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EFFECTS OF MODELED TERMS
IN THE REYNOLDS-STRESS TRANSPORT EQUATIONS

BY

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B.S., New Mexico State University, 1992

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ABSTRACT

A discussion of the origins and interpretation of various previously proposed models for the terms in the incompressible Reynolds-stress equation is given. It is hoped that the interpretations will provoke thoughts that will help in the future modeling of Reynolds-stress transport equations. Different forms of the closed Reynolds-stress equation have been solved numerically for the given mean velocity field of the wake flow behind a flat plate at a Reynolds number of 1000. A finite-difference/finite-volume collocation scheme was used to approximate the spatial derivatives, which were implemented in a time-marching scheme. The numerical time integration produced values for the six independent Reynolds-stress components, the turbulent kinetic energy decay rate, and the turbulent length scale for each of the models tried. The results of the different cases were compared and some conclusions were drawn on the effects of the various investigated modeled terms.

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1. INTRODUCTION

The Navier-Stokes equations are assumed for present purposes to describe all fluid motion arising from a detailed set of initial and boundary conditions. The concept of turbulence arises from the fact that not all boundary and initial conditions are known exactly. The portions of the initial and boundary conditions that are not accounted for are the features that initiate turbulence. Turbulence is the portion of fluid motion that is not explicitly known at any given instance in time. The Navier-Stokes equations govern both the turbulent and nonturbulent motions, and in this thesis two different ways of solving the Navier-Stokes equations to observe the average turbulence characteristics will be compared.

The determination of the mean flow variables and characterization of the energy associated with the turbulence are important because so many of the flows occurring in nature and in engineering applications are turbulent. The boundary layer of the earth's atmosphere, its jet stream, ocean currents, and the photospheres of stars are all examples of turbulent flows in nature. Boundary layers on the surface of an aircraft wing, most combustion processes, and flow patterns of natural gas and oil in pipelines are all examples of turbulence in engineering applications [1]. Within the context of this thesis, the fact that many wake flows are turbulent is particularly important.

Some examples of applications where the understanding of turbulent wake flows is important are seen in the design of aircrafts, ships, and cars, where the knowledge of turbulent wake flows is imperative in order to improve performance. In the development of motor vehicles it is important to understand how to modify turbulent wake flow patterns in order to limit drag. In the development of military submarines it is necessary to understand how to limit the turbulent wake in order to make the submarine harder to

detect. These applications are just a few examples of the reasons that the study of turbulence in a wake is particularly important.

In order to quantitatively describe turbulence, it is necessary to define a set of variables representing the strength of the random fluctuations at any location. The velocities and pressure can be represented as the sum of a mean part and a fluctuating part in the form, $L = \bar{L} + l'$, where L is the instantaneous variable, $\bar{L}$ is the mean value of L (the ensemble average of L), and l' is the deviation from the mean value.

The ensemble average of the product of two velocity fluctuations will be called a Reynolds-stress tensor,

$$R_{ij} = \overline{u_i' u_j'} . \quad (1.1)$$

The mean total kinetic energy per unit mass in a fluid can be described as the sum of the mean flow kinetic energy and mean turbulent kinetic energy. The mean total kinetic energy per unit mass is designated by $\overline{KE}$, the mean turbulent kinetic energy is designated by K , and the mean flow kinetic energy is designated by KE_{mean} .

$$\overline{KE} = \frac{1}{2} (\overline{U_i^2}) \quad (1.2)$$

$$\overline{KE} = \frac{1}{2} (\overline{(\bar{u}_i + u_i')^2}) \quad (1.3)$$

$$\overline{KE} = \frac{1}{2} (\overline{\bar{u}_i \bar{u}_i + 2\bar{u}_i u_i' + u_i' u_i'}) = \frac{1}{2} (\overline{\bar{u}_i \bar{u}_i + u_i' u_i'}) \quad (1.4)$$

$$\overline{KE} = KE_{mean} + K \quad (1.5)$$

where:

$$KE_{mean} = \frac{1}{2} \bar{u}_i \bar{u}_i \text{ and } K = \frac{1}{2} \overline{u_i' u_i'} .$$

The development above makes use of the rule that the ensemble average of a fluctuation is zero and the ensemble average of a mean value multiplied by a fluctuation is also zero. It can be seen, from equation 1.5, that the mean turbulent kinetic energy is the sum of the diagonal components of the Reynolds-stress tensor.

The off-diagonal components of the Reynolds-stress tensor are measures of the correlation between a velocity fluctuation in one direction and the velocity fluctuations in an orthogonal direction. Unlike the diagonals, the off-diagonals can have a positive or negative value. If the value of an off-diagonal term is zero, the value indicates that there is no correlation between velocity fluctuations in the two directions. For any given velocity fluctuation value in one direction, the fluctuation in the other direction will have an equal probability of being one value as any other.

One method for finding the values of the Reynolds-stress is by using a transport equation to describe their changes in time and space. A transport equation for the Reynolds-stress tensor can be derived from the Navier-Stokes equations. The transport equation involves ensemble averages of products of velocity and pressure fluctuations. It is desirable to be able to calculate the Reynolds-stress without knowing the individual velocity fluctuations, because this capability would reduce the resolution needed for any computer simulation. Models have been suggested for the transport equation. They are meant to enable the Reynolds-stress and mean flow to be solved, based on ensemble average quantities, which can be explicitly calculated.

The concepts of Reynolds-stress and Reynolds averaging were introduced by Sir Osborne Reynolds. In the same time frame, Boussinesq (1877) introduced the first eddy viscosity model, but it was not until after 1920 that the first successful calculation of a practical turbulent flow was made, using Reynolds averaging and an eddy viscosity model. Prandtl introduced the concept of the mixing length for use in determining the eddy viscosity. These models all suffered from the inability to account for flow history,

because they were all point functional models. In 1945 Prandtl linked the eddy viscosity to the turbulent kinetic energy, which required a separate transport equation. This is said to be the first of the one-equation turbulence models [2].

In later years multiple transport models for various turbulence quantities were used to solve the Navier-Stokes equations for turbulent flows. The development of two-equation turbulence transport models began in 1945 when Chou recognized the importance of a second turbulence transport equation. In 1951 Rotta, also realizing the need for two transport equations, laid the foundation for full Reynolds-stress turbulence equation closure [3]. Utilizing some of the statistical ideas of A. M. Komogorov (1941), Rotta managed to close the Reynolds-stress transport equations [2]. In the early 1970s researchers such as Daly and Harlow (1970), Donaldson (1971), Naot, Shavit, and Wolfstein (1972) followed Rotta's example in the creation of two-equation Reynolds-stress models [4]. These models attempt to provide simulation of the dissipation, diffusion, and redistribution properties appearing in the unmodeled transport equations. They tend to provide reasonable representation of the specific flows for which the parameters were calibrated but do not do an adequate job of modeling more general flows.

One of these earlier Reynolds-stress models was discussed by Daly and Harlow, in *The Physics of Fluids* (1970), "Transport Equations of Turbulence." The closure technique introduced in this paper was the beginning of a new thrust in the development and implementation of second-order closure models, brought on by the availability of high-speed computers [2]. Daly and Harlow utilized the concept of a "gradient diffusion hypothesis" [4]. The model discussed is a two-equation model of the Reynolds-stress transport equations and the corresponding turbulent energy dissipation equation. This paper laid many of the foundations for the models to be analyzed in this thesis.

Soon after, researchers including Launder, Reece, Rodi, Glusko, Davydov, and Bradshaw were developing models and theory that extended these foundations for much

of the turbulence equation modeling being done today. In 1974, B.E. Launder, G. J. Reece and W. Rodi used a model of turbulence in which the Reynolds-stress are determined from the solution of transport equations [4]. These transport equations retained terms that had been neglected by previous authors. The dissipation equation was modeled much like the decay equation of later authors.

In 1992, a model written for variable density flow was documented by Besnard, Harlow, Rauenzahn, and Zemach. This model combines several techniques of the past (Daly and Harlow's gradient hypothesis; Launder, Reece, and Rodi's conservative pressure terms; and Rotta's nonconservative pressure term) and some new ideas to give a new version of the Reynolds-stress transport equation and a corresponding decay rate equation [5]. This approach provides some of the basis for the analysis to be done in this thesis.

There have been various models suggested to eliminate the need for calculation of the fluctuating values in the turbulence transport equations; each of them includes its own physical implications and effects when applied in a given flow field. This thesis is written to give interpretation and analysis of some of these models and to look at their effects when applied to the near wake region of a flow past a flat plate.

For this purpose a computer program has been written to solve the modeled equations for the flat-plate geometry. The program uses a time-marching technique until it reaches a stochastically steady solution and produces the Reynolds-stress values, the turbulent length scale, and the rate at which the turbulent kinetic energy is decaying at any point in space.

As a basis for comparison we use a set of data produced by a highly resolved computer code written by D. Joshi (1992) [6] that calculates averages over long time spans in order to compute the desired quantities. This calculation of details is a technique used to solve

for the turbulence quantities, also based on the Navier-Stokes equations; this procedure is called Direct Numerical Simulation (DNS).

The limitations of DNS calculations are imposed by limited time averaging and limited resolution. Joshi's DNS averaged quantities were averaged sufficiently long, after an initial start up period, to remove most of the asymmetrical features in the diagonal Reynolds-stresses and most of the nonantisymmetrical features of the R_{12} component. This particular DNS calculation used a grid which was able to resolve all but about three percent of the turbulent stresses [6] (this remaining three percent are referred to as subgrid stresses and were neglected). When the data from this simulation was compared to experimental data, at higher Reynolds numbers produced by Bradbury and Moss (1975), "fair" agreement [6] was shown. The calculated mean velocities agree well with the experimental data and the magnitudes of the turbulence quantities are higher than the experimental values. Joshi explained the differences between the DNS results and the experimental values as a result of the difference in the Reynolds numbers between the calculation and the experiment. The DNS calculation is used instead of experimental results, as the basis for comparison, because of the larger amount of information that can be obtained from DNS calculations.

The set of DNS data provides long time-averaged quantities for the mean velocity components and six independent Reynolds-stress components. The DNS data provides initial condition for the Reynolds-stress model program. Any divergence from this initial condition can be interpreted as showing that the DNS data is not a solution for whatever model is being tested. If other initial conditions were used, there exists the possibility that the solution is approaching the DNS data but has not reached a point where this approach could be recognized. The DNS data also provides inlet conditions for the Reynolds-stress model program.

For easier isolation of the solution of the turbulence transport equations, the mean flow velocity field is held fixed. For the interpretation of results, our working hypothesis is that the DNS data are sufficiently close to the true Navier-Stokes solutions. The discrepancies between the turbulence field produced by the Reynolds-stress transport equations and the turbulence field created by the DNS can only be attributed to the model.

To compare the implications of different models, a base case was implemented in the computer program and serves as a basis for comparison. Variations to the diffusion and decay terms in the subsequent runs will allow the effects of those variations to be compared with the Base Case results and the DNS data.

A possible source of error arises from the length scale that is to be used in the inlet. The DNS code did not compute the decay or the length scale for this problem. This leaves room for error in the assignment of such values at the inlet in the Reynolds-stress models. The absence of a scale to accompany the inlet boundary conditions for the Reynolds-stress creates a problem because not only is this value needed for the diffusion coefficient, but it sets the inlet value for the scalar turbulent kinetic energy decay rate. Different scales will be applied to the Base Case in order to determine how sensitive the Reynolds-stress models are to the inlet scale.

2. TURBULENCE TRANSPORT EQUATIONS

In the first section of this chapter the Reynolds-stress transport equations are introduced. The exact transport equations, as derived directly from the Navier-Stokes equations, are given along with a brief discussion of their derivation. The concept of turbulent kinetic energy decay rate is described in this section also. There are terms in the exact transport equation that need to be modeled in order to obtain closure as described in the second section of this chapter.

Sections 2.3 through 2.8 describe the terms in the Reynolds-stress transport equation. These sections are meant to give some physical interpretation to each of the terms and describe the role that the term should play in the transport of the Reynolds-stress. Some discussion is given to the techniques used to model the pressure-velocity correlations, the triple correlation, and the decay term. The full tensor decay term is dependent on the turbulent kinetic energy decay rate, which has its own transport equation. This second transport equation is described briefly in section 2.9.

2.1 The Unmodeled Equations

A transport equation for the Reynolds-stress can be derived directly from the Navier-Stokes equations in the form

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t} + \frac{\partial}{\partial x_n} (u_n R_{ij}) + R_{in} \frac{\partial u_j}{\partial x_n} + R_{jn} \frac{\partial u_i}{\partial x_n} + \frac{\partial (u_i' u_j' u_k')}{\partial x_k} = \\ - \left[\frac{\partial (u_i' \phi')}{\partial x_j} + \frac{\partial (u_j' \phi')}{\partial x_i} \right] + \overline{\phi' \frac{\partial u_i'}{\partial x_j}} + \overline{\phi' \frac{\partial u_j'}{\partial x_i}} \\ + \nu \frac{\partial^2 R_{ij}}{\partial x_k^2} - 4 \nu D_{ij} \end{aligned} \quad (2.1)$$

Equation 2.1 can be derived by decomposing the instantaneous velocity and pressure, U_i and $\Phi \equiv P/\rho$, into mean values, u_i and $\bar{\phi}$, and fluctuating values, u_i' and ϕ' (the mean velocity $\bar{u}_i$ being now written as u_i for simplicity). In order to derive equation 2.1, the mean and fluctuating quantities are inserted into the Navier-Stokes equations for incompressible flow. The i direction momentum equation is multiplied by u_j' , and the j direction momentum equation is multiplied by u_i' . The two equations are added. Taking the ensemble average of the entire equation gives equation 2.1.

This Reynolds-stress transport equation shows the time rate of change in the ij^{th} component of the Reynolds-stress tensor, $R_{ij} = \overline{u_i' u_j'}$, in terms of unknown ensemble averages of products of velocity fluctuations, pressure fluctuations, mean velocities, u_i , molecular viscosity, ν , and the dissipation rate tensor, D_{ij} . The dissipation tensor, D_{ij} , is defined as

$$D_{ij} = \frac{1}{2} \left[\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k} \right]. \quad (2.2)$$

The turbulent dissipation quantity, $4\nu D_{ij}$, represents the rate at which the turbulence is being dissipated and the energy is being converted to heat. This loss rate at the fine scales may not be the same as the loss rate from the most important parts of the turbulence. The turbulent length scale, s , is the length scale of the principle energy-containing eddies and is of the same order of magnitude as the integral length scale. The length scale appropriate for the eddies that are being dissipated to heat is of the order of the Kolmogorov scales. At high enough Reynolds numbers there are orders of magnitudes between these two scales. If the Reynolds number is high enough, the Reynolds-stress transport equations should not depend on the molecular viscosity.

There is a transport mechanism not described by equation 2.1. There is cascade of R_{ij} from bigger scales to smaller scales. It is most useful to use the Reynolds-stress transport

equation to describe the turbulence at the larger length scales where the majority of the energy resides. This scale, s , is the scale that tends to hold the dominant turbulence characteristics. The majority of the turbulent energy at these scales is transferred to smaller scales, where it resides temporarily before being converted to heat by the molecular viscosity. The transport of energy from larger scales to smaller ones can be described by a decay term. The inclusion of the decay terms representing the turbulence that is being cascaded away from scale s to smaller scales, $-2\varepsilon_{ij}$, and from even smaller scales to the smallest scales, $+2\varepsilon_{ij}$, gives

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t} + \frac{\partial}{\partial x_n} (u_n R_{ij}) + R_{in} \frac{\partial u_j}{\partial x_n} + R_{jn} \frac{\partial u_i}{\partial x_n} + \frac{\partial (\overline{u_i' u_j' u_k'})}{\partial x_k} = \\ - \left[\frac{\partial (\overline{u_j' \phi'})}{\partial x_i} + \frac{\partial (\overline{u_i' \phi'})}{\partial x_j} \right] + \overline{\phi' \frac{\partial u_i'}{\partial x_j}} + \overline{\phi' \frac{\partial u_j'}{\partial x_i}} + \nu \frac{\partial^2 R_{ij}}{\partial x_k^2} - 2\varepsilon_{ij} \quad (2.3) \\ + 2\varepsilon_{ij} - 4\nu D_{ij} \end{aligned}$$

At equilibrium, the cascade to the smallest scales, $+2\varepsilon_{ij}$, has almost the same magnitude but different sign as $-4\nu D_{ij}$. These two terms would cancel each other out. We assume that the turbulence is sufficiently close to equilibrium so that these two terms can be omitted,

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t} + \frac{\partial}{\partial x_n} (u_n R_{ij}) + R_{in} \frac{\partial u_j}{\partial x_n} + R_{jn} \frac{\partial u_i}{\partial x_n} + \frac{\partial (\overline{u_i' u_j' u_k'})}{\partial x_k} = \\ - \left[\frac{\partial (\overline{u_j' \phi'})}{\partial x_i} + \frac{\partial (\overline{u_i' \phi'})}{\partial x_j} \right] + \overline{\phi' \frac{\partial u_i'}{\partial x_j}} + \overline{\phi' \frac{\partial u_j'}{\partial x_i}} - 2\varepsilon_{ij} \quad (2.4) \end{aligned}$$

This equation describes the transport of the Reynolds-stress at the energy-containing scale, s , for flows that have high Reynolds numbers (high enough that the effects of molecular viscosity can be neglected). The factor of 2 is introduced because it is desirable

for ε_{ij} to be the decay rate of the turbulent kinetic energy K . It can be seen that $2\varepsilon_{ij}$ must have the units of length squared over time cubed since it is the change in Reynolds-stress over the change in time. A characteristic time can be identified as $s/\sqrt{K}$. When the Reynolds-stress is divided by this time scale, we get a term with proper dimensions and tensor form to represent $2\varepsilon_{ij}$. Using $\varepsilon \equiv \varepsilon_{ij}$ we get

$$\varepsilon = \frac{K^{3/2}}{s} . \quad (2.5)$$

2.2 Closure Modeling

There are various forms of turbulence representations, all of which are aimed at circumventing the need to directly solve the Navier-Stokes equations for turbulent flows. Their need arises because of the limitations of current computer memories and speeds. It would be almost impossible to solve complex flows with adequate resolution to calculate all of the fluctuating velocities. Turbulence tends to have activity over several magnitudes of scales in three spatial directions. For example, in a situation with a turbulent Reynolds number of ten thousand, the large eddies could very easily be five hundred to a thousand times as large as the ones that are affected by the molecular viscosity. In order to resolve all of the fluctuations, there needs to be at least a thousand nodes in each spatial direction. Large amounts of memory are needed for such a large calculation. This type of resolved calculation also requires more speed than is currently available, since the transport equation must be applied at all nodes at all time steps or iterations. For this reason DNS calculations are currently feasible only at very low Reynolds numbers. The DNS calculations used in this thesis are a coarse approximation to the unattainable ideal.

All of the representations must appear in closed form. Closed form or closure refers to having the terms in the equations be functions of values that can be explicitly determined.

In other words, the necessity for closure dictates that there will not be more unknowns than equations. We model the Reynolds-stress equation to obtain closure, in order to avoid the need for explicit calculation of the velocity fluctuations.

The only variables that we have explicit methods to obtain are the average quantities for velocities, decay rate, and Reynolds-stress. In equations 2.4 there are three basic types of higher-order moments that are not explicitly solved with the independent variables of mean velocity, mean velocity gradients, R_{ij} and ϵ_{ij} . The first of these higher-order moments is the ensemble averaged gradient of a product of a pressure fluctuation and a velocity fluctuation. The second type is the correlation between a pressure fluctuation and gradients of velocity fluctuations. The third type of higher-order moment needing modeling is the correlation of multiple velocity fluctuations.

In the development of turbulence models in general, there is a set of requirements that lend guidance to their proper formulation .

1. One of these modeling requirements is dimensional consistency. The modeled term must retain the same dimensions as the unmodeled term. This requirement dictates that any models used for the terms in the Reynolds-stress equation must have dimensions of length squared divided by time cubed. These are the dimensions corresponding to the time rate of change of energy per unit mass. The dimensions for any modeled terms in the decay rate transport equation must be length squared divided by time to the fourth power.
2. If the unmodeled term or combination of terms has no contribution to the total turbulent kinetic energy, then the modeled term cannot have any contribution either. This means that if the value of the unmodeled term is zero when the free indices are contracted, the contraction of the modeled terms must also be zero.

3. If the unmodeled term is conservative, then the modeled term must be conservative also. The increase in a conservative quantity in any arbitrary control volume will be equal to the total influx of the quantity minus the total outflux.
4. The modeled term must be tensor invariant, and if the unmodeled term is symmetric in its indices then its model must be also. Tensor invariance demands that the modeled terms be in proper tensor notation and exhibit tensor properties.
5. A model for a term in the turbulence equations must exhibit Galilean invariance. Under this requirement the term must retain the same properties no matter what inertial frame it is expressed in. Galilean invariance requires that the modeled terms be invariant under Galilean group transformations, which allow for any arbitrary transitional velocity.
6. Another of these requirements is that of realizability [1]. The constraint of realizability as first proposed by Schumann (1977) and rigorously proven by Lumley (1978), states that $R_{\alpha\alpha} \geq 0$ ($R_{\alpha\beta} = \overline{u_\alpha' u_\beta'}$), and that $R_{\alpha\alpha} R_{\beta\beta} - R_{\alpha\beta}^2 \geq 0$ [2]. Here the Greek letters indicate no summation. This constraint indicates that a Reynolds-stress model should yield positive component energies, and thus as $R_{\alpha\alpha}$ vanishes its time rate of change must vanish as well.

In addition to the basic modeling requirements, it is also a goal to represent as wide a range of transient flow problems as possible with a single representation [3]. This universality would require that model terms and parameters not be functions of initial and boundary conditions. In addition, as mentioned, point functional Reynolds-stress models have particular problems with universality. The model must include the capability to convect, diffuse, create, and destroy quantities as well as exchange components among the nonisotropic directions.

For very strong turbulence, the transport equations should perform independent of molecular viscosity, ν . The effects of the turbulent decay, creation, diffusion, convection,

and reappportionment should dominate. These ideas agree with the fact that the turbulent viscosity is of the order of $.09sK^{1/2}$. As the turbulent energy and scale go up, the turbulent viscosity will rise and be more important than the molecular viscosity. The effective Reynolds number, $Re_{ef} = UL / (\nu + \nu_T)$, becomes dominated by the turbulent viscosity. This is in contrast to low-intensity turbulence, which is appreciably affected by molecular viscosity. In this case not only is the effective Reynolds number affected by the molecular viscosity, but the scales at which the viscosity takes effect are much closer in size to those containing the majority of the energy.

2.3 The Inertial and Advection Terms

The first term in equation 2.4 is the inertial term,

$$\frac{\partial R_{ij}}{\partial t} \quad (2.6)$$

This term shows the rate of change of the Reynolds-stress components with respect to time. It does not need modeling and allows for the development of the Reynolds-stress with time.

The second term in equation 2.4,

$$\frac{\partial}{\partial x_n} (u_n R_{ij}), \quad (2.7)$$

is the advection term, which comes from the nonlinear advection term in the Navier-Stokes equations.

This term contains only average quantities. For this reason it is not necessary to model it because these quantities are explicitly computed. The advection of the Reynolds-stress represents the difference in the amount of a particular component of the Reynolds-stress

that is being carried into and out of a differential volume by the average velocity. If there is more Reynolds-stress being carried into the control volume than carried out of the control volume, there will be an increase in the amount of Reynolds-stress in the control volume.

2.4 The Creation Term

The third and fourth terms in equation 2.4 are the two creation terms,

$$R_{in} \frac{\partial u_j}{\partial x_n} + R_{jn} \frac{\partial u_i}{\partial x_n} . \quad (2.8)$$

These terms also originate from the advection term in the Navier-Stokes equation. The creation terms do not need modeling since they contain only Reynolds-stress components and mean velocity gradients. These terms represent the transformation of mean flow energy to turbulent energy.

The presence of turbulence in a mean shear flow leads to additional rotation of the turbulence and mixing of the free stream. This process increases the amount of turbulence energy but removes energy from the mean flow because it takes mean flow energy to turn the turbulence and mix the free stream layers. The principle sources of turbulent energy consist of an off-diagonal term Reynolds-stress multiplied by a mean velocity gradient.

An excellent example of the actions being described by this type of term is given by the observation of an infinite shear flow with velocities in the x direction and a positive velocity gradient in the y direction. The primary creation for R_{11} for this type of flow consists of a term of the form

$$2R_{12} \frac{\partial u_1}{\partial y} . \quad (2.9)$$

The value of R_{12} is negative in the presence of a positive x velocity gradient in the y direction, because a positive fluctuation of a fluid element in the y direction pushes slower-moving fluid into a faster-moving mean flow. This represents a negative fluctuation in the x direction, as shown in figure 2.1.

The negative value of R_{12} will be multiplied by a positive velocity gradient and form a positive source to R_{11} (due to the negative sign that is on the term). This also produces a sink in the energy of the mean flow equations, which can be proven to be exactly the source in the turbulence kinetic energy equation.

Figure 2.1. A positive x velocity gradient in the y direction produces a negative R_{12} value. As a particle at a point (x, y) moves with a velocity fluctuation in the y direction shown by the vertical dashed line, u_2' , it carries a velocity in the x direction that will be different from the mean velocity of its new location. The difference is a velocity fluctuation in the x direction, shown by the horizontal dashed line, u_1' .

In order to give some physical explanation for why the creation of Reynolds-stress should be proportional to components of the Reynolds-stress, the actions in the shear layer can be examined again. We can imagine a fluid element, initially sitting in a fluid moving to the left with respect to a given reference velocity. The element is carried in a positive y direction by a fluctuation u_2' . As explained earlier, the value of R_{12} for the case of a positive velocity gradient is negative. This means that a fluid element fluctuating in a positive y direction will most likely have a negative fluctuation in the x direction relative to the surroundings it moves in to. If the value of R_{12} has a larger negative magnitude, the particle will either move farther into the fluid above it (making it appear that the fluctuation is higher) or will have a stronger negative fluctuation compared to its surroundings. The increase in R_{12} would increase the creation of the R_{11} component of the Reynolds-stress, since the magnitude of the difference between the fluid elements velocity and the velocity of the surrounding fluid will increase.

2.5 Conservative Pressure Velocity Correlation Term

There are two different forms of terms involving correlations of pressure fluctuations in equations 2.4, of which the first is

$$-\left[\frac{\partial(u_i' \phi')}{\partial x_j} + \frac{\partial(u_j' \phi')}{\partial x_i} \right] \quad (2.10)$$

and the second is

$$\phi' \left[\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right]. \quad (2.11)$$

The first term is conservative because it is the derivative of a flux. This means that the term does not contribute to any Reynolds-stress component in one area without removing the same amount of that particular component from some other part of the domain. The only way that this term can change the amount of Reynolds-stress in the domain is by receiving or giving through the domain boundaries.

In order to understand some physical implications of this term, we can look at its contribution to R_{11} in particular. The contribution for this Reynolds-stress component, from the conservative pressure velocity correlation term, can be written as

$$-2 \left[\frac{\partial (\overline{u_1' \phi'})}{\partial x_1} \right]. \quad (2.12)$$

Figure 2.2. The difference between the work fluctuation done to the control volume on the left side and the work fluctuation done by the control volume on the right side is a contribution to R_{11} , and the difference between the fluctuating work done on the control volume on the bottom and the work fluctuation done by the top of the control volume is a contribution to R_{22} .

The term in equation 2.12 can be observed as the difference between the rate of work per unit mass being done by pressure fluctuations on the left and right side of a fluid element.

If, on the average, the amount of work being done on the left side is equal to the amount of work being done by the right side, then there will not be any change in the R_{11} component of the turbulent energy. This condition would be the same as having a zero gradient in the ensemble-averaged correlation between the velocity fluctuation and pressure fluctuation. If there is a negative gradient of this correlation, then the rate at which energy is being supplied to the element is greater than the rate at which the element is supplying energy to the other elements. The result is an increase in the R_{11} component of the turbulent energy in the element. The opposite is true if the gradient is positive.

In order to observe some of the physical significance of the contributions of this term to the off-diagonal term Reynolds-stress components, we can look at the contribution of

$$-\frac{\partial(\overline{u_2' \phi'})}{\partial x_1} \quad (2.13)$$

to R_{12} . In order for this term to contribute to the change in R_{12} , the difference between the correlation, $(\overline{u_2' \phi'})$, on one side and the other side of a fluid element must induce a change in the velocity fluctuation in the x direction. We can imagine a control volume with pressure and velocity fluctuations on the right and left sides. It is assumed that the values on the left and right sides of the element are far enough apart to be uncorrelated and that the values of the fluctuations in the middle are somewhere in between the values on the sides of the control volume. The values of the fluctuations on the right side of the element are given an average value of zero associated with any given fluctuation on the left side, since values at the two sides are uncorrelated. At one instant in time there is a positive pressure fluctuation and a positive velocity fluctuation in the y direction on the left side of the control volume. The difference in the pressure fluctuations cause a velocity fluctuation in the x direction to become more positive, feeding the growth of R_{12} . The correlation between the positive velocity fluctuations in the x and y directions becomes

more positive also since the value of u_2' will be positive in the center of the volume. At the next instant in time the pressure fluctuation and velocity fluctuation on the left side might have negative values (the product is still positive). The velocity fluctuation in the x direction will tend to become less positive. R_{12} will become more positive (less negative) because the value of u_2' is negative.

2.6 Nonconservative Pressure Velocity Correlation Term

The second type of term involving a pressure fluctuation is nonconservative. This term,

$$\phi' \left[\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right], \quad (2.14)$$

cannot increase the amount of energy in any location because it has a vanishing contraction but may transmit energy between diagonal components. The term in equation 2.14 is usually modeled in two separate parts, the "rapid pressure term" [4],

$$\begin{aligned} & -C_1 \left[R_{ik} \frac{\partial u_j}{\partial x_k} + R_{jk} \frac{\partial u_i}{\partial x_k} - \frac{2}{3} R_{ik} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right] \\ & -C_2 \left[R_{ik} \frac{\partial u_k}{\partial x_j} + R_{jk} \frac{\partial u_k}{\partial x_i} - \frac{2}{3} R_{ik} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right] - C_3 K \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \end{aligned} \quad (2.15)$$

and the "return to isotropy term",

$$-C_{IR} \frac{\epsilon}{K} \left[R_{ij} - \frac{2}{3} \delta_{ij} K \right] \quad (2.16)$$

This particular form of the rapid pressure term was proposed by Launder, Reece, and Rodi. The return to isotropy was originally proposed by Rotta [4]. In order to

understand the need for two such terms, one can look at a physical interpretation of equation 2.14.

The terms in 2.15 and 2.16 have a contraction of zero. The vanishing contraction of 2.14 precludes any type of model that could change the total amount of turbulent energy in a given location. The only transmission of energy allowed under this constraint is among diagonal components of the Reynolds-stress.

Figure 2.2. A perturbed vortex sheet between two walls. The diagram illustrates the fact that in any particular x location there is a consistent positive or negative fluctuation in streamwise velocity. The deflection of the vortex sheet, which causes the streamwise velocity fluctuations, results in pressure fluctuations above and below the sheet. The pressure fluctuations cause cross-stream velocity fluctuations, which deflect the sheet even more.

In an attempt to understand how the modeled term should behave, a heuristic analysis can be made. An example of the behavior of this term is a perturbed vortex sheet with x -direction velocity jump δu in the y direction.

Figure 2.2 shows the vortex sheet between two walls with a positive mean velocity above the sheet and a negative mean velocity below the sheet. Once this sheet is slightly perturbed in the y direction, the perturbation will grow and sustain itself. For this example, we will assume that the mean velocity above the vortex sheet is the same magnitude as the negative mean value below the sheet. The perturbation in the vortex sheet causes there to be portions of the vortex sheet that are closer to the top wall and locations where the vortex sheet is closer to the bottom wall.

If we consider an x location where the vortex sheet is bulged upward, the velocity above the sheet is greater than the mean flow above the sheet because the area between the sheet and the upper boundary is smaller than at a location where there was no bulge. The magnitude of the instantaneous velocity below the sheet is smaller than the mean flow below the sheet, because the flow area at this location is larger than it would be if there was no bulge upward in the sheet. The increased velocity above the sheet and the decreased velocity below the sheet both represent positive velocity fluctuations in the x direction. The increased velocity above the sheet causes a negative pressure fluctuation, and the decreased negative velocity below the sheet causes a positive pressure fluctuation. The difference in the pressure fluctuations above and below the sheet causes the sheet to bulge even more than it had previously. This movement in the sheet is accompanied by a velocity fluctuation in the y direction. R_{22} energy is increased in this process. The size of the initial perturbation can be visualized as a measure of the initial strength of fluctuations in the y direction. We have discussed the correlation between this positive y velocity fluctuation and the positive x velocity fluctuation that accompanies it in relation to the pressure fluctuations.

An equation,

$$\frac{DR_{22}}{Dt} = -CR_{12} \frac{\partial u_1}{\partial x_2}, \quad (2.17)$$

for a relationship between the creation of R_{22} to the velocity fluctuations and the mean velocity jump has been postulated by previous authors.

The effect shown above is not accounted for by the primary creation terms. When only the primary creation terms are examined, the nonvanishing components are

$$\frac{DR_{11}}{Dt} = -R_{12} \frac{\partial u_1}{\partial x_2}, \quad \frac{DR_{12}}{Dt} = -R_{22} \frac{\partial u_1}{\partial x_2}. \quad (2.18)$$

These resulting creation terms are not in agreement with the Bernoulli type effect described above. The remaining creation must come from the pressure velocity correlation term.

A general return to isotropy trend is also caused by the pressure strain correlation. One heuristical example of this need is in the case of a flow that originates with Reynolds-stress in primarily one direction. This Reynolds-stress represents fluctuating velocities in that direction. Since two adjacent points will not have the same instantaneous velocities at any instant in time, there will be concentrations of high and low pressure fluctuations. The negative pressure fluctuations are at a location where the particles in the instantaneous flow field appear to be spreading out from the point and the positive pressure fluctuations are where the particles in the instantaneous flow would appear to be colliding. These high and low pressure zones are going to draw or push in directions normal to the fluctuations as well as parallel to the fluctuations. In this way the pressure and velocity fluctuations have together combined to damp the fluctuations in one direction and increase the fluctuations in the others. This would, without the effects of an anisotropic source, cause the turbulence to become more isotropic.

We can make these qualitative remarks more precise. As outlined by Launder, Reece, and Rodi, the explicit appearance of pressure fluctuations can be eliminated from the unmodeled equations by taking the divergence of the fluctuating velocity equation [4]. A

Poisson's equation can be solved with a Green's function integral, multiplied by $\partial u_i' / \partial x_j$ and ensemble averaged to give

$$\overline{\phi \frac{\partial u_i'}{\partial x_j}} = \frac{1}{4\pi} \int_{vol} \left\{ \left[\frac{\partial^2 u_i' u_m'}{\partial x_l \partial x_m} \right] \frac{\partial u_i'}{\partial x_j} \right]_y + 2 \left[\frac{\partial u_l}{\partial x_m} \right]_y \frac{\partial u_m'}{\partial x_l} \left[\frac{\partial u_i'}{\partial x_j} \right]_x \right\} \frac{dvol}{|x-y|} + S_{ij} . \quad (2.19)$$

In equation 2.19, S_{ij} is the surface integral, which will be negligible if not near a boundary [4]. The first term is referred to as $T_{ij,1}$, and the second is referred to as $T_{ij,2}$. $T_{ij,1}$ involves only fluctuating quantities, and $T_{ij,2}$ involves a mean velocity gradient.

In the equation above, it was proposed by Launder, Reece, and Rodi to look to the symmetrized second term in the integral $T_{ij,2} + T_{ji,2}$ to produce creation of Reynolds-stress proportional at the mean shear flow. Rotta expressed the fact that the term $T_{ij,2}$ could be expressed in the form [4]

$$T_{ij,2} = \frac{\partial u_l}{\partial x_m} a_{ij}^{mi} \quad (2.20)$$

where

$$a_{ij}^{mi} = -\frac{1}{2\pi} \int \frac{\partial^2 (u_m)_x (u_i)_y}{\partial \xi_l \partial \xi_j} \frac{dvol}{d|x-y|} . \quad (2.21)$$

ξ_l s are the Cartesian components of the vector $x - y$. This equation was derived under the assumption that all second derivatives of the velocity field are small and the field may be taken as nearly homogeneous [4]. Utilizing this form as a guideline together with the fact that the contraction of the term must be zero, the fact that the indices m and i should be interchangeable as well as l and j , and the basic generalized form for a fourth-order tensor, Launder Reece and Rodi have written a model of the form found in equation 2.22.

$$\begin{aligned}
(T_{ij,2} + T_{ji,2}) = & -C_1 \left[R_{ik} \frac{\partial u_j}{\partial x_k} + R_{jk} \frac{\partial u_i}{\partial x_k} - \frac{2}{3} R_{lk} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right] \\
& -C_2 \left[R_{ik} \frac{\partial u_k}{\partial x_j} + R_{jk} \frac{\partial u_k}{\partial x_i} - \frac{2}{3} R_{lk} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right] - C_3 K \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]. \quad (2.22)
\end{aligned}$$

The model shown in equation 2.22 includes all possible forms of the model but is often reduced to equation 2.23,

$$(T_{ij,2} + T_{ji,2}) = -C_{2R} \left[R_{ik} \frac{\partial u_j}{\partial x_k} + R_{jk} \frac{\partial u_i}{\partial x_k} - \frac{2}{3} R_{lk} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right]. \quad (2.23)$$

This has been done by various modelers and is justified because the first term in equation 2.22 is dominant and exhibits the desired physical significance. This term tends to isotropize the turbulence production tensor [4].

Rotta proposed the model for the sum of $T_{ji,1} + T_{ij,1}$.

$$T_{ij,1} + T_{ji,1} = -C_{1R} \frac{\varepsilon}{K} \left[R_{ij} - \frac{2}{3} \delta_{ij} K \right] \quad (2.24)$$

This return to isotropy term has two basic manifestations. The first of these is the one applied to the diagonals, equalizing the energy contained in the diagonal Reynolds-stress components. The second is the action performed on the off-diagonals. The return to isotropy serves as a sink for these terms. This term describes the physical effect of mixing up the flow in such a way as to remove the correlation between different fluctuating components of the velocity. As the flow moves away from the region of primary creation, the turbulence becomes more mixed up and the significance of this term grows. The shear structures in the mean flow are not as strong; and the off-diagonal term Reynolds-stress components, which have been advected and diffused downstream, are acted upon by the return to isotropy. The positive and negative correlations that

characterize these off-diagonal components begin to mix and cancel one another out as the flow approaches isotropic conditions.

2.7 The Triple Correlation and Diffusion Terms

The triple correlation, which arises from the nonlinear advective term in the Navier-Stokes equation, is

$$\frac{\partial(\overline{u_i' u_j' u_k'})}{\partial x_k} \quad (2.25)$$

This term acts as a diffuser of turbulence, tending to spread out turbulent energies. This term does not cause the energy to be converted into heat energy and does not tend to redistribute energy among nonisotropic components. It primarily diffuses energy in a spatial manner. Table 2.1 gives the various forms of diffusion that will be addressed in this thesis. Each model is diffusive in form and originates from the triple correlation.

Diffusion Type	Diffusion Model
Simple Diffusion	$\frac{2}{3} C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K^2}{\epsilon} \frac{\partial R_{ij}}{\partial x_m} \right]$
D.H. Diffusion	$C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\epsilon} R_{mn} \frac{\partial R_{ij}}{\partial x_n} \right]$
BHR Diffusion	$\sum^{sym} \frac{3}{2} C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\epsilon} R_{in} \frac{\partial R_{mj}}{\partial x_n} \right]$

Table 2.1 The diffusion terms tested in this thesis.

The models take different forms and thus have different physical meanings. It is important that the implications of these differences be understood both from a physically intuitive standpoint and from an application standpoint. The derivations of the various terms are outlined here, and their physical differences are explored. One way to look at the triple correlation is to imagine it as the instantaneous advection of the fluctuation of the product of two velocity fluctuations. A fluctuation of the product of two velocity fluctuations can be defined as

$$R_{ij}' = u_i' u_j' - \overline{u_i' u_j'} = u_i' u_j' - R_{ij}. \quad (2.26)$$

The fluctuation of u_k' , which is carrying R_{ij}' , is said to be a random value. We can suppose that the random value is of the same magnitude as the square root of the turbulent kinetic energy. R_{ij}' is being carried in a so-called random walk pattern. This is not unlike molecular motion. It is the same random walk motion that is responsible for the diffusion of a dye in water or the mixing of one gas into another. The amount of movement in the resulting direction of motion tends to be linearly dependent on the so-called step size, which in this case is of order s . It also tends to be dependent on the number of these steps taken in a given time.

A diffusion contribution to the time rate of change of the amount of some scalar L in a control volume is the difference in the flux entering and leaving the control. In turbulence the flux in a direction is also dependent on a diffusion coefficient, which is a function of the scale of the energy-containing eddies and the strength of the correlations between the fluctuating velocities, since these values will dictate how far and fast the diffusion should occur. The coefficient is proportional to the scale and to the average absolute magnitude of the velocity fluctuations. This can be shown by using the fact that the mean distance a particle travels in a random walk pattern, D , is

$$D \equiv s\sqrt{N} , \quad (2.27)$$

where N is the number of steps taken in a characteristic time.

The diffusion coefficient is of the order D^2/t , where t is the characteristic time scale, $t = s/\sqrt{K}$. The number of steps taken in the characteristic time is

$$N = \frac{s}{t\sqrt{K}} . \quad (2.28)$$

When we combine equation 2.27 and 2.28, we can see that the diffusion coefficient should be proportional to $s\sqrt{K}$ for the isotropic case where the magnitudes of the velocity fluctuations are assumed to be similar in all directions.

It is possible to develop a transport equation for the triple correlation by multiplying the i momentum equation by $u'_j u'_k$ and performing similar multiplications on the other two momentum equations. The velocities can be decomposed into mean and fluctuating parts and the ensemble averages taken. In this transport equation, it is discovered that there is a term containing $\overline{u'_i u'_j u'_k u'_l}$. In general, the transport equation for an n^{th} correlation introduces an $(n+1)^{\text{th}}$ correlation. This is why a model for the triple correlation must be introduced. There are various ways attempted for the modeling of this term. One, described by Daly and Harlow, utilizes a gradient approximation for the correlation of fluctuations of a scalar with fluctuations in the velocity. The transport of a scalar, L , can be written,

$$\frac{\partial L}{\partial t} + u_k \frac{\partial L}{\partial x_k} = - \frac{\partial}{\partial x_k} \left[\beta \frac{\partial L}{\partial x_k} \right] , \quad (2.29)$$

with a convection and diffusion term (β is the laminar-flow diffusion coefficient). This equation is combined with full momentum equation for a fluid,

$$\frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} = -\frac{\partial \phi}{\partial x_j} + \frac{\partial}{\partial x_k} \left[\nu \frac{\partial L}{\partial x_k} \right]. \quad (2.30)$$

When the equations are added to one another, the velocities and pressures are decomposed, ensemble averages are taken, and the equations are allowed to go to steady state, the result is

$$R_{ik} \frac{\partial L}{\partial x_k} = \overline{L' \frac{\partial \phi'_i}{\partial x_k}} + (\nu + \beta) \left[\overline{\frac{\partial L'}{\partial x_k} \frac{\partial u'_i}{\partial x_k}} \right]. \quad (2.31)$$

For the lowest order approximation, the arbitrary scalar L is not significantly correlated to pressure. Under these assumptions, equation 2.31 takes the form

$$R_{ik} \frac{\partial L}{\partial x_k} = \alpha \left[\overline{\frac{\partial L'}{\partial x_k} \frac{\partial u'_i}{\partial x_k}} \right]. \quad (2.32)$$

α is a diffusion coefficient having units of length squared over time, which has been shown to be proportional to $s\sqrt{K}$. The derivatives can be approximated using values of u'_i and L' divided by some finite length scale if it is assumed that their derivatives are constant over that distance, the turbulent length scale,

$$\left[\overline{\frac{\partial L'}{\partial x_k} \frac{\partial u'_i}{\partial x_k}} \right] = \frac{C}{s^2} \overline{u'_i L'}. \quad (2.33)$$

Equation 2.33 is a subset of the more general postulate,

$$\frac{\partial A'}{\partial x_k} \frac{\partial B'}{\partial x_k} = \frac{C}{s^2} (A' B'). \quad (2.34)$$

C can be solved by applying empirical data that are applicable to the particular variable L .

A direct result of equations 2.32 and 2.33 is

$$\overline{u_i' L'} = -\frac{s^2}{C\alpha} R_{ik} \frac{\partial L}{\partial x_k}. \quad (2.35)$$

In this form, any correlation of a fluctuating variable and a component of the fluctuating velocity can be expressed as a function of the gradient for the variable and the Reynolds-stress tensor.

When the triple correlation is broken up as in 2.25, it is possible to use equation 2.31 to get several different forms of diffusion terms to model the triple correlation. The most obvious is derived by first dividing the triple correlation as

$$\overline{u_i' u_j' u_k'} = \overline{R_{ij}' u_k'} = -\frac{s^2}{\alpha} \left[R_{kl} \frac{\partial R_{ij}}{\partial x_l} \right]. \quad (2.36)$$

Under close inspection of the first step of the derivation, it can be seen that this step violates the symmetry rule because the left side of the equation is symmetrical in i, j , and k , and the right side is not. This technique continues to be used because of the much higher complexity encored when the division is made fully symmetric. The following step is to take the derivative of both sides of equation 2.36,

$$\frac{\partial(\overline{u_i' u_j' u_k'})}{\partial x_k} = -\frac{\partial}{\partial x_k} \left[\frac{s^2}{\alpha} R_{kl} \frac{\partial R_{ij}}{\partial x_l} \right], \quad (2.37)$$

and then make the appropriate substitutions for the diffusion coefficient,

$$\frac{\partial(\overline{u_i' u_j' u_k'})}{\partial x_k} = -C_{DR} \frac{\partial}{\partial x_k} \left[\frac{s}{K^{1/2}} R_{kl} \frac{\partial R_{ij}}{\partial x_l} \right]. \quad (2.38)$$

This model, which will be referred to as the Daly Harlow diffusion term (D.H. diffusion term), is diffusive in form and shows the expected dependence on $s\sqrt{K}$ when the

turbulence is isotropic. As the turbulent energy and therefore the Reynolds-stress and scale decrease, the strength of this term decreases drastically.

This type of diffusion can be visualized if we consider the diffusion of an arbitrary scalar, L , in the form

$$-C_{DR} \frac{\partial}{\partial x_i} \left[\frac{s}{K^{1/2}} R_{ki} \frac{\partial L}{\partial x_k} \right]. \quad (2.39)$$

The type of diffusion that is made possible by the presence of an off-diagonal term Reynolds-stress in the diffusion coefficient can be illustrated by looking at the diffusion of a scalar, L , in the presence of a negative x -direction velocity gradient in the y direction.

Figure 2.4 Diffusion described by the D.H. diffusion term in the presence of a mean flow gradient. The intensity of the shading depicts contours of L in a negative y -direction gradient. The Reynolds-stress component, R_{12} , in the diffusion coefficient tells us about the presence of motion in the x direction.

The particular component of the diffusion term that we are illustrating is

$$-C_{DR} \frac{\partial}{\partial x_1} \left[\frac{s}{K^{1/2}} R_{12} \frac{\partial L}{\partial x_2} \right]. \quad (2.40)$$

It has been previously explained that, in this flow arrangement, the value of R_{12} will be positive. For this example we will assume that L has a negative gradient in the y direction. This gradient is going to induce a flux of the scalar in the positive y direction. The positive value of R_{12} in the diffusion coefficient tells us that any positive fluctuations in the y direction will on average be accompanied by a fluctuation in the positive x direction. The difference in the flux in the x direction from the left to the right side of the element in question constitutes the increase in the amount of L in the element.

Another form of the diffusion model for the triple correlation can be obtained by inserting the first part of the Boussinesq approximation as

$$-C_{DR} \frac{\partial}{\partial x_k} \left[\frac{s}{\sqrt{K}} \frac{R_{nn} \delta_{kl}}{3} \frac{\partial R_{ij}}{\partial x_l} \right]. \quad (2.41)$$

In the equation 2.39, the s^2/α of equation 2.37 has been replaced by $C_{DR} s/\sqrt{K}$ since we can use the Boussinesq value for the turbulent diffusion coefficient of $Cs\sqrt{K}$. After simplifying equation 2.39 we get

$$-\frac{2C_{DR}}{3} \frac{\partial}{\partial x_l} \left[s\sqrt{K} \frac{\partial R_{ij}}{\partial x_l} \right]. \quad (2.42)$$

This model has the desired dependence on $s\sqrt{K}$. This form of the diffusion of R_{ij} is dependent on the derivative in the l direction of a flux that is proportional to a gradient in the l direction. The D.H. model is dependent on the derivative in the k direction of a flux that is proportional to the gradient of R_{ij} in the l direction.

Another form of the diffusion model for the triple correlation term can be derived by splitting up the triple correlation as

$$\overline{u_i' u_j' u_k'} = \frac{1}{2} [\overline{R_{ik}' u_j'} + \overline{R_{jk}' u_i'}] = -\frac{s^2}{2\alpha} \left[R_{jl} \frac{\partial R_{ik}}{\partial x_l} + R_{il} \frac{\partial R_{jk}}{\partial x_l} \right]. \quad (2.43)$$

The derivative is taken on both sides,

$$\frac{\partial \overline{u_i' u_j' u_k'}}{\partial x_k} = -\frac{\partial}{\partial x_k} \left[\frac{s^2}{2\alpha} \left[R_{jl} \frac{\partial R_{ik}}{\partial x_l} + R_{il} \frac{\partial R_{jk}}{\partial x_l} \right] \right], \quad (2.44)$$

and appropriate simplifications are made,

$$\frac{\partial \overline{u_i' u_j' u_k'}}{\partial x_k} = \sum^{sym} -C_{DR} \frac{\partial}{\partial x_k} \left[\frac{s^2}{\sqrt{K}} R_{il} \frac{\partial R_{jk}}{\partial x_l} \right], \quad (2.45)$$

The use of two very similar terms in the breakup of the triple correlation is required because of the need to retain symmetry between the free indices in the model, although like the D.H. model the derivation of this model appears to violate the symmetry rule in the first step of the derivation.

This diffusion term adds one more complexity. The contribution to the time rate of change of R_{ij} is no longer a function of the gradient of that component. The flux is now dependent on the gradients of R_{ik} and of R_{jk} .

Another direction that has been tried for the model of the triple correlation was to split the term as shown below [3]. The derivation of this model does abide by the symmetry rule.

$$\overline{u_i' u_j' u_k'} = \frac{1}{3} (\overline{u_i' R_{jk}} + \overline{u_j' R_{ik}} + \overline{u_k' R_{ij}}) \quad (2.46)$$

Using equation 2.35, equation 2.46 can be written in the form

$$-\frac{s^2}{3c} \left[R_{il} \frac{\partial R_{jk}}{\partial x_l} + R_{jl} \frac{\partial R_{ik}}{\partial x_l} + R_{kl} \frac{\partial R_{ij}}{\partial x_l} \right] + W_{ijk} \quad (2.47)$$

The problem that occurs with this method is that, where there is no mean flow, it is possible to get a negative diffusion of the energy components. This is contrary to the expected physical performance since it would move energy from locations with low energy to locations with higher energy. Because of this apparent negative diffusive property, Harlow and Hirt chose not to use this modeling technique, and it will not be addressed any further here.

2.8 Decay Term

In the Reynolds-stress transport equation, the decay of a Reynolds-stress component is given by ε_{ij} . In the current model, only the decay of the turbulent kinetic energy ε is explicitly calculated, and we therefore need a decay term that will approximate ε_{ij} using the scalar ε .

There are two decay terms that are accepted and widely used. The first is the isotropic decay term,

$$\varepsilon_{ij} = \frac{1}{3} \varepsilon \delta_{ij} . \quad (2.48)$$

This formulation is based on the assumption that each of the diagonal components of the Reynolds-stress tensor should decay at the same rate. The value of ε is the decay rate of the turbulent kinetic energy, whereas the term in 2.48 is half of the modeled decay of a Reynolds-stress component. The value of $1/3$ takes into account that the sum of the decay for the three diagonals should be equal to the decay for the turbulent kinetic energy.

The δ_{ij} indicates that only the diagonal Reynolds-stress components are effected by this term. This formulation treats the decay of turbulence as if there is no reduction in the magnitude of the off-diagonal term Reynolds-stress components and therefore no decline in the magnitudes of correlations of orthogonal velocity fluctuations.

The second form of the decay term that is common is the nonisotropic decay term,

$$\varepsilon_{ij} = -\frac{R_{ij}}{R_{ii}} \varepsilon . \quad (2.49)$$

In this term, instead of dividing up the total decay of the turbulent kinetic energy equally among the diagonal elements, the decay is shared among the Reynolds-stress components depending on the ratio of that particular component to the contraction of the Reynolds-stress.

If one looks closely at the return to isotropy term, it can be seen that the off-diagonal part of it is of the same form as this nonisotropic decay term. This is not double counting the reduction of the off-diagonal term Reynolds-stress components, because the two separate terms describe two separate processes. The return to isotropy term's contribution to the off-diagonal term Reynolds-stress components refers to the annihilation of the cross velocity correlations due to the nonconservative pressure correlation. This process can be thought of as a rotation and mixing of the turbulence structures. The mixing process causes the off-diagonal components to cancel one another out (as described in section 2.6). The decay term's effects on the off-diagonal term Reynolds-stress components refer to effects of the splitting of large eddies into smaller ones. Large eddies shear against one another and cause each other to break down into smaller eddies. This process is the cascading of the turbulence from large-scale structures to smaller ones. As the turbulence cascades, the cross correlations become associated with smaller scales as well.

2.9 Decay Rate Equation

The turbulent decay rate, ε , is calculated using a transport equation that is built primarily from the rules of modeling listed earlier and physical observations,

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial(u_k \varepsilon)}{\partial x_k} + c_{1\varepsilon} \frac{\varepsilon}{K} R_{mn} \frac{\partial u_m}{\partial x_n} = c_{D\varepsilon} \frac{\partial}{\partial x_n} \left[\frac{K^2}{\varepsilon} \frac{\partial \varepsilon}{\partial x_n} \right] - c_{2\varepsilon} \frac{\varepsilon^2}{K} . \quad (2.50)$$

The equation shows advection, creation, diffusion, and decay terms, most of which are devised from making an analogy between the decay rate transport equation and the turbulent kinetic energy equation. The advection term is fairly simple, acting similarly to that of the Reynolds-stress advection term. The creation term resembles the creation term of the Reynolds-stress equation. If the indices are contracted on the Reynolds-stress creation term and the term is divided in half to form a term for the creation of the turbulent kinetic energy, the form would be

$$R_{mn} \frac{\partial u_m}{\partial x_n} . \quad (2.51)$$

The creation of ε is presumed to be of the same form. This form can be used for the epsilon equation when it is made into a creation of a rate of change in energy, by dividing by the characteristic turbulent time scale, $s\sqrt{K}$ or K/ε . The form of this model is used under the assumption that the rate of creation of ε is proportional to the rate of creation of turbulent kinetic energy. The resulting ε creation term, including a constant of proportionality that is empirical, is shown in equation 2.50.

Previous authors proposed that the decay rate diffusion term should have the same form as the diffusion term in the turbulent kinetic energy equation. The rate of diffusion coefficient of ε is equal to the diffusion coefficient of the turbulent kinetic energy divided by a turbulent Prandtl number, often chosen to equal 1.3.

The Reynolds-stress diffusion terms that are being addressed in this thesis are listed in the table 2.2 with the corresponding diffusion terms for the turbulent decay rate equation. The simple and D.H. diffusion terms for the decay equation can be derived

first by converting the Reynolds-stress diffusion term to a turbulent kinetic energy term (contracting the indices). The next step is to replace the turbulent kinetic energy inside the innermost derivative of the diffusion term with the scalar decay and dividing by the turbulent Prandtl number. The results are shown in table 2.2.

Diffusion Type	Reynolds-stress Diffusion	Decay Diffusion
Simple Diffusion	$\frac{2}{3} C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K^2}{\epsilon} \frac{\partial R_{ij}}{\partial x_m} \right]$	$\frac{2}{3} \frac{C_{DR}}{1.3} \frac{\partial}{\partial x_m} \left[\frac{K^2}{\epsilon} \frac{\partial \epsilon}{\partial x_m} \right]$
D.H. Diffusion	$C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\epsilon} R_{mn} \frac{\partial R_{ij}}{\partial x_n} \right]$	$\frac{C_{DR}}{1.3} \frac{\partial}{\partial x_m} \left[\frac{K}{\epsilon} R_{mn} \frac{\partial \epsilon}{\partial x_n} \right]$
BHR Diffusion		
nonisotropic decay	$\sum^{sym} \frac{3}{2} C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\epsilon} R_{in} \frac{\partial R_{mj}}{\partial x_n} \right]$	$\frac{3C_{DR}}{1.3} \frac{\partial}{\partial x_m} \left[\frac{K}{\epsilon} R_{in} \frac{\partial}{\partial x_n} \left[\frac{R_{mi}}{R_{ll}} \epsilon \right] \right]$
isotropic decay		$\frac{C_{DR}}{1.3} \frac{\partial}{\partial x_m} \left[\frac{K}{\epsilon} R_{mn} \frac{\partial \epsilon}{\partial x_n} \right]$

Table 2.2. The diffusion terms tried in the Reynolds-stress transport equation and their corresponding decay rate diffusion terms.

In "Turbulence Transport Equations for Variable Density Turbulence and Their Relationship to Two-field Models" [5], the authors describe a model originally documented by Besnard, Harlow, and Rauenzahn (BHR) at an earlier point in time. In the article [5], the D.H. diffusion term, which was present in the original BHR report, that was present in the original documentation was omitted. In this article the diffusion for the epsilon equation, which was not developed in the original BHR, is of the same form as the simple diffusion.

The simple diffusion term in the decay rate transport equation was not used because of the apparent disagreement with the diffusion term in the Reynolds-stress equation.

Instead we attempted to find a form of diffusion for the scalar decay rate that would agree more closely with the diffusion model placed in the Reynolds-stress equation.

The development of an equivalent diffusion term in the decay equation to the BHR diffusion term in the Reynolds-stress equation is not as straightforward as the simple and D.H. diffusion terms, since flux in the diffusion term for R_{ij} is not dependent on the gradient of R_{ij} alone. This disallows the simple replacement of the R_{ij} with ε since the contraction of i and j in the complex Reynolds-stress diffusion does not yield a diffusion dependent on the gradient of K . This creates a need for a diffusion term that is dependent on the gradient ε_{ij} , of

$$\sum^{sym} \frac{3}{2} C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\varepsilon} R_{in} \frac{\partial \varepsilon_{mj}}{\partial x_n} \right]. \quad (2.52)$$

Since we do not have an explicit transport equation for ε_{ij} , we refer to the decay term used in the Reynolds-stress equation and utilize the same type of approximation for ε_{ij} .

When we use the isotropic decay term in the Reynolds-stress equation, then we must be consistent and use this same approximation in the decay rate diffusion term. The first step in the development of this term is to contract the free indices,

$$3C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\varepsilon} R_{in} \frac{\partial \varepsilon_{mi}}{\partial x_n} \right]. \quad (2.53)$$

The next step is to apply the isotropic decay relation for ε_{im} ,

$$3C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\varepsilon} R_{in} \frac{\partial}{\partial x_n} \left[\frac{\delta_{im}}{3} \varepsilon \right] \right]. \quad (2.54)$$

The term can be simplified to the following state and is now a function of only the scalar ε ,

$$C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\varepsilon} R_{mn} \frac{\partial \varepsilon}{\partial x_n} \right]. \quad (2.55)$$

This term shows a diffusion term that is now a function of the derivative in the m direction of a flux.

The other form the decay rate diffusion can take is found by utilizing the nonisotropic decay form. The development of this form comes about in a similar manner, by contraction of the free indices and then substituting in the nonisotropic decay relationship,

$$3C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\varepsilon} R_{in} \frac{\partial}{\partial x_n} \left[\frac{R_{mi}}{R_{ll}} \varepsilon \right] \right]. \quad (2.53)$$

This ε diffusion terms that are described for use with the BHR diffusion in the Reynolds-stress equation can be reduced to the same form as the simple diffusion model, since the flux is dependent on something other than the scalar ε . This forces us to conclude that since the forms are not equivalent the diffusion rate will not necessarily differ by the given constant of 1.3.

The decay term in the ε equation is also assumed to be of the same form as the decay in the turbulent kinetic energy equation divided by the turbulent time scale.

3. DESCRIPTION OF MODELS FOR VARIOUS TEST CASES

This chapter discusses the different forms of the models that will be used to investigate the results of various decay and diffusion terms.

Name of Test Case	Decay Term	Diffusion Term	Inlet Length Scale (Plate Heights)	C_{1R}	Modeled Equations
Base Case	Nonisotropic	Simple	3	2.4	3.1, 3.2
Isotropic Decay Base Case (IDBC)	Isotropic	Simple	3	2.4	3.3, 3.2
D.H. Nonisotropic Case	Nonisotropic	D.H. Diffusion	3	2.4	3.4, 3.5
D.H. Isotropic Case	Isotropic	D.H. Diffusion	3	2.4	3.6, 3.5
BHR Nonisotropic Case	Nonisotropic	BHR Diffusion	3	2.4	3.7, 3.8
BHR Isotropic	Isotropic	BHR Diffusion	3	2.4	3.9, 3.10
Higher C_{1R} Case	Nonisotropic	Simple	3	4.8	3.1, 3.2
Scale 5 Case	Nonisotropic	Simple	5	2.4	3.1, 3.2
Scale $f(y)$ Case	Nonisotropic	Simple	$f(y)$	2.4	3.1, 3.2
Scale 1 Case	Nonisotropic	Simple	1	2.4	3.1, 3.2

Table 3.1 The specifications for the models and parameters for each of the test cases.

Test case setups are also described for the observation of the results of different input scales and different values of the coefficient on the return to isotropy term. Each case is shown in table 3.1. The models described in this chapter are used to model the turbulence of the wake behind a flat plate for a Reynolds number of 1000.

Figure 3.1 The flow around the flat plate gives a turbulent wake at $Re = 1000$.

The flat plate is infinite in the spanwise direction and was treated with periodic boundary conditions in the DNS calculation.

3.1 The Base Case Model

The Reynolds-stress equation that is going to be used for the Base Case Model is

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t} + \frac{\partial}{\partial x_n} (u_n R_{ij}) + R_{in} \frac{\partial u_j}{\partial x_n} + R_{jn} \frac{\partial u_i}{\partial x_n} = \frac{2C_{DR}}{3} \frac{\partial}{\partial x_n} \left[s\sqrt{K} \frac{\partial R_{ij}}{\partial x_n} \right] \\ - C_{1r} \frac{\varepsilon}{K} (R_{ij} - \frac{2}{3} \delta_{ij} K) - C_{2R} \left[R_{ik} \frac{\partial u_j}{\partial x_k} + R_{jk} \frac{\partial u_i}{\partial x_k} - \frac{2}{3} R_{ik} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right] - \frac{R_{ij}}{K} \varepsilon \end{aligned} \quad (3.1)$$

The decay rate equation that is going to be used in the Base Case is

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial (u_k \varepsilon)}{\partial x_k} + C_{1\varepsilon} \frac{\varepsilon}{K} R_{mn} \frac{\partial u_m}{\partial x_n} = C_{D\varepsilon} \frac{\partial}{\partial x_n} \left[\frac{K^2}{\varepsilon} \frac{\partial \varepsilon}{\partial x_n} \right] - C_{2\varepsilon} \frac{\varepsilon^2}{K} \quad (3.2)$$

The simple Reynolds-stress diffusion model (2.42) and the simple decay rate diffusion (shown in table 2.2) are used in this case because they take a form that has been shown to have some validity in the $K - \varepsilon$ models. The equations are in tensor notation where the

indices 1, 2, and 3 are the x (streamwise), y (cross-stream), and z (spanwise) directions respectively. The nonisotropic decay term (2.49) is used in the Base Case instead of the isotropic decay term (2.48) because of its ability to accommodate nonisotropic actions in the manner described in section 2.8.

The values of the constants except for C_{1R} are shown in table 3.2 and are the same in all test cases. The approximate size of the Von Karman vortex scale is used as a guide in the selection of an inlet scale. An inlet scale of 3 was used in the base case, based on the assumption that the inlet scale should be approximately the same size as the dominant eddies of the flow.

Term	Coefficient	Value
Diffusion Term	C_{DR}	.135
Return to isotropy	C_{1R}	2.4
Rapid Pressure Term	C_{2R}	.75
Decay Rate Creation	$C_{1\epsilon}$	1.55
Decay Rate Decay	$C_{2\epsilon}$	1.75
Decay Rate Diffusion	$C_{D\epsilon}$	$C_{DR}/1.3$

Table 3.2 The constants for the Base Case Reynolds-stress and Decay Rate models.

3.2 Isotropic Decay Base Case

A test case in which the nonisotropic decay term is replaced by the isotropic decay term (2.48) is used to compare the relative influence of the two decay terms. The other features are identical to those of the Base Case. By leaving the remaining features the same in the Base Case and IDBC, the effects of the decay term become more isolated, and also the IDBC provides a second test case as a base for comparison when the diffusion terms are analyzed. This isotropic decay Base Case, "IDBC", uses the governing equation

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t} + \frac{\partial}{\partial x_n} (u_n R_{ij}) + R_{in} \frac{\partial u_j}{\partial x_n} + R_{jn} \frac{\partial u_i}{\partial x_n} = \frac{2C_{DR}}{3} \frac{\partial}{\partial x_n} \left[s\sqrt{K} \frac{\partial R_{ij}}{\partial x_n} \right] \\ - C_{1r} \frac{\varepsilon}{K} (R_{ij} - \frac{2}{3} \delta_{ij} K) - C_{2R} \left[R_{ik} \frac{\partial u_j}{\partial x_k} + R_{jk} \frac{\partial u_i}{\partial x_k} - \frac{2}{3} R_{ik} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right] - \frac{2}{3} \delta_{ij} \varepsilon \end{aligned} \quad (3.3)$$

and again uses equation 3.2 for the decay rate equation

3.3 Test Cases Involving the D.H. Diffusion Term

There are the two test cases run with the purpose of observing the effects of the D.H. diffusion term. The first of these, the D.H. Nonisotropic Case, consists of the Base Case with the simple diffusion replaced with the D.H. diffusion (2.38) in the Reynolds-stress transport model. The simple diffusion term in the decay rate equation is replaced by the D.H.-type decay diffusion term (shown in table 2.2). The remaining features of the Base Case are kept the same. The D.H. Nonisotropic model is

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t} + \frac{\partial}{\partial x_n} (u_n R_{ij}) + R_{in} \frac{\partial u_j}{\partial x_n} + R_{jn} \frac{\partial u_i}{\partial x_n} = C_{DR} \frac{\partial}{\partial x_k} \left[\frac{s}{K^{1/2}} R_{kl} \frac{\partial R_{ij}}{\partial x_l} \right] \\ - C_{1r} \frac{\varepsilon}{K} (R_{ij} - \frac{2}{3} \delta_{ij} K) - C_{2R} \left[R_{ik} \frac{\partial u_j}{\partial x_k} + R_{jk} \frac{\partial u_i}{\partial x_k} - \frac{2}{3} R_{ik} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right] - \frac{R_{ij}}{K} \varepsilon \end{aligned} \quad (3.4)$$

and

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial (u_k \varepsilon)}{\partial x_k} + C_{1\varepsilon} \frac{\varepsilon}{K} R_{mn} \frac{\partial u_m}{\partial x_n} = C_{D\varepsilon} \frac{\partial}{\partial x_m} \left[\frac{K}{\varepsilon} R_{mn} \frac{\partial \varepsilon}{\partial x_n} \right] - C_{2\varepsilon} \frac{\varepsilon^2}{K} \quad (3.5)$$

The second model used to observe the D.H. diffusion, the "D.H. Isotropic Case", is generated by replacing the simple diffusion terms with the D.H. diffusion terms in the IDBC equations. The Reynolds-stress equation for this model is

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t} + \frac{\partial}{\partial x_n} (u_n R_{ij}) + R_{in} \frac{\partial u_j}{\partial x_n} + R_{jn} \frac{\partial u_i}{\partial x_n} = C_{DR} \frac{\partial}{\partial x_k} \left[\frac{s}{K^{1/2}} R_{kl} \frac{\partial R_{ij}}{\partial x_l} \right] \\ - C_{1r} \frac{\varepsilon}{K} (R_{ij} - \frac{2}{3} \delta_{ij} K) - C_{2R} \left[R_{ik} \frac{\partial u_j}{\partial x_k} + R_{jk} \frac{\partial u_i}{\partial x_k} - \frac{2}{3} R_{lk} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right] - \frac{2}{3} \delta_{ij} \varepsilon \end{aligned} \quad (3.6)$$

and the decay rate equation is the same as equation 3.5.

3.4 Test Cases Involving the BHR Diffusion Term

There are two test cases used for the observation of the influence of the more complex BHR diffusion terms. The first of these test cases, the BHR Nonisotropic Case, is derived from the Base Case by replacing the simple diffusion term in the Reynolds-stress Equations with the BHR diffusion term,

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t} + \frac{\partial}{\partial x_n} (u_n R_{ij}) + R_{in} \frac{\partial u_j}{\partial x_n} + R_{jn} \frac{\partial u_i}{\partial x_n} = \sum^{sym} \frac{3}{2} C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\varepsilon} R_{in} \frac{\partial R_{mj}}{\partial x_n} \right] \\ - C_{1r} \frac{\varepsilon}{K} (R_{ij} - \frac{2}{3} \delta_{ij} K) - C_{2R} \left[R_{ik} \frac{\partial u_j}{\partial x_k} + R_{jk} \frac{\partial u_i}{\partial x_k} - \frac{2}{3} R_{lk} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right] - \frac{R_{ij}}{K} \varepsilon \end{aligned} \quad (3.7)$$

For the decay rate equation we need to substitute in a consistent diffusion term. The nonisotropic BHR form diffusion term is substituted for the simple diffusion in the decay rate equation,

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial (u_k \varepsilon)}{\partial x_k} + C_{1\varepsilon} \frac{\varepsilon}{K} R_{mn} \frac{\partial u_m}{\partial x_n} = 3 C_{D\varepsilon} \frac{\partial}{\partial x_m} \left[\frac{K}{\varepsilon} R_{in} \frac{\partial}{\partial x_n} \left[\frac{R_{mi}}{R_{ll}} \varepsilon \right] \right] - C_{2\varepsilon} \frac{\varepsilon^2}{K} \quad (3.8)$$

The second test case, used to observe the BHR diffusion, is the BHR Isotropic Case. In this case, the simple diffusion in the Reynolds-stress equation of the IDBC is replaced with the BHR diffusion term and the simple decay diffusion is replaced by the isotropic

version of the BHR decay diffusion. The Reynolds-stress equation for this model for this case is

$$\begin{aligned} \frac{\partial R_{ij}}{\partial t} + \frac{\partial}{\partial x_n} (u_n R_{ij}) + R_{in} \frac{\partial u_j}{\partial x_n} + R_{jn} \frac{\partial u_i}{\partial x_n} = \sum^{sym} \frac{3}{2} C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\epsilon} R_{in} \frac{\partial R_{mj}}{\partial x_n} \right] \\ - C_{1r} \frac{\epsilon}{K} (R_{ij} - \frac{2}{3} \delta_{ij} K) - C_{2R} \left[R_{ik} \frac{\partial u_j}{\partial x_k} + R_{jk} \frac{\partial u_i}{\partial x_k} - \frac{2}{3} R_{ik} \frac{\partial u_l}{\partial x_k} \delta_{ij} \right] - \frac{2}{3} \delta_{ij} \epsilon \end{aligned} \quad (3.9)$$

and the decay rate transport equation is

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial (u_k \epsilon)}{\partial x_k} + C_{1\epsilon} \frac{\epsilon}{K} R_{mn} \frac{\partial u_m}{\partial x_n} = C_{D\epsilon} \frac{\partial}{\partial x_m} \left[\frac{K}{\epsilon} R_{mn} \frac{\partial \epsilon}{\partial x_n} \right] - C_{2\epsilon} \frac{\epsilon^2}{K} \quad (3.10)$$

3.5 Test Cases Involving Larger Values of the Return to Isotropy Constant

There arises a need to investigate some the effects of the constant for the return to isotropy term, C_{1R} . The governing equations for the case that illustrates the effects of the return to isotropy constant are produced by replacing its value in the Base Case model by 4.8. All other features are kept constant.

3.6 Test Cases Involving Variations of the Inlet Scale

The inlet scale is not explicitly known from the DNS and can therefore only be estimated. It is important to be aware of how sensitive the Reynolds-stress distribution is to this inlet length scale. In order to compare the effects of the inlet scale a series of test cases are performed with identical features to the Base Case except the inlet scale is altered. This comparison includes the effects of an inlet scale of 5, 3 (Base Case), 1 and

$f(y)$. $f(y)$ is a variable inlet scale, shown in Figure 3.1 as the combination of two Gaussian curves with an offset from $y = 0$.

Figure 3.1 shows the distribution of the inlet length scale when $f(y)$ is utilized.

3.7 Best Combination Case

For a final comparison of the best combination of the terms in the model, a test case was run with the raised value of C_{1r} , the simple diffusion, the nonisotropic decay term, and the inlet scale $f(y)$.

4. THE NUMERICAL METHODOLOGY

The turbulence equations are solved using an explicit time-marching technique in a two-dimensional spatial arrangement. The stochastically steady state solution is found by utilizing a time-marching scheme. The explicit finite difference-approximation of the Reynolds-stress equations is

$$R_{ij}^{n+1} = R_{ij}^n + \Delta t \left\{ -\frac{\partial}{\partial x_n} (u_n^n R_{ij}^n) - R_{in}^n \frac{\partial u_j^n}{\partial x_n} - R_{jn}^n \frac{\partial u_i^n}{\partial x_n} + \frac{2C_{DR}}{3} \frac{\partial}{\partial x_n} \left[s^n \sqrt{K^n} \frac{\partial R_{ij}^n}{\partial x_n} \right] - C_{1r} \frac{\varepsilon^n}{K^n} \left(R_{ij}^n - \frac{2}{3} \delta_{ij} K^n \right) - C_{2R} \left[R_{ik}^n \frac{\partial u_j^n}{\partial x_k} + R_{jk}^n \frac{\partial u_i^n}{\partial x_k} - \frac{2}{3} R_{ik}^n \frac{\partial u_l^n}{\partial x_k} \delta_{ij} \right] - \frac{R_{ij}^n}{K^n} \varepsilon^n \right\} \quad (4.1)$$

The super scripts n and $n+1$ indicate values at time $n\Delta t$ and time $(n+1)\Delta t$. Under the explicit time-marching scheme, in which the mean flow velocity field is held fixed, the Reynolds-stress and decay rate at time $n+1$, R_{ij}^{n+1} , and ε^{n+1} are solved as functions of u_i^n , R_{ij}^n , s^n , and ε^n , except in the decay of ε term. This term is calculated using a local point implicit algorithm. The finite difference approximation of decay rate equation used for the local-point implicit algorithm is

$$\varepsilon^{n+1} = \Delta t \left[-\frac{\partial (u_k^n \varepsilon^n)}{\partial x_k} - C_{1\varepsilon} \frac{\varepsilon^n}{K^n} R_{mn}^n \frac{\partial u_m^n}{\partial x_n} + C_{D\varepsilon} \frac{\partial}{\partial x_n} \left[\frac{(K^n)^2}{\varepsilon^n} \frac{\partial \varepsilon^n}{\partial x_n} \right] + C_{2\varepsilon} \frac{\varepsilon^n \varepsilon^{n+1}}{K^n} \right] + \varepsilon^n \quad (4.2)$$

The value of ε at time $n+1$ can be solved for in the manner,

$$\varepsilon^{n+1} = \frac{\Delta t \left[-\frac{\partial (u_k^n \varepsilon^n)}{\partial x_k} - C_{1\varepsilon} \frac{\varepsilon^n}{K^n} R_{mn}^n \frac{\partial u_m^n}{\partial x_n} + C_{D\varepsilon} \frac{\partial}{\partial x_n} \left[\frac{K^{n2}}{\varepsilon^n} \frac{\partial \varepsilon^n}{\partial x_n} \right] \right] + \varepsilon^n}{\left[1 + \frac{C_{2\varepsilon} \Delta t \varepsilon^n}{K^n} \right]} \quad (4.3)$$

The variables are spatially discretized on a collocated grid, in which the values of the Reynolds-stress components, decay rate, mean velocities, and turbulent scale are all located at the same location (at the center of a their control volume). The grid spacing is uniform in the x and y directions throughout the domain.

The derivatives in the z direction are assumed to be zero in this case in which the average quantities are independent of their location in the z direction, therefore we limit calculations to one x, y plane.

Both finite-volume and finite-difference schemes are used in the calculation of the terms involving derivatives. The derivatives in the advection, creation, and rapid pressure terms are calculated using high-order finite-difference schemes; and the diffusion terms are calculated using a finite-volume technique, which performs better with the variable diffusion coefficient.

A fifth-order upwind difference scheme is used for the discretization of the advection terms in the Reynolds-stress equation and the decay rate equation. The terms are put in nonconservative form (allowed because the fluid is incompressible), then the following algorithm is applied for the advection of any scalar L [9].

if $u_{l(ix,iy)} > 0$ then

$$\left[u_l \frac{\partial L}{\partial x} \right]_{(ix,iy)} = u_{l(ix,iy)} \frac{1}{120\Delta x} \left(-6L_{(ix+2,iy)} + 60L_{(ix+1,iy)} + 40L_{(ix,iy)} - 120L_{(ix-1,iy)} + 30L_{(ix-2,iy)} - 4L_{(ix-3,iy)} \right)$$

if $u_{l(ix,iy)} < 0$ then

$$\left[u_l \frac{\partial L}{\partial x} \right]_{(ix,iy)} = u_{l(ix,iy)} \frac{1}{120\Delta x} \left(4L_{(ix+3,iy)} - 30L_{(ix+2,iy)} + 120L_{(ix+1,iy)} - 40L_{(ix,iy)} - 60L_{(ix-3,iy)} + 6L_{(ix-4,iy)} \right)$$

(4.4)

N_x is the number of points in the x direction, and ix and iy reference particular nodes in the x and y direction. This scheme is implemented at the interior points, $ix=4$ to N_x-3 . At the point $ix=N_x-2$ a fourth-order central-difference scheme is applied.

$$\left[\frac{\partial L}{\partial x} \right]_{(N_x-2, iy)} = \frac{1}{12\Delta x} (L_{(N_x-4, iy)} - 8L_{(N_x-3, iy)} + 8L_{(N_x-1, iy)} - L_{(N_x, iy)}) . \quad (4.5)$$

At the point $ix=N_x-1$, a fourth-order unsymmetric scheme is used.

$$\left[\frac{\partial L}{\partial x} \right]_{(ix, iy)} = \frac{1}{12\Delta x} (3L_{(ix, iy)} + 10L_{(N_x-1, iy)} - 18L_{(N_x-2, iy)} + 6L_{(N_x-3, iy)} - L_{(N_x-4, iy)}) . \quad (4.6)$$

Near the inlet, the advection term is computed with a lower-order pseudo-donor cell derivative technique in the x direction. The lower-order scheme is used near the left boundary, where the recirculation in the mean flow can cause negative velocities at the inlet.

The derivatives in the creation and rapid pressure terms are calculated using a fourth-order scheme for the derivative of the mean flow. These derivatives are calculated as

$$\left[\frac{\partial L}{\partial x} \right]_{(ix, iy)} = \frac{1}{12\Delta x} (L_{(ix-2, iy)} - 8L_{(ix-1, iy)} + 8L_{(ix+1-1, iy)} - L_{(ix+2, iy)}) . \quad (4.7)$$

This fourth-order central difference scheme is used from $ix=3$ to N_x-2 . At $ix=2$ and N_x-1 the same type of algorithm is used as shown in 4.6. The derivatives in the y direction are evaluated in the same manner.

The diffusion terms are calculated using a second-order finite-volume scheme (a slightly different scheme for each different diffusion term). The finite-volume approximation for the simple diffusion term is

$$\begin{aligned}
& \frac{2}{3} C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K^2}{\varepsilon} \frac{\partial L}{\partial x_m} \right] = \\
& \frac{1}{3} C_{DR} \left\{ \frac{1}{\Delta x} \left[\left(\frac{K_{(ix+1,iy)}^2}{\varepsilon_{(ix+1,iy)}} + \frac{K_{(ix,iy)}^2}{\varepsilon_{(ix,iy)}} \right) \frac{1}{\Delta x} (L_{(ix+1,iy)} - L_{(ix,iy)}) \right. \right. \\
& \quad \left. \left. - \left(\frac{K_{(ix,iy)}^2}{\varepsilon_{(ix,iy)}} + \frac{K_{(ix-1,iy)}^2}{\varepsilon_{(ix-1,iy)}} \right) \frac{1}{\Delta x} (L_{(ix,iy)} - L_{(ix-1,iy)}) \right] \right. \\
& \quad + \frac{1}{\Delta y} \left[\left(\frac{K_{(ix,iy+1)}^2}{\varepsilon_{(ix,iy+1)}} + \frac{K_{(ix,iy)}^2}{\varepsilon_{(ix,iy)}} \right) \frac{1}{\Delta y} (L_{(ix,iy+1)} - L_{(ix,iy)}) \right. \\
& \quad \left. \left. - \left(\frac{K_{(ix,iy)}^2}{\varepsilon_{(ix,iy)}} + \frac{K_{(ix,iy-1)}^2}{\varepsilon_{(ix,iy-1)}} \right) \frac{1}{\Delta y} (L_{(ix,iy)} - L_{(ix,iy-1)}) \right] \right\} \quad (4.8)
\end{aligned}$$

L can be replaced by R_{ij} for the Reynolds-stress diffusion term and ε for the decay rate diffusion term.

The finite-volume approximation for the D.H. diffusion terms is shown at the end in the Appendix.

The BHR diffusion terms are different. There are two different forms of the decay rate diffusion. Equation 4.10 in the Appendix shows the finite volume approximation of the BHR diffusion for the Reynolds-stress equation, where the summation symbol indicates that the term where i and j are reversed is added to the term shown.

Equation 4.11 in the Appendix shows the finite volume approximation of the decay rate for the case where the nonisotropic decay term is used in the Reynolds-stress equation. The decay rate diffusion for the case in which the isotropic decay term is used has the same finite-volume form as shown for the D.H. diffusion term, equation 4.9 in the Appendix.

The flow that was simulated by the DNS is the flow around a flat plate. The flat plate is located 4.5 plate heights downstream of the inlet, and the exit of the domain is 15.5

plate heights downstream of the plate. The plate was centered in the cross-stream direction, which is twelve plate heights high (5.5 plate heights above the plate and 5.5 plate heights below the plate), and the plate is assumed to have no thickness.

Figure 4.1 An x, y plane region of flow. The bold line represents the plate. The solid line is the boundaries of the flow for the DNS calculation. The dotted line outlines the region considered for the Reynolds-stress model calculation.

In order to get sufficient resolution by the grid, the domain was cut down to a smaller region behind the wake. As shown in figure 4.1, the DNS domain has a large enough cross-stream width to allow the flow field to form with small boundary effects. When the steady mean flow was found in the DNS calculation, it showed that outside a subdomain, (as shown by the dotted lines in figure 4.1), the mean flow is basically uniform with very little variation in the x and y directions. The Reynolds-stress patterns also are almost exclusively inside this smaller subdomain. In uniform flow there are no substantial gradients and thus very little creation of turbulence. Because the Reynolds-stress is

primarily within this subdomain, the Reynolds-stress model only needs to be computed inside this subdomain.

Because the performance of the Reynolds-stress model was to be observed in the near wake, the region upstream of the plate could be eliminated. In order to remove the problem of the modeling actions of the turbulence quantities near the surface of the plate, the subdomain in the streamwise direction was chosen to start a half plate height downstream of the plate. The exit of the subdomain was chosen to be ten plate heights downstream of this point. This region (ten plate heights long) encompassed all the strong actions of the near wake region in the DNS calculation.

5. COMPARISON OF DNS AND TURBULENCE MODEL RESULTS

In this chapter the results of the various test cases will be compared with each other and ultimately to the DNS results. The first section describes the comparison between the Base Case and the DNS calculation. In general the Reynolds-stress patterns calculated by the Base Case tend to be restricted to a smaller region in the y direction than the DNS Reynolds-stress.

The second section in this chapter presents a comparison between the Base Case and a test case in which the nonisotropic decay term in the Base Case is replaced by the isotropic decay term (IDBC). The objective of this comparisons is to allow observation of the differences between the two decay terms. The comparison reveals that the IDBC produces results that have larger discrepancies from the DNS results than the Base Case did. In particular the structure of R_{33} has a severely pinched structure, which is more exaggerated than in the Base Case.

Section 5.3 describes the comparisons used to investigate the D.H. diffusion term. This comparison involved the D.H. Nonisotropic Case and the D.H. Isotropic Case. The comparisons are made for both the isotropic and nonisotropic cases. The results show that the D.H. diffusion term does not produce as much diffusion in the y direction as the simple diffusion term does. This makes the models involving this term even worse compared to the DNS.

Section 5.4 describes the problems encountered in the investigation of the effects of the BHR diffusion term, involving the BHR Nonisotropic case and the BHR Isotropic case. The models involving this form of term became unstable and are not able to produce steady-state results.

The fifth section describes the results of a test case in which the coefficient of the return to isotropy term is doubled, the Higher C_{1R} Case. The effect of the raised value of this coefficient is to lower the maximum values of R_{11} and R_{22} and raise the maximum value of R_{33} . The pinched look in the R_{33} structure is reduced in the Higher C_{1R} Case, making it more like the DNS R_{33} structure.

The effects of the inlet scale on the Reynolds-stress structures are analyzed in section 5.6. The results of this comparison show that the Reynolds-stress Models are relatively insensitive to the inlet length scale as long as it is close to or larger than the dominant length scale of the resolved vortex street.

The last section in this chapter describes turbulence structures resulting from the Best Combination Case. These results are not still the same as the DNS results but do show some improvement over the Base Case.

In the geometry being analyzed there are no net shears in the z direction because the flow is independent of location in this third direction. Due to the lack of shears in the z direction, the values of R_{13} , and R_{23} are zero. The quantities used for the comparison are the values of R_{11} , R_{22} , R_{33} , R_{12} , $(\bar{s})$, and ε . These values are presented in the form of contour plots.

The contour plots are given in figures 5.1 through 5.40. Each contour plot is labeled with three values. The first of the three numbers is the minimum contour level plotted. The second number is the maximum contour level plotted. The third value given for each plot is the difference between the values indicated by two consecutive contour levels, the contour increment. It should be noted that these minimum and maximum contour levels are not necessarily the minimum and maximum values, but they do convey the limits between which these values must fall. The minimum value is between the value of the minimum contour level and the minimum contour level minus the contour increment.

Figure 5.1 DNS R_{11}
(min.=.02, max.=.24, inc.=.02)

Figure 5.2 DNS R_{22}
(min.=.02, max.=.50, inc.=.02)

Figure 5.3 DNS R_{33}
(min.=.005, max.=.11, inc.=.005)

Figure 5.4 DNS R_{12}
(min.=-.12, max.=.12, inc.=.02)

Figure 5.5 Base Case R_{11}
(min.=.02, max.=.28, inc.=.02)

Figure 5.6 Base Case R_{22}
(min.=.02, max.=.26, inc.=.02)

Figure 5.7 Base Case R_{33}
(min.=.005, max.=.06, inc.=.005)

Figure 5.8 Base Case R_{12}
(min.=.16, max.=.16, inc.=.02)

Figure 5.9 IDBC R_{11}
(min.=.02, max.=.32, inc.=.02)

Figure 5.10 IDBC R_{22}
(min.=.02, max.=.32, inc.=.02)

Figure 5.11 IDBC R_{13}
(min.=.005, max.=.06, inc.=.005)

Figure 5.12 IDBC R_{12}
(min.=-.16, max.=.16, inc.=.02)

Figure 5.13 Base Case scale
(min.=1.2, max.=3.2, inc.=.20)

Figure 5.14 IDBC scale
(min.=1.2, max.=3.2, inc.=.20)

Figure 5.15 Base Case ε
(min.=.005, max.=.065, inc.=.005)

Figure 5.16 IDBC ε
(min.=.005, max.=.07, inc.=.005)

Figure 5.17 D.H. Nonisotropic Case R_{11}
(min.=.02, max.=.34, inc.=.02)

Figure 5.19 D.H. Nonisotropic Case R_{33}
(min.=.005, max.=.07, inc.=.005)

Figure 5.18 D.H. Nonisotropic Case R_{22}
(min.=.02, max.=.26, inc.=.02)

Figure 5.20 D.H. Nonisotropic Case R_{12}
(min.=.16, max.=.16, inc.=.02)

Figure 5.21 D.H. Isotropic Case R_{11}
(min.=.02, max.=.36, inc.=.02)

Figure 5.22 D.H. Isotropic Case R_{22}
(min.=.02, max.=.30, inc.=.02)

Figure 5.23 D.H. Isotropic Case R_{33}
(min.=.005, max.=.06, inc.=.005)

Figure 5.24 D.H. Isotropic Case R_{12}
(min.=-.16, max.=.16, inc.=.02)

Figure 5.25 D.H. Nonisotropic Case scale
(min.=1.2, max.=3.2, inc.=.02)

Figure 5.26 D.H. Isotropic Case scale
(min.=1.2, max.=3.2, inc.=.20)

Figure 5.27 D.H. Nonisotropic Case ϵ
(min.=.005, max.=.09, inc.=.005)

Figure 5.28 D.H. Isotropic Case ϵ
(min.=.005, max.=.09, inc.=.005)

Figure 5.29 Higher C_{IR} Case R_{11}
(min.=.02, max.=.22, inc.=.02)

Figure 5.30 Higher C_{IR} Case R_{22}
(min.=.02, max.=.22, inc.=.02)

Figure 5.31 D.H. Higher C_{IR} Case R_{33}
(min.=.005, max.=.09, inc.=.005)

Figure 5.32 Higher C_{IR} Case R_{12}
(min.=-.12, max.=.12, inc.=.02)

Figure 5.33 Base Case (inlet scale=3) scale
(min.=1.2, max.=3.2, inc.=.20)

Figure 5.34 Inlet Scale=5 Case scale
(min.=1.5, max.=5.0, inc.=.50)

Figure 5.35 Inlet Scale= $f(y)$ Case scale
(min.=1.5, max.=6.5, inc.=.50)

Figure 5.36 Inlet Scale=1 Case scale
(min.=.90, max.=1.45, inc.=.05)

Figure 5.37 Best Comparison Case R_{11}
(min.=.02, max.=.24, inc.=.02)

Figure 5.38 Best Comparison Case R_{22}
(min.=.02, max.=.22, inc.=.02)

Figure 5.39 Best Comparison Case R_{33}
(min.=.005, max.=.09, inc.=.005)

Figure 5.40 Best Comparison Case R_{12}
(min.=-.12, max.=.12, inc.=.02)

The maximum value of a quantity is between the maximum contour level and the maximum contour level plus the contour increment.

5.1 The Base Case, DNS Comparison

The first test case that will be compared to the DNS data is the case involving the simple diffusion, the isotropic decay, and the inlet scale set to a constant value of 3 plate heights. The Reynolds-stress components for the DNS are shown in figures 5.1 through 5.4, along with the corresponding Base Case results in figures 5.5 through 5.8.

The maximum R_{11} values for the Base Case and DNS are not in the same location, as seen in figures 5.1 and 5.5. The locations of the maximum DNS R_{11} values are about .9 plate heights from the inlet and are in the central region of each shear layer. The Base Case maxima are about 1.9 plate heights from the inlet and confined to the inner edge of each shear layer. The value of the DNS R_{11} at the location of the DNS maximum is about .24. This is close to the Base Case maximum R_{11} , value .22, but the Base Case value continues to increase in the streamwise direction for about another plate height to a value of .28. The DNS and Base Case values are lower along the centerline and outside the shear layer and are high in the areas of high shear. This is consistent with the shear layers being a location of high creation and the other regions relying on diffusional processes. The structures of the Base Case are more elongated in the streamwise direction, where the structures in the DNS case extend outside the shear layer farther and do not extend downstream as strongly. The Base Case turbulence transport model is lacking in the processes needed to move the turbulence in the cross-stream direction.

The largest magnitude difference between the Base Case and the DNS results is seen in the values of R_{22} . There is almost a factor of 2 difference between the DNS maximum, .50, and the Base Case maximum R_{22} , .26. The location of the maximums, seen in

figures 5.2 and 5.6, are approximately the same, with the Base Case maximum being only about a quarter plate height farther down-stream. The maximums are occurring near the location of mean flow streamline reattachment, where the strongest velocities in the cross-stream direction collide on the centerline. This occurs approximately two plate heights inside the domain.

The difference in the locations of the R_{11} and R_{22} maximums in the DNS, Base Case and IDBC test cases can be explained by looking at the creation terms involved.

$$R_{11} = R_{11} \frac{\partial u_1}{\partial x} + R_{12} \frac{\partial u_1}{\partial y} \quad (5.1)$$

$$R_{22} = R_{12} \frac{\partial u_2}{\partial x} + R_{22} \frac{\partial u_2}{\partial y} \quad (5.2)$$

In the shear layer R_{11} has its primary creation coming from the second term in equation 5.2. The value of this term is going to be highest in the shear layer where not only is there a strong transverse velocity gradient but there is a strong value of R_{12} . The smaller gradient in the x direction of the x -direction velocity is going to limit the size of the first term in equation 5.1. R_{22} is going to be strongest in the region close to the reattachment of the mean flow streamlines. This is because of the large y -direction gradient of the y -direction velocity at this location. The value of R_{12} will not be large in this area because there is very little shearing going on near the center, therefore the value of the first term will be significantly smaller near the reattachment. Neither the first or second terms in equation 5.2 will have large magnitudes in the shear layer since there is very little velocity in the y direction and very little gradient of this small velocity.

The Base Case and DNS R_{22} distributions show a different structure. The DNS results show a much wider pattern than the Base Case especially in the area about two plate heights inside the domain. The values of the gradients in the y direction are very similar

in the region of high shear. In the x directions the gradients are not the same. The magnitude of the downstream gradient of R_{22} in the Base Case is about half of the gradient of R_{22} in the DNS calculation in the positive x direction. This is explained by the fact that the decay of a Reynolds-stress component has a strong proportional dependence on the magnitude of that component of the Reynolds-stress tensor. The decay term has more significance as fluid moves downstream from the reattachment area. The decay term plays a very large role in the determination of the Reynolds-stress at the outlet of the domain.

There are large differences in the magnitudes and qualitative structures of the R_{33} values, shown in figures 5.3 and 5.7. The maximum value, .11, of the DNS case occurs about one plate height downstream from the inlet, along the centerline. This location in the Base Case has a R_{33} value of about .5. The maximum Base Case R_{33} value, .60, is only about half of the maximum value for the DNS case and occurs at about 3.5 plate heights down stream from the inlet along the centerline. The most obvious deviation in the structure occurs about 1.5 plate heights downstream from the inlet. The Base Case overall structure narrows at this location and then widens out again. Although not completely visible in figure 5.3, the DNS structures show their highest amount of widening at the location about 1.5 plate heights down stream of the inlet.

There are no primary creation terms for R_{33} since there are no mean flow gradients in the z direction and there is no mean flow in this direction either. The large difference in total R_{33} energy can only be caused by the decay rate term, the return to isotropy term, and the nonconservative pressure term, because these are the only terms capable of creating or destroying R_{33} in this geometry

The DNS and Base Case plots of R_{12} , shown in figures 5.4 and 5.8 respectively, show some discrepancies qualitatively and quantitatively. The maximum absolute values of the DNS R_{12} , .12, is in the central part of the shear layer about 1.5 plate heights from the inlet. The Base Case value of R_{12} in this area is about .12 but this is not its maximum.

The maximum absolute value of the Base Case R_{12} , .16, is farther downstream, about 2 plate heights from the inlet and on the inside edge of the shear layer. The structures of the Base Case are more elongated on the streamwise direction and are primarily confined to the region between the shear layers.

5.2 Base Case Compared with IDBC

In order to observe the effects of the nonisotropic decay term in comparison with the isotropic decay term, the Base Case was modified only in the decay term. Any differences in the results between the original Base Case and this isotropic decay Base Case can be attributed to the decay term. The Reynolds-stress contour plots for this test case are shown in figures 5.9 through 5.12.

In isotropic decay Base Case, IDBC, the structures of R_{11} are very similar to that of the Base Case, as seen from figure 5.9. The biggest difference in the R_{11} distributions of the two cases is that the maximum R_{11} for the IDBC is .32 where the maximum R_{11} in the Base Case is .28. This is probably caused by the isotropic decay term acting on the R_{11} and R_{22} components less strongly than the nonisotropic decay term. To demonstrate this type of behavior we can look at a circumstance in which the magnitudes of R_{11} and R_{22} are roughly three times the magnitude of R_{33} .

For the isotropic decay term with R_{11} and R_{22} both close to three times R_{33} :

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \frac{1}{3} \epsilon$$

For the nonisotropic decay term:

$$\epsilon_{22} \cong \epsilon_{11} = \frac{R_{11}}{R_{ii}} \epsilon = \frac{3}{7} \epsilon \quad (5.3)$$

$$\epsilon_{33} = 2 \frac{R_{33}}{R_{ii}} \epsilon = \frac{1}{7} \epsilon$$

As seen in these equations, there is a difference in the magnitudes of the decay rate for R_{11} when different decay models are used. The lower amount of decay in the isotropic case would allow the value of R_{11} to grow to higher magnitudes. This effect is accentuated even because, at the place of maximum R_{11} , the value of R_{22} is not as high. The location of the .02 contour is at almost the same location on the R_{11} contour plots for the Base Case and IDBC, but the gradients are larger in the shear layers of the IDBC case in order to reach the higher maximum in the inside of the shear layer.

The contour plot of R_{22} for the IDBC is shown in figure 5.10. The structure of R_{22} is also very similar between the Base Case and the IDBC; however, the maximum magnitudes at the point of mean flow streamline reattachment are different. The IDBC has a maximum of .32 and the Base Case has a maximum of .26. This difference can be explained in a similar manner as the difference in magnitudes of the R_{11} maximums. The value of R_{22} at the reattachment position is much higher than the other diagonal components and thus has a much higher decay in the nonisotropic decay Base Case than the isotropic decay Base Case.

The IDBC shows a large difference from the Base Case in the structure of R_{33} . The structure in the IDBC case, shown in figure 5.11, is dominated by the isotropic decay. At the locations of the maximum values for R_{11} and R_{22} , R_{33} has a large decay rate when the isotropic decay is used. Again, this reflects the fact that the isotropic decay sets all of the diagonal Reynolds-stress decay rates equal, and the R_{11} and R_{22} require high decay rates at these locations in order to deal with their high creations. These high decay rates are applied to R_{33} as well. The higher decay value of the IDBC R_{33} creates a structure that appears to be "pinched" near the positions downstream of the reattachment location (around 2 plate heights downstream). This pinched structure is even more unlike the structure of the DNS than is the Base Case. The point of maximum R_{33} (with a value of

.035) is shifted downstream in the IDBC results from the location of the maximum (value of .055) in the Base Case, and is significantly lower.

The structures of R_{12} are very similar between the Base Case and the IDBC. The maximum values are close to one another, and the structures are very similar as shown in figures 5.8 and 5.12.

The structures of the length scales, shown in figures 5.13 and 5.14, are very similar in form except for a slightly lower length scale in the region of streamline reattachment in the IDBC. This is associated by a slightly higher maximum decay rate at this location in the IDBC case, shown in figures 5.15 and 5.16.

5.3 Comparisons Involving The D.H. Diffusion Term

In order to investigate the effects of the D.H. diffusion term, two tests are performed. The simple diffusion terms in the IDBC and Base Case are replaced with D.H. diffusion terms. The Reynolds-stress contour plots for the D.H. Nonisotropic case are shown in figures 5.17 through 5.20 and for the D.H. isotropic case in figures 5.21 through 5.24.

The distribution and magnitudes of R_{11} , shown in figures 5.5 and 5.17, are different for the Base Case and the D.H. Nonisotropic Case. There is a very sharp gradient in the y direction in the shear layer. The Reynolds-stress structure does not spread much past the y location where the minimum contour originates at the inlet. The R_{11} energy is held inside this limit (about one plate height above and below the center line). The DNS and Base Case R_{11} values spread out very quickly past this limit (although the Base Case does not spread as much as the DNS). The lack of spreading in the D.H. Nonisotropic Case can be understood in reference to the terms in the D.H. diffusion for R_{11} .

$$\frac{\partial R_{11}}{\partial t} = \dots C_{DR} \frac{\partial}{\partial x_k} \left[\frac{s}{K^{1/2}} R_{kl} \frac{\partial R_{11}}{\partial x_l} \right] \quad (5.4)$$

$$\frac{\partial R_{11}}{\partial t} = \dots C_{DR} \left\{ \frac{\partial}{\partial x_1} \left[\frac{s}{K^{1/2}} \left[R_{11} \frac{\partial R_{11}}{\partial x_1} + R_{12} \frac{\partial R_{11}}{\partial x_2} \right] \right] + \frac{\partial}{\partial x_2} \left[\frac{s}{K^{1/2}} \left[R_{21} \frac{\partial R_{11}}{\partial x_1} + R_{22} \frac{\partial R_{11}}{\partial x_2} \right] \right] \right\} \quad (5.5)$$

In the shear layer (near the limit of the R_{11} distribution), there is a lack of diffusion in the y direction. The second term in equation 5.5, the term responsible for diffusion in the y direction, contains two different types of fluxes, the first of these uses the off-diagonal in the diffusion coefficient to indicate that some diffusion in the y direction should be expected from a gradient in the x direction. This term is very small since there is little if any gradient in R_{11} in the x direction in the area about one plate height above or below the centerline near the inlet. The second flux depends on the gradient in the y direction, a very large gradient; but the diffusion coefficient is very small at this location since R_{22} is almost zero, much as in the Base Case. With these two fluxes being very small, the diffusion in the area is therefore minimal in the y direction. The small amount of spreading is probably due to the diverging advection from the inlet. The maximum value on the inside of the shear layer is .34 in the D.H. diffusion case and .28 in the Base Case. This higher value would be expected when there is limited diffusion in the y direction.

In order to make a complete comparison, the D.H diffusion term was also implemented in the IDBC case. When the IDBC R_{11} and the D.H. Isotropic Case R_{11} results are compared, the same types of trends are seen as are seen for the Base Case and D.H. Nonisotropic Case. The maximum R_{11} is higher in the D.H. Isotropic Case than in the IDBC: .36 for the D.H. Isotropic Case and .32 for the IDBC, much as in the nonisotropic comparison, as seen in figures 5.9 and 5.21. There is a lack of diffusion in the y direction in the D.H. isotropic case, which limits the spread of the Reynolds-stress past the y location about one plate height from the centerline. The decay term has very little effect on the diffusion properties of the D.H. diffusion model.

The R_{22} structures are much more alike in the cases with the D.H. diffusion and the simple diffusion than the R_{11} structures are. This is not necessarily a consequence of the

diffusion model working different for this component since the R_{22} structure of the Base Case and IDBC were already much more constricted, remaining inside one plate height from the centerline. The good comparison is merely a reflection of the fact that the Base Case and IDBC R_{22} values were confined to a narrow region near the center line already. The DNS R_{22} extends out much farther, reaching two plate heights from the center fairly soon after the inlet. The fact that the R_{22} is small in the second of the two flux terms contributing to diffusion in the y direction does not cause a discrepancy in this case since it is diffusing R_{22} . Even if this R_{22} were replaced, in this diffusion coefficient by the full turbulent kinetic energy, it would be diffusing a very low value of R_{22} a plate height from the centerline. The maximum is the same for the Base Case and D.H. Nonisotropic Case, .26. The maximums are close for the IDBC and D.H. Isotropic Case, .32 and .30.

This comparison in which the D.H. diffusion was used does give one set of results that are closer to the DNS than the Base Case is. The structure of the R_{33} , shown in figures 5.19 and 5.23, looks less pinched in the cases with the D.H. diffusion term. The D.H. Nonisotropic Case shows some pinching at about two plate heights downstream but it is much less severe than in the Base Case. This is true for the isotropic comparison also, but in both applications of the D.H. diffusion the structures do not diffuse as much in the y direction downstream as the Base Case and IDBC do. The lack of diffusion in the y direction is probably for similar reasons as the lack of diffusion in the y direction of R_{11} and R_{22} .

The R_{12} results in the cases that utilized D.H. diffusion term, shown in figures 5.20 and 5.24, are fairly similar to the R_{12} results with the simple diffusion. The maximum magnitudes remain essentially unchanged when the diffusion terms are changed. The structure is fairly similar, except that in the cases with the D.H. diffusion, the R_{12} is confined to a smaller region close to the center line (as shown by all of the D.H. diffusion Reynolds-stress components).

The scale structures in the cases with D.H. diffusion are much more complicated near the inlet, but downstream the trends and magnitudes are similar to the Base Case and IDBC. These scales are shown in figures 5.25 and 5.26.

The decay rate in the D.H. cases, shown in figures 5.27 and 5.28, are also confined to the area inside one plate height away from the centerline, but the maximum decay rate is much larger than the maximums in the results that utilize simple diffusion. This reflects the fact that the diffusion in the ϵ equation is of the same form as in the Reynolds-stress equation and acts in a similar manner. The higher maximum and lack of diffusion in the y direction can be explained in a similar way to the way it was explained for R_{11} .

5.4 The BHR Diffusion Model

The third diffusion term is the complex form found in BHR. This diffusion term was implemented in the Reynolds-stress model for the cases with isotropic decay and nonisotropic decay. In each case, the corresponding diffusion, described in table 2.2, was implemented in the decay rate equation. The models were tested with the normal time step and a time step of one quarter of that time step. In both the isotropic decay and nonisotropic decay cases at both time step values, the simulation became unstable and the turbulent kinetic energy grew at a very fast rate. This turbulent kinetic energy eventually grows to the point where the numerical diffusion stability criterion is violated.

Since the value of the turbulent kinetic energy grows so fast, it would appear that the decay rate at that location is not strong enough. This suggests that the growth of the turbulent kinetic energy is caused by not maintaining the proper relationship between the diffusion in the decay rate equation and the diffusion in the Reynolds-stress equation. This tells us that the criteria described in chapter 2 for the relationship between the BHR diffusion in the Reynolds-stress equation and corresponding diffusion in the decay rate

equation are not adequate. It may be necessary to appeal to a full spectral model to obtain a correct relationship.

5.5 The Effects of Doubling the Return to Isotropy Term Coefficient

The effect of doubling the return to isotropy coefficient is to take more energy from the R_{11} and R_{22} components and put that energy in the R_{33} component. The contour plots of these quantities are shown in figures 5.29 through 5.32. The structure of the R_{11} and R_{22} does not change very much except that the magnitudes in both cases is smaller, .22 for both as compared with the Base Case. This reduction of the maximums for the two dominant diagonal components is accompanied by an increase in the magnitude, .09 from .06, and change in structure of R_{33} . The R_{33} distribution loses the pinched look, and the location of the maximum shifts towards the inlet, more like the DNS than the Base Case.

5.6 The Effect of the Inlet Scale

There is another source of possible error besides the model itself. The DNS results do not provide the author with the inlet scale or inlet decay rate. The inlet decay rate is needed for the terms in the decay rate equation and for the diffusion term in the Reynolds-stress equation. Use of incorrect scale has the potential of causing large discrepancies between the modeled Reynolds-stress values and the DNS values. For this reason, an analysis of the effects of the inlet scale was performed. The Base Case (inlet scale of 3 plate heights) was compared to similar test cases with the inlet scale set to 1 plate height, 5 plate heights, and a function $f(y)$, which varies between 3 and 6.5 plate heights.

Observations of the effects of inlet scale can be seen by looking at the contour plots of scale for the four pertinent cases, figures 5.33 through 5.36. The comparison of figures

5.33 and 5.43, the Base Case scale, and the case with the inlet scale of 5 reveals that beyond 2 plate heights past the inlet the contour plot shows very little difference. There is a higher gradient at the inlet near the centerline in the case with inlet scale of 5, but this appears to be an adjustment that the model is performing on the scale. This adjustment range is the model's way of maintaining continuity in the scale from the prescribed inlet value to the value that the model calculates. Beyond about 1.5 plate heights of adjustment in the streamwise direction the contour plots look very similar; and near the center line they have very similar magnitudes, slightly above 2.5 for the location on the centerline about 2 plate heights from the inlet. In the contour plots of scale for the Base Case and for the Scale 5 case, there are relatively sharp gradients that lie at the edges of the DNS wake. On the outside of the gradient the scale does not fall below the inlet scale. This attribute of the scale contour reflects the fact that, as the turbulence gets weaker, the predominant scale increases. This is the only location where the inlet scale appears to play a large role in the scale downstream. It indicates that a large scale is appropriate near the top and bottom boundaries. The R_{11} values from the case with inlet scale of 5 are larger than the Base Case values (having a maximum of about .32) about two plate heights downstream from the inlet and a half a plate height from the center line. The general structure of the R_{11} appears to follow the same trend as in the Base Case. The maximum value of R_{22} are similar, .28; and the structures in general follows the same trends. The values of R_{33} and R_{12} remain similar between the two cases.

In order to try and alleviate some of the need for such large adjustment gradients near the inlet while preserving the large-scale effects outside the wake, the scale that is not constant with y was used. The results, shown in figure 5.35, show the same type of pattern as the cases with an inlet scale of 5 and the Base Case. There is a small adjustment range near the inlet (not as much adjustment as exhibited as in the case with inlet scale of

5), and the flow near the boundary has a length scale greater than the maximum scale at the inlet.

The case in which the inlet scale in the Base Case is replaced by a value of 1 exhibits a very different behavior from all previously described cases, as shown in figure 5.36. The contour plot of the scale does not show the same type of structure. The contours do not make the fast adjustment to a value around 2.5 near the inlet, and the wake is not surrounded by the large gradient. Instead, the structure on the inside of the wake steadily increases inside the wake region and has a maximum near the centerline close to the outlet. When the Reynolds-stress values of this case are compared to the Base Case there are also large differences. The structure of R_{11} has a maximum of .20, compared to a value of .28 for the Base Case, and the location of these R_{11} maximums are closer to the inlet. The flow structure as a whole is much more narrow than the Base Case R_{11} structure. These differences are believed to be caused by excessive decay. With the large decay at the inlet, the Reynolds-stress has no chance to build up in the region close to the inlet in contrast to Base Case. The value of R_{22} is also small compared to the Base Case value and much smaller than the DNS. The maximum R_{22} value is only .20, and the R_{22} structure is especially narrow.

In the turbulent wake region the scale of the turbulence appears to be fairly insensitive to the prescribed inlet scale as long as this inlet scale is not much lower than around 2.5 to 3 plate heights.

5.7 The Best Combination

A test case was run with a combination of features from the previous cases, in order to obtain the best results possible with the current set of models. This best combination case was implemented with the simple diffusion term, the nonisotropic decay term, the inlet

scale of $f(y)$, and the value of the coefficient for the return to isotropy term set at 4.8. The selection of these specifications and terms was made based on the previous test cases.

The distribution of R_{11} , as seen in figure 5.37, in the best combination case is slightly wider in the y direction than in the previous test cases but is not as wide as the DNS R_{11} distribution. The maximum values of R_{11} are in different locations, the DNS maximum being about one plate height downstream from the inlet; and the best combination maximum being almost two plate heights downstream from the inlet but having nearly the same magnitude. R_{11} has a maximum value of .24 in the both best combination results and the DNS result.

The R_{22} distribution, shown in figure 5.38, shows a pattern very much like that of the Base Case but narrower than the DNS. The best combination R_{22} maximum lies in approximately the same location as the Base Case maximum (about two plate heights downstream along the inlet on the center line) and is about .22, whereas the DNS maximum is .50 and the Base Case maximum is .26. The structure is more narrow in the best combination run than the DNS, which is true in all of the test cases.

The R_{33} structure, as seen in figure 5.39, looks qualitatively better than in any of the other test cases when compared to the DNS results. The pinched look is removed, and the location of the maximum is shifted closer to the inlet, more like the DNS R_{33} . The maximum R_{33} is higher than in the other test cases, making it closer to the DNS.

The narrowness of the Reynolds-stress patterns even in this best combination case forces one to look harder at the models being used. The current models have creation of turbulence only in the immediate vicinity of a velocity gradient, whereas we know that the instabilities induced by the shear layer can be felt at least a half of the predominant scale away from the shear layer, as shown in the potential flow solution for the Kelvin Helmholtz instability [9]. In a Kelvin Helmholtz instability, the effects of the disturbances

whose wave lengths are much larger than the width of the shear layer are felt far outside of the shear layer.

6. CONCLUSIONS

This thesis exhibits that the meaning of terms in the turbulence transport models have been studied as well as the relationship between the models and these physical meanings. The models were implemented into a computer code and used to simulate the turbulence in the flow around a flat plate at Reynolds number of 1000. The results from these computer calculations were compared to each other and to a set of DNS data. The comparisons were analyzed to learn more about the basic nature of turbulence and the process of modeling it. There are four major conclusions that can be drawn from the analysis performed in this thesis.

The first of the conclusions concerns the inadequacy of spatial diffusion models in general for spreading the turbulence. In the test cases utilizing either the simple diffusion or the D.H. diffusion, the turbulence structures remained much closer to the center line and did not spread in the cross-stream direction as much as in the DNS. There was not enough transport of the turbulence perpendicular to the axis of flow. There must be another process that creates turbulence in locations that do not have primary creation. There is a need for a term that would show a creation of turbulence at locations of some distance from the primary creation regions. This term would show the effect of the pressure in an incompressible fluid, which should be felt far beyond the location of the mean flow shear.

The second of the conclusions is demonstrated by the failure of the models containing the BHR diffusion term. A mismatch between the diffusion of the turbulent decay rate and the Reynolds-stress components allowed the turbulent kinetic energy to rise to a point of instability. This mismatch demonstrates the need for a new development of the decay rate equation possibly starting from the fundamental spectral turbulence equations.

The analysis of the effects of the inlet scale provides the third major conclusion. Various length scales were applied at the inlet, and their effects on the turbulence length scale inside the domain was observed. This analysis revealed that the turbulence adjusted itself very quickly. In the middle of the domain there was very little difference in the turbulence quantities in the test cases with different inlet length scales. A different behavior was observed if the inlet scale was significantly smaller than the dominant turbulence scale of the flow. The low scale resulted in a large decay rate, which did not allow the turbulence to adjust like the other cases.

The fourth conclusion that can be drawn from the work done in this thesis concerns the concept of double counting the annihilation of the off-diagonal term Reynolds-stress components. The fact that the isotropic decay term did not perform as well as the nonisotropic decay term gives reason to believe that the treatment of the reduction of the off-diagonals as two separate processes has merit. The nonisotropic decay term represents removal of Reynolds-stress components from the dominant scale. This term's contribution to the off-diagonals describes the effect of cascading the off-diagonals, whereas the return to isotropy term describes the effect of mixing the off-diagonals. These two processes take the same form in the transport equations but are both important.

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(4.9)

where

$$Kb\epsilon_{x(i,j)} = \frac{1}{2} \left[\frac{K_{(i,j)}^2}{\epsilon_{(i,j)}} + \frac{K_{(i+1,j)}^2}{\epsilon_{(i+1,j)}} \right] \quad \text{and} \quad Kb\epsilon_{y(i,j)} = \frac{1}{2} \left[\frac{K_{(i,j)}^2}{\epsilon_{(i,j)}} + \frac{K_{(i,j+1)}^2}{\epsilon_{(i,j+1)}} \right]$$

The scalar L can be replaced by R_i or ε .

The finite volume approximation for the BHR Reynolds-stress diffusion term is

$$\begin{aligned}
& \sum^{sym} \frac{3}{2} C_{DR} \frac{\partial}{\partial x_m} \left[\frac{K}{\varepsilon} R_{in} \frac{\partial R_{mj}}{\partial x_n} \right] = \\
& \sum^{sym} C_{DR} \left\{ \frac{1}{\Delta x} \left[Kb \varepsilon x_{(ix,iy)} \left[\left(\frac{R_{i1(ix+1,iy)} + R_{i1(ix,iy)}}{2} \right) \frac{1}{\Delta x} (R_{j1(ix+1,iy)} - R_{j1(ix,iy)}) \right. \right. \right. \\
& + \left(\frac{R_{i2(ix+1,iy)} + R_{i2(ix,iy)}}{2} \right) \frac{1}{2\Delta y} \left(\frac{R_{j1(ix+1,iy+1)} + R_{j1(ix,iy+1)}}{2} - \frac{R_{j1(ix+1,iy-1)} + R_{j1(ix,iy-1)}}{2} \right) \left. \right. \\
& - Kb \varepsilon x_{(ix-1,iy)} \left[\left(\frac{R_{i1(ix,iy)} + R_{i1(ix-1,iy)}}{2} \right) \frac{1}{\Delta x} (R_{j1(ix,iy)} - R_{j1(ix-1,iy)}) \right. \\
& + \left(\frac{R_{i2(ix,iy)} + R_{i2(ix-1,iy)}}{2} \right) \frac{1}{2\Delta y} \left(\frac{R_{j1(ix,iy+1)} + R_{j1(ix-1,iy+1)}}{2} - \frac{R_{j1(ix,iy-1)} + R_{j1(ix-1,iy-1)}}{2} \right) \left. \right. \left. \right] \\
& + \frac{1}{\Delta y} \left[Kb \varepsilon y_{(ix,iy)} \left[\left(\frac{R_{i2(ix,iy+1)} + R_{i2(ix,iy)}}{2} \right) \frac{1}{\Delta y} (R_{j2(ix,iy+1)} - R_{j2(ix,iy)}) \right. \right. \\
& + \left(\frac{R_{i1(ix,iy+1)} + R_{i1(ix,iy)}}{2} \right) \frac{1}{2\Delta x} \left(\frac{R_{j2(ix+1,iy+1)} + R_{j2(ix+1,iy)}}{2} - \frac{R_{j2(ix-1,iy+1)} + R_{j2(ix-1,iy)}}{2} \right) \left. \right. \\
& - Kb \varepsilon y_{(ix,iy-1)} \left[\left(\frac{R_{i2(ix,iy)} + R_{i2(ix,iy-1)}}{2} \right) \frac{1}{\Delta y} (R_{j2(ix,iy)} - R_{j2(ix,iy-1)}) \right. \\
& + \left(\frac{R_{i1(ix,iy)} + R_{i1(ix,iy-1)}}{2} \right) \frac{1}{2\Delta x} \left(\frac{R_{j2(ix+1,iy)} + R_{j2(ix+1,iy-1)}}{2} - \frac{R_{j2(ix-1,iy)} + R_{j2(ix-1,iy-1)}}{2} \right) \left. \right. \left. \right] \left. \right\} \quad (4.10)
\end{aligned}$$

where the summation symbol indicates that the case where i and j are reversed is added to the case shown.

The discretized form of the BHR decay rate diffusion with the non isotropic decay term is

$$\begin{aligned}
& \frac{3C_{DR}}{1.3} \frac{\partial}{\partial x_m} \left[\frac{K}{e} R_{in} \frac{\partial}{\partial x_n} \left[\frac{R_{mi}}{R_{ll}} \epsilon \right] \right] = \\
& 2 \frac{C_{DR}}{1.3} \left\{ \frac{1}{Dx} \left[Kb \epsilon_{x(ix,iy)} \left[\left(\frac{R_{i1(ix+1,iy)} + R_{i1(ix,iy)}}{2} \right) \frac{1}{\Delta x} (W_{x(ix+1,iy)} - W_{x(ix,iy)}) \right. \right. \right. \\
& + \left. \left(\frac{R_{i2(ix+1,iy)} + R_{i2(ix,iy)}}{2} \right) \frac{1}{2\Delta y} \left(\frac{W_{x(ix+1,iy+1)} + W_{x(ix,iy+1)}}{2} - \frac{W_{x(ix+1,iy-1)} + W_{x(ix,iy-1)}}{2} \right) \right] \\
& - Kb \epsilon_{x(ix-1,iy)} \left[\left(\frac{R_{i1(ix,iy)} + R_{i1(ix-1,iy)}}{2} \right) \frac{1}{\Delta x} (W_{x(ix,iy)} - W_{x(ix-1,iy)}) \right. \\
& + \left. \left(\frac{R_{i2(ix,iy)} + R_{i2(ix-1,iy)}}{2} \right) \frac{1}{2\Delta y} \left(\frac{W_{x(ix,iy+1)} + W_{x(ix-1,iy+1)}}{2} - \frac{W_{x(ix,iy-1)} + W_{x(ix-1,iy-1)}}{2} \right) \right] \right] \\
& + \frac{1}{\Delta y} \left[Kb \epsilon_{y(ix,iy)} \left[\left(\frac{R_{i2(ix,iy+1)} + R_{i2(ix,iy)}}{2} \right) \frac{1}{\Delta y} (W_{y(ix,iy+1)} - W_{y(ix,iy)}) \right. \right. \\
& + \left. \left(\frac{R_{i1(ix,iy+1)} + R_{i1(ix,iy)}}{2} \right) \frac{1}{2\Delta x} \left(\frac{W_{y(ix+1,iy+1)} + W_{y(ix+1,iy)}}{2} - \frac{W_{y(ix-1,iy+1)} + W_{y(ix-1,iy)}}{2} \right) \right] \\
& - Kb \epsilon_{y(ix,iy-1)} \left[\left(\frac{R_{i2(ix,iy)} + R_{i2(ix,iy-1)}}{2} \right) \frac{1}{\Delta y} (W_{y(ix,iy)} - W_{y(ix,iy-1)}) \right. \\
& + \left. \left(\frac{R_{i1(ix,iy)} + R_{i1(ix,iy-1)}}{2} \right) \frac{1}{2\Delta x} \left(\frac{W_{y(ix+1,iy)} + W_{y(ix+1,iy-1)}}{2} - \frac{W_{y(ix-1,iy)} + W_{y(ix-1,iy-1)}}{2} \right) \right] \right] \left. \right\} \quad (4.11)
\end{aligned}$$

where

$$W_{x(ix,iy)} = \frac{R_{i1(ix,iy)} \epsilon_{(ix,iy)}}{2K_{(ix,iy)}} \text{ and } W_{y(ix,iy)} = \frac{R_{i2(ix,iy)} \epsilon_{(ix,iy)}}{2K_{(ix,iy)}}$$