WHEEL ROLLING CONSTRAINTS AND SLIP IN MOBILE ROBOTS*

Shashank Shekhar

Oak Ridge National Laboratory
Robotics and Process Systems Division
P.O. Box 2008
Oak Ridge, TN 37831-6305

*This research was supported in part by the U.S. Air Force Material Command (AFMC) San Antonio Air Logistics Center, Robotics and Automation Center of Excellence (SA/ALC-RACE) under Interagency Agreement 2146-H055-A1 with the U.S. Department of Energy, in part by the Naval Air Warfare Center Aircraft Division under Interagency Agreement 2072-E123-A1 with the U.S. Department of Energy, under contract DE-AC05-96OR22464 with Lockheed Martin Energy Research Corp., and in part by an appointment to the Oak Ridge National Laboratory Postdoctoral Research Associates Program administered jointly by the Oak Ridge National Laboratory and the Oak Ridge Institute of Science and Education.
DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, make any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.
DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.
Wheel Rolling Constraints and Slip in Mobile Robots

Shashank Shekhar
Robotics and Process Systems
Oak Ridge National Laboratory
Oak Ridge, TN 37831-6305

Abstract—It is widely accepted that dead reckoning based on the rolling with no slip condition on wheels is not a reliable method to ascertain the position and orientation of a mobile robot for any reasonable distance. We establish that wheel slip is inevitable under the dynamic model of motion using classical results on the accessibility and controllability in nonlinear control theory and an analytical model of rolling of two linearly elastic bodies.

I. Introduction

The mobility literature of wheeled mobile robots with fixed, centered, off-centered, and omnidirectional wheels is traditionally founded on the equations of motion derived from the rolling with no slip constraint on the wheels. Together with the wheel orientation encoders that are used to infer the configuration (end-point) of the mobile robot, these constraints are convenient in reducing the order of the state-space description of the mobile robot. However, dead-reckoning error is substantial for large distances. It renders the reduced state-space model and the corresponding dead-reckoning method of inferring the configuration of the mobile robot, at best, questionable.

Our objective in this paper is to explore the implications of imposing the rolling with no slip condition using classical results on the accessibility and controllability in nonlinear control theory [NV90]. When the rolling constraints are imposed, they allow forces at the wheel-ground interface to be transmitted up to the frictional bound with perfect rolling contact. The analytical theory of two bodies in rolling contact, however, establishes a definite slip associated with the traction forces at the wheel-ground interface. We consider that the traction forces at the wheel-ground interface are determined under the following conditions:

Hypothesis 1:

a. The rolling bodies are linearly elastic,

b. Quasi-identity relation on the elastic properties of the two bodies in contact holds. (This includes the case when the two bodies are elastically similar and approximates the situation when one body, say a rubber wheel, is incompressible, and the other body, say the concrete ground, is relatively rigid.)

c. The area of contact between the two bodies is symmetric about the direction of the rolling of the wheels.

The conditions we identify are roughly the following:

1. If the constraints in the lateral (sideways) and longitudinal (rolling) directions of a wheel transmit traction forces determined under the conditions of Hypothesis 1, then only mobile robots with off-centered wheels can, in general, preserve the kinematic constraints imposed by the wheels. The state of rest is, however, an equilibrium point of the dynamic system.

2. If the constraints in the lateral (sideways) direction of the wheels are satisfied, then preserving longitudinal direction constraint of rolling with no slip with wheel-ground traction determined under the conditions of Hypothesis 1 implies that the base of those mobile robots with fixed, centered, and omnidirectional wheels cannot change its state from the state of rest — a case of zero accessibility (and controllability) for the base of the mobile robot.

In effect, we identify conditions for which wheel slip is inevitable. A key aspect of our study is the analytical formulation of the theory of rolling of two linearly elastic bodies in contact. The origin of such studies is founded in the law of friction of Coulomb-Amontons, the analytic models of deformation of a three-dimensional half-space elastic body due to a concentrated load of Boussinesq (1885) and Cerruti (1889), and Hertz's theory (1882) of two elastic surfaces with curvature in contact. Application of these theories to the study of rolling contact between two bodies was initiated by Carter (1926), who gave solutions of a two-dimensional problem, i.e., when the extents of the rolling objects lie in a plane. Subsequently, Fromm (1927), Johnson (1958), de Pater (1956), Kalker (1987), Haines and Ollerton (1964), and Heinrich and Desoyer (1967) have extended the solutions under various other assumptions; see a review article by Kalker [Kalker 79].

Previously, Alexander and Maddocks [AM92] considered wheel scrubbing. It arises from inconsistent positioning and orientation of the wheels with respect to the kinematic mobility of the base of the mobile robot. They also offer an analytical justification of the phenomena of sideways lurching with uneven rolling friction conditions on the wheels using a minimum work principle on their quasi-static model of motion. The wheel slip we consider here and the implication on its existence subsumes kinematic consistency. It is, therefore, different from their wheel scrubbing phenomena. A recent paper of Balakrishna and Ghosal [BG95] considered a model of the traction forces arising from a rolling tire. Their model of traction force and wheel slip arises from an empirical model of tire mechanics analysis. An analytical model of a rolling tire, in the sense we present here for two linearly elastic rolling bodies, is a difficult problem [Kalker90]. Their empirical model, however, incorporates essential aspects of the analytical theory of rolling under Hypothesis 1 that we consider. The numerical simulation results, therefore, exhibit the presence of wheel slip, a conclusion we prove based entirely on an analytical theory. Our primary results are theorems 6 and 8.

II. Kinematics

This section introduces the kinematic constraints imposed by the nature and configuration of various types of wheels of the mobile robot. The following sub-sections consider the form of the specific instances of the kinematic constraints of a wheel type of a mobile robot. The kinematic model we derive is based on a model of a zero width non-deformable planar circle rolling with no slip on the ground. The subsequent analysis and results of the paper, however, are not restricted by this intermediate step in deriving the model.

A. Kinematic Model of Motion of the Base of Wheeled Mobile Robot

First consider \( S \), the plane of motion of the base of the wheeled mobile robot. Let \( F \) be a choice of a coordinate system in the plane so that \( F: S \to \mathbb{R}^2; p \mapsto (F^1(p), F^2(p)) \). Let the configuration of the base of the mobile robot, an element of \( SE(2) \), be denoted \( X_t = \ldots \).
\( (x, y, \theta)^T \) in the choice of coordinate system \( \mathcal{F} \). Let the velocity of the base of the mobile robot in the plane of motion at the configuration \( x_1 \) be denoted \( x_1 = (x, y, \theta) \). It is easy to verify that the velocity of the base of the mobile robot \( X M^M \) in the moving reference frame \( \mathcal{M} \) is related to \( x_1 \) by \( X M^M = \hat{X} x_1 \), where \( \hat{X} = \hat{X}(x_1) \) is a homogenized orthogonal transformation matrix of three-by-three of the form

\[
\hat{X}(\theta) = \begin{bmatrix}
\cos(\theta) & \sin(\theta) & 0 \\
-\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

B. Kinematic Constraints Imposed by Wheels

Our model of a wheeled mobile robot is a generalized model of such robots considered by Campion et al. [CBD 93]. A wheeled mobile robot has either conventional type wheel or an omnidirectional type wheel. A conventional type wheel has a given axis about which the wheel can rotate and is driven. It is of the following three categories: (i) fixed, (ii) centered orientable, and (iii) off-centered orientable. An omnidirectional wheel can rotate about an arbitrary axis of rotation in the plane of motion of the base of the mobile robot and is (usually) driven about one given axis in that plane. The configuration of a mobile robot with an arbitrary combination of wheels is described by the following: \( X_1 \), the three bases of the coordinate, \( x_2 \), the vector of angular orientations of the plane containing the off-centered wheel, \( x_3 = (\phi_f, \phi_c, \phi_o, \phi_d) \), the angular orientations of the fixed, centered, off-centered, and omnidirectional wheels, respectively, about their driven directions, \( x_4 = (\phi_d) \), an appropriate choice of angular velocities of the omnidirectional wheels about directions complementary to the directions of their drive, and \( x_5 \), the orientations of the plane containing centered wheel.

If the number of fixed, centered, off-centered, and omnidirectional wheels are \( N_f, N_c, N_o, \) and \( N_d \), respectively, then the dimension of an element \( X_1 = (x_1, x_2, x_3, x_4, x_5) \) describing the configuration of the mobile robot is \( 3 + N_f + N_c + N_o + N_d = 3 + N_f + 2(N_c + N_o + N_d) \). Consider that the origin of the choice of coordinate system \( C_{\mathcal{F}_f} \), \( \mathcal{F}_f \in \{ f, c, o, d \} \), \( \mathcal{F}_f \in \{ f, c, o, d \} \), the choice of coordinate frame of the \( j \)-th wheel of type \( f \) with the origin on the wheel axle above the center point of wheel-ground contact for fixed, centered, and omnidirectional wheels and at the pivot of the arm of off-centered wheels is given by \( (l, \alpha) \) in polar coordinates in the choice of coordinate system \( \mathcal{M} \) and the radial line \( \alpha \) is the \( x \)-axis of \( C_{\mathcal{F}_f} \) (see Fig. 1). Similarly, the origin of the choice of slip coordinate frames \( \mathcal{M}_{\mathcal{F}_f} \) (or the origin of \( \mathcal{M}_{\mathcal{F}_f} \)) is \((d, \beta - \hat{f})\) in the choice of coordinate system \( \mathcal{M}_{\mathcal{F}_f} \) and the radial line \( \beta - \hat{f} \) is the \( x \)-axis of the frame \( \mathcal{M}_{\mathcal{F}_f} \). Let \( \gamma \) be the angle that the direction of complementary rolling \( \hat{f} \) of an omnidirectional wheel makes with the direction of \( \phi \), the axis about which the wheel is driven. The three scalar components of the constraints imposed by the wheels are as follows:

\[
\begin{align*}
\{ \cos(\delta) \sin(\delta) & \quad l \cos(\delta - \alpha) + d \cos(\alpha + \beta + \gamma - \delta) \} \hat{X} x_1 \\
- r \sin(\alpha + \beta - \delta) \hat{\phi} & \quad + d \hat{\phi} \cos(\alpha + \beta - \delta), \\
- [\sin(\delta) \cos(\delta) & \quad l \cos(\delta - \alpha) + d \sin(\alpha + \beta + \gamma - \delta) \} \hat{X} x_1 \\
+ d \hat{\phi} \sin(\alpha + \beta - \delta) & \quad + r \hat{\phi} \cos(\alpha + \beta + \gamma - \delta)
\end{align*}
\]

where \( r \) is the radius of the wheel about the driven direction \( \phi \), and \( r' \) is the radius of the omnidirectional wheel about the complementary direction \( \hat{\phi} \) and \( \delta \) is a quantity determined by equating

\[
- \sin(\delta) \cos(\delta) \{ l \cos(\delta - \alpha) + d \sin(\alpha + \beta + \gamma - \delta) \} \hat{X} x_1
\]

to zero in this instantiation so that the \( y \)-velocity in the slip coordinate frame \( \mathcal{M}_{\mathcal{F}_f} \) is zero. For fixed wheels \( d \) = 0 and \( \beta \) is a constant, for centered wheels \( d \) = 0. For centered and off-centered wheels \( \beta \) is a state variable, a component of \( x_3 \), and \( x_4 \), respectively. For fixed, centered, and off-centered wheels, the component containing \( \phi \) does not appear and \( \gamma \) = 0. For omnidirectional wheels \( \beta \) is a constant.

The three scalar constraints in Eq. (1) for each wheel restrict the motion of the base of the mobile robot at the center point of wheel-ground contact in the \( x_2 \), \( y_2 \), and \( \theta_2 \) directions of the slip coordinate system \( \mathcal{M}_{\mathcal{F}_f} \). For convenience, the slip coordinate frame \( z \)-direction will also be called the longitudinal direction, the \( y \)-direction as the lateral direction, and the \( \hat{\phi} \)-direction as the rotational direction. In this terminology, the scalar kinematic constraints for each wheel are also called **longitudinal**, **lateral**, and **rotational** constraints due to the \( e_j^{th} \)-wheel. Let the longitudinal, lateral, and rotational constraints for all the wheels be collected in the form \( J_{\mathcal{F}_f} x_2, J_{\mathcal{F}_f} x_4, \) and \( J_{\mathcal{F}_f} x_5 \), respectively, where

\[
J_{\mathcal{F}_f} = \begin{bmatrix}
J_{\mathcal{F}_f} x_2 & 0 & 0 \\
J_{\mathcal{F}_f} x_4 & 0 & 0 \\
J_{\mathcal{F}_f} x_5 & 0 & 0
\end{bmatrix},
\]

for \( p \in \{ f, c, o, d \} \) and constraints in the fourth column due to the angular velocity of the wheels are further expanded into four subcomponents corresponding to the fixed, centered, off-centered, and omnidirectional wheels. The functional dependency of the terms in the Jacobians are:

\[
J_{\mathcal{F}_f} x_2 = \begin{bmatrix}
J_{\mathcal{F}_f} x_{21} & 0 & 0 \\
J_{\mathcal{F}_f} x_{24} & 0 & 0 \\
J_{\mathcal{F}_f} x_{25} & 0 & 0
\end{bmatrix},
\]

\[
J_{\mathcal{F}_f} x_4 = \begin{bmatrix}
J_{\mathcal{F}_f} x_{41} & 0 & 0 \\
J_{\mathcal{F}_f} x_{42} & 0 & 0 \\
J_{\mathcal{F}_f} x_{43} & 0 & 0
\end{bmatrix},
\]

\[
J_{\mathcal{F}_f} x_5 = \begin{bmatrix}
J_{\mathcal{F}_f} x_{51} & 0 & 0 \\
J_{\mathcal{F}_f} x_{52} & 0 & 0 \\
J_{\mathcal{F}_f} x_{53} & 0 & 0
\end{bmatrix}.
\]

III. Rolling contact of two elastic bodies

The theory of frictional rolling of two bodies addresses the problem of determining the traction forces at the wheel-ground contact. A large
fraction of this literature is dedicated to the rolling of tires founded in the empirical models of tire mechanics. We, however, limit our study to planar linearly elastic wheels. In particular, the two bodies in rolling contact are assumed to follow our Hypothesis 1.

The analytical theory of frictional rolling of two linearly elastic bodies associates a definite slip called creep associated with the traction forces in the area of contact. We show a certain new symmetry in the creep-force relation. The remainder of this section reviews other symmetries with the elastic quasi-identity assumption of Hypothesis 1 given by Kalker [Kalker 67]. These relations, in effect, allow us to infer the traction forces at zero slip velocity. Though the new symmetry we show does not require the quasi-identity assumptions, we also need another symmetry that is valid only with the quasi-identity assumption. Therefore, in general, our conclusions on wheel slip remain valid only with the quasi-identity assumption.

A. Creep-Force Relation Problem Definition

Consider a linearly elastic circular body, denoted \( \alpha \), rolling on a planar linearly elastic material. Let the velocity of the center of the wheel axle, \( \alpha \), be in the \( z \)-direction. Let \( J_{e,j} \) and \( J_{e,j} \) be defined in Sect. II-B. Let \( J_{\{x,y\}} \) and \( J_{\{x,y\}} \) represent the rigid slip of the wheel at the wheel-ground interface in the \( x \), \( y \), and \( z \) directions, respectively, of the slip coordinate frames. Define \( \nu_{e,j} \), the longitudinal creepage, \( \nu_{e,j} \), the lateral creepage, and \( \nu_{e,j} \), the spin for the wheels as

\[
\nu_{e,j}(x,y,z) = \frac{\mathbf{r}_{e,j} \cdot \mathbf{X}_{e,j}}{V_{e,j}},
\]

where \( \mathbf{r}_{e,j} = \frac{\mathbf{X}_{e,j}}{V_{e,j}} \) is the magnitude of the \( x \)-direction velocity of the point on the axle of the wheel (recall that by the choice of the coordinate frame. Let \( \nu_{e,j} \), \( \nu_{e,j} \), and \( \nu_{e,j} \) be defined in Eqs. (2) and (3), respectively. The terms \( \mathbf{J}_{\{x,y\}} \mathbf{X}_{e,j} \) represent the rigid slip of the wheel at the wheel-ground interface in the \( x \), \( y \), and \( z \) directions, respectively, of the slip coordinate frames. Define \( \nu_{e,j} \), the longitudinal creepage, \( \nu_{e,j} \), the lateral creepage, and \( \nu_{e,j} \), the spin for the wheels as

\[
\nu_{e,j}(x,y,z) = \frac{\mathbf{r}_{e,j} \cdot \mathbf{X}_{e,j}}{V_{e,j}},
\]

where \( \mathbf{r}_{e,j} = \frac{\mathbf{X}_{e,j}}{V_{e,j}} \) is the magnitude of the \( x \)-direction velocity of the point on the axle of the wheel (recall that by the choice of the coordinate frame. Let \( \nu_{e,j} \), \( \nu_{e,j} \), and \( \nu_{e,j} \) be defined in Eqs. (2) and (3), respectively. The terms \( \mathbf{J}_{\{x,y\}} \mathbf{X}_{e,j} \) represent the rigid slip of the wheel at the wheel-ground interface in the \( x \), \( y \), and \( z \) directions, respectively, of the slip coordinate frames. Define \( \nu_{e,j} \), the longitudinal creepage, \( \nu_{e,j} \), the lateral creepage, and \( \nu_{e,j} \), the spin for the wheels as

\[
\nu_{e,j}(x,y,z) = \frac{\mathbf{r}_{e,j} \cdot \mathbf{X}_{e,j}}{V_{e,j}},
\]

where \( \mathbf{r}_{e,j} = \frac{\mathbf{X}_{e,j}}{V_{e,j}} \) is the magnitude of the \( x \)-direction velocity of the point on the axle of the wheel (recall that by the choice of the coordinate frame. Let \( \nu_{e,j} \), \( \nu_{e,j} \), and \( \nu_{e,j} \) be defined in Eqs. (2) and (3), respectively. The terms \( \mathbf{J}_{\{x,y\}} \mathbf{X}_{e,j} \) represent the rigid slip of the wheel at the wheel-ground interface in the \( x \), \( y \), and \( z \) directions, respectively, of the slip coordinate frames. Define \( \nu_{e,j} \), the longitudinal creepage, \( \nu_{e,j} \), the lateral creepage, and \( \nu_{e,j} \), the spin for the wheels as

\[
\nu_{e,j}(x,y,z) = \frac{\mathbf{r}_{e,j} \cdot \mathbf{X}_{e,j}}{V_{e,j}},
\]

where \( \mathbf{r}_{e,j} = \frac{\mathbf{X}_{e,j}}{V_{e,j}} \) is the magnitude of the \( x \)-direction velocity of the point on the axle of the wheel (recall that by the choice of the coordinate frame. Let \( \nu_{e,j} \), \( \nu_{e,j} \), and \( \nu_{e,j} \) be defined in Eqs. (2) and (3), respectively. The terms \( \mathbf{J}_{\{x,y\}} \mathbf{X}_{e,j} \) represent the rigid slip of the wheel at the wheel-ground interface in the \( x \), \( y \), and \( z \) directions, respectively, of the slip coordinate frames. Define \( \nu_{e,j} \), the longitudinal creepage, \( \nu_{e,j} \), the lateral creepage, and \( \nu_{e,j} \), the spin for the wheels as

\[
\nu_{e,j}(x,y,z) = \frac{\mathbf{r}_{e,j} \cdot \mathbf{X}_{e,j}}{V_{e,j}},
\]

where \( \mathbf{r}_{e,j} = \frac{\mathbf{X}_{e,j}}{V_{e,j}} \) is the magnitude of the \( x \)-direction velocity of the point on the axle of the wheel (recall that by the choice of the coordinate frame. Let \( \nu_{e,j} \), \( \nu_{e,j} \), and \( \nu_{e,j} \) be defined in Eqs. (2) and (3), respectively. The terms \( \mathbf{J}_{\{x,y\}} \mathbf{X}_{e,j} \) represent the rigid slip of the wheel at the wheel-ground interface in the \( x \), \( y \), and \( z \) directions, respectively, of the slip coordinate frames. Define \( \nu_{e,j} \), the longitudinal creepage, \( \nu_{e,j} \), the lateral creepage, and \( \nu_{e,j} \), the spin for the wheels as

\[
\nu_{e,j}(x,y,z) = \frac{\mathbf{r}_{e,j} \cdot \mathbf{X}_{e,j}}{V_{e,j}},
\]

where \( \mathbf{r}_{e,j} = \frac{\mathbf{X}_{e,j}}{V_{e,j}} \) is the magnitude of the \( x \)-direction velocity of the point on the axle of the wheel (recall that by the choice of the coordinate frame. Let \( \nu_{e,j} \), \( \nu_{e,j} \), and \( \nu_{e,j} \) be defined in Eqs. (2) and (3), respectively. The terms \( \mathbf{J}_{\{x,y\}} \mathbf{X}_{e,j} \) represent the rigid slip of the wheel at the wheel-ground interface in the \( x \), \( y \), and \( z \) directions, respectively, of the slip coordinate frames. Define \( \nu_{e,j} \), the longitudinal creepage, \( \nu_{e,j} \), the lateral creepage, and \( \nu_{e,j} \), the spin for the wheels as

\[
\nu_{e,j}(x,y,z) = \frac{\mathbf{r}_{e,j} \cdot \mathbf{X}_{e,j}}{V_{e,j}},
\]

where \( \mathbf{r}_{e,j} = \frac{\mathbf{X}_{e,j}}{V_{e,j}} \) is the magnitude of the \( x \)-direction velocity of the point on the axle of the wheel (recall that by the choice of the coordinate frame. Let \( \nu_{e,j} \), \( \nu_{e,j} \), and \( \nu_{e,j} \) be defined in Eqs. (2) and (3), respectively. The terms \( \mathbf{J}_{\{x,y\}} \mathbf{X}_{e,j} \) represent the rigid slip of the wheel at the wheel-ground interface in the \( x \), \( y \), and \( z \) directions, respectively, of the slip coordinate frames. Define \( \nu_{e,j} \), the longitudinal creepage, \( \nu_{e,j} \), the lateral creepage, and \( \nu_{e,j} \), the spin for the wheels as

\[
\nu_{e,j}(x,y,z) = \frac{\mathbf{r}_{e,j} \cdot \mathbf{X}_{e,j}}{V_{e,j}},
\]

where \( \mathbf{r}_{e,j} = \frac{\mathbf{X}_{e,j}}{V_{e,j}} \) is the magnitude of the \( x \)-direction velocity of the point on the axle of the wheel (recall that by the choice of the coordinate frame. Let \( \nu_{e,j} \), \( \nu_{e,j} \), and \( \nu_{e,j} \) be defined in Eqs. (2) and (3), respectively. The terms \( \mathbf{J}_{\{x,y\}} \mathbf{X}_{e,j} \) represent the rigid slip of the wheel at the wheel-ground interface in the \( x \), \( y \), and \( z \) directions, respectively, of the slip coordinate frames. Define \( \nu_{e,j} \), the longitudinal creepage, \( \nu_{e,j} \), the lateral creepage, and \( \nu_{e,j} \), the spin for the wheels as

\[
\nu_{e,j}(x,y,z) = \frac{\mathbf{r}_{e,j} \cdot \mathbf{X}_{e,j}}{V_{e,j}},
\]
posed in Eq. (9) as the quasi-identical creep-force law. This separation of vertical and the tangential problem in quasi-identity allows several other symmetries in the creep-force law [Kalker 67] including a specialization of the Proposition 2 we proved earlier. We mention one other:

Proposition 4 (Kalker, 1967) The traction symmetry relations \( \lambda_{\alpha \beta} \delta \mathbf{e}_{\alpha} \mathbf{e}_{\beta} \mathbf{e}_{\alpha} \mathbf{e}_{\beta} \mathbf{e}_{\alpha} \mathbf{e}_{\beta} = \lambda_{\alpha \beta} \delta \mathbf{e}_{\alpha} \mathbf{e}_{\beta} \mathbf{e}_{\alpha} \mathbf{e}_{\beta} \mathbf{e}_{\alpha} \mathbf{e}_{\beta} \mathbf{e}_{\alpha} \mathbf{e}_{\beta} \) verify the quasi-identity creep-force law when the contact area \( C_{\alpha \beta} (x, y) \).

Corollary 5 (Kalker, 1967) With no longitudinal slip, the longitudinal traction in quasi-identical problem disappears when the contact area \( C_{\alpha \beta} (x, y) = C_{\alpha \beta} (x, -y) \), i.e., \( \lambda_{\alpha \beta} (0, \delta \mathbf{e}_{\alpha} \mathbf{e}_{\beta} \mathbf{e}_{\alpha} \mathbf{e}_{\beta} \mathbf{e}_{\alpha} \mathbf{e}_{\beta} \) = 0.

IV. Equation of motion without and with constraints

The dynamic model of mobile robot is obtained by Euler-Lagrange formulation subject to the external forces applied at the actuated joints, and the forces at the wheel-ground interface. Let the vector of forces at the wheel-ground interface are denoted as \( \mathbf{F}_{\alpha \beta} \).

The first type of constraint arises from the fixed and centered wheels. The second type arises from those of the off-centered wheels, and the third type arises from the lateral constraints.

The forces at the wheel-ground interface are denoted \( \mathbf{F}_{\alpha \beta} \), verify the quasi-identity creep-force law when the contact area \( C_{\alpha \beta} (x, y) \).

The configuration parameters \( \mathbf{X} = (x_1, x_2, x_3, x_4, x_5) \) together with \( \mathbf{X} = (x_6, x_7, x_8) \), the independent parameterization of the velocities in Eq. (14), form the state space of the wheeled mobile robot.

With a nonsingular kinematic parameterization in Eq. (14), the equations of motion in Eq. (10) reduce to a set of first-order differential equations on the state space of the form

\[
\dot{\mathbf{X}} = f(\mathbf{X}) + g_1(\mathbf{X}) \mathbf{r}_1 + g_2(\mathbf{X}) \mathbf{r}_2 + g_3(\mathbf{X}) \mathbf{r}_3.
\]

V. Can lateral and longitudinal constraints be preserved?

Consider the situation when neither lateral (sideways) nor the longitudinal (rolling) direction constraints are imposed on the equations.

\[
P = \begin{bmatrix} \mathbf{R}^T \Sigma & 0 & 0 \\ -\mathbf{J}_{32} \Sigma & 0 \\ 0 & -\mathbf{J}_{11} \Sigma & 0 \\ 0 & 0 & I \end{bmatrix}.
\]
of motion of a mobile robot with fixed, centered, off-centered, and omnidirectional wheels; i.e., the traction forces in either of these directions are determined by the creep-force law of Sect. III.

Theorem 6: In general, only mobile robots with off-centered wheels can preserve lateral and longitudinal constraints.

Proof. The state-space is defined by the set of independent coordinates \( \{X_1, X_2, X_3, X_4, X_5, X_6\} \). The dynamic model of the mobile robot without lateral and longitudinal constraints imposed by the wheels is given by equation (10). When the lateral and longitudinal constraints are preserved, it follows from the Corollaries 3 and 5 that wheel-ground traction forces \( \lambda_2 \) and \( \lambda_3 \) are zero.

Consider the off-centered wheels. If the lateral constraints are preserved, the longitudinal constraints in equation Eq. (2) for the off-centered wheels reduce to the form \( \dot{J}_{2ac}(X_1, X_2) \dot{X}_1 + \dot{J}_{2oc}(X_3) = 0 \). The Jacobian of the lateral and longitudinal constraints are

\[
\begin{align*}
D_{X_1} \dot{J}_{2ac} X_1 &+ D_{X_2} \dot{J}_{2ac} X_2 \dot{X}_2 + \dot{J}_{2oc} X_3, \\
D_{X_1} \dot{J}_{2ac} X_1 \dot{X}_1 &+ D_{X_2} \dot{J}_{2ac} X_2 \dot{X}_2 + \dot{J}_{2oc} X_3. 
\end{align*}
\]

The equations of motion in Eq. (10) must preserve these constraints. It follows that the \( \tau_{ac} \) and \( \tau_{oc} \) are determined uniquely as a function of the state. A similar construction for the fixed and centered wheels results in state-dependent constraints that are not satisfied in general. For instance, only when \( \phi_3 \), the angular velocity of the base of the mobile robot, is zero, can the fixed and centered wheels preserve the lateral constraints. Another construction for the omnidirectional wheels results in the state-dependent \( \tau_{ac} \) arising from the lateral and longitudinal constraints, respectively.

Corollary 7: If a mobile robot with off-centered wheels preserves the lateral and the longitudinal constraints, then the state of rest is an equilibrium point.

Proof. At the state of rest, the spin is zero. Hence, by Corollary 3, the angular traction vanishes. It is easy to verify that the drift term and the state-dependent inputs \( \tau_{ac} \) and \( \tau_{oc} \) determined in the proof of Theorem 6 all vanish at the state of rest.

VI. Local controllability with lateral constraints

Consider a control system of the form \( \dot{X} = f(X) + \Sigma g_i(X)u_i \) defined on an open subset \( S \subset \mathbb{R}^n \) of the state space where \( f, g_i: V \to TV \) are smooth functions of the control inputs \( \{u_1, \ldots, u_p\} \subset \mathbb{R}^p \). Let \( C^m(V, TV) \) be the space of smooth vector fields on \( V \) with the Lie algebra defined by the standard Lie bracket \([\cdot, \cdot]\) of two vector fields. Let the accessibility algebra \( \mathcal{C} \) be the smallest subalgebra of the Lie algebra \( C^m(V, TV) \) containing the vector fields \( f \) and the \( g_i \)'s. If the dim\(C(X)\) at a point \( X \) is equal to the dimension of the state space, then we must have fixed \( f \) or the state of rest, the elements \( DBr(k-1)[0]_X = 0 \) and \( DBr(k-1)[0]_X \) vanish.

Consider an arbitrary bracket \( Br_r \) of length \( k \) and the leading term \( f' \) expressed in terms of \( Br_r(k-1) = Br_r(k-1)f' - Df'(Br_r(k-1)) \). The drift vector \( f'_{X=0} = 0 \). The system \( f'_{X=0} = 0 \) and \( Br_r(k-1)_X = 0, j \neq 4 \) on the subset \( X_3 = 0 \) corresponding to the zero angular velocity of the plane of the centered wheel \( Df'_{X=0} = 0, j \neq 4 \). By the assumption of the induction \( Br_r(k-1)_X = 0 \), it follows that \( Br_r(k)_X = 0 \). Hence, the accessibility algebra does not contribute any element to the \( X_1 \) and \( X_4 \) subspaces.

Consider the case when the leading term of a bracket of length \( k \) is \( g_2 \). An arbitrary bracket \( Br_r(k)_X \) of length \( k \) and the leading term \( g_2 \) expressed in terms of \( Br_r(k-1) = Br_r(k-1)g_2 - Dg_2(Br_r(k-1)) \). Since \( Dg_2 \equiv 0 \) and the sixth term of \( g_2 \) is the only non-zero constant term, it follows that \( Br_r(k-1)_X = 0 \) and \( Br_r(k-1)_X = 0 \). Hence, \( Br_r(k)_X = 0 \). Hence, \( Br_r(k)_X = 0 \).

Let us consider the derivative \( D_{X_1} \ldots D_{X_{2n}} X_r(k)(i, j) \) of length \( k \), \( i \in \{1, \ldots, 6\} \).

\[
\begin{align*}
D_{X_1} \ldots D_{X_{2n}} X_r(k)(i, j) &\equiv \sum_{p=1}^2 D_{X_1} \ldots D_{X_{2n}} X_r(k-1) D_p X_j \\
&= D_p DBr(k-1)_X g_j X \\
&= D_p DBr(k-1)_X g_j X. 
\end{align*}
\]

where \( 2^k \) is a set of subsets of the set \( A \) and \( p \in 2^k \), \( \sum \) is the sum of \( p \) is the complement of \( A \). The structure of the control system is zero at the state of rest in the subsystem parameterizing the configuration and the velocity of the base of the mobile robot.

Proof. The accessibility algebra \( \mathcal{C} \) is defined by brackets of the form \([X_1, \ldots, X_{2n}] \), where \( X_i \notin \{f, g_1, g_2, \ldots\} \), \( i \in \{1, \ldots, 2n\} \). Our proof is inductive on the Lie brackets of increasing length of the aforementioned kind.

First, consider the equations of motion with lateral constraints in Eq. (15). The Lagrangian forces due to the lateral constraints have been eliminated and therefore forces up to the frictional bound can be transmitted. In addition, if we assume that the longitudinal constraints of rolling with no slip conditions are preserved, then \( \nu_{slip} = 0 \) and we obtain a set of equations resulting from those in Eq. (16) by substituting \( \nu_{slip} = 0 \) implied by the Corollary.

There are no off-centered wheels. The state \( X_4 \), the free-wheeling direction of the omnidirectional wheels, is decoupled from the rest of the equations of motion. Hence, we drop the states \( X_2 \) and \( X_3 \) and control input \( \tau_{oc} \) and by renumbering obtain the control equations whose functional dependency is as follows:

\[
\begin{align*}
J'1(X) &= J'4(X_1, X_3, X_4), \\
J'2(X) &= J'5(X_2), \\
J'3(X) &= J'5(X_3), \\
J'4(X) &= J'5(X_4). 
\end{align*}
\]

VII. Conclusion

We have considered the dynamic model of motion of a mobile robot with an arbitrary combination of conventional fixed wheels, centered wheels, off-centered wheels, and omnidirectional wheels. It is a standard practice in mechanics to reduce the number of independent coordinates describing the state of the mobile robot by considering that the wheels undergo rolling with no slip motion. We, however, developed the dynamic model without incorporating the rolling with no slip condition.

Every model, therefore, including an analytical model of the traction forces generated by the rolling of wheels under a set of assumptions given in Hypothesis 1 for the linearly elastic wheels.

We established that the lateral (sideways) slip of wheels for a straight line (large curvature) trajectory of the base of the mobile robot is
likely to be small, we imposed the lateral constraints. We considered mobile robots with a combination of wheels of the conventional fixed, centered, or omnidirectional type. We showed that the base of the mobile robot has zero accessibility and controllability as long as the angular orientation of the plane of the centered wheels has zero velocity. Therefore, the fixed, centered, and omnidirectional wheels cannot preserve the longitudinal (rolling direction) constraints. An example in [Sh 96] shows the necessity of the minor condition of zero velocity of the plane of centered wheels on the zero controllability result. The base of a mobile robot with off-centered wheels can also change its configuration by a crab-like motion when the lateral constraints are imposed.

In summary, wheel slip is inevitable according to the proposed model. The wheel slip we establish in the paper may account for parts of the error in dead reckoning. Our ongoing work will address this issue.

References


