The Second-Order Tune Shift with Amplitude
for Octupole-induced Resonances in Storage Ring

The purpose of this note is to analyze the octupole-induced resonances, to lowest order, in a synchrotron and storage ring. When the Hamiltonian with octupole term is transformed to action-angle variables, it is found that the amplitude-dependent tune shift terms are composed of two types: terms of second-order in betatron oscillation amplitude of a particle and terms of fourth-order in oscillation amplitude. Obtaining fourth-order terms requires complicated analysis even with the first-order perturbation theory employed. Treatment of this analysis will be the subject of a subsequent note. Second-order terms are straightforward and simple to calculate, and therefore we treat them here first.

The Hamiltonian for a general octupole field in a storage ring is given by:

$$H_1 = \frac{eA_s}{c} = \frac{1}{4!B\rho} \text{Re}[\left(\frac{\partial^3 B_y}{\partial x^3} + i \frac{\partial^3 B_z}{\partial y^3}\right)(x + iy)^4],$$

where $A_s$ is the vector potential and $B\rho$ is the magnetic rigidity.

Normal octupole component:

$$H_1 = \frac{1}{24B\rho} \frac{\partial^3 B_y}{\partial x^3} (x^4 - 6x^2y^2 + y^4).$$

Skew octupole component:

$$H_1 = \frac{1}{6B\rho} \frac{\partial^3 B_z}{\partial y^3} (x^3y - xy^3).$$

Total scaled Hamiltonian, including both quadrupole and normal octupole terms,
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is then given by:

\[ H = \frac{p_z^2}{2} + \frac{K_xx^2}{2} + \frac{p_y^2}{2} + \frac{K_yy^2}{2} + \frac{B'''}{24B\rho}(x^4 - 6x^2y^2 + y^4), \]  

(4)

where \( B''' = \partial^3 B_y/\partial x^3 \).

The equations of motion corresponding to the octupole term are:

\[ \Delta x' = -\frac{B'''_l}{6B\rho}(x^3 - 3xy^2) \]

(5)

\[ \Delta y' = \frac{B'''_l}{6B\rho}(3x^2y - y^3). \]

We now perform the canonical transformation to action-angle variables via the generating function:

\[ F(x, y, \phi_x, \phi_y; s) = -\frac{x^2}{2\beta_x}(\tan \phi_x + \alpha_x) - \frac{y^2}{2\beta_y}(\tan \phi_y + \alpha_y), \]

(6)

where \( \alpha \) and \( \beta \) are the usual Twiss parameters:

\[ \alpha_x = -\frac{1}{2}\frac{d\beta_x}{ds}, \quad \frac{d\alpha_x}{ds} = \beta_x K_x - \gamma_x \quad ; z = x, y. \]

(7)

The old variables can then be expressed in terms of action and angle variables,

\[ z = \sqrt{2\beta_x J_z} \cos \phi_x, \]

\[ p_z = -\frac{z}{\beta_x}(\tan \phi_x + \alpha_x) = -\frac{\sqrt{2\beta_x J_z}}{\beta_x} \cos \phi_x(\tan \phi_x + \alpha_x). \]

(8)

It is easy to see that the actions \( J_x \) and \( J_y \) are constants of the motion for the unperturbed Hamiltonian. They are given by:

\[ J_x = \frac{(\beta_x p_x + \alpha_x z)^2 + z^2}{2\beta_x} = \frac{\epsilon_x}{2}, \]

(9)

where \( \epsilon_x \) is the emittance of a beam in the \( x \)-plane. The new Hamiltonian is then
determined from
\[ h = H + \frac{\partial F(x, y, \phi_x, \phi_y)}{\partial s}. \]  

As a result, the linear term of the new Hamiltonian is
\[ h_0 = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y}. \]

and the octupole term is
\[ h_1 = \frac{B'''}{24 B \rho} (x^4 - 6x^2y^2 + y^4) \]
\[ = \frac{B'''}{24 B \rho} \left[ (2\beta_x J_x)^2 \cos^4 \phi_x - 6(2\beta_x J_x)(2\beta_y J_y) \cos^2 \phi_x \cos^2 \phi_y \right. \]
\[ + \left. (2\beta_y J_y)^2 \cos^4 \phi_y \right] \equiv V(J_x, J_y, \phi_x, \phi_y; s). \]

By using
\[ \cos^4 \phi_x = \frac{\cos 4\phi_x}{8} + \frac{\cos 2\phi_x}{2} + \frac{3}{8} \]
\[ \cos^2 \phi_x = \frac{\cos 2\phi_x + 1}{2} \]  
and
\[ \cos 2\phi_x \cos 2\phi_y = \frac{1}{2} [\cos 2(\phi_x + \phi_y) + \cos 2(\phi_x - \phi_y)] \] ,

\[ V \] can be rewritten as:
\[ V(J_x, \phi; s) = \frac{B'''}{48 B \rho} \left[ \beta_x^2 J_x^2 (\cos 4\phi + 4 \cos 2\phi + 3) \right. \]
\[ - 6\beta_x \beta_y J_x J_y \{ \cos 2(\phi_x + \phi_y) + \cos 2(\phi_x - \phi_y) + 2 \cos 2\phi_x \}
\[ + 2 \cos \phi_y + 2 \} + \beta_y^2 J_y^2 (\cos 4\phi_y + 4 \cos 2\phi_y + 3)]]. \]

In the above equation, the terms that are independent of \( \phi \) introduce the lowest-order tune shift with amplitude (which is the second-order in oscillation
amplitude). The $\phi$-dependent terms are then the object of the canonical perturbation theory, which leads to the fourth-order tune shift with amplitude. This will be described in a subsequent note. Here we consider the $\phi$-independent terms only.

$$V_0 = \frac{B'''_{zz}}{16B\rho} \beta_z^2 J_z^2 - \frac{B'''_{zy}}{4B\rho} \beta_z \beta_y J_z J_y + \frac{B'''_{yy}}{16B\rho} \beta_y^2 J_y^2.$$  \hfill (16)

From this we can directly extract the tune shifts, which are given by

$$2\pi \Delta \nu_x = \frac{\partial V_0}{\partial J_z} = \frac{B'''_{zz}}{8B\rho} \beta_z^2 J_z - \frac{B'''_{zy}}{4B\rho} \beta_z \beta_y J_y$$

$$2\pi \Delta \nu_y = \frac{\partial V_0}{\partial J_y} = -\frac{B'''_{zy}}{4B\rho} \beta_z \beta_y J_z + \frac{B'''_{yy}}{8B\rho} \beta_y^2 J_y.$$  \hfill (17)

In order to relate the above expressions to those given by Collins [1], who reached the same formula by a different approach, we define:

$$m \equiv \frac{B'''_{zz}}{6B\rho} \beta_z^2 = \frac{B'''_{zz}}{6B\rho} \delta(s-s_k)\beta_z^2$$

$$\tilde{m} \equiv \frac{B'''_{zz}}{6B\rho} \beta_y^2 = \frac{B'''_{zz}}{6B\rho} \delta(s-s_k)\beta_y^2$$

and

$$a^2 = 2J_z, \quad b^2 = 2J_y.$$  \hfill (19)

Finally, summing over all the octupoles around the ring, we have:

$$2\pi \Delta \nu_x = a^2 (3/8) \Sigma m - b^2 (3/4) \Sigma m$$

$$2\pi \Delta \nu_y = -a^2 (3/4) \Sigma m + b^2 (3/8) \Sigma \tilde{m}.$$  \hfill (20)
Reference