi\dot{F} = -\frac{a-1}{(t-1)(t-a)} = \frac{1}{t-1} - \frac{1}{t-a},$$

$$\boxed{\pi F = \ln\frac{1-t}{1-t/a}.}$$

The flux into the side of the pole, for $-\infty \leq t \leq 0$, is

$$\boxed{A_S = \frac{1}{\pi}\ln(a).}$$

We describe the excess flux into the poleface by

$$\Delta A_P = F(0) - F(1-\varepsilon) - (z(0) - z(1-\varepsilon)) \quad \text{follows} \quad \varepsilon \to 0\,,$$

$$\pi \Delta A_P = \ln \frac{1-1/a}{\varepsilon} + \left(1 + D \ln \frac{\sqrt{a}+1}{\sqrt{a}-1} - \ln \frac{2}{\varepsilon/2}\right),$$

$$\frac{a-1}{\varepsilon a}\frac{\varepsilon}{4} = \frac{a-1}{4a},$$

$$\boxed{\Delta A_P = \frac{1}{\pi}\left(1 + \ln \frac{a-1}{4a} + D \ln \frac{\sqrt{a}+1}{\sqrt{a}-1}\right).}$$

The definition of ΔA_P means that the flux into the pole surface is the same as the uniform flux into a pole whose width is increased, on both sides, by the product of the half-gap and the expression for ΔA_P. The definition of A_S means that the total flux into each side of the pole equals the product of the scalar potential of the pole and the expression for A_S.

From our expression for D, and $a - 2D\sqrt{a} + 1 = 0$, we have

$$\boxed{\sqrt{a} = D + \sqrt{D^2 - 1}.}$$

We may now eliminate a from A_S and ΔA_P. Thus,

$$\boxed{A_S = \frac{2}{\pi} \ln(D + \sqrt{D^2 - 1}),}$$

and further,

$$1/\sqrt{a} = D - \sqrt{D^2 - 1}\,,$$

$$\frac{\sqrt{a}+1}{\sqrt{a}-1} = \frac{D+1+\sqrt{D^2-1}}{D-1+\sqrt{D^2-1}} = \frac{\sqrt{D+1}}{\sqrt{D-1}} \cdot \frac{\sqrt{D+1}+\sqrt{D-1}}{\sqrt{D-1}+\sqrt{D+1}} = \sqrt{\frac{D+1}{D-1}},$$

$$\frac{a-1}{4a} = \frac{\sqrt{a}-\sqrt{1/a}}{4\sqrt{a}} = \frac{\sqrt{D^2-1}}{2(D+\sqrt{D^2-1})} = \frac{1}{2(1+1/\sqrt{1-1/D^2})},$$

$$\boxed{\Delta A_P = \frac{1}{\pi}\left(1 + \frac{D}{2} \ln \frac{D+1}{D-1} - \ln\left(2\left(1 + \frac{D}{\sqrt{D^2-1}}\right)\right)\right).}$$

Flux Transport Along Axial Direction of Electro-Magnetic Wiggler

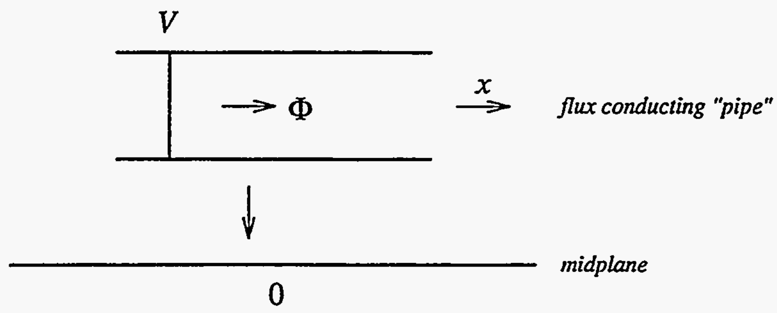


Figure 1.

Status characterized by status vector $v = \begin{pmatrix} V \\ \Phi \end{pmatrix}$, where V is the scalar potential with respect to the midplane, and Φ is the flux transported to the right. Going "downstream", V and Φ change because of the $\int H ds$ "loss" in iron (and due to small gaps), and because of flux going to the midplane. Over a short distance,

$$\frac{d\Phi}{dx} = -V \cdot \varepsilon\,, \tag{1}$$

with ε to 0th approximation (detailed later in this note) is given by

$$\varepsilon = \frac{W}{h}\,, \tag{2}$$

with h having the value of the half-gap, and W being the width over which the flux "escapes" to the midplane.

$$\frac{dV}{dx} = -\Phi \cdot k_2\,, \tag{3}$$

April, 1993. Note 0129u-w.

with k_2 in 0th approximation (also detailed later) given by

$$k_2 = \frac{1}{a\mu} = \gamma/a\,, \tag{4}$$

where a is the cross-section area of the flux "duct", and μ is the permeability. The voltage drop due to small gaps perpendicular to the flux flow will be added later. Within the section with constants k_2 and ε, we get

$$V'' = -k_2\Phi' = Vk^2 \quad \text{with} \quad k^2 = \varepsilon k_2\,. \tag{5}$$

The solution within the uniform section of length x is

$$V = \alpha C + \beta S \quad \text{with} \quad C = \cosh kx \ \ \text{and} \ \ S = \sinh kx\,,$$

$$\Phi = -V'/k_2 = -(\alpha S + \beta C)k/k_2\,,$$

$$v(x) = \begin{pmatrix} C & S \\ -Sk/k_2 & -Ck/k_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -k_2/k \end{pmatrix} v(0)\,,$$

$$v(x) = M \cdot v(0)\,, \quad v = \begin{pmatrix} V \\ \Phi \end{pmatrix} \text{ and } M = \begin{pmatrix} C & -Sk_2/k \\ -Sk/k_2 & C \end{pmatrix}. \tag{6}$$

By reversing the direction arrow of Φ, i.e., by re-defining the sign of Φ, the off-diagonal minus signs disappear. The sequence of sections with different properties are taken into account by multiplying their matrices. v remains unchanged when crossing the interface from one section to the next unless there is a (steering) coil, or a local field clamp, thus introducing an additive Δv when going through that interface.

It is clear that $1/k$ is the important scaling distance that describes how transported flux decays.

Structure of Solution to Simple Problem.

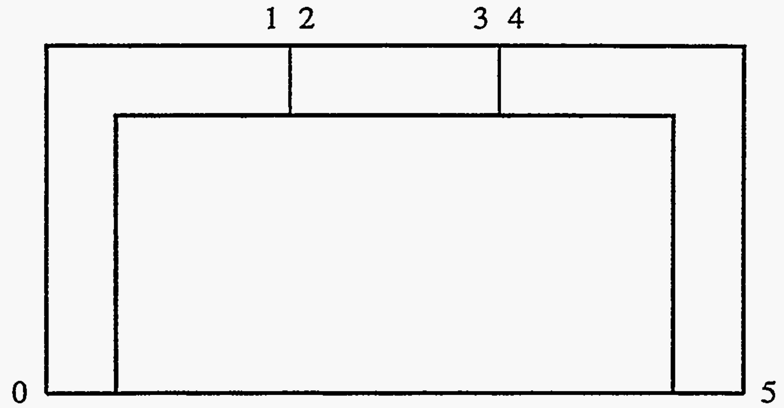


Figure 2.

There are field clamps at each end, i.e. at point 0 and point 5. There are $\pm\Delta V$ coils at the interfaces between points 1 and 2, and between points 3 and 4. The status vectors $v_0 = \begin{pmatrix} 0 \\ \Phi_0 \end{pmatrix}$ and $v_5 = \begin{pmatrix} 0 \\ \Phi_5 \end{pmatrix}$ describe that the points 0 and 5 are located in the midplane, and that they contain the to-be-determined values Φ_0 and Φ_5 which represent the fluxes going to the midplane through the field clamps. Of similar interest are the Φ-components of v_2 and v_4.

Given $\Delta v_0 = \begin{pmatrix} \Delta v_0 \\ 0 \end{pmatrix}$, we describe the coil(s) by

$$v_2 = M_{01} v_0 + \Delta v_0 \,. \tag{7.1}$$

$$v_4 = M_{23} v_2 - \Delta v_0 = M_{23} M_{01} v_o + (M_{23} - I)\Delta v_0 \,, \tag{7.2}$$

where I is the unit matrix.

$$v_5 = M_{45} v_4 = \underbrace{M_{05}}_{a_{ik}} v_0 + \underbrace{M_{45}(M_{23} - I)}_{b_{ik}} \Delta v_0 \,, \tag{7.3}$$

where a_{ik} and b_{ik} are elements of these matrices, and thus

$$v_5 = \Phi_0 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \Delta V_0 = \begin{pmatrix} 0 \\ \Phi_5 \end{pmatrix} .$$

$$\boxed{\Phi_0 = -\Delta V_0 b_{11}/a_{12} .} \tag{7.4}$$

$$\Phi_5 = \Phi_0 a_{22} + \Delta V_0 b_{21} = \Delta V_0 (b_{21} - a_{22} b_{11}/a_{12})$$

$$\boxed{\Phi_5 = \Delta V_0 (a_{12} b_{21} - a_{22} b_{11})/a_{12} .} \tag{7.5}$$

With Φ_0 and Φ_5 now known, (7.1) and (7.2) give the flux produced by the coils in the section delimited by points 2 and 3.

Details of k_2.

One has to be careful to use the correct value for μ. If the field associated with this flux is parallel to the pre-existing field, one has to use $\mu = \dfrac{dB}{dH}$. If it is perpendicular to the pre-existing flux, one must use $\mu = \dfrac{B}{H}$ which is the "normal μ".

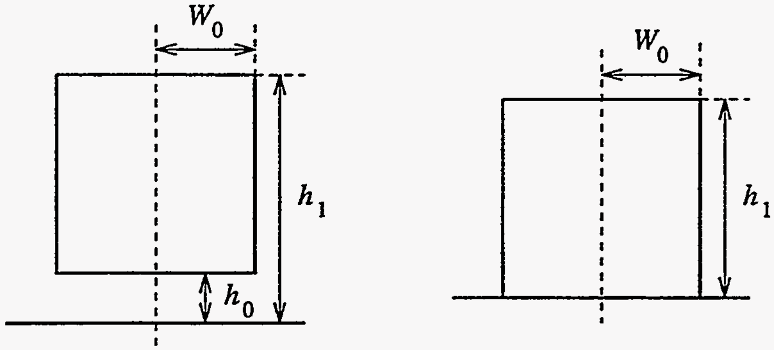
Now we must look at the effect of a thin gap over a large area. ΔV across that gap from flux Φ is gotten from $\Phi = \displaystyle\int \frac{\Delta V \, da}{g}$ and thus $\Delta V = \dfrac{\Phi}{\displaystyle\int \frac{da}{g}}$. If a gap-less length L of μ is associated with this gap, the total ΔV is given by

$$\Delta V = \Phi \left(\frac{1}{\int \frac{da}{g}} + \gamma \frac{L}{A} \right) = \Phi L \left(\frac{\gamma}{a} + \frac{1/L}{\int \frac{dg}{a}} \right) ,$$

$$\boxed{k_2 = \frac{\gamma}{a} + \frac{1/L}{\int \frac{da}{g}} .} \tag{8}$$

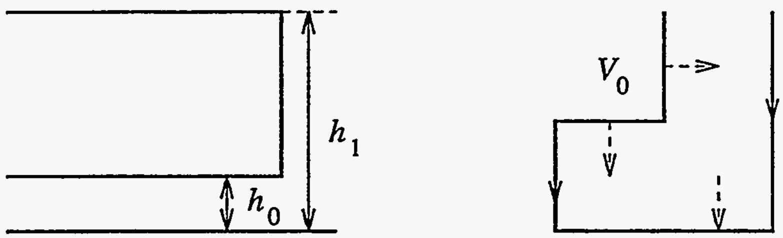
Details of ε.

Only the general approach and the results derived in a separate note are given here. There are three contributions to ε: flux from the top, from the sides, and from the poles facing the midplane.



(a) (b)

Figures 3(a) and 3(b).



(c) (d)

Figures 3(c) and 3(d).

To get the flux into the top per unit length in direction perpendicular to the paper plane, we use as a model the solid block that touches the midplane of Figure 3(b). For the flux into each side, we use the geometry of Figure 3(c) and calculate the flux into the side. If the side has "pole structure" we take it into account with an excess voltage drop coefficient approximation (if necessary). For flux from the poles to the midplane, we calculate the flux for the geometry of Figure 3(d), and we use the excess flux coefficient for a solid block of Figure 3(c) to correct the width $2W_0$ of the cross-section shown in Figure 3(a).

Results for the Geometry of Figure 3(d).

$$\boxed{\varepsilon_P = \frac{\Phi(\lambda/4)}{V_0}\frac{4}{\lambda}(W_0 + \Delta W_0)\,2,} \tag{9.1}$$

with $\dfrac{\Phi(\lambda/4)}{V_0}$ calculated by POISSON or an analytical program.

We calculate ΔW_0 from the geometry of Figure 3(c), with

$$\boxed{D = h_1/h_0\,,} \tag{9.2}$$

$$\boxed{\Delta W_0 = h_0\frac{1}{\pi}\left(1 + \frac{D}{2}\ln\left(\frac{D+1}{D-1}\right) - \ln\left(2\left(1 + \frac{D}{\sqrt{D^2-1}}\right)\right)\right).} \tag{9.3}$$

The contribution from the flux into the top is

$$\varepsilon_T = \frac{2}{\pi}\ln\frac{1+a_1}{1-a_1}\,, \tag{10.1}$$

where a_1 is determined from

$$\frac{h_1}{W_0} = \frac{E(b) - a_1^2 K(b)}{E(a_1) - b^2 K(a_1)} \tag{10.2}$$

with $b^2 = 1 - a_1^2$, and

$$E(a_1) = \int_0^{\pi/2}\sqrt{1 - a_1^2\sin^2\varphi}\,d\varphi \quad \text{and} \quad K(a_1) = \int_0^{\pi/2}\frac{d\varphi}{\sqrt{1 - a_1^2\sin^2\varphi}}. \tag{10.3}$$

The flux into the sides contributes

$$\varepsilon_S = \frac{4}{\pi}\ln\left(D + \sqrt{D^2-1}\right)\,,$$

with D given by (9.2).

(11.1) assumes smooth sides, i.e., the excess potential drop is ignored. It should be noted that the area a in (8) is smaller than the cross-section shown in Figure 3(a), the latter includes the poles, while the former does not.

We make here further clarifications on units. If we were to deal with a uniform field over a width W of a flat pole, at distance h_0 from the midplane, ε would be exactly $\varepsilon = \frac{W}{H}$. That is, $\frac{d\Phi}{dx}$ and V have the same dimensions, meaning that either $\mu_0 = 4\pi \cdot 10^{-7}$ is incorporated in the vector potential V, or μ_0 is left out of the definition of Φ. The meaning of ε is the flux per unit length in the axial direction of the structure on potential V, divided by V.

ε_S with excess potential drop is given by

$$\varepsilon_S = \frac{4}{\pi} \ln\left(D_1 + \sqrt{D_1^2 - 1}\right) \tag{11.2}$$

with

$$D_1 = \frac{h_1 + \frac{2}{\pi}\Delta L}{h_0 + \frac{2}{\pi}\Delta L}. \tag{11.3}$$

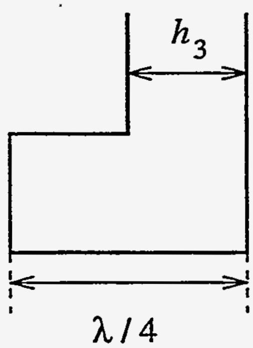


Figure 4.

$$\alpha = \frac{\lambda/4}{h_3} \quad \text{and} \quad \Delta L = \frac{h_3}{\pi}\left((\alpha+1)\ln(1+1/\alpha) + (\alpha-1)\ln(1-1/\alpha)\right). \tag{11.4}$$

The effect of ΔL will be very small under most circumstances. The excess flux potential drop is too small to be of concern for ε_T.

3D Scalar Potential for Saturation-Caused Fields in the Insertion Device

This entails the same approach as for the case of $\mu = \infty$, except that the condition $\partial V/\partial x = 0$ at $y = h$ is to be dropped:

$$V = \sum \cos nk_z z \cdot g_n(x, y)\,,$$

$$\nabla^2 V = 0 \quad \Longrightarrow \quad \nabla^2 g_n = n^2 k_z^2 g_n\,.$$

We introduce $nk_z x = u$, and $nk_z y = v$:

$$\nabla^2_{u,v} g = g\,. \tag{1}$$

We construct $g(u, v)$ that has the following properties: odd in y, $g(-v) = -g(v)$, and gives field approximating $\cosh \varepsilon u - 1$ for $y = 0$. ε is arbitrary, real or imaginary, and the field equals 0 for $u = 0$ when letting $\varepsilon \to 0$ at end.

We try $g = \cosh \varepsilon u \sinh av$. To satisfy (1):

$$\varepsilon^2 + a^2 = 1 \quad \text{and thus} \quad \boxed{a = \sqrt{1 - \varepsilon^2}}$$

has to hold. We add a function of v, such that g'_v is proportional to $\cosh \varepsilon u - 1$ for $v = 0$. The only odd function of v that will satisfy this requirement and also satisfy (1) is $-a \sinh v$, thus

$$\boxed{g = \cosh \varepsilon u \sinh av - a \sinh v\,.} \tag{2}$$

One can use the superposition of such functions with different ε, but this would probably not be practical.

The expansion for $\varepsilon \to 0$ is

$$\boxed{g = \frac{\varepsilon^2}{2}\left(u^2 \sinh v - v \cosh v + \sinh v\right).} \tag{3}$$

For $v = 0$, we obtain the expected sextupole field:

$$g'_v(u, 0) = \frac{\varepsilon^2}{2} u^2\,. \tag{4}$$

At the pole, where $v_h = nk_z h$,

$$g'_u(u, v_h) = \frac{\varepsilon^2}{2} 2u \sinh v_h\,, \tag{5}$$

It is this field in the x-direction that is responsible for the sextupole field in the midplane.

February, 1992. Note 0125u-w.

(5) allows us to make an estimate of the saturation effects in the midplane during the design phase. Thus,

$$\frac{g'_v(u,0)}{g'_u(u,v_h)} = \frac{u}{2\sinh v_h} = \frac{nk_z x}{2\sinh nk_z h} = \frac{x}{2h}\cdot\frac{nk_z h}{\sinh nk_z h}. \tag{6}$$

It is interesting to note that every additional expansion of (2) in ε^2 leads to a new solution to (1) describing the fields in the midplane to the highest orders $\sim x^4, x^6$, etc.

To check on (3), its expansion in k_z up to the 3rd order terms in $\{u, v\}$ gives, as expected,

$$\begin{aligned} g &= \frac{\varepsilon^2}{2}\left(u^2 v - v\left(1+\frac{v^2}{2}-\left(1+\frac{v^2}{6}\right)\right)\right) \\ &= \frac{\varepsilon^2}{2}\left(u^2 v - \frac{v^3}{3}\right) \\ &= \frac{\varepsilon^2}{6}\Im(u+iv)^3. \end{aligned} \tag{7}$$

Scalar Potential for 3D Insertion Device Fields

In the 2D case,

$$V = \sum_{n=\mathrm{odd}} b_n \cos nk_z z \cdot \sinh nk_z y \quad \text{with} \quad k_z = \frac{2\pi}{\lambda}\,. \tag{1}$$

In order to simplify matters, we drop the sum, and re-introduce it at the end.
The effects of lateral ends are equally periodic in z, thus $nk_z \Rightarrow k_z$, and

$$V_{LE} = \cos k_z z \cdot g(x,y)\,. \tag{2}$$

Where $g(x,y)$ is valid only in the vacuum region of the $\{x,y\}$ space.
Further, we have that

$$\nabla^2 V = 0 \quad \Longrightarrow \quad \nabla^2 g = k_z^2 g\,, \tag{3}$$

where g is the Fourier expansion coefficient as a function of x, y,
At the pole surface, for integer μ, $z = \mu\lambda/2$ and $y = h =$ half gap, $B_x = B_y = 0$. We expand g in a Fourier series in y. We have that $g \sim \sin mk_y y$ for $k_y = \pi/h$, and

$$g = \sum a_m \sin mk_y y \cdot \cosh k_m x\,,$$

with $k_m^2 = k_z^2 + m^2 k_y^2$. We use $a_m = b_0 b_m / \cosh k_m W$, where W is half the pole width, and we expect b_m to be only weakly dependent on W.

$$\begin{cases} V = \displaystyle\sum_{n=\mathrm{odd}} b_{n_0} \cos nk_z z \cdot (\sinh nk_z y + g_n), & \\ g_n = \displaystyle\sum_{m=1} \frac{b_{nm} \sin mk_y y \cdot \cosh k_{nm} x}{\cosh k_{nm} W}, & \\ k_{nm}^2 = n^2 k_z^2 + m^2 k_y^2, & \text{with } k_z = 2\pi/\lambda,\ k_y = \pi/h, \\ k_{nm}^2 = m^2 k_y^2 \left(1 + \dfrac{n^2}{m^2}\left(\dfrac{k_z}{k_y}\right)^2\right), & \text{with } \dfrac{k_z}{k_y} = \dfrac{2h}{\lambda} = \dfrac{\text{gap}}{\lambda}. \end{cases} \tag{4}$$

Under most circumstances, $\dfrac{k_z}{k_y} \leq .5$. For $n = 1$, $k_{nm} \approx mk_y$.

February, 1992. Note 0124u-w.

We want to measure $\partial B_y/\partial x$. We move a B_y-coil in the x direction that is "long" in

In the region of interest, only the case of $m = 1$ is of importance. That is, the dominant term is

$$g_1 = \frac{b_{11} \sin k_y y \cdot \cosh k_{11} x}{\cosh k_{11} W}. \tag{5}$$

We may now proceed to conclude that

(6)

$$-H_y = k_z \sum n b_{n_0} \cos n k_z z \left(\cosh n k_z y + \sum b_{nm} \frac{m k_y}{n k_z} \cdot \frac{\cos m k_y y \cdot \cosh k_{nm} x}{\cosh k_{nm} W} \right).$$

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Make all measurements as function of z, filter out random errors, and then do the harmonic analysis by measuring the quantities derived from $\sinh n k_z y + g_n$. To measure field components, measure B_y at $y = 0$ for a number of values of x close enough to the lateral edge to get values of b_{n1} and b_{n2}. Then measurements of B_x close to the lateral ends are made, at $y \simeq h/2$, to check the validity of $V(x, y, z)$. If agreement is reached, an investigatation of whether b_{nm} are more easily obtained from B_x measurements is to be done. To verify the model, compare the measurements at individual points, without the harmonic analysis, to the model calculations.

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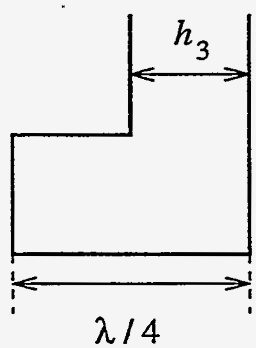

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February, 1992. Note 0125u-w.

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February, 1992. Note 0124u-w.

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Gradient Measurement in Insertion Device

The beam is in the z direction. The midplane is the $\{x, z\}$ plane. We use a vibrating coil to measure $\partial B_y/\partial x$.

As a general mechanical design principle, make the wanted resonance frequency and its harmonics different from the resonance frequencies of other vibrating modes.

We want to measure $\partial B_y/\partial x$. We move a B_y-coil in the x direction that is "long" in z and short in x. The problem arises that this motion may excite vibration in the y direction, adding a $\partial B_y/\partial y$ signal. A better way to collect the same information is to measure $\partial B_x/\partial y$, by vibrating a B_x-coil in the y direction such that it is "long" in z and short in y. Possible contamination due to $\partial B_x/\partial z$ drops out in the Fourier analysis in z.

February, 1992. Note 0120u-w.

Undulator Trajectory and Radiation

We begin with the following definitions:

$$\ddot{\mathbf{r}}\gamma m = -e(\mathbf{e}_x\dot{x} + \mathbf{e}_y\dot{y} + \mathbf{e}_z\dot{z}) \times \mathbf{e}_y B,$$

$$\ddot{x}\gamma m = e\dot{z}B = e\dot{z}A'_z,$$

$$\dot{x}\gamma m = eA(z),$$

$$dt\beta c = ds = dz\sqrt{1+x'^2}, \quad \dot{x} = x'\frac{\beta c}{\sqrt{1+x'^2}},$$

$$A = B_0\int \cos kz dz = \frac{B_0}{k}\sin kz$$

$$\frac{x'}{\sqrt{1+x'^2}} = \varepsilon \sin kz, \quad \varepsilon = \frac{B_0/k}{\beta\gamma mc/e} = \frac{K}{\gamma},$$

$$\frac{1}{x'^2} + 1 = \frac{1}{\varepsilon^2 \sin^2 kz}, \quad \boxed{x' = \frac{es}{\sqrt{1-\varepsilon^2\sin^2 kz}}.}$$

Thus,

$$J = \int \dot{x}e^{i\varphi}dt,$$

where

$$\varphi = \omega\left(t - \frac{z}{c}\right) = \frac{\omega}{\beta c}(\beta ct - \beta z) = \frac{\omega}{\beta c}\int\left(\sqrt{1+x'^2} - \beta\right)dz,$$

$$1 + x'^2 = \frac{1}{1-\varepsilon^2\sin^2 kz},$$

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} \simeq 1 - \frac{1}{2\gamma^2}.$$

and furthermore,

$$\sqrt{1+x'^2} - \beta = 1 + \frac{\varepsilon^2\sin^2 kz}{2} - 1 + \frac{1}{2\gamma^2} = \frac{1}{2\gamma^2}(1 + \varepsilon^2\gamma^2\sin^2 kz),$$

March, 1990. Note 0101u-w.

and

$$\varphi = \frac{\omega}{2\beta c\gamma^2}\int (1 + K^2 \sin^2 kz)dz = \boxed{\frac{\omega}{2\beta c\gamma^2}\int \left(1 + \frac{K^2}{2} - \frac{K^2}{2}\cos 2kz\right) dz,}$$

where, for $\beta \approx 1$,

$$\varphi = \frac{\omega}{2c\gamma^2}\left(1 + \frac{K^2}{2}\right) z - \frac{\omega}{2c\gamma^2}\frac{K^2}{2}\frac{\sin 2kz}{2k}.$$

Therefore,

$$J = \int x' dz \cdot e^{i\varphi}, \quad \text{with} \quad \frac{\omega}{c} = k_L,$$

$$J = \frac{\varepsilon}{2i}\int e^{i\left(\frac{k_L}{2\gamma^2}\left(1+\frac{K^2}{2}\right)-k\right)z - i\left(\frac{k_L}{2\gamma^2}\frac{K^2}{2}\frac{\sin 2kz}{2k}\right)} dz,$$

where

$$\Delta k = \left(\frac{k_L}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) - k\right),$$

$$e^{iu\sin x} = \sum J_n(u)e^{inx}, \quad \text{with} \quad u = \frac{k_L K^2}{8\gamma^2 k} \quad \text{and} \quad x = kz.$$

Thus,

$$J = \frac{\varepsilon}{2i}\int e^{i(\Delta kz - 2nkz)} J_n(u)dz.$$

Further, from

$$\frac{k_L(1 + K^2/2)}{2\gamma^2} - (2n+1)k = 0,$$

and solving for $\frac{k_L}{k}$, we have

$$u = \frac{(n + 1/2)K^2/2}{1 + K^2/2}.$$

Mathematical Representation of Undulator and Wiggler Fields

Undulator and wiggler fields that are not uniform in the transverse direction are usually derived from

$$V = V \cosh k_1 x \sinh k_2 y \cos z \quad \text{with} \quad k_1^2 + k_2^2 = k^2. \tag{1}$$

Starting with $\nabla^2 V$ in cylindrical co-ordinates, we have

$$r^2 \nabla^2 V = \left(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \varphi^2} + r^2 \frac{\partial^2}{\partial z^2} \right) V = 0.$$

Assuming, without loss of generality, midplane symmetry, we write

$$V = \sum F_n \sin n\varphi \cos kz, \tag{2.1}$$

thus getting

$$\left(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} - n^2 - k^2 r^2 \right) F_n = 0. \tag{3}$$

and therefore,

$$F_n = a_n I_n(kr). \tag{2.2}$$

An interesting consequence is that whether one uses (1) or (2), one would get the same fields and pole shapes for a sufficiently small kr.

$$\sum a_n I_n(kr) \sin n\varphi = V_0 \cosh(k_1 r \cos\varphi) \sinh(k_2 r \sin\varphi), \tag{4}$$

and, in particular, this means that

$$a_n I_n(kr)\pi = V_0 \int_0^{2\pi} \cosh(k_1 r \cos\varphi) \sinh(k_2 r \sin\varphi) \sin n\varphi d\varphi. \tag{5}$$

This must hold in particular for $kr \ll 1$, i.e. by comparing the lowest order term in r and executing the trivial integrations, one gets a_n which then leads to an extremely interesting integral representation of $I_n(kr)$.

July, 1986. Note 0055u-w.

Publications of Klaus Halbach

(October 1994)

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