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Measurement of Sub-Picosecond Bunch Profiles
Using Coherent Transition Radiation*

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Measurement of Sub-Picosecond Bunch Profiles Using Coherent Transition Radiation*

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Abstract

A technique for measuring the longitudinal profile of sub-picosecond electron bunches based on autocorrelation of coherent transition radiation is reviewed. The technique uses sub-millimeter/far-infrared Michelson interferometry to obtain the autocorrelation of transition radiation emitted from a thin conducting foil placed in the beam path. The theory of coherent radiation from a charged particle beam passing through a thin conducting foil is presented for normal and oblique incidence. Michelson interferometric analysis of this radiation is shown to provide the autocorrelation of longitudinal bunch profile. The details of a noninvasive technique for measuring longitudinal bunch profile using coherent diffraction radiation are discussed.

INTRODUCTION

Transition radiation is a well-known phenomenon useful for measuring various charged-particle beam parameters including energy, emittance, and transverse profile (1). Typically, these measurements are made in the visible region where the radiation, due to individual particle effects, is incoherent.

In recent years, a technique which utilizes coherent transition radiation for measuring longitudinal profile of short bunches has come into use (2,3). This technique uses Michelson interferometry to obtain the autocorrelation of coherent transition radiation emitted by short bunches passing through a thin foil. The autocorrelation of the radiation is a direct measure of the autocorrelation of the longitudinal bunch profile. For picosecond to femtosecond bunch lengths, the measurements are made in the submillimeter to far-infrared range where the bunches radiate collectively.

In this paper, a review based on reference (2) of the theoretical aspects of this longitudinal bunch profile measurement technique is presented. In addition, a noninterceptive technique for measuring bunch profile with coherent diffraction radiation is outlined (4). A companion paper(5) in this proceedings addresses experimental results of the coherent transition radiation bunch length measurement technique.

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RADIATION FROM A CONDUCTING FOIL

In order to present a clear picture of the origin and properties of coherent transition radiation, the backward radiation from an arbitrary charge distribution striking a conducting foil is derived here in some detail.

Consider a beam of charged particles with velocity βc in the $\hat{z}$ direction striking a conducting foil located at the $z = 0$ plane in the cylindrical coordinate system (Figure 1). The foil is assumed to be perfectly conducting and the beam filamentary along the z axis. For radiation wavelengths large compared to particle spacing, the beam is well approximated by a moving line charge distribution $q_\ell(t - z/\beta c)$, where $q_\ell(t)$ is the charge distribution of the beam measured as a function of time as the beam passes the $z = 0$ plane. Accordingly, the coherent beam current is defined as $I_0(t - z/\beta c) = \beta c q_\ell(t - z/\beta c)$.

Figure 1. Image method for radiation from a thin foil.

In order to satisfy boundary conditions at the conducting plane, surface currents are induced in response to the fields of the incident beam. These currents in turn radiate electromagnetic energy. As indicated in Fig. 1, a convenient technique for solving the foil radiation problem makes use of image theory. Here, the conducting foil is removed and an image current $I_0(t + z/\beta c)$ is inserted for $z \geq 0$ so that:

$$I(z, t) = \begin{cases} I_0(t + z/\beta c) & z \geq 0 \\ I_0(t - z/\beta c) & z \leq 0 \end{cases} \quad (1)$$

With this current, it is clear that only the z component of vector potential exists and from symmetry, the vector potential and the fields derived from it

are independent of ϕ .

The vector potential must satisfy the homogeneous wave equation for $\rho > 0$:

$$\nabla^2 A_z(\rho, z, t) - \frac{1}{c^2} \frac{\partial^2 A_z(\rho, z, t)}{\partial t^2} = 0 \quad \rho > 0 \quad (2)$$

subject to the excitation condition:

$$\lim_{\rho \rightarrow 0} \int_0^{2\pi} H_\phi(\rho, z, t) \rho d\phi = I(z, t) \quad (3)$$

The procedure for solving (2) is greatly simplified by introducing the two dimensional Fourier transform:

$$\tilde{f}(\rho, \eta, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\rho, z, t) e^{-j(\omega t + \eta z)} dt dz \quad (4)$$

which has an inversion given by:

$$f(\rho, z, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(\rho, \eta, \omega) e^{j(\omega t + \eta z)} d\omega d\eta \quad (5)$$

Applying (4) to (1), (2) and (3) results in a statement of the problem in the transform domain:

$$\frac{\partial^2 \tilde{A}_z(\rho, \eta, \omega)}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \tilde{A}_z(\rho, \eta, \omega)}{\partial \rho} + (k_0^2 - \eta^2) \tilde{A}_z(\rho, \eta, \omega) = 0 \quad (6)$$

$$\lim_{\rho \rightarrow 0} \int_0^{2\pi} \tilde{H}_\phi(\rho, \eta, \omega) \rho d\phi = \frac{-j2k\bar{I}_0(\omega)}{\eta^2 - k^2} \quad (7)$$

where : $k = k_0/\beta = \omega/\beta c$

$\bar{I}_0(\omega)$ is the frequency spectrum of the current defined by:

$$\bar{I}_0(\omega) = \int_{-\infty}^{\infty} I_0(t) e^{-j\omega t} dt \quad (8)$$

with inversion:

$$I_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{I}_0(\omega) e^{j\omega t} d\omega \quad (9)$$

Equation (6) is Bessel's equation of order zero. Anticipating outward propagating wave characteristics, a solution to (6) is:

$$\tilde{A}_z(\rho, \eta, \omega) = c(\eta, \omega) H_0^{(2)}(\rho \sqrt{k_0^2 - \eta^2}) \quad (10)$$

The ϕ component of the $\vec{H}$ field can be derived from the vector potential using:

$$\tilde{H}_\phi(\rho, \eta, \omega) = -\frac{1}{\mu_0} \frac{\partial \tilde{A}_z(\rho, \eta, \omega)}{\partial \rho} \quad (11)$$

Using equations (10), (11) and (7) and the small argument approximation for $H_1^{(2)}(x)$ yields the following expression for $c(\eta, \omega)$:

$$c(\eta, \omega) = \frac{-\mu_0 \bar{I}_0(\omega)}{2} \left(\frac{k}{\eta^2 - k^2} \right) \quad (12)$$

Therefore, from (10):

$$\tilde{A}_z(\rho, \eta, \omega) = \frac{-\mu_0 \bar{I}_0(\omega)}{2} \left(\frac{k}{\eta^2 - k^2} \right) H_0^{(2)} \left(\rho \sqrt{k_0^2 - \eta^2} \right) \quad (13)$$

Taking the $\eta \rightarrow z$ part of the inversion given in (5) yields:

$$\bar{A}_z(\rho, z, \omega) = \frac{-\mu_0 \bar{I}_0(\omega)}{4\pi} \int_{-\infty}^{\infty} \left(\frac{k}{\eta^2 - k^2} \right) H_0^{(2)} \left(\rho \sqrt{k_0^2 - \eta^2} \right) e^{j\eta z} d\eta \quad (14)$$

Equation (14) gives, in integral form, the exact frequency domain expression for the vector potential. For the purpose of evaluating the fields in the radiation zone (ρ and z large), the integral in (14) may be approximated by the method of stationary phase giving:

$$\bar{A}_z(r, \theta_s, \omega) = \frac{-j\mu_0 \bar{I}_0(\omega) e^{-jk_0 r}}{2\pi r} \left(\frac{k}{k_0^2 \cos^2 \theta_s - k^2} \right) \quad k_0 r \gg 1 \quad (15)$$

where: $z = -r \cos \theta_s$, $\rho = r \sin \theta_s$

As indicated in Fig. 1, r and θ_s are spherical coordinates with θ_s measured from the $-z$ axis. (axis of specular reflection).

The electric and magnetic fields may be obtained in the typical manner using:

$$\vec{H}(r, \theta_s, \omega) = \frac{1}{\mu_0} \nabla \times \vec{A}(r, \theta_s, \omega) \quad (16)$$

$$\vec{E}(r, \theta_s, \omega) = -j\omega \vec{A}(r, \theta_s, \omega) + \frac{\nabla \nabla \cdot \vec{A}(r, \theta_s, \omega)}{j\omega \mu_0 \epsilon_0} \quad (17)$$

Resolving (15) into $\hat{r}$ and $\hat{\theta}_s$ components, substituting into (16) and (17) and retaining only $1/r$ terms, expressions for the radiation fields are found:

$$\bar{E}_{\theta_s}(r, \theta_s, \omega) = \frac{Z_0 \bar{I}_0(\omega) e^{-jk_0 r}}{2\pi r} \left(\frac{\beta \sin \theta_s}{1 - \beta^2 \cos^2 \theta_s} \right) \quad (18)$$

$$\bar{H}_{\phi_s}(r, \theta_s, \omega) = \frac{\bar{E}_{\theta_s}(r, \theta_s, \omega)}{Z_0} \quad (19)$$

$$\text{where: } Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\Omega$$

Equations (18) and (19) are the frequency domain field expressions for backward transition radiation emitted from a beam striking a conducting foil.

Several important features of coherent transition radiation are evident from equations (18) and (19). It is seen that the frequency spectrum of the radiation is identical to that of the beam current, at least for a foil of infinite extent. It is this property that makes the autocorrelation measurement, to be described, possible. Bandpass properties of finite foils will be addressed in a later section. Another important characteristic of transition radiation is its spatial distribution. From (18), it is clear that the angular distribution of energy or power density is given by the function:

$$S^2(\theta_s) = \left(\frac{\beta \sin \theta_s}{1 - \beta^2 \cos^2 \theta_s} \right)^2 \quad (20)$$

This function has a single, very sharp maximum at $\theta_s = 1/\beta\gamma$. Therefore, for relativistic beams, virtually all of the radiation is in the vicinity of this extremely small angle. In this case, an excellent approximation for $S^2(\theta_s)$ is:

$$S^2(\theta_s) \approx \left(\frac{\theta_s}{1/\gamma^2 + \theta_s^2} \right)^2 \quad \gamma \text{ large} \quad (21)$$

From (21) it is noted that the energy or power density is proportional to γ^2 at $\theta_s \approx 1/\gamma$.

For the autocorrelation technique to be described, we require $I_0(t)$ to be a simple periodic current corresponding to a train of bunches passing through the foil. By use of equations (18) and (19) and the definition of the Poynting vector, the total average power per unit solid angle radiated by a periodic beam current, $I_0(t)$, striking a foil can be found:

$$\frac{dP}{d\Omega} = \frac{P_{Z_0}}{4\pi^2} S^2(\theta_s) \quad \text{watts/steradian} \quad (22)$$

where:

$$P_{Z_0} = \frac{Z_0}{T} \int_T I_0^2(t) dt \quad T = \text{period of } I_0(t) \quad (23)$$

The quantity P_{Z_0} is recognized as the rms power dissipated by $I_0(t)$ in a 377Ω (free space) resistor. By integrating (22) over the backward radiation half space, the total radiated power for a relativistic beam is obtained:

$$P \approx \frac{P_{Z_0} \ln \gamma}{2\pi} \quad \text{watts} \quad (24)$$

The frequency spectrum of the radiated power consists of discrete lines at integer multiples of $1/T$ with amplitudes proportional to the square of the Fourier transform of the bunch profile. In this case, it is clear that the critical frequency components for determining bunch profile and length are in the $1/\tau_b$ region. This region covers the far-infrared/submillimeter range for the approximate range of bunch lengths, $.03 \text{ psec} \leq \tau_b \leq 3 \text{ psec}$.

AUTOCORRELATION OF BUNCH PROFILE THROUGH MICHELSON INTERFEROMETRY

A simple system for obtaining the autocorrelation of the beam current and therefore bunch profile is shown in Fig. 2. Here, the beam current, $I_0(t)$, passes through a thin conducting foil at an incident angle of 45° . Backward transition radiation is then emitted about the axis of specular reflection and directed into an infrared/submillimeter wave Michelson interferometer. By measuring power at the output port of the interferometer as a function of Δ in the delay path, the autocorrelation, or equivalently, the power spectrum of $I_0(t)$ may be obtained. It is important to note that in general, the transition radiation field expressions for oblique incidence are considerably more complicated than those for normal incidence. However, as shown in the Appendix, for large γ , the field expressions about the axis of specular reflection are well approximated by the normal incidence expressions derived in the previous section.

The interferometer, illustrated in Fig. 2, basically consists of a fixed mirror, a movable mirror and a splitter/combiner. These elements are arranged so that the incoming radiation is split into two beams. One of the beams is then delayed by a distance Δ before recombination takes place at the output port. As indicated in Fig. 2, the reflection and transmission coefficients for the splitter/combiner are designated as S_{11} and S_{21} , respectively. Both mirrors are assumed to have reflection coefficients of 1. If the divergence of the radiation is small (γ large), the total electric field at the power detector is given by:

Figure 2. Michelson Interferometer.

$$\vec{E}(r, \theta, \omega) = \vec{E}_1 + \vec{E}_2 = \frac{S_{11}S_{21}Z_0\bar{I}_0(\omega)}{2\pi r} e^{-jk_0 r} (1 + e^{-jk_0 \Delta}) S(\theta) \hat{\theta} \quad (25)$$

Dropping the overall phase factors in (25) and transforming to the time domain yields:

$$E_\theta(r, \theta, t) = \frac{|S_{11}S_{21}|Z_0S(\theta)}{2\pi r} [I_0(t) + I_0(t - \tau)] \quad (26)$$

where $\tau = \Delta/c$. Because the radiation is confined to a cone of half angle $1/\gamma$, it may be assumed that the detector measures the total available power. In this case, the total power detected as a function of τ becomes:

$$P_d(\tau) = \frac{|S_{11}S_{21}|^2 \ln \gamma}{\pi} \left[P_{Z_0} + \frac{Z_0}{T} \int_T I_0(t) I_0(t - \tau) dt \right] \quad (27)$$

where P_{Z_0} is defined in (23).

Clearly, the second term in the brackets of Eq. (27) represents the auto-correlation of the beam current, which in turn is proportional to the auto-correlation of the bunch profile repeated periodically with period T . Thus by measuring power as the movable mirror is moved, over a distance equal to the bunch length, the longitudinal bunch profile is obtained. The resolution of this technique, for large foils, is limited only by the dispersive properties of the interferometer and the response of the power detector as discussed in (5). The effect of finite foil dimensions on resolution is discussed in the next section.

DIFFRACTION RADIATION — A NONINVASIVE ALTERNATIVE

In some cases, perhaps in electron storage rings, it is desirable to have a noninvasive technique for measuring longitudinal bunch profile. This requirement can be easily accommodated by replacing the solid foil with a foil which contains a circular aperture centered on the beam axis as shown in Fig. 3. The most straightforward technique for finding the backward radiated fields in this case treats the conductor as a scatterer or backward diffractor of the incident fields of the beam.

Figure 3. Geometry for diffraction radiation.

In the far field, i.e., $r \gg 2b^2/\lambda$, Fraunhofer diffraction theory can be used to find the fields radiated by the conductor. Here, b is the outer radius of the conductor which could conceivably be as large as the inner radius of a circular beam chamber. Based on the results of the previous analysis for a solid foil, the radiated fields are expected to be of most interest in the region about $\theta_s = 1/\gamma$. Thus, for relativistic beams, the far-field scattered vector potential can

be found from the incident vector potential of the beam at the conductor using the paraxial Fraunhofer diffraction integral:

$$\bar{A}_{zs}(r, \theta_s, \omega) = \frac{-jk_0 e^{-jk_0 r}}{r} \int_a^b \rho' J_0(k_0 \rho' \sin \theta_s) \bar{A}_{zi}(\rho', \omega) d\rho' \quad (28)$$

The incident vector potential of the beam has a $\hat{z}$ component only given by the well known expression:

$$\bar{A}_{zi}(\rho', \omega) = \frac{\mu_0 \bar{I}_0(\omega)}{2\pi} K_0\left(\frac{k_0 \rho'}{\beta \gamma}\right) \quad (29)$$

Substituting (29) into (28) and imposing the condition $a < \gamma\lambda/2\pi < b$, the scattered vector potential in the $\theta_s \approx 1/\gamma$ region is obtained:

$$\begin{aligned} \bar{A}_{zs}(r, \theta_s, \omega) \approx & \frac{-j\mu_0 \bar{I}_0(\omega)}{2\pi r k_0} \left(\frac{1}{\theta_s^2 + 1/\gamma^2} \right) \\ & \times \left[J_0(k_0 a \theta_s) - \sqrt{2} \cos(k_0 b \theta_s) e^{-\frac{k_0 b}{\gamma}} \right] \quad a < \gamma\lambda/2\pi < b \quad (30) \end{aligned}$$

The radiation fields are then found from (16) and (17):

$$\begin{aligned} \bar{E}_{\theta_s}(r, \theta_s, \omega) \approx & \frac{Z_0 \bar{I}_0(\omega)}{2\pi r} e^{-jk_0 r} \left(\frac{\theta_s}{\theta_s^2 + 1/\gamma^2} \right) \\ & \times \left[J_0(k_0 a \theta_s) - \sqrt{2} \cos(k_0 b \theta_s) e^{-\frac{k_0 b}{\gamma}} \right] \quad a < \gamma\lambda/2\pi < b \quad (31) \end{aligned}$$

$$\bar{H}_{\phi_s}(r, \theta_s, \omega) = \frac{\bar{E}_{\theta_s}(r, \theta_s, \omega)}{Z_0} \quad (32)$$

For $b \rightarrow \infty$ and $a \rightarrow 0$, (31) and (32) reduce to expressions (18) and (19) for the solid foil in the paraxial approximation.

In the ideal case, an exact measure of longitudinal bunch profile with the autocorrelation technique requires the spectrum of the radiated fields to be identical to that of the beam current, $\bar{I}_0(\omega)$. Clearly, from the square-bracket term in (31), this is not the case for finite b , and $a \neq 0$. To see how this term comes about, consider the effective radius of the incident field of the beam, $\rho_e \approx \gamma\lambda/2\pi$. For short wavelengths (high frequencies) the incident fields are concentrated around the beam axis. If a is larger than $\gamma\lambda/2\pi$, the incident fields pass through the aperture with very little energy scattered by the conductor. This is the origin of the low-pass term, $J_0(k_0 a \theta_s)$, in (31). Similarly, if b is small

compared to $\gamma\lambda/2\pi$ for long wavelengths, very little low frequency energy is scattered. This is the origin of the cosine/exponential high-pass term in (31). Together, these terms form the bandpass function in the bracket:

$$H(\omega) = J_0(k_0 a \theta_s) - \sqrt{2} \cos(k_0 b \theta_s) e^{-\frac{k_0 b}{\gamma}} \quad (33)$$

An examination of (33) at $\theta_s = 1/\gamma$ indicates that the high and low frequency cutoff points occur approximately, for $\lambda/2\pi = a/\gamma$ and $\lambda/2\pi = b/\gamma$ respectively. In addition, $H(\omega)$ rolls off at a rate of approximately 12 dB per octave at very low frequencies and exponentially at very high frequencies. The low frequency rolloff will distort the measured bunch profile with tilt. Some admittedly subjective but conservative reasoning indicates that the tilt effect will be acceptable for a bunch of duration, τ_b , if the following condition on b is met:

$$b > 2\pi\gamma\ell_b \quad \ell_b = c\tau_b \quad (34)$$

The high frequency rolloff will limit the resolution of the technique. Again, some conventional, albeit subjective, reasoning based on rise-time/bandwidth concepts indicates a minimum resolution of:

$$x_\varepsilon = \frac{2\pi a}{\gamma} \quad (35)$$

As pointed out at the end of the previous section, the bandpass characteristics of the interferometer and power detector must also be taken into account when estimating resolution. In any case, if conditions (34) and (35) are met, the diffraction technique should perform as well as the solid foil technique. Finally, it is mentioned that the comments in the appendix regarding oblique incidence apply in the diffraction radiation case as well.

SUMMARY

A review of the theory of coherent transition radiation and its application in longitudinal bunch profile measurements has been presented. Through far-infrared/submillimeter wave Michelson interferometry, the autocorrelation of longitudinal bunch profile may be obtained to a high degree of resolution using coherent transition radiation. In addition, it has been shown that the noninvasive technique, diffraction radiation, holds great promise as a standard longitudinal bunch diagnostic.

APPENDIX

Referring to Fig. A, consider a line charge distribution, $\rho_1(z_1, t) = q_\ell(t + z_1/\beta c)$, travelling along the z_1 axis at an angle θ_i from the normal to the foil. Corresponding to ρ_1 , there will be an image charge distribution, $\rho_2(z_2, t) = -q_\ell(t + z_2/\beta c)$, travelling toward the foil along the z_2 axis. Following the conventions in the main text, the corresponding currents are:

$$I_1(z_1, t) = -\beta c q_\ell(t + z_1/\beta c) = -I_0(t + z_1/\beta c) \quad (\text{A.1})$$

$$I_2(z_2, t) = \beta c q_\ell(t + z_2/\beta c) = I_0(t + z_2/\beta c) \quad (\text{A.2})$$

Figure A. Image geometry for oblique incidence

Repeating the analysis in the main text for each current individually yields the following vector potentials:

$$\vec{\bar{A}}_1(r_1, \theta_1, \omega) = \frac{-j\mu_0 \bar{I}_0(\omega) e^{-jk_0 r_1}}{4\pi r_1} \left(\frac{1}{k + k_0 \cos \theta_1} \right) \hat{z}_1 \quad (\text{A.3})$$

$$\vec{A}_2(r_2, \theta_2, \omega) = \frac{-j\mu_0 \bar{I}_0(\omega) e^{-jk_0 r_2}}{4\pi r_2} \left(\frac{1}{k + k_0 \cos \theta_2} \right) \hat{z}_2 \quad (\text{A.4})$$

The total vector potential is the sum of (A.3) and (A.4).

The vector potential may be referred to the axis of specular reflection, z_s , and its associated coordinate system by making the following coordinate and unit vector transformations:

$$\begin{pmatrix} r_1 = r_2 = r_s = r \\ \cos \theta_2 = -\cos \theta_s \\ \cos \theta_1 = \cos \theta_s \cos 2\theta_i - \sin \theta_s \cos \phi_s \sin 2\theta_i \\ \hat{z}_2 = -\hat{z}_s \\ \hat{z}_1 = -\hat{z}_s \cos 2\theta_i - \hat{x}_s \sin 2\theta_i \end{pmatrix} \quad (\text{A.5})$$

Substituting these transformations into (A.3) and (A.4) results in a total vector potential possessing $\hat{x}_s$ and $\hat{z}_s$ components given as follows:

$$\bar{A}_{x_s}(r, \theta_s, \phi_s, \omega) = \frac{j\mu_0 \bar{I}_0(\omega)}{4\pi r} \left[\frac{\sin 2\theta_i}{k + k_0 (\cos \theta_s \cos 2\theta_i - \sin \theta_s \cos \phi_s \sin 2\theta_i)} \right] e^{-jk_0 r} \quad (\text{A.6})$$

$$\bar{A}_{z_s}(r, \theta_s, \phi_s, \omega) = \frac{-j\mu_0 \bar{I}_0(\omega)}{4\pi r} \left[\frac{1}{k - k_0 \cos \theta_s} + \frac{\cos 2\theta_i}{k + k_0 (\cos \theta_s \cos 2\theta_i - \sin \theta_s \cos \phi_s \sin 2\theta_i)} \right] e^{-jk_0 r} \quad (\text{A.7})$$

For normal incidence ($\theta_i = 0$), $\bar{A}_{x_s}$ goes to zero and $\bar{A}_{z_s}$ reduces to equation (15).

It is clear from (A.6) and (A.7) that the fields show some ϕ variation about the axis of specular reflection for the case of oblique incidence. However, for large γ , a careful study of the fields resulting from (A.6) and (A.7) reveals that the first term in the brackets of (A.7) greatly dominates all other terms so that the vector potential is given approximately by:

$$\bar{A}_{z_s}(r, \theta_s, \omega) \approx \frac{-j\mu_0 \bar{I}_0(\omega)}{4\pi r} \left[\frac{1}{k - k_0 \cos \theta_s} \right] e^{-jk_0 r} \quad (\text{A.8})$$

Equation (A.8) in the small angle approximation becomes:

$$\bar{A}_{z_s}(r, \theta_s, \omega) \approx \frac{-j\mu_0 \bar{I}_0(\omega)}{2\pi r k_0} \left(\frac{1}{1/\gamma^2 + \theta_s^2} \right) e^{-jk_0 r} \quad (\text{A.9})$$

This expression agrees with (15) in the main text for small θ_s . Therefore, for relativistic beams, the fields for oblique incidence, are well approximated by normal incidence fields.

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