A Formula for the Integration of Radiation
Using Yoshida's Lie Methods*

Etienne Forest
KEK
101 Oho, Tsukuba, Ibaraki, Japan

and

Z. Parsa
Brookhaven National Laboratory
Upton, NY 11973

and

*This work was performed under the auspices of the U.S.
Department of Energy under Contract No. DE-AC02-98CH10886.

October 1998

CENTER FOR ACCELERATOR PHYSICS

BROOKHAVEN NATIONAL LABORATORY
BROOKHAVEN SCIENCE ASSOCIATES

Under Contract No. DE-AC02-98CH10886 with the
UNITED STATES DEPARTMENT OF ENERGY

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency, contractor or subcontractor thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency, contractor or subcontractor thereof.
DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.
A FORMULA FOR THE INTEGRATION OF RADIATION USING YOSHIDA’S LIE METHODS

Étienne Forest
KEK, 1-1 Oho, Tsukuba, Ibaraki, Japan 305-0801
E-mail: forest@kekwa.kek.jp

Zohreh Parsa
Brookhaven National Laboratory, Upton, New York, 11973
E-mail: parsa@bnl.gov

We present our earlier formula 1 and show how the nonsymplectic operator for the classical radiation can be imported in a generalized Yoshida’s formula. The challenge is to “break” the force into exactly solvable parts. Then the entire machinery of Lie integrators takes over.

1 Application to Non-Symplectic Problems: Radiation

Yoshida’s method is based on the existence of a Lie operator of some sort and a quadratic time-reversible approximation of the exact map. Thus Ruth’s 2 as well as Yoshida’s 3 integrators have little to do with the symplectic structure per se. There are more connected to the Lie representation of operators and clearly apply to non-symplectic systems.

A useful non-symplectic application of Yoshida’s methods in accelerators is in the domain of (classical) radiation in electron rings. If a computer code computes the change in energy of an electron due to radiation at each time steps, one will be able to determine various useful properties of the ring such as the phase at the cavities and the damping coefficients. Typically one computes the new closed orbit with radiation and the matrix around it using something like a truncated power series algebra package (TPSA package i.e. “DA”, for example see 4,5). The eigenvalues of this matrix will give us the damping time of the ring under consideration.

One can ask why bother with symplectic formalisms when there exist in the literature high order integration schemes which would work very well on a non-Hamiltonian force? The answer to this lies in the smallness of the radiation effects in a ring. On a turn by turn basis a particle will rarely lose more than one percent of its energy; typically it loses about one part in ten thousand. It turns out that, if this effect is added on top of a symplectic scheme, it will be easily detected even if the scheme itself is relatively inaccurate. If, on the other hand, we use a nonsymplectic scheme, we would need a highly accurate scheme in order to resolve the nonsymplectic inaccuracies introduced by the
integration of the Hamiltonian system from the actual radiation. By adding radiation on top of a symplectic scheme we can continue to use the Talman philosophy of “exact” tracking through “inexact” re-fitted elements.

How does this work? Consider a beam element (magnet) represented by a Hamiltonian $H$ and approximated in the tracking code by a time-reversible quadratic approximation $S_2(s)$. This is general enough to encompass Ruth’s integrator and the whole sequence of Yoshida’s high order formulas. In passing we point out the obvious: if the element without radiation is exactly solved, it is certainly a “quadratic approximation.” For example in standard kick codes, bends and quadrupoles are often solved exactly within the framework of large machine Hamiltonians (the solution being in terms of matrices for transverse phase space and quadratic polynomials for the longitudinal variable $\ell$).

The effect of radiation can be added as a force $\tilde{F}_{\text{rad}}(\tilde{x}, \tilde{p})$ which we will assume to be $s$-independent within a particular magnet. Then it is clear that a new quadratic approximation of the Lie map can be written as follows:

$$ S_2^{\text{rad}}(s) = \exp \left( \frac{s}{2} \tilde{F}_c \cdot \tilde{\nabla} \right) S_2(s) \exp \left( \frac{s}{2} \tilde{F}_c \cdot \tilde{\nabla} \right). $$

The operators are all assumed to be in canonical variables as reflected by the superscript “c”. A first order solution of the transfer map associated with the operator $\tilde{F}_c \cdot \tilde{\nabla}$ can be derived easily. We will proceed now with a sketch of such the derivation (see 6 or 7 for more details).

Following Sand’s the change in the relative momentum deviation $\delta$ (we are assuming an ultrarelativistic electron) is given by

$$ \frac{d\delta}{d\ell} = K_c (1 + \delta)^2 \frac{\tilde{B}_\perp \cdot \tilde{B}_\perp}{Brho^3}; \quad K_c = 1.40789357 \times 10^{-5} E_0^3. $$

The reference energy of the electron $E_0$ is measured in Gev. The field $\tilde{B}_\perp$ is the component of the local magnetic field perpendicular to the direction of propagation. The quantity $\tilde{B}_\perp \cdot \tilde{B}_\perp$ can be easily computed from the value of the field given along the unit vectors of a cylindrical frame of reference. Needless to say that these quantities should be available to a well-written tracking code.

For an ultrarelativistic particle phase space can be described by the set $(x, p_x, y, p_y, \delta, \ell)$ where the transverse momenta and the energy $\delta$ are scaled by the design momentum $p_0$. Since our variable of integration is a distance $s$ along the magnet, we need to convert the derivative with respect to $\ell$ into a derivative with respect to $s$. This is done using the underlying Hamiltonian of the magnet under consideration:

$$ \frac{d\delta}{ds} = \frac{d\delta}{d\ell} \frac{d\ell}{ds}. $$
During the radiation process, an ultrarelativistic particle will emit a photon in the forward direction only. Thus, the usual directions \( \frac{dz}{ds} \) and \( \frac{du}{ds} \) are left unchanged while the transverse momenta actually change. Therefore, to the symplectic integrator of step size \( \Delta s \), we must add the radiative terms:

\[
\frac{d\delta}{d\ell} [\ell, H] = -\frac{d\delta}{d\ell} \frac{\partial H}{\partial \delta}
\]

The above set of equations constitutes a first order solution of the transfer map associated with the radiation process. Although it can be part of a multi-step Yoshida integrator, terms proportional to it will not converge with the rate predicted by the theory.

This situation can be improved by using the non-canonical variables \((\frac{dz}{ds}, \frac{du}{ds})\) instead of the canonical variables \((p_z, p_u)\). In non-canonical variables, the radiative operator has a simple form because only the energy \( \delta \) is changed by the process:

\[
F^{n-c} \cdot \vec{v} = K_c(1 + \delta)^2 \frac{\vec{B}_1 \cdot \vec{B}_1}{Brho^2} \frac{\partial H}{\partial \delta} \Delta s.
\]

Furthermore the quantity \( \frac{\partial H}{\partial \delta} \) which expresses the change in path length cannot depend on the energy when expressed in terms of the non-canonical variables \((\frac{dz}{ds}, \frac{du}{ds})\). Thus equation 1 can be re-written as

\[
S^{rad}_2(s) = \exp \left( \frac{s}{2} \tilde{F} \cdot \vec{v} \right) S_2(s) \exp \left( \frac{s}{2} \tilde{F} \cdot \vec{v} \right) = C \exp \left( \frac{s}{2} \tilde{F}^{n-c} \cdot \vec{v} \right) C^{-1} S_2(s) C^{-1} \exp \left( \frac{s}{2} \tilde{F}^{n-c} \cdot \vec{v} \right) C,
\]

where \( C \) is a change of variables from \((p_z, p_u)\) to \((\frac{dz}{ds}, \frac{du}{ds})\). The operator \( \exp \left( \Delta s \tilde{F}^{n-c} \cdot \vec{v} \right) \) changes only the variable \( \delta \) according to the relation:

\[
\delta'(\Delta s) = \exp \left( \Delta s \tilde{F}^{n-c} \cdot \vec{v} \right) \delta = \left( \frac{1}{1 + \delta} - K_c \frac{\vec{B}_1 \cdot \vec{B}_1}{Brho^2} \frac{\partial H}{\partial \delta} \Delta s \right)^{-1} - 1.
\]
Because equation 9 is an exact solution of the radiative operator, equation 8, if used in one of the high order integrators previously discussed, will behave appropriately and preserve the expected rate of convergence of the integrator.

Conclusion

We have shown how the nonsymplectic operator for the classical radiation can be imported in a generalized Yoshida formula. Probably the same can be done with the stochastic term affecting the moments.