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REPORT

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Nonlinear Tearing Mode Interactions and Mode Locking in Reversed Field Pinches

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Abstract

The nonlinear interaction of a set of tearing instabilities and plasma flow is studied in a cylindrical plasma. An analytic theory of mode locking is developed which includes the effects of the localized electromagnetic torques, plasma inertia and cross-field viscosity. The calculation is specialized for the case of mode locking on the Madison Symmetric Torus (MST) reversed field pinch. In MST plasmas, a set of $m = 1$ tearing instabilities become phase locked and form a toroidally localized, rotating magnetic disturbance. An evolution equation for the phase velocity of this magnetic disturbance is derived which accounts for two types of electromagnetic torques. The external torques describe the interaction of the tearing modes with static magnetic perturbations located outside the plasma region. The interior torques describe the nonlinear interaction of three tearing modes which satisfy a wave number resonance condition. For conditions typical of MST, the internal torques dominate the external torques, which suggest the nonlinear interaction of tearing instabilities play a prominent role in the momentum degradation and mode locking.

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I. Introduction

Tearing instabilities play a prominent role in describing many physics aspects of reversed field pinches (RFP's).¹ In particular, tearing modes are believed to play a primary role in describing the anomalous transport of particles and energy in the core region of RFP's, sawteeth, anomalous current transport and the MHD dynamo induced toroidal field reversal.

The central role of this work is to provide a theoretical framework to describe mode locking on RFP's. Mode locking in the context of observations on the Madison Symmetric Torus (MST)² RFP, refers to the cessation of the mode rotation of the primary tearing instabilities. The underlying physical processes involved in RFP mode locking are similar to that in tokamaks. The interaction of tearing modes and resonant magnetic perturbations emanating from a source outside the vicinity of the rational surface produce electromagnetic torques on the plasma. These torques are localized to the vicinity of the rational surface. Magnetic field errors and resistive shells can be sources of the resonant perturbations in tokamaks and RFP's. In the present work, it is suggested that there are additional electromagnetic torques present in RFP's due to the nonlinear interaction of a number of tearing instabilities.

A number of features of RFP tearing modes are distinct from tearing activity on tokamaks. The dominant tearing modes in the RFP are resonant in the core region of the plasma with typical mode numbers $m = 1$, $n \sim 2R/a$, where m and n are the poloidal and toroidal mode numbers and R/a is the aspect ratio. For MST, this corresponds to toroidal mode numbers $n \approx 5-10$. The modes co-rotate and become phased locked so that they form a localized magnetic disturbance that propagates toroidally with the plasma rotation.³⁻⁵

The tearing instabilities produce an MHD dynamo which sustains the field reversed configuration longer than electrical resistivity should allow. A

novel feature of the dynamo activity is that the relaxations often occur in discrete bursts which are referred to as the RFP sawteeth.⁶⁻⁹ The sawtooth crash is preceded by a dramatic rise in the dominant tearing mode fluctuation amplitudes and has the effect of flattening the normalized current profile after the crash. The redistribution of the plasma current causes an inward shifting of the plasma column to smaller major radius which dynamically produces radial magnetic field errors in a poloidal gap of the conducting shell. A schematic representation of the field error evolution is shown in Fig. 1. Associated with the sawtooth crash and the dynamically generated field error is the cessation of the bulk plasma rotation and the propagation of the magnetic structure.^{5,8-9}

Mode locking is also observed in tokamaks where the dynamics of a single magnetic helicity can account for the phenomena.¹⁰⁻¹³ After the mode locking occurs in tokamaks, the magnetic island associated with the tearing mode is observed to grow to large amplitude and in many cases, the plasma disrupts and the discharge is terminated. On the other hand, in MST, locking events always occur at the sawtooth crash and in two varieties.⁸ The most common locking event does not terminate the discharge. In this temporary locking case, after the sawtooth crash and mode locking, the plasma slowly reaccelerates at a rate comparable to the confinement time of the plasma and the field error in the gap decreases gradually. In the permanent locking case, the field error in the gap is sustained after the crash and is observed to grow in time. When permanent locking occurs, the plasma is prevented from reaccelerating, substantial confinement degradation is observed and the discharge terminates, but not disruptively.⁵

Theoretical models for mode locking of a single helicity tearing mode in a tokamak plasma are well developed.¹⁴⁻¹⁸ The general approach to tearing mode theory is to separate the plasma into two regions. In the bulk of the plasma the

magnetic perturbations are determined by the marginal ideal magnetohydrodynamic (MHD) equations neglecting plasma inertia and dissipation processes. This description breaks down in the vicinity of the rational surface of the mode where the exterior equations become singular. The inner layer is resolved by including the effects of inertia and dissipation processes. The two solutions are matched using the theory of asymptotics.

The effect of external magnetic perturbations can be introduced by modifying the boundary conditions of the exterior linear solution. The field error induces eddy currents in the vicinity of the rational surface. That part of the eddy current that is in phase with the naturally occurring magnetic island at the rational surface affects the amplitude of the magnetic island. The part of the eddy current that is out of phase with the naturally occurring magnetic island yields electromagnetic forces in the layer. Mode locking theories account for these electromagnetic forces which attempt to cause the plasma to cease rotating at the resonant rational surface and force the mode to take on the phase of the external field error. These forces are counteracted by inertial and viscous forces acting in the vicinity of the rational surface which attempt to keep the mode rotating at its 'natural' frequency, the frequency of a naturally occurring tearing instability. The point at which these forces are comparable determines the locking threshold.

Many of the features of mode locking on MST can be explained qualitatively with the single helicity theories developed for tokamaks.^{8-9,19} However, some aspects of the mode locking phenomena are not well understood. Namely, efforts to offset the sawtooth crash induced field errors by adding a static field error at the gap of equal and opposite magnitude do not completely eliminate the flow deceleration.⁵ The field error evolution for this case is shown in Figure 2. Mode locking occurs at the sawtooth crash despite the fact that there

is no net field error at the gap and no locking occurs between sawteeth when there is an applied field error. This suggests that not all of the electromagnetic torque effects have been properly accounted for.

A thesis of this study is that there are two types of electromagnetic torques which can act upon the magnetic disturbance. The "external" torque accounts for the interaction of the tearing modes with externally occurring resonant magnetic field perturbations. The description of the external forces is basically a multiple mode generalization of the single helicity mode locking theory developed for the tokamak. The "interior" electromagnetic torques result from the nonlinear interaction of three tearing modes that satisfy a mode number resonance condition. The interior torque can be accounted for by considering nonlinear modifications to the outer region solutions of the marginal ideal MHD equations used in the asymptotic matching problem of tearing mode theory. Although the formalism can be used to describe any three modes that satisfy a resonance condition, it is assumed that the most prominent interaction involves two core resonant $m = 1$ modes and the $m = 0$ mode, since this seems to be the most relevant for MST plasmas.

In order to make analytic progress, a number of assumptions are used. The tearing modes that are studied are assumed to be nonlinearly large and that the constant- ψ assumption of tearing mode theory can be used. This treatment is not valid for the case of forced reconnection,²⁰⁻²³ where the mode or modes under consideration are tearing stable, and the source for the reconnection and island formation is driven by external boundary conditions. The internal torques are also operative when one or more of the interacting modes is tearing stable, however then a different theoretical treatment is needed. Further, it is assumed the nonlinear mode coupling can be accounted for in the "exterior" region of tearing mode theory. A 'no-slip' condition is used where the plasma flow is

prohibited from flowing through the magnetic island separatrices. The deviations in the mode frequencies are related to changes in the rotation rate of the island region. The plasma rotation profile is assumed to be coupled across field lines by viscosity. For the most part, we will concentrate on toroidal flow evolution, since the electromagnetic torques are larger in the toroidal direction than in the poloidal direction for RFP plasmas.

In the following section, the basic theoretical model is introduced and the amplitudes of the electromagnetic and viscous torques acting in the vicinity of a tearing mode rational surface are derived. In Section III, the model is specialized to derive an evolution equation for the phase velocity of the magnetic disturbance whose components are a set of $m = 1$ tearing instabilities. Section IV discusses implications for the experimental observations in light of the theoretical calculation.

II. Theoretical Model

Periodic cylindrical geometry is used throughout the calculation with $r, \theta, \phi = z/2\pi R$ constituting an orthogonal coordinate system where $2\pi R$ is the length of the cylinder and equilibrium quantities will be functions of r alone. In RFP ordering, the equilibrium poloidal and toroidal magnetic fields are comparable and the safety factor $q = rB_z/RB_\theta$ is ordered small due to the small aspect ratio assumption $r/R \ll 1$. Reversed field pinch q -profiles are generally monotonically decreasing with radius and go to zero at some radius. The magnetic surface where the toroidal field vanishes is referred to as the reversal surface. The plasma pressure is neglected.

The most important tearing instabilities in RFP's are generally $m = 1$ tearing modes with toroidal mode numbers $n \cong 5-9$. These modes have their

rational surfaces residing in the core region. The $m = 0, n = 1$ mode is assumed to be the most important resonant harmonic at the reversal surface, where $q = 0$.

A. Tearing Mode Theory

The perturbed magnetic fields are written

$$\delta \mathbf{B}_{mn} = \delta \mathbf{B}_{mn}(r) \exp[i m \theta - i n \phi + i \zeta_{mn}] + \text{c. c.}, \quad (1)$$

where $d\zeta_{mn}/dt = \omega_{mn}$ is the rotation frequency of the magnetic perturbation. It is convenient to use a magnetic potential to describe the magnetic perturbation.

We write

$$\delta \mathbf{B}_{mn} = -\nabla \times \mathbf{b}_{mn} \Psi_{mn} \quad (2)$$

where $\mathbf{b}_{mn}$ is the unit vector in the direction of the equilibrium magnetic field at the rational surface given by $q = m/n$. For the core tearing modes, the perturbed current is written approximately from Ampere's Law as

$$\delta \mathbf{J}_{mn} \equiv \mathbf{b}_{mn} \nabla_{\perp}^2 \Psi_{mn}, \quad (3)$$

while the perturbed current for the $m = 0$ mode is

$$\delta J_{0n} = b_{0n} \left[\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (r \Psi_{0n}) - \frac{n^2}{R^2} \Psi_{0n} \right]. \quad (4)$$

This form for the magnetic potential yields the usual tearing mode assumptions that $\delta \mathbf{J}_{mn} \equiv \delta J_{\parallel mn} \mathbf{b}_{mn}$ and $\delta \mathbf{B}_{mn} \cdot \mathbf{b}_{mn} \equiv 0$ at the rational surface.

The cylindrical tearing mode stability properties are characterized by the exterior asymptotic solution where the ideal MHD forces determine the eigenfunction shape. The linearized version of the quasineutrality condition $\nabla \cdot \mathbf{J} = 0$ for Ψ_{mn} , ignoring inertia, plasma pressure and dissipation for modes resonant in the core region is given by

$$(m - nq) \left[\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \psi_{mn} - \frac{m^2}{r^2} \psi_{mn} - \frac{n^2}{R^2} \psi_{mn} \right] - \frac{mB\lambda_0}{B_0} \psi_{mn} = 0, \quad (5)$$

where $\lambda_0 = (d/dr)(J_{||}/B)$, q , B , and B_0 are the equilibrium values of the normalized current gradient, safety factor, magnetic field strength and poloidal magnetic field. A similar equation exists for the $m = 0$ mode.

Note that Eq. (5) is singular at the rational surface $q = m/n$. Near the rational surface, new physics is added to the model to resolve the singularity. For the purposes of this work, we will assume that the inner region solution is characterized by the 'constant- ψ ' assumption which states that the value of the radial magnetic field within the nonideal layer is constant.²⁴ For the forced reconnection case this assumption is not valid and a different treatment is needed.²² However, since the locking properties of pre-existing magnetic structures is of interest, the constant- ψ approximation is justified.

Since the exterior region solution has a singularity at the rational surface, the general solution to Eq. (5) has a jump discontinuity in the first derivative of ψ_{mn} . The value of this jump derivative is characterized in tearing mode theory by the value of Δ' , defined by²⁴

$$\Delta'_{mn} = \frac{1}{\psi_{mn}(r_{mn})} \frac{d\psi_{mn}}{dr} \Big|_{r_{mn}^+} - \frac{d\psi_{mn}}{dr} \Big|_{r_{mn}^-}, \quad (6)$$

where the radial labels $r_{mn}^+ \equiv r_{mn} + w_{mn}/2$ and $r_{mn}^- \equiv r_{mn} - w_{mn}/2$ refer to the radial location of the outer edge of the magnetic island associated with ψ_{mn} of width w_{mn} . Physically, the value of Δ'_{mn} refers to the value of the current sheet density distributed in the inner region solution. The value of this quantity is used to determine the stability properties of the nonlinear magnetic island and the nonlinear $\mathbf{J} \times \mathbf{B}$ forces imposed in the magnetic island region.

Since Eq. (5) is linear, the general solution can be characterized by the value of the magnetic perturbation at its rational surface, $\psi_{mn}(r_{mn}) \equiv \psi^{mn}$. For

the case of a free reconnection where ψ_{mn} satisfies the boundary conditions $\psi_{mn}(0) = \psi_{mn}(a) = 0$ at the origin and the location of the conducting wall, the exterior solution is given by

$$\begin{aligned}\psi_{mn}(r) &= \psi^{mn} F_I(r) & \text{for } 0 < r < r_{mn}^- \\ &= \psi^{mn} F_{II}(r) & \text{for } r_{mn}^+ < r < a,\end{aligned}\quad (7)$$

where $F_I(r)$ and $F_{II}(r)$ are solutions of Eq. (5) with $F_I(0) = 0$, $F_I(r_{mn}) = F_{II}(r_{mn}) = 1$, and $F_{II}(a) = 0$. The value of Δ'_{mn} derived from Eq. (7) will be termed $\Delta'_{mn \text{ mode}} = dF_{II}/dr(r_{mn}^+) - dF_I/dr(r_{mn}^-)$. This quantity determines the inherent stability property of the tearing mode.

The effect of a static external field error is to change the boundary conditions in the solution of Eq. (5).^{15-18,25-27} A finite value of the perturbed magnetic field at the boundary is written $\delta B_{mn}(a) = \delta B_{mn}^{\text{vac}} \exp[i m \theta - i n \phi + \zeta_{mn}^{\text{vac}}]$. Note that while the spatial variation of the boundary condition resonates with the rational surface $q = m/n$, in general, the time dependence of the vacuum error field solution does not correspond to the natural frequency of the mode embodied in $\zeta_{mn}(t) = \zeta_{mn}^{\text{vac}}$, the phase difference between the mode and the field error. The general solution of Eq. (5), with the amended boundary condition is given by

$$\begin{aligned}\psi_{mn}(r) &= \psi^{mn} F_I(r) & \text{for } 0 < r < r_{mn}^- \\ &= \psi^{mn} F_{II}(r) + \psi_{mn}^{\text{vac}} e^{i(\zeta_{mn}^{\text{vac}} - \zeta_{mn})} F_{III}(r) & \text{for } r_{mn}^+ < r < a,\end{aligned}\quad (8)$$

where $F_I(r)$ and $F_{II}(r)$ satisfy the same properties as described below Eq. (7), and $F_{III}(r)$ is the inhomogeneous solution which satisfies the boundary conditions $F_{III}(r_{mn}) = 0$ and $F_{III}(a) = 1$. The quantity ψ_{mn}^{vac} characterizes the strength of the static field perturbation at the boundary.

The presence of the inhomogeneous boundary condition now changes the value of Δ'_{mn} . Using Eqs. (6) and (8), we now have

$$r_{mn}\Delta'_{mn} = r_{mn}\Delta'_{mn \text{ mode}} + C_{mn}^{\text{vac}} \frac{\psi_{mn}^{\text{vac}}}{\psi_{mn}} e^{i(\zeta_{mn}^{\text{vac}} - \zeta_{mn})}, \quad (9)$$

where $C_{mn}^{\text{vac}} = r_{mn} dF_{\text{III}}(r_{mn})/dr$. The second term describes a current sheet in the inner layer which is proportional to the field error amplitude and oscillates in time at the rotation rate of the magnetic island relative to the fixed perturbation.

The presence of a slightly resistive wall can also affect the tearing mode. Eddy currents generated in the wall due to a rotating magnetic island can induce sheet currents in the layer.^{14,17} Although we will not include this effect in the calculation, the theory can be easily modified. As a practical manner, however, this effect is quite small on MST due to the high electrical conductivity of the thick aluminum wall.

Up to this point, the exterior region has been characterized by linear ideal MHD theory. However, it is possible that nonlinear effects can also play a role. To account for this we include nonlinear contributions to the exterior region equation. From $\nabla \cdot \mathbf{J} = 0$, Ψ_k satisfies

$$B_0 \nabla \frac{\delta J_{\parallel k}}{B} = - \sum_{k'} \delta B_{k'} \nabla \frac{\delta J_{\parallel k-k'}}{B}, \quad (10)$$

for each mode k , where k represents the mode with poloidal and toroidal mode numbers m and n . For the purposes of this study, we assume that the nonlinear contributions are small; thus we will only go to first order in the nonlinear terms. The appendix describes the procedure for constructing the perturbation series.

Physically, the nonlinear mode coupling can produce modifications to the mode structure. In general, these modifications describe an induced current

sheet in the layer of the resonant surface of the mode. Using the results of the Appendix, the value of this induced current as measured by Δ' is given by

$$r_{mn}\Delta'_{mn} = r_{mn}\Delta'_{mn \text{ mode}} + C_{mn}^{\text{vac}} \frac{\psi_{mn}^{\text{vac}}}{\psi_{mn}} e^{i(\zeta_{mn}^{\text{vac}} - \zeta_{mn})} + \sum_{k'} C(k, k', k-k') \frac{\psi_{k'}^k \psi_{k-k'}^k}{\psi_k} \exp[-i\zeta_k + i\zeta_{k'} + i\zeta_{k-k'}] , \quad (11)$$

where the sum contains the contributions from the three wave resonances. Notice that, in general the induced current sheet from the nonlinearities oscillates in time as described by the time dependence of the exponential phase factors.

B. Island Evolution Equation

A non-zero value of ψ^{mn} indicates magnetic reconnection has occurred. Associated with this reconnection is the formation of a magnetic island. The width of the island associated with ψ^{mn} is

$$w_{mn} = 4 \sqrt{\left| \frac{\psi^{mn}}{nB_0 q'} \right|} , \quad (12)$$

where B_0 and $q' = dq/dr$ are evaluated at the rational surface.

The evolution of a nonlinear constant- ψ magnetic island is given by a Rutherford evolution equation in resistive MHD theory:²⁸

$$\frac{dw_{mn}}{dt} = k_0 \frac{\eta}{\mu_0} \text{Re}(\Delta'_{mn}) = k_0 \frac{\eta}{\mu_0} [\Delta'_{mn \text{ mode}} + \frac{D_{mn}}{w_{mn}}] , \quad (13)$$

where k_0 is a positive constant, and η/μ_0 is the resistive diffusion coefficient. The term D_{mn} is made up of contributions from the static field errors and the nonlinear effects:

$$D_{mn} = \frac{16}{|nq'rB_0|} [C_{mn}^{\text{vac}} \psi_{mn}^{\text{vac}} \cos(\zeta_{mn} - \zeta_{mn}^{\text{vac}})]$$

$$+ \sum_{\mathbf{k}'} C(\mathbf{k}, \mathbf{k}', \mathbf{k}-\mathbf{k}') \psi^{\mathbf{k}'} \psi^{\mathbf{k}-\mathbf{k}'} \cos(-\zeta_{\mathbf{k}} + \zeta_{\mathbf{k}'} + \zeta_{\mathbf{k}-\mathbf{k}'})] , \quad (14)$$

where $\psi^{\mathbf{k}'} \psi^{\mathbf{k}-\mathbf{k}'}$ and C are taken to be real. Notice that only the in-phase part of the induced currents can affect the magnetic island width.

Typically for RFP operation, a set of $m = 1$ tearing instabilities are unstable; $\Delta'_{\text{mode}} > 0$. Nonlinearly, each mode grows to a large amplitude which depends upon the saturation processes of the instability. For these unstable tearing instabilities, the contributions contained in D_{mn} are generally of higher order and do not substantially affect the mode amplitude. However, the mode coupling and field error perturbations can substantially affect the rotation properties of the instability. This will be described in the following sub-sections.

For modes with $\Delta'_{\text{mode}} < 0$, the effects embodied in D_{mn} can cause the formation of magnetic islands driven by the external or nonlinear perturbations. However, then the constant- ψ theory described above is not valid and other descriptions of the island formation are needed.²⁰⁻²² The description of this mechanism is outside the scope of this work.

C. Electromagnetic Torques

The integrated toroidal and poloidal electromagnetic torques on the plasma can be calculated from the expression¹⁷

$$T_{\text{EMz}} = \int dV R \hat{z} \cdot \delta \mathbf{J} \times \delta \mathbf{B} , \quad (15)$$

$$T_{\text{EM}\theta} = \int dV r \hat{\theta} \cdot \delta \mathbf{J} \times \delta \mathbf{B} . \quad (16)$$

As has been shown previously,¹⁷ there is no net electromagnetic torque in the region governed by ideal MHD; the only region in which there is a contribution is in the singular inner region. If the radial integration is restricted to a region that encompasses a single rational surface at $r = r_{\mathbf{k}}$ where the harmonic $\Psi_{\mathbf{k}}$ is

resonant, the only contribution to the torque comes from the $\delta \mathbf{J} \times \delta \mathbf{B}$ forces due to the resonant magnetic perturbation.

As the rational surface is approached, the perturbed current becomes singular. A useful approximate expression for the perturbed current is $\delta \mathbf{J}_{\parallel \mathbf{k}} \equiv \Delta'_k \delta(r-r_k) \Psi_k$. Actually, the perturbed sheet current is resolved and distributed within the nonideal layer. This current, however, satisfies the property that the product of the current sheet and the layer width is determined by the appropriate asymptotic matching condition. As such it is not necessary to compute the details of the current layer, only to know its integrated value. The toroidal electromagnetic torque localized to a region surrounding the rational surface is given by

$$T_{EMz}^k = \int_{r_{k-}}^{r_{k+}} dr r R \int d\theta \int d\phi R \hat{z} \delta \mathbf{J} \times \delta \mathbf{B} = 8\pi^2 R n \text{Im}(r_k \Delta'_k |\Psi^k|^2). \quad (17)$$

The poloidal electromagnetic torque is related to the toroidal electromagnetic torque by¹⁷

$$T_{EM\theta}^k = -\frac{m}{n} T_{EMz}^k. \quad (18)$$

Since RFP's normally have $q \ll 1$, the poloidal torque tends to be much smaller than the toroidal torque. Using Eq. (11), the toroidal torque at r_k is given by

$$T_{EMz}^k = 8\pi^2 R n [-C_{mn}^{\text{vac}} \psi_{mn}^{\text{vac}} \psi^{mn} \sin(\zeta_{mn} - \zeta_{mn}^{\text{vac}}) + \sum_{k'} C(k, k', k-k') \psi^k \psi^{k'} \psi^{k-k'} \sin(-\zeta_k + \zeta_{k'} + \zeta_{k-k'})], \quad (19)$$

where the first term represents the torque imparted on the island from the external vacuum field error and the sum describes the effects of the mode coupling.

D. Viscous Torques

A phenomenological cross field viscosity will be introduced into the one fluid momentum equation.¹⁵⁻¹⁸ The effect of the viscosity will be to apply a force in the vicinity of the magnetic island when the island rotation velocity deviates from its 'natural' value. The natural value of the island frequency is assumed to be given dominantly by the Doppler shift of the equilibrium flow velocity.

The plasma within the island region is assumed to be rotating at the same velocity as the phase velocity of the mode. The profile of the rotation profile between r_{k+} and r_{k-} is taken to be a spatial constant. The deviation of the rotation velocity is denoted

$$\Delta v_k = v_k - v_{k0} \quad (20)$$

where v_k is the instantaneous value of the flow profile within the island region and v_{k0} is the natural rotation value which is calculated from the equilibrium momentum equation without the electromagnetic effects. This simplification essentially models the island regions as radial slabs of width w_{mn} centered about r_{mn} .

As in previous mode locking theories, the regions of the plasma between the rational surfaces will be coupled via viscous diffusion. In this region, the change in the toroidal velocity is governed by

$$\rho \frac{\partial \Delta v_z}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} r v \rho \frac{\partial \Delta v_z}{\partial r} \quad (21)$$

where v is the viscous diffusion coefficient and ρ is the mass density. A similar equation can be written for Δv_θ . Generally, the viscous torque is discontinuous across the island region. The value of this torque at the rational surface $q = m/n$ is given by

$$T_{vz}^k = 4\pi^2 R^2 (\rho v r \frac{\partial \Delta v_z}{\partial r}) \Big|_{r=r_k^-}^{r=r_k^+} . \quad (22)$$

Each tearing mode instability has its own separate rotation velocity. The deviation in the toroidal rotation velocity at the mode rational surface is denoted Δv_{zk} . For the case of a set of closely packed magnetic islands, the viscous diffusion time between two neighboring sets of island chains is usually the shortest time in the system. In this limit, the toroidal viscous torque on the island at r_k is given by

$$T_{vz}^k = -4\pi^2 R^2 (\rho v) \Big|_{r=r_k} \left[\frac{\Delta v_{zk} - \Delta v_{zj}}{L_{k-}} + \frac{\Delta v_{zk} - \Delta v_{zl}}{L_{k+}} \right] , \quad (23)$$

where Δv_{zj} (Δv_{zl}) is the deviation in the toroidal rotation velocity at the rational surface immediately interior (exterior) to the rational surface at r_k , and

$$L_{k-} = (\rho v) \Big|_{r=r_k} \int_{r_j^+}^{r_k^-} dr \frac{1}{\rho v r} \equiv \frac{(r_k^-) - (r_j^+)}{r_k} , \quad (24)$$

$$L_{k+} = (\rho v) \Big|_{r=r_k} \int_{r_k^+}^{r_l^-} dr \frac{1}{\rho v r} \equiv \frac{(r_l^-) - (r_k^+)}{r_k} , \quad (25)$$

where the last expressions in (24) and (25) describe the radial extent between the edges of two neighboring magnetic island regions. The approximate expressions are valid provided $L_k \ll 1$. The physics of this limit is that as island separatrices of two incommensurate helicities approach either other, the rotation of the two neighboring surfaces are strongly viscously coupled together. In order to avoid very large viscous torques as island overlap ensues, the plasma rotation at the neighboring rational surfaces must equilibrate, i. e. $v_{zk} = v_{zj}$ (and $\Delta v_{zk} = \Delta v_{zj}$). An equivalent explanation is that as magnetic island overlap occurs, magnetic stochasticity produces large anomalous viscosity in the overlap region which prohibits significant radial gradients in the rotation profile.

Inertial effects become important in determining the viscous torque when the characteristic oscillation frequency is comparable to the viscous time scale between two neighboring islands, $\omega \equiv v / [(r_k -) - (r_j +)]^2$. When ω exceeds this value, the rotation profile develops a boundary layer of width $\sqrt{v/\omega}$ around the island region. In this limit, the characteristic value for L_k is $\sqrt{v/\omega} / r_k$. Typically, this inertial boundary layer is quite large in comparison to the distance between closely packed magnetic islands and therefore doesn't play much role in determining the viscous interaction. However, the inertial layer will determine the viscous drag on two well separated magnetic island regions.

III. Magnetic Disturbance Evolution Equation and Locking

In the previous section, a set of electromagnetic and viscous torques acting on the plasma in the vicinity of a rational surface was derived. This calculation was based upon the assumption of a set of nonlinearly evolving tearing instabilities. In this section, the theory is specialized to describe mode locking on MST. The modes of interest will be a collection of $m = 1$ instabilities which are resonant in the core region. For notational simplicity these modes will be denoted by the toroidal mode number only, Ψ_n . Additionally, the $m = 0$ mode will play a role and is denoted Ψ_1 .

A. Torques on Individual Tearing Instabilities

The electromagnetic torques experienced by each rational surface is the sum of the exterior and interior contributions described in the previous section. Using Eq. (19), the toroidal electromagnetic torque on the magnetic island at surface r_n is

$$T_{EMz}^n = 8\pi^2 R n [-C_n^{vac} \psi_n^{vac} \psi_n^n \sin \zeta_n$$

$$\begin{aligned}
& + C_n \psi^n \psi^{n+1} \psi^1 \sin(\zeta_{n+1} - \zeta_n - \zeta_1) \\
& - C_{n-1} \psi^n \psi^{n-1} \psi^1 \sin(\zeta_n - \zeta_{n-1} - \zeta_1)] , \tag{26}
\end{aligned}$$

where the coefficient C_n is the coupling coefficient between the three modes n , $n+1$ and 1. As discussed in the Appendix, the coupling coefficient has the same value in the expressions of the internal torque contributions of each of the three modes. The mode amplitudes and the coefficients C_n are taken to be real and positive, with the arbitrariness of the sign taken up in the choice of phase of the $m = 0$ mode. Additionally, the phases of the external field errors are taken to be the same for all the $m = 1$ modes. This is consistent with the experimental situation on MST since the field error contributions all come from the same source, the poloidal gap in the wall.

The equation of motion for the deviation in the toroidal velocity at r_n can be derived by considering the toroidal projection of the momentum equation:

$$\rho \frac{d\Delta v_z}{dt} = \hat{z} \cdot \mathbf{J} \times \mathbf{B} - \frac{dp}{dz} + \nabla \cdot (\rho \mathbf{v} \nabla \Delta v_z) . \tag{27}$$

Integrating this equation over a volume with the radial extent given by the island width yields

$$(4\pi^2 R^2) \rho_n r_n w_n \frac{d\Delta v_{zn}}{dt} = T_{EMz}^n + T_{vz}^n , \tag{28}$$

where T_{EMz}^n is given by (26) and the viscous torque T_{vz}^n is given by

$$T_{vz}^n = -4\pi^2 R^2 (\rho v) \Big|_{r=r_n} \left[\frac{\Delta v_{zn} - \Delta v_{zn+1}}{L_{n-}} + \frac{\Delta v_{zn} - \Delta v_{zn-1}}{L_{n+}} \right] . \tag{29}$$

In the limit that neighboring islands start to overlap, the dominant term in Eq. (29) is the viscous torque. In this limit, the evolution equation for a single island rotation rate is given by

$$\Delta v_{zn} - \Delta v_{zn+1} = O(L_n) \tag{30}$$

for all nearest neighbor pairs Δv_{zn} and Δv_{zn+1} . As discussed below Eq. (25), this describes the strong viscous coupling between two nearly overlapped magnetic islands. Since this is typically the case for core tearing modes in the RFP, it is concluded that the phase velocity of the differing tearing modes is the same. Further, the phase velocity of the tearing modes and the flow profile in the core region would be closely coupled together. This prediction is in qualitative agreement with observations where the flow mode and flow velocities track each other remarkably well before, during and after the locking phase.⁸

A similar equation to (28) for the evolution of the $m = 0$ mode can also be derived:

$$(4\pi^2 R^2) \rho_1 r_1 w_1 \frac{d\Delta v_{z1}}{dt} = T_{EMz}^1 + T_{vz}^1, \quad (31)$$

where the electromagnetic torques are given by

$$T_{EMz}^1 = 8\pi^2 R [-C_1^{\text{vac}} \psi_1^{\text{vac}} \psi_1^1 \sin(\zeta_1 - \zeta_1^{\text{vac}}) + \sum_n C_n \psi^n \psi^{n+1} \psi^1 \sin(\zeta_{n+1} - \zeta_n - \zeta_1)] , \quad (32)$$

where ζ_1^{vac} is the phase of the external field error associated with the $m = 0$ mode.

The internal torques are summed over all the nearest neighbor interactions of the core tearing modes. The viscous torque on the $m = 0$ mode is

$$T_{vz}^n = -4\pi^2 R^2 (\rho v) |_{r=r_1} \left[\frac{\Delta v_{z1} - \Delta v_{zN}}{L_{1-}} + \frac{\Delta v_{z1}}{L_{1+}} \right] , \quad (33)$$

where Δv_{zN} is the deviation in the toroidal flow velocity of the core tearing mode centered closest to the edge. The last term in the viscous torque represents the force on the boundary. In the viscous limit, L_{1+} is given by

$$L_{1+} = (\rho v) |_{r=r_1} \int_{r_1+}^a \frac{1}{\rho v r} dr , \quad (34)$$

where this result is derived assuming that the deviation in the toroidal flow profile satisfies the boundary condition $\Delta v_z(a) = 0$. Associated with this viscous torque on the $m = 0$ island is an equal and opposite torque on the plasma boundary wall.

B. Magnetic Disturbance Evolution Equation

The dominant force on any one core tearing mode is the viscous interaction between nearest neighbor islands. As such it is difficult to discern a mode locking condition for any one tearing mode. However, an evolution equation for modes' phase velocity can be derived. To make further progress, we use the result of Eq. (30) to assume all phase velocities of every $m = 1$ mode is the same:

$$R \frac{d\zeta_c}{dt} = \Delta V_{zc} = \Delta v_{zn-1} = \Delta v_{zn} = \Delta v_{zn+1} = \dots \quad (35)$$

where $d\zeta_c/dt$ denotes the toroidal component of the phase velocity of the core tearing modes. Experimentally it is observed that not only do the core tearing modes have the same phase velocity, but also a fixed relative phase between them.^{5,8-9} This assumption will be used in what follows. In particular, the phase factors under the phase locking assumption are given by

$$\zeta_{n+1} - \zeta_n - \zeta_1 = (n+1) \int dt \frac{\Delta v_{zn+1}}{R} - n \int dt \frac{\Delta v_{zn}}{R} - \zeta_1 = \zeta_c - \zeta_1 \quad (36)$$

Note that the deviations in the poloidal velocities do not enter this expression. An equation for ζ_c can be derived by integrating the toroidal projection of the flux surfaced averaged momentum, Eq. (27), over a radial slice that encompasses all the core tearing modes:

$$(4\pi^2 R^3) \bar{\rho} \bar{r} \Delta \frac{d^2 \zeta_c}{dt^2} = \sum_n T_{EMz}^n + \sum_n T_{vz}^n, \quad (37)$$

where the average mass density and radius of the core region are denoted by $\bar{\rho}$ and $\bar{r}$ and the radial extent of this region is Δ . The only viscous torque acting on the magnetic disturbance is

$$\sum_n T_{vz}^n = -4\pi^2 R^2 (\rho v) \Big|_{r=r_{N+}} \left[\frac{\Delta v_c - \Delta v_{z1}}{L_{N+}} \right], \quad (38)$$

where r_{N+} is largest radius of the core region. The characteristic length scale of the radial derivative of the velocity profile, L_{N+} is given by the smaller of

$$L_{N+} \equiv \frac{1}{r_{N+}} \sqrt{\frac{vR}{\Delta v_c - \Delta v_{z1}}}, \quad (39)$$

which describes a boundary layer determined by the relative flow between the two regions, or

$$L_{N+} = (\rho v) \Big|_{r=r_{N+}} \int_{r_{N+}}^{r_1^-} \frac{1}{\rho v r} dr \equiv \frac{(r_1^-) - (r_{N+})}{r_{N+}}, \quad (40)$$

which is analogous to the viscous expressions in Eqs. (24) and (25).

The net electromagnetic torque can be calculated by summing over the electromagnetic torques of Eq. (26),

$$\begin{aligned} \sum_n T_{EMz}^n = 8\pi^2 R \Big[\sum_n -n C_n^{\text{vac}} \psi_n^{\text{vac}} \psi_n^n \sin \zeta_n \\ - C_n \psi_n^n \psi^{n+1} \psi^1 \sin(\zeta_c - \zeta_1) \Big]. \end{aligned} \quad (41)$$

Notice that because of the assumption of phase locking, the sum of the internal torques all have the same phase dependence.

An equation for the difference in the phase velocity $\Delta\zeta = \zeta_c - \zeta_1$ can be derived by a linear combination of Eq. (31) and Eq. (37). This yields a harmonic oscillator-type equation

$$\frac{d^2 \Delta \zeta}{dt^2} + A \sin(\Delta \zeta) + \sum_n B_n \sin \zeta_n = F[\Delta \Omega_0 - \frac{d\Delta \zeta}{dt}] + F_1 \frac{d\Delta \zeta_1}{dt}, \quad (42)$$

where

$$A = \left(\frac{1}{\rho r \Delta} + \frac{1}{\rho_1 r_1 w_1} \right) \sum_n \frac{2C_n}{R^2} \psi^n \psi^{n+1} \psi^1, \quad (43)$$

$$B_n = \frac{1}{\rho r \Delta} n \frac{2C_n^{\text{vac}}}{R^2} \psi_n^{\text{vac}} \psi^n \text{ for } m=1 \text{ modes}, \quad (44)$$

$$B_1 = - \frac{1}{\rho_1 r_1 w_1} 2C_1^{\text{vac}} \psi_1^{\text{vac}} \psi^1, \quad (45)$$

$$F = \frac{(\rho v)|_{r=r_{N+}}}{L_{N+} \rho r \Delta} + \frac{(\rho v)|_{r=r_1-}}{L_1 \rho_1 r_1 w_1} \equiv \frac{(\rho v)|_{r=r_{N+}}}{L_{N+}} \left[\frac{1}{\rho r \Delta} + \frac{1}{\rho_1 r_1 w_1} \right], \quad (46)$$

$$F_1 = \frac{v|_{r=r_1+}}{r_1 w_1 L_{1+}}, \quad (47)$$

where $\Delta \Omega_0$ is the difference in the natural rotation rates of the core region and the edge. The sum over n in Eq. (42) includes the contribution from the $m=0$ mode with $\zeta_1 - \zeta_1^{\text{vac}}$ being the argument for the sin.

A further assumption that could be made is that the toroidal flow rotation at the $q=0$ surface is negligible with respect to the flow rotation in the core. There seems to be some experimental evidence to indicate that this is typically the case in MST.²⁹ Additionally, by neglecting the poloidal flow contribution to the phase velocity of the core modes, $\zeta_n \equiv n\zeta_c$, an ordinary differential equation for $\Delta \zeta$ can be derived.

$$\frac{d^2 \Delta \zeta}{dt^2} + A \sin(\Delta \zeta) + \sum_n B_n \sin[n(\Delta \zeta + \zeta_1)] = F[\Omega_0 - \frac{d\Delta \zeta}{dt}], \quad (48)$$

where $\Omega_0 = v_{c0}/R$ is the natural frequency of the core region and ζ_1 is now a fixed phase. Since the rational surface of the $m=0$ mode is taken to be

stationary, the external torque on the $m = 0$ mode is constant in time (assuming that the vacuum $m = 0$ contribution is stationary).

The key new piece of physics in this calculation is the inclusion of the internal torques embodied in the term $A \sin(\Delta\zeta)$. In the absence of internal torques, an evolution equation for the core phase velocity is given by

$$\frac{d^2\zeta_c}{dt^2} + \sum_n B_n \sin(n\zeta_c) = F_c \left[\Omega_0 - \frac{d\zeta_c}{dt} \right], \quad (49)$$

where

$$F_c \equiv \frac{(\rho r)|_{r=r_N+}}{\bar{\rho} r \Delta} \sqrt{\frac{v \Delta v_c}{R}}. \quad (50)$$

In this limit the electromagnetic term is just a linear sum of the single helicity external forces.

C. Mode Deceleration

Equation (48) describes the evolution of the relative flow velocity of the phase locked core tearing modes. This is the fundamental equation for investigating the observed mode locking phenomena on RFP's. Mode deceleration occurs when the electromagnetic terms exceed the inertia and viscous forces. When this threshold is exceeded, the phase locked disturbance no longer propagates toroidally.

There are two limits that can be examined. For the case where the inertia exceeds the viscous damping force, the right side of Eq. (48) can be neglected. Assuming that the growth rate of the tearing modes and the field errors is slow compared to the rotation rate, Eq. (48) is the undamped pendulum equation and can be integrated to leading order. The instantaneous flow velocity is given by

$$\frac{d\Delta\zeta}{dt} = \pm \sqrt{\Omega^2 + 2A\cos(\Delta\zeta) + \sum_n \frac{2B_n}{n} \cos[n(\Delta\zeta + \zeta_1)]}, \quad (51)$$

where Ω is a constant in time to leading order. Figure 3 shows an example of the phase space solutions described by (51). Propagating solutions exist for an integration constant $\Omega^2 > 2A + \sum 2B_n/n$. In this limit, the electromagnetic torques are insufficient to cause the decay of the average rotation rate.

To next order, the viscous torque plays a role in establishing the value of Ω . Multiplying (48) by $d\Delta\zeta/dt$ and using (51) to define Ω , an evolution equation for Ω on the slow viscous timescale is derived:

$$\frac{d\Omega^2}{dt} = 2F[\Omega_0 \frac{d\Delta\zeta}{dt} - (\frac{d\Delta\zeta}{dt})^2], \quad (52)$$

where $d\Delta\zeta/dt = d\Delta\zeta/dt(\Omega, \zeta)$. Averaging this equation over the phase angle ζ for fixed Ω yields an asymptotic steady state solution for the averaged rotation rate $\langle d\Delta\zeta/dt \rangle = \oint d\zeta d\Delta\zeta/dt(\Omega, \zeta)$,

$$\langle \frac{d\Delta\zeta}{dt} \rangle = \Omega_0 \left[\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4\langle \tilde{\Omega}^2 \rangle}{\Omega_0^2}} \right], \quad (53)$$

where $\tilde{\Omega} = d\Delta\zeta/dt - \langle d\Delta\zeta/dt \rangle$.

Locking occurs when the magnetic forces exceed the inertia. This limit is given by $\Omega = \Omega_x$, where Ω_x is given by

$$\Omega_x^2 = -\text{Min}\{2A\cos(\Delta\zeta) + \sum_n \frac{2B_n}{n} \cos[n(\Delta\zeta + \zeta_1)]\}. \quad (54)$$

The value Ω_x is the separatrix of the harmonic oscillator defined at the minimum value of the electromagnetic term with respect to the angle $\Delta\zeta$. For $\Omega < \Omega_x$, the magnetic disturbance no longer propagates toroidally. Once inside the separatrix, the viscous force causes the solution to decay to a fixed point solution given by

$$F\Omega_0 = A\sin(\Delta\zeta) + \sum_n B_n \sin[n(\Delta\zeta + \zeta_1)] , \quad (55)$$

where (40) is used in the evaluation of F .

In the opposite limit, $F > \Omega$, the viscosity determines the locking threshold and is given by

$$F\Omega_0 = \text{Max}\{A\sin(\Delta\zeta) + \sum_n B_n \sin[n(\Delta\zeta + \zeta_1)]\} , \quad (56)$$

where the factor L_{N+} in Eq. (46) is given by the smaller of the two expressions in (39) and (40). At the locking threshold, the magnetic disturbance decelerates on a time scale approximately given by

$$t_{\text{deceleration}} \equiv \frac{A + \sum_n B_n}{\Omega_0} . \quad (57)$$

Generally, the mode unlocking threshold is smaller than the locking threshold. Thus, there is a hysteresis effect in the onset of the reacceleration relative to the threshold of the deceleration. As shown in Eq. (55), in the locked magnetic structure state, inertia plays no role in determining the relative phase angle. In order to unlock, the magnetic perturbations must satisfy

$$\text{Max}\{\sin(\Delta\zeta)A + \sum_n B_n \sin[n(\Delta\zeta + \zeta_1)]\} < F\Omega_0 , \quad (58)$$

where the viscous value of L_{N+} (40) is used in computing the viscous force F .

A similar analysis can be used on (49) to determine the locking threshold when the internal electromagnetic contributions are negligible.

IV. Discussion

An analytic theory has been developed to describe the evolution properties of the phase velocities of a set of tearing instabilities. The mode phase velocity is determined by the inertia of the tearing unstable region, viscous and electromagnetic forces. Both internally and externally produced electromagnetic

torques influence the flow evolution. The external contribution of the electromagnetic torques results from the resonant interaction of a magnetic island with a field perturbation lying outside the plasma region. The internal electromagnetic torques result from the nonlinear interaction of pre-existing tearing instabilities that satisfy a spatial resonance condition.

The theory is specialized to describe the dominant nonlinear tearing instability interaction in the MST reversed field pinch. In this example, an evolution equation for the phase velocity of a set of phased locked rotating $m = 1$ core tearing modes is derived. Typically, these modes are strongly viscously coupled together since their associated magnetic islands are overlapped or nearly overlapped. As such it is quite difficult to discern a mode locking threshold for a single helicity tearing mode. However, it is possible to understand the dynamics of the phase velocity of the magnetic disturbance which is constructed from the phase locked set of tearing modes. Equation (48) describes an evolution equation of the magnetic disturbance, which is acted on by electromagnetic and viscous forces.

When the electromagnetic forces become large enough, the phase velocity of the magnetic disturbance is altered. The external electromagnetic forces attempt to decelerate the phase velocity such that the magnetic islands in the plasma have the same phase as the vacuum field errors. The internal electromagnetic forces attempt to force the dominant nonlinear interactions to satisfy a three wave phase resonance condition. For the case of two $m = 1$ core instabilities interacting with an $m = 0$ mode resonant at the reversal surface, the internal torque attempts to make the plasma rotate as a rigid rotor at the phase velocity of the $m = 0$ mode. These electromagnetic forces are counteracted by plasma inertia and viscosity which attempt to keep the modes rotating at their natural frequency.

Subject to a couple of simplified assumptions outlined in Sec. III, mode locking thresholds are determined in two limits. In the inertial limit, the required locking electromagnetic torque is given by the approximate expression

$$\sum_n \frac{\bar{r}}{\Delta} \frac{\delta B_n^{\text{vac}}}{B} \frac{\delta B_n}{B} + \sum_n \frac{\bar{r}^2}{4r_1^2} \sqrt{|q'R|} \frac{\delta B_n}{B} \frac{\delta B_{n+1}}{B} \sqrt{\frac{\delta B_1}{B}} \equiv \frac{\Omega^2 R^2}{v_A^2}, \quad (59)$$

where the first sum is the external contribution and the second sum is the internal contribution assuming that the dominant nonlinear coupling is between two $m = 1$ modes and the $m = 0, n = 1$ mode. A more precise definition of this criterion is given in the expression for Eqs. (43), (44) and (54). For the purposes of this solution, the coupling factors are approximated by $C \equiv 1/Br_1$ and $w_1 \ll \bar{\Delta}$.

In the viscous limit, the required fluctuation amplitude for locking is

$$\sum_n \frac{n\bar{r}}{\Delta} \frac{\delta B_n^{\text{vac}}}{B} \frac{\delta B_n}{B} + \sum_n \frac{\bar{r}^2}{4r_1^2} \sqrt{|q'R|} \frac{\delta B_n}{B} \frac{\delta B_{n+1}}{B} \sqrt{\frac{\delta B_1}{B}} \equiv \sqrt{v} \frac{\Omega^{3/2} R^2}{w_1 v_A^2}, \quad (60)$$

where the inertial form of the boundary layer, Eq. (39), is used in estimating the viscous force. More precise versions of this criterion are obtained from Eqs. (38)-(40) and (56).

The unlocking is presumed to occur after the flow profile has had sufficient time to relax. In order to unlock and have the plasma rotation spin up again, the fluctuation amplitude must be reduced to the value

$$\sum_n \frac{n\bar{r}}{\Delta} \frac{\delta B_n^{\text{vac}}}{B} \frac{\delta B_n}{B} + \sum_n \frac{\bar{r}^2}{4r_1^2} \sqrt{|q'R|} \frac{\delta B_n}{B} \frac{\delta B_{n+1}}{B} \sqrt{\frac{\delta B_1}{B}} \equiv v \frac{\Omega_0 R^2}{w_1 v_A^2 \Delta r}, \quad (61)$$

where Δr is the distance between the edge of the core region and the reversal surface. This value of the required fluctuation amplitude should be smaller than the locking threshold conditions given by (59) and (60). If this is not the case, Eq. (61) also represents the approximate locking threshold condition as well.

For typical MST discharges, $\delta B_n/B \cong 1\%$ and the estimated value of the vacuum field error at the appropriate rational surface during the sawtooth crash is $\delta B_n^{\text{vac}}/B \cong 10^{-4}$.⁵ The two sets of electromagnetic torques are comparable when the fluctuation level of the $m = 0$ mode is only $\delta B_1/B \cong 10^{-4}$. This level of fluctuation amplitude is very difficult to measure experimentally, particularly when the rotation at the $q = 0$ surface is quite small. Additionally, it may also be that the $m = 0$ mode is itself locked to an external source since there is very little inertia at $q = 0$ and the rational surface is close to the edge. If the $m = 0$ is locked and stays locked through the sawtooth crash, the flow profile would only be allowed to change due to the locking interior to the $q = 0$ surface. However, changes in the flow profile also has to compete with the fact that the anomalous transport properties change abruptly at the sawtooth crash.

A significant difference between the effects of external and internal torques is that there is no net poloidal force acting on the core tearing modes from the internal forces. The two neighboring $m = 1$ modes impart an equal and opposite poloidal force upon each other and no poloidal force on the $m = 0$ mode. The external contribution give a poloidal torque m/n smaller than the toroidal torque on the core tearing modes. Therefore, any change in the poloidal rotation at the sawtooth crash must come from the effects of external effects or another class of internal torques that do not involve the $m = 0$ mode. In tokamak plasmas, strong neoclassical flow damping is present and may play some role in counteracting the poloidal component of the electromagnetic forces on a magnetic island.¹⁷ In RFP's, there is a smaller mod-B variation along a flux surface than in a tokamak and it is anticipated that the poloidal flow damping rate would be correspondingly smaller. However, there is no quantitative estimate of the flow damping rate or its effect on mode locking in RFP's.

In summary, it is suggested that the nonlinear interaction of tearing instabilities can dramatically alter the flow profile in reversed field pinches. The physics of this effect is similar to the effect a resonant magnetic field error can have on the bulk flow rotation when it interacts with a rotating tearing mode. It is postulated that the internal torques play a prominent role in the observed flow deceleration associated with sawteeth crashes on MST. In particular, this may provide an explanation for the observed mode deceleration after the crash in discharges where offset applied field errors were used to eliminate external electromagnetic torques. Additionally, it is also possible that the nonlinear interaction of the tearing modes play a role in the observed phase locking of the core tearing modes on MST. This will be the subject of future work.

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Appendix: Nonlinear interaction of tearing modes

The interaction of the tearing instabilities is accounted for through nonlinear mode couplings that appear in the exterior region of tearing mode theory. The ideal MHD equation written in the limit of negligible plasma pressure inertia and dissipation is given by

$$\mathbf{B}_0 \cdot \nabla \frac{\delta J_{\parallel k}}{B} = - \sum_{\mathbf{k}'} \delta \mathbf{B}_{\mathbf{k}'} \cdot \nabla \frac{\delta J_{\parallel \mathbf{k}-\mathbf{k}'}}{B}, \quad (\text{A1})$$

where the notation $\mathbf{k}$ represents the wave number associated with poloidal mode number m and toroidal mode number n , $\mathbf{k}' = (m', n')$, and $\mathbf{k}-\mathbf{k}' = (m-m', n-n')$.

The left side of (A1) is equivalent to the linear operator written in Eq. (5), while the terms on the right side are the nonlinear contributions.

For simplicity consider the interaction of three modes denoted $\Psi_{\mathbf{k}} = \psi_{\mathbf{k}}(r)\exp[i(m\theta - n\phi + i\zeta_{\mathbf{k}}(t))]$, $\Psi_{\mathbf{k}'} = \psi_{\mathbf{k}'}(r)\exp[i(m'\theta - n'\phi + i\zeta_{\mathbf{k}'}(t))]$ and $\Psi_{\mathbf{k}-\mathbf{k}'} = \psi_{\mathbf{k}-\mathbf{k}'}(r)\exp[i(m-m')\theta - i(n-n')\phi + i\zeta_{\mathbf{k}-\mathbf{k}'}(t)]$. Equation (A1) is now written

$$\frac{B_\theta}{rB}(m-nq)\nabla_*^2\psi_{\mathbf{k}} - k_\perp\lambda_0\psi_{\mathbf{k}} = \exp[-i\zeta_{\mathbf{k}} + \zeta_{\mathbf{k}'} + \zeta_{\mathbf{k}-\mathbf{k}'}]\{k_\perp'\psi_{\mathbf{k}'}(r)\frac{d}{dr}\frac{\delta J_{\parallel\mathbf{k}-\mathbf{k}'}}{B} - (k-k')_\perp\frac{d\psi_{\mathbf{k}'}}{dr}\frac{\delta J_{\parallel\mathbf{k}-\mathbf{k}'}}{B}\}, \quad (\text{A2})$$

where $k_\perp = b_{\mathbf{k}} \times \hat{\mathbf{r}} \cdot (m\nabla\theta - n\nabla z)$, $(k-k')_\perp = b_{\mathbf{k}'} \times \hat{\mathbf{r}} \cdot [(m-m')\nabla\theta - (n-n')\nabla z]$, and the operator ∇_*^2

$$\nabla_*^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{m^2}{r^2} - \frac{n^2}{R^2}, \quad (\text{A3})$$

for core tearing modes and

$$\nabla_*^2 = \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r - \frac{n^2}{R^2} \quad (\text{A4})$$

for the mode resonant at the reversal surface. Similar expressions can be derived for $\psi_{\mathbf{k}'}$ and $\psi_{\mathbf{k}-\mathbf{k}'}$ with a nonlinear contribution of the other two modes appearing in the same fashion as in (A2). The phase factors that enter in the expressions indicate that in general the nonlinear terms oscillate in time with respect to the time dependence of $\psi_{\mathbf{k}}$.

The exterior equation is now solved using a perturbation theory assuming that the nonlinear terms are higher order. Therefore, to lowest order the solutions to $\psi_{\mathbf{k}}$, $\psi_{\mathbf{k}'}$ and $\psi_{\mathbf{k}-\mathbf{k}'}$ are given by their linear forms as described in Eq. (7). As in Sec. IIA, the lowest order linear solution is characterized by the value of the magnetic perturbation at its associated rational surface, $\psi^{\mathbf{k}} = \psi_{\mathbf{k}}(r_{\mathbf{k}})$. The next order solution is calculated by solving the inhomogeneous equation (A2),

using the linear solutions for $\psi_{k'}$ and $\psi_{k-k'}$ on the right hand side. The solution to (A2), valid to first order in ψ , can be written

$$\begin{aligned}\psi_k &= \psi^k F_I(r) + \exp[-i\zeta_k + \zeta_{k'} + \zeta_{k-k'}] \psi^k \psi^{k-k'} G_I(r) \text{ for } 0 < r < r_{k-} \\ &= \psi^k F_{II}(r) + \exp[-i\zeta_k + \zeta_{k'} + \zeta_{k-k'}] \psi^k \psi^{k-k'} G_{II}(r) \text{ for } r_{k+} < r < a, \quad (A5)\end{aligned}$$

where F_I and F_{II} satisfy the boundary conditions listed below Eq. (7) and $G_I(0) = G_I(r_{k-}) = G_{II}(r_{k+}) = G_{II}(a) = 0$. Since the nonlinearities of the form considered here do not resolve the singular nature of the solution near the rational surface, the general solution of (A5) has a discontinuous first derivative. Denoting this discontinuity as in Eq. (6), we derive expressions for Δ' for each mode:

$$r_k \Delta'_k = r_k \Delta'_{k \text{ mode}} + C(k, k', k-k') \frac{\psi^{k'} \psi^{k-k'}}{\psi^k} \exp[-i\zeta_k + i\zeta_{k'} + i\zeta_{k-k'}], \quad (A6)$$

$$\begin{aligned}r_{k'} \Delta'_{k'} &= r_{k'} \Delta'_{k' \text{ mode}} \\ &+ C(k', k, k-k') \frac{\psi^k \psi^{k-k'}}{\psi^{k'}} \exp[-i\zeta_{k'} + i\zeta_k - i\zeta_{k-k'}], \quad (A7)\end{aligned}$$

$$\begin{aligned}r_{k-k'} \Delta'_{k-k'} &= r_{k-k'} \Delta'_{k-k' \text{ mode}} \\ &+ C(k-k', k, -k') \frac{\psi^k \psi^{k'}}{\psi^{k-k'}} \exp[-i\zeta_{k-k'} + i\zeta_k - i\zeta_{k'}], \quad (A8)\end{aligned}$$

where the coefficients $C = dG_{II}(r_{k+})/dr - dG_I(r_{k-})/dr$. We have omitted possible contributions from amended boundary conditions as described in Eq. (9), but these contributions will be kept in the more complete expressions.

As explained in Section IIC, the imaginary part of Δ' contributes to the electromagnetic torque imparted on the plasma at the rational surface. Making use of Eq. (17), the values of the poloidal and toroidal portions of the

electromagnetic torques at each of the rational surfaces due to the nonlinear interaction can be derived:

$$T_{EMz}^k = 8\pi^2 R n C(k, k', k-k') \times \text{Im}[\psi^{k*} \psi^{k'} \psi^{k-k'} \exp(-i\zeta_k + i\zeta_{k'} + i\zeta_{k-k'})], \quad (\text{A9})$$

$$T_{EMz}^{k'} = 8\pi^2 R n' C(k', k, k'-k') \times \text{Im}[\psi^{k*} \psi^k \psi^{k-k'} \exp(-i\zeta_{k'} + i\zeta_k - i\zeta_{k-k'})]. \quad (\text{A10})$$

$$T_{EMz}^{k-k'} = 8\pi^2 R (n-n') C(k-k', k, -k') \times \text{Im}[\psi^{k-k'*} \psi^k \psi^{k'} \exp(-i\zeta_{k-k'} + i\zeta_k - i\zeta_{k'})]. \quad (\text{A11})$$

Similar expressions are derived for $T_{EM\theta}^k$ which satisfies (18).

Since these forces are entirely contained within the plasma volume, there must be no net toroidal and poloidal torques on the plasma. We therefore require the properties,

$$T_{EMz}^k + T_{EMz}^{k'} + T_{EMz}^{k-k'} = 0, \quad (\text{A12})$$

$$T_{EM\theta}^k + T_{EM\theta}^{k'} + T_{EM\theta}^{k-k'} = 0. \quad (\text{A13})$$

These conditions are satisfied if

$$-n C(k, k', k-k') + n' C(k', k, k'-k') + (n-n') C(k-k', k, -k') = 0, \quad (\text{A14})$$

$$-m C(k, k', k-k') + m' C(k', k, k'-k') + (m-m') C(k-k', k, -k') = 0. \quad (\text{A15})$$

Equations (A14) and (A15) can only be satisfied if $C(k, k', k-k') = C(k', k, k'-k') = C(k-k', k, -k')$, i. e., the coupling coefficients in the Δ' expressions are equivalent.

Since the nonlinear interactions are higher order in Ψ , all sets of three-wave couplings can be handled independently. As such, the general expression for Δ'_k to first order in Ψ is

$$r_k \Delta'_k = r_k \Delta'_k \text{ mode}$$

$$+ \sum_{k'} C(k, k', k-k') \frac{\psi^{k'} \psi^{k-k'}}{\psi^k} \exp[-i\zeta_k + i\zeta_{k'} + i\zeta_{k-k'}]. \quad (\text{A16})$$

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Figure Captions

1. The time dynamics of the magnetic field errors produced in the gap of MST. At the sawtooth crash, a radial field error is produced. For the more common temporary locking cases, the field error decays to zero after the crash.
2. A schematic representation of the field error in the gap for cases where a static field error is added in an attempt to eliminate the field error at the sawtooth crash. Despite the absence of a field error at the sawtooth crash, mode deceleration occurs.
3. An example of the phase space diagram for the magnetic disturbance phase velocity in the inertia dominated limit, Eq. (51). The contours represent constants of motion of the fast rotation time scale. The top three contours represent propagating solutions where the electromagnetic torques are insufficient to prevent mode locking. The closed contour represents a surface inside the separatrix which indicates the electromagnetic terms are large enough to prevent the modes from rotating. The viscous effects, which are not included in this figure, determine which contour the solution asymptotes to in time.

Figure 1

Figure 2

Figure 3

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