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THROUGH FOURIER TRANSFORM HETERODYNE

Author(s):

Bradly J. Cooke
Amy E. Galbraith
Bryan E. Laubscher
Nicholas L. Olivas
Roy M. Goeller
Andrew C. Grubler
Maureen Cafferty
Scott D. Briles

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COHERENT ELECTROMAGNETIC FIELD IMAGING THROUGH FOURIER TRANSFORM HETERODYNE

Bradly J. Cooke, Amy E. Galbraith[†], Bryan E. Laubscher, Nicholas L. Olivas, Roy M. Goeller,
Andrew C. Grubler[‡], Maureen Cafferty, and Scott D. Briles

Los Alamos National Laboratory
P.O. Box 1663, MS-D448
Los Alamos, NM 87545
505-667-8512

Abstract: *We present a detection process capable of directly imaging the transverse amplitude, phase, and if desired, Doppler shift of coherent electromagnetic fields. Based on coherent detection principles governing conventional heterodyned RADAR/LIDAR systems, Fourier Transform Heterodyne (FTH) incorporates transverse spatial encoding of the local oscillator for image capture. Appropriate selection of spatial encoding functions, or basis set, allows image retrieval by way of classic Fourier manipulations. Of practical interest: (i) imaging is accomplished on a single element detector requiring no additional scanning or moving components, and (ii) a wide variety of appropriate spatial encoding functions exist that may be adaptively configured in real-time for applications requiring optimal detection. In this paper, we introduce the underlying principles governing FTH imaging, followed by demonstration of concept via a simple experimental setup based on a HeNe laser and a 69 element spatial phase modulator.*

Key words: *coherent electromagnetic field imaging, imaging RADAR/LIDAR, adaptive imaging*

1. Introduction

The ability to image both amplitude and phase of mm-RF through optical coherent electromagnetic fields enables interesting adaptations of current RADAR/LIDAR and communications technology & applications. Field imaging permits the correction of phase-front distortion imposed by target, atmosphere, and receiver optics. A coherent receiver capable of adaptively tracking and correcting atmospheric induced phase-front error allows the realization of large aperture, long-range communication systems.¹ Real-time adaptive processing algorithms, prompted by direct transverse field encoding, might be put to use in the realization of novel RADAR/LIDAR systems queuing on a specific target(s) shape, size, velocity and trajectory.

[†] Electrical and Computer Engineering Department, University of Arizona

[‡] United States Naval Academy, P.O. Box 12939, Annapolis, MD 21412

Figure 1. Basic square-law detection. A square-law detection process will result in the observer imaging the intensity of the field, with all phase information lost.

Electromagnetic field imaging requires a detection process in which both spatial amplitude and phase information is preserved. Eyes, film, CCD cameras, and other photo-sensitive elements image light in a process referred to as squared modulus or square-law detection.² Referring to **Figure 1**, a coherent source such as a laser or RF transmitter illuminates a target to be imaged. The scattered image field, Φ_{Image} , is collected at the receiving aperture and relayed to the observer. A square-law detection process will result in the observer imaging the *intensity*³ of the field,

$$I_i \propto |\Phi_{Image}|^2, \quad [1]$$

with all phase information lost. Preservation of *both* spatial amplitude and phase suggests a detection process that sidesteps the limitations imposed by the square-law operator.

In this paper, we introduce a heterodyned imaging detection process that circumvents the square-law limitations, and hence; is capable of directly imaging the transverse amplitude, phase, and if needed, the Doppler shift of coherent electromagnetic fields: **Section 2** introduces the underlying principles governing FTH imaging, while **Section 3** demonstrates the concept via a simple experimental setup based on a HeNe laser and a 69 element spatial phase modulator. Finally, practical considerations are discussed in **Section 4**.

2. Fourier Transform Heterodyne

Fourier Transform Heterodyne, or FTH, expands on the coherent detection principles governing conventional heterodyned RADAR/LIDAR systems, and incorporates transverse spatial encoding of the local oscillator for image retrieval. What follows in this section is a summary of the principle concepts underlying FTH. A more formal derivation is provided in **Appendix A**.

Referring to **Figure 2**, the observer depicted in **Figure 1** has been replaced by the FTH system of interest. The basic steps involved in the capture of the image field, Φ_{image} , with the FTH detection process follow.

Figure 2. Fourier transform heterodyne (FTH). Based on coherent detection principles governing conventional heterodyned systems, FTH applies Fourier projection techniques through transverse spatial encoding ($\Phi_{m,n}$) of the local oscillator for electromagnetic field imaging.

FTH Field Imaging:

- Heterodyne the image field, Φ_{Image} , with a local oscillator (l.o.) whose transverse amplitude and phase is sequentially modulated with a known set of reference spatial phase functions, $\Phi_{Ref} = \Phi_{m,n}$, that form a valid basis set.
- Measure the magnitude and phase of the intermediate frequency (I.F.) detector current and form a complex Fourier coefficient, $C_{l,m,n}$, for each spatial function, $\Phi_{m,n}$. As will be derived presently, the heterodyned square-law detection process can be perceived as a Fourier projection operator with the I.F. current conveying the magnitude and phase of the projection. The index l represents the Doppler index of the center frequency ($\Delta\omega$) of the l^{th} bandpass filter.
- Reconstruct the image field from the basis functions, $\Phi_{m,n}$, and the measured $C_{l,m,n}$:

$$\Phi_{Image}(x, y, \Delta\omega_l) = \Phi_{Image}(x, y, l) = \sum_m \sum_n C_{l,m,n} \Phi_{m,n}(x, y). \quad [2]$$

The Fourier projection properties inherent to heterodyned square-law detection processes are subsequently summarized. From **Figure 2**, the dependence of the induced detector current, I , given the detector's quantum efficiency, η , and the spatial distribution of the image and reference local-oscillator electromagnetic fields, Φ_{Image} and Φ_{Ref} respectively, over the detector surface, s , is evaluated through Poynting's relation^{4,5,6}

where

$$I = k \iint_S \eta (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} \quad [\text{Ampere}], \quad [3]$$

$$\mathbf{E} = \Phi_{Image} + \Phi_{Ref}, \quad \mathbf{H} = \mathbf{H}_{Image} + \mathbf{H}_{Ref}, \quad k = \text{constant}. \quad [4]$$

Assuming the image and reference fields are transverse electromagnetic (TEM) and of the form

$$\Phi_{Image} = \rho(x, y) e^{-i\omega t - i\phi(x, y)}, \quad [5]$$

$$\Phi_{Ref} = A e^{-i\omega t - i\theta(x, y)}, \quad [6]$$

and with the reference fields forming a complete orthonormal basis set such that

$$\iint_S e^{i\theta(k_x, k_y, x, y)} e^{-i\theta(k'_x, k'_y, x, y)} dx dy = \delta(k_x - k'_x) \delta(k_y - k'_y), \quad [7]$$

the detector current at the intermediate frequency, $\Delta\omega = \omega - \omega'$, reduces to

$$I(k_x, k_y, \Delta\omega) = 2kA\eta \sqrt{\frac{\mu}{\epsilon}} \text{Re} \left[\iint_S \Phi_{Image}(x, y) e^{i\omega' t + i\theta(k_x, k_y, x, y)} dx dy \right] \quad [8]$$

$$= 2kA\sqrt{\frac{\mu}{\epsilon}} \beta(k_x, k_y) \cos(\Delta\omega t + \alpha(k_x, k_y)) \quad [\text{Ampere}]. \quad [9]$$

Complex Fourier coefficients, $C(k_x, k_y) \propto \beta(k_x, k_y) e^{i\alpha(k_x, k_y)}$, can then be formed from the I.F. current's complex phase and magnitude, resulting in the following transform pair relating the image field, complex Fourier coefficient and reference basis set:

$$C(k_x, k_y) = \iint_S \Phi_{Image}(x, y) e^{i\theta(k_x, k_y, x, y)} dx dy \Leftrightarrow \Phi_{Image}(x, y) = \iint_S C(k_x, k_y) e^{-i\theta(k_x, k_y, x, y)} dk_x dk_y \quad [10]$$

In general, spatial variations may be present in both the reference beam and detection quantum efficiency. Taking these variations into account, Equation 8 would be rewritten as

$$I(k_x, k_y, \Delta\omega) = 2k\sqrt{\frac{\mu}{\epsilon}} \text{Re} \left[\iint_S \eta(x, y) \Phi_{Image}(x, y) \Phi_{Ref}(x, y)^* dx dy \right], \quad [11]$$

$$\text{where } \Phi_{Ref}(x, y) = \rho'(x, y) e^{-i\omega' t - i\theta(x, y)}. \quad [12]$$

Finally, if the image field is heterodyned with a discrete set of reference fields ($\Phi_{Ref} \rightarrow \Phi_{m,n}$), then **Equation 11** reduces to the following Fourier expansion pair

$$\Phi_{Image}(x, y, \Delta\omega_l) = \Phi_{Image}(x, y, l) = \sum_m \sum_n C_{l,m,n} \Phi_{m,n}(x, y), \quad [13]$$

$$C_{l,m,n} \Rightarrow I_{l,m,n} \propto \iint_s \Phi_{Image} \Phi_{m,n}^* dx dy, \quad [14]$$

where the Doppler index l , representing the center frequency of the l^{th} I.F. bandpass filter, has been included.

Note that the detecting surface, s , can be either a single element or partitioned into multiple elements and coherently summed,

$$\iint_s \Phi_{Image} \Phi_{m,n}^* dx dy \Rightarrow \sum_{s_i} \iint_{s_i} \Phi_{Image} \Phi_{m,n}^* dx dy. \quad [15]$$

This property is of practical interest in the development of synthetic apertures and very-large-array (VLA) receivers.

3. Experiment

A simple FTH experiment designed to demonstrate the FTH concept has been constructed. Schematically illustrated in **Figure 3** with photographed apparatus in **Figure 4**, the experiment implements a Zeeman split HeNe laser ($\Delta f = 250$ kHz) for source and local-oscillator lines, and a 69 element spatial phase modulator to encode the 36 term basis used in the experiment.

A description of the FTH experimental procedure follows:

- The two orthogonally polarized laser lines ($\Delta f = 250$ kHz) at the output of the HeNe laser are split with the polarizing beam splitter into two separate beams paths.
- The first beam illuminates the target forming the image, Φ_{Image} , while the second beam passes through the 69-element spatial modulator generating the n^{th} reference term, Φ_n , $n = 0, 1, \dots, 35$.
- The two beams are recombined and are heterodyned at the single-element silicon detector generating a 250 kHz intermediate frequency current I_n .
- A digital oscilloscope measures the 250 kHz I.F. current's phase and magnitude from which the Fourier coefficients, C_n , are formed (see **Figure 5**).
- Once all 36 basis coefficients, C_n , $n = 0, 1, \dots, 35$, are sequentially measured, the resulting magnitude and phase of the detected image, Φ_{Image} , are displayed through **Equation 13** on the computer screen.

Figure 3. Schematic diagram of FTH experiment.

Figure 4. Experimental apparatus.

Figure 5. A digital oscilloscope measures the 250 kHz I.F. current's phase and magnitude from which the Fourier coefficients, C_n , are formed.

The 2-D spatial phase modulator, implemented to encode the reference beam phase front, is the commercially produced Meadowlark Optics Hex69 spatial light modulator (SLM). The Meadowlark SLM is a 69 pixel, two-dimensional array of hexagon shaped, liquid-crystal variable retarders developed for real time programmable phase masking applications. As Meadowlark Optics conveniently supplies the software to generate the first 36 terms of the Zernike⁷ polynomial expansion set, the Zernike set was adopted for the experiment and Equation 12, in circular coordinates, takes on the form

$$\Phi_{ref} \rightarrow \Phi_n = \rho'(r, \theta) e^{-i\omega't - i2\pi Z_n(r, \theta)}, \quad n = 0, 1, \dots, 35, \quad [16]$$

with the full 36 Z_n (Zernike polynomials) listed in Appendix B. Figure 6 catalogues the set of beam targets employed in the experiment. Note that commensurate with the rather limited 36 term expansion set, simple target geometry was observed during the experiment.

Figure 6. Beam targets: A) No target, B) $\frac{1}{2}$ Block, C) $\frac{1}{3}$ Center Block, D) $\frac{1}{2}$ Dielectric Block (glass microscope slide with transmission $T = 90\%$ and phase shift $\Delta\theta = 0.9\pi$).

Figure 7. Measured spectral coefficients for no target. Included are plane-wave and Gaussian ($\omega = 1/2$) coefficients for baseline comparison (coefficients have been normalized) with the Gaussian model providing the best fit.

For the purpose of providing a comparative baseline between measurement and theory, the complex coefficients were directly evaluated from Equation 14, giving

$$C_n = \int_0^{2\pi} \int_0^1 T(r, \theta) \rho(r, \theta) \rho'(r, \theta) e^{2\pi i Z_n(r, \theta)} r dr d\theta, \quad n = 0, 1, \dots, 35, \quad [17]$$

with:

$T(r, \theta) \equiv$ beam targets of Figure 6,

$\rho(r, \theta) = \rho'(r, \theta) = 1$ for a plane wave, and

$\rho(r, \theta) = \rho'(r, \theta) = e^{-\frac{r^2}{\omega^2}}$ for a simple Gaussian beam⁸.

Plotted in Figure 7 are measured spectral coefficients (no target) along with the calculated spectral coefficients for the simple plane wave and Gaussian ($\omega = 1/2$) models. As it provided the best fit, the Gaussian model was evaluated for the remaining targets of Figure 6, generating baseline comparison with measurements.

The theoretically predicted and experimentally measured amplitude, phase, and intensity (energy) for the beam targets of **Figure 6** are plotted in **Figures 8 – 11**. Both calculated and measured coefficients were normalized and a constant phase offset was added to the measured coefficients to equalize pedestal/piston - *no other data processing was introduced*. From **Equation 13**, the field image was expanded with

$$\Phi_{Image}(r, \theta) = \sum_n C_n \Phi_n(r, \theta), \quad n = 0, 1, \dots, 35, \quad [18]$$

and the intensity

$$Intensity(r, \theta) = |\Phi_{Image}(r, \theta)|^2. \quad [19]$$

It is important to remember that the primary purpose of the experiment is the demonstration of the underlying principles governing FTH imaging. Images shown in **Figures 8–11** do indeed indicate that the basic mechanisms expected, such as Fourier projection, are present and valid. However, care should be taken when interpreting image content as the actual image essentially is filtered through the relatively low-resolution 36-term expansion window. Sharp amplitude and phase structure in the imaged field tends to excite energy into higher spatial modes that may fall outside of the 36-term window and result in image distortion (see **Appendix C**).

An unfortunate consequence of raising the Zernike polynomials to an exponent, as prescribed by **Equation 12**, is reflected in the fact that the 36-term basis set is no longer orthogonal. A non-orthogonal basis set will result in the off diagonal coupling of the expansion coefficients which, depending on coupling strength and image spectrum, can in turn degrade image fidelity. The magnitude of off diagonal coupling of **Equation 16** is on the order of a rather high 5% - 10%. The coefficients can be decoupled through the following linear transform⁹

$$C \rightarrow \Lambda^{-1} C, \quad [20]$$

where:

$$\Lambda = \begin{bmatrix} \langle Z_0, Z_0 \rangle & \dots & \langle Z_0, Z_{35} \rangle \\ \vdots & \ddots & \vdots \\ \langle Z_{35}, Z_0 \rangle & \dots & \langle Z_{35}, Z_{35} \rangle \end{bmatrix}, \quad [21]$$

$$\text{and } \langle Z_p, Z_q \rangle = \int_0^{2\pi} \int_0^1 e^{-2\pi i Z_p(r, \theta)} e^{2\pi i Z_q(r, \theta)} r dr d\theta. \quad [22]$$

The transform is sensitive to systematic errors including source coherence, pixel distribution and geometry, and SLM phase drift that introduce deviations from the ideal basis set of **Equation 16** (a strong endorsement for the use of orthogonal basis sets), and hence difficult to apply in practice. However, as shown in **Figure 12**, the transform was successful in bringing out the phase structure of **Figure 11**. Within the limitations imposed by the 36-term expansion, the formation of the phase shift discontinuity caused by the microscope slide edge is visible down the center of the phase image.

A) Gaussian Model

B) Measured

Amplitude

Amplitude

Phase

Phase

Intensity

Intensity

Figure 8. No target. A) Theoretically predicted and B) experimentally measured amplitude, phase, and intensity.

A) Gaussian Model

B) Measured

Amplitude

Amplitude

Phase

Phase

Intensity

Intensity

Figure 9. $\frac{1}{2}$ Block. A) Theoretically predicted and B) experimentally measured amplitude, phase, and intensity.

A) Gaussian Model

B) Measured

Amplitude

Amplitude

Phase

Phase

Intensity

Intensity

Figure 10. 1/3 Center Block. A) Theoretically predicted and B) experimentally measured amplitude, phase, and intensity.

A) Gaussian Model

B) Measured

Amplitude

Amplitude

Phase

Phase

Intensity

Intensity

Figure 11. 1/2 Dielectric Block. A) Theoretically predicted ($T = 90\%$, $\Delta\theta = 0.9\pi$) and B) experimentally measured amplitude, phase, and intensity.

A) Gaussian Model

B) Measured

Amplitude

Amplitude

Phase

Phase

Phase

Phase

(Top View)

(Top View)

Figure 12. 1/2 Dielectric Block - Diagonalized. A) Theoretically predicted ($T = 90\%$, $\Delta\theta = 0.9\pi$) and B) experimentally measured amplitude and phase. Within the limitations imposed by the 36 term Zernike expansion, the formation of the phase shift discontinuity caused by the microscope slide edge is visible down the center of the phase image.

Figure 13. Improved FTH receiver implementation based on knowledge gained from the experimental apparatus. Note that a reflective spatial modulator is used in this example.

4. Practical Considerations

An improved FTH receiver implementation, based on knowledge gained from the simple experimental apparatus of **Figure 4** is diagramed in **Figure 13**. Practical consideration is given to active phase-front correction, phase locked detection, pulsed mode correlated double sample, and pure phase basis functions.

4.1. Active Phase-Front Correction

The subsystem generating the spatial phase-fronts will require systematic monitoring and calibration for the removal of phase distortion introduced, for example, by alignment error or modulator drift. Inserting an interferometer into the phase-front beam path is one approach to direct wavefront monitoring. Referring to the phase-front generation subsystem of **Figure 13**, as the spatial modulator (SM) scans through pre-programmed phase functions, the interferometer array monitors the phase-front interference distribution and provides corrective feedback through the SM controller. Note that each SM element must be programmable within a spatial phase-shift of at least $\pm \pi$ (for the wavelength of interest) or larger depending on the feedback algorithm implemented.

4.2. Phase Locked Detection

Practical FTH systems must acquire Fourier projected coefficients at a sufficient rate of speed. Detection of coefficients requires the precision measurement of the I.F. phase and magnitude.

Figure 14. FTH imaging resolution can be limited by the fundamental coherence properties governing laser and RF sources. Correlated sampling algorithms can aid in mitigating the limitations imposed by a source's finite coherence time, τ_c .

Precision high-speed acquisition can be accomplished through techniques employing I.F. phase-locking¹⁰ for measurement of magnitude and phase (α and β). Phase lock detection will dramatically improve acquisition speed and accuracy over the current experimental configuration (Figure 5).

4.3. Pulsed Mode Correlated Double Sample

FTH imaging resolution can be limited by the fundamental coherence properties governing laser and RF sources. The degradation is a manifestation of transverse amplitude and phase fluctuations on the order of the coherence time¹¹, τ_c , which is intrinsic to a given coherent source. In essence, source variations cannot be distinguished from image induced modulations, thereby limiting resolution.

What follows is a description of a correlated sampling algorithm that can aid in mitigating the limitations imposed by a source's finite coherence. Correlated double sampling (CDS), as applied to FTH, is the process of removing source fluctuations from the target by correlating the outgoing transmitted pulse to the return, target-modulated pulse, through a common reference. For example, a pulsed transmitter-receiver system acquiring one coefficient $C_{m,n}$ per pulse would implement the following algorithm.

- Measure the Fourier projection coefficient, $C_{Xm,n}$, from the basis function $\Phi_{m,n}$ of the outgoing transmitted pulse. The measurement must be performed within the sources coherence time τ_c .
- Measure the Fourier projection coefficient, $C_{Rm,n}$, from the basis function $\Phi_{m,n}$ of the target-modulated return pulse (after a transit time τ_t).
- Subtract the common transverse amplitude and phase error:

$$C_{m,n} \Phi_{m,n} = (C_{Rm,n} - C_{Xm,n}) \Phi_{m,n} \rightarrow C_{m,n} = C_{Rm,n} - C_{Xm,n}.$$

As both the outgoing and return pulses are compared to a common reference, it is essential that the reference field is properly conditioned (for example, spatially filtered) and stationary over the transit time τ_t . Note that as an additional benefit to CDS, any stationary spatial phase or amplitude structure in the local oscillator is correlated out.

4.4. Basis Functions

FTH receivers, in general, may implement basis functions in which both amplitude and phase are modulated. Complex modulators are practical only if they can achieve sufficiently high modulation rates and are readily calibrated. More efficient are systems in which only the phase is modulated through a refractive or reflective spatial modulator. From **Equation 12**, pure phase modulation requires the phase function argument

$$\theta(x, y) = i \ln \left(\frac{1}{\rho'(x, y)} \Phi_{Ref}(x, y) \right), \quad [23]$$

remain real for synthesis with a refractive or reflective spatial modulator. For example, the familiar two-dimensional Cartesian exponential Fourier series (x, y normalized to a unit square)

$$\Phi_{m,n}(x, y) = e^{-i 2 \pi (m x + n y)}, \quad [24]$$

possesses the real phase function

$$\theta(x, y) = 2 \pi (m x + n y). \quad [25]$$

Implementation of **Equation 25** suggests a spatial phase modulator generating an x, y phase pattern across the modulator with slope proportional to m, n respectively. Furthermore, as coherent fields are longitudinally periodic, the spatial phase “wraps around” every 2π , limiting modulation range to $\pm \pi$.

5. Conclusion

We have introduced a detection process capable of directly imaging the transverse amplitude and phase of coherent electromagnetic fields. The detection process, Fourier Transform Heterodyne (FTH), incorporates transverse spatial encoding of the local oscillator along with Fourier projection principles for field imaging. Appropriate selection of spatial encoding functions allows image retrieval by way of classic Fourier manipulations. The underlying principles governing FTH imaging were demonstrated via a simple experimental setup based on a HeNe laser and a 69 element spatial phase modulator. The experimental data, when compared to theoretical predictions, do indeed indicate that the basic FTH mechanisms expected are present, validating the concept.

Appendix A: Heterodyne Phase Projection

From Section 2, the dependence of the induced detector current I , the detector quantum efficiency η , and signal and local oscillator (l.o.) electromagnetic fields over the detector surface s are represented by

$$I = k \iint_s \eta (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s},$$

where $k = \text{constant}$, $\mathbf{E} = \mathbf{E}_s + \mathbf{E}_{lo}$, and $\mathbf{H} = \mathbf{H}_s + \mathbf{H}_{lo}$ (signal + local oscillator). Expanding and projecting the field components through the surface of the detector

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & E_y & E_z \\ H_x & H_y & H_z \end{vmatrix} = (E_y H_z - E_z H_y) \hat{x} - (E_x H_z - E_z H_x) \hat{y} + (E_x H_y - H_x E_y) \hat{z}$$

results in

$$I = k \iint_{x_o y_o} \eta(x, y) (E_x H_y - H_x E_y) dx dy.$$

Assuming the fields are transverse electromagnetic (TEM) and carrying only the x axis of polarization (along E_x)

$$\begin{aligned} I &= k \iint_{x_o y_o} \eta(x, y) (E_{s_x} + E_{lo_x}) (H_{s_y} + H_{lo_y}) dx dy, \\ &= k \iint_{x_o y_o} \eta(x, y) (E_{s_x} H_{s_y} + E_{lo_x} H_{lo_y} + E_{s_x} H_{lo_y} + E_{lo_x} H_{s_y}) dx dy. \end{aligned}$$

If the signal is now bandpassed at the heterodyne intermediate frequency (I.F.), the DC terms are filtered out and the resulting I.F. signal becomes

$$I_{IF} = k \int \int_{x_o y_o} \eta(x, y) (E_{s_x} H_{lo_y} + E_{lo_x} H_{s_y}) dx dy.$$

Furthermore, since the fields are TEM then $H_y = \sqrt{\frac{\epsilon}{\mu}} E_x$ and

$$I_{IF} = 2k \sqrt{\frac{\epsilon}{\mu}} \int \int_{x_o y_o} \eta(x, y) E_{s_x} E_{lo_x} dx dy.$$

A similar term for the E_y polarization can be derived.

For signal and local oscillator fields of the form

$$\begin{aligned} E_s(x, y) &= \rho(x, y)_s e^{-i\omega t - i\phi(x, y)}, \\ E_{lo}(x, y) &= \rho(x, y)_{lo} e^{-i\omega' t - i\theta(x, y)}, \end{aligned}$$

results in

$$\begin{aligned} I_{IF} &= 2k \sqrt{\frac{\epsilon}{\mu}} \operatorname{Re} \left[e^{i\Delta\omega t} \iint_s \eta(x, y) \rho(x, y)_s \rho(x, y)_{lo} e^{-i\phi(x, y)} e^{i\theta(x, y)} dx dy \right] \\ &= 2k \sqrt{\frac{\epsilon}{\mu}} \operatorname{Re} [e^{i\Delta\omega t} \beta e^{i\alpha}] \\ &= 2k \sqrt{\frac{\epsilon}{\mu}} \beta \cos(\Delta\omega t + \alpha) \end{aligned}$$

where $\Delta\omega = |\omega - \omega'|$ is the heterodyne intermediate frequency.

Appendix B: Zernike Polynomials

$$Z_0 = 1$$

$$Z_1 = r \cos(\theta)$$

$$Z_2 = r \sin(\theta)$$

$$Z_3 = 2r^2 - 1$$

$$Z_4 = r^2 \cos(2\theta)$$

$$Z_5 = r^2 \sin(2\theta)$$

$$Z_6 = (3r^2 - 2)r \cos(\theta)$$

$$Z_7 = (3r^2 - 2)r \sin(\theta)$$

$$Z_8 = 6r^4 - 6r^2 + 1$$

$$Z_9 = r^3 \cos(3\theta)$$

$$Z_{10} = r^3 \sin(3\theta)$$

$$Z_{11} = (4r^2 - 3)r^2 \cos(2\theta)$$

$$Z_{12} = (4r^2 - 3)r^2 \sin(2\theta)$$

$$Z_{13} = (10r^4 - 12r^2 + 3)r \cos(\theta)$$

$$Z_{14} = (10r^4 - 12r^2 + 3)r \sin(\theta)$$

$$Z_{15} = 20r^6 - 30r^4 + 12r^2 - 1$$

$$Z_{16} = r^4 \cos(4\theta)$$

$$Z_{17} = r^4 \sin(4\theta)$$

$$Z_{18} = (5r^2 - 4)r^3 \cos(3\theta)$$

$$Z_{19} = (5r^2 - 4)r^3 \sin(3\theta)$$

$$Z_{20} = (15r^4 - 20r^2 + 6)r^2 \cos(2\theta)$$

$$Z_{21} = (15r^4 - 20r^2 + 6)r^2 \sin(2\theta)$$

$$Z_{22} = (35r^6 - 60r^4 + 30r^2 - 4)r \cos(\theta)$$

$$Z_{23} = (35r^6 - 60r^4 + 30r^2 - 4)r \sin(\theta)$$

$$Z_{24} = (70r^8 - 140r^6 + 90r^4 - 20r^2 + 1)$$

$$Z_{25} = r^5 \cos(5\theta)$$

$$Z_{26} = r^5 \sin(5\theta)$$

$$Z_{27} = (6r^2 - 5)r^4 \cos(4\theta)$$

$$Z_{28} = (6r^2 - 5)r^4 \sin(4\theta)$$

$$Z_{29} = (21r^4 - 30r^2 + 10)r^3 \cos(3\theta)$$

$$Z_{30} = (21r^4 - 30r^2 + 10)r^3 \sin(3\theta)$$

$$Z_{31} = (56r^6 - 105r^4 + 60r^2 - 10)r^2 \cos(2\theta)$$

$$Z_{32} = (56r^6 - 105r^4 + 60r^2 - 10)r^2 \sin(2\theta)$$

$$Z_{33} = (126r^8 - 280r^6 + 210r^4 - 60r^2 + 5)r \cos(\theta)$$

$$Z_{34} = (126r^8 - 280r^6 + 210r^4 - 60r^2 + 5)r \sin(\theta)$$

$$Z_{35} = (252r^{10} - 630r^8 + 560r^6 - 210r^4 + 30r^2 - 1)$$

Appendix C: Amplitude and phase progression, $\Delta\theta = 0$ to 2π over 36-term (diagonalized) expansion window, for 1/2 dielectric block ($T = 90\%$, Gaussian model).

$\Delta\theta = 0, 2\pi$

Amplitude

Phase

$\Delta\theta = 1/2\pi$

Amplitude

Phase

$\Delta\theta = 3/4\pi$

Amplitude

Phase

$\Delta\theta = \pi$

Amplitude

Phase

$\Delta\theta = 5/4\pi$

Amplitude

Phase

$\Delta\theta = 3/2\pi$

Amplitude

Phase

References

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- ¹ M. C. Roggemann and B. Welsh, *Imaging Through Turbulence*, CRC Press, 1996.
- ² J. W. Goodman, *Introduction to Fourier Optics*, McGraw-Hill, 1968.
- ³ G. O. Reynolds, J. B. DeViles, G. B. Parrent and Brian J. Thompson, *Physical Optics Notebook Tutorials In Fourier Optics*, SPIE and American Institute of Physics, pp. 94-96, 1989.
- ⁴ J. D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, 2nd edition, 1975.
- ⁵ S. C. Cohen, "Heterodyne Detection: Phase Front Alignment, Beam Spot Size, and Detector Uniformity," *Applied Optics*, vol. 14, no. 8, pp. 9153-1958, Aug. 1975.
- ⁶ S. Fowler, G. W. Kamerman, and G. Lawson, "Analysis of Heterodyne Efficiency for Coherent Laser Radars," *SPIE Applied Laser Radar Technology*, vol. 1936, pp. 137-146, 1993.
- ⁷ J. Y. Wang and D. E. Silva, "Wave-Front Interpretation With Zernike Polynomials," *Applied Optics*, vol. 19, no. 9, pp. 1510-1513, May 1980.
- ⁸ K. Chang, *Handbook of Microwave and Optical Components Vol. 3*, pp. 28-32, John Wiley & Sons, 1990.
- ⁹ O. K. Ersoy, *Fourier-Related Transforms, Fast Algorithms and Applications*, Prentice Hall, 1997.
- ¹⁰ F. M. Gardner, *Phaselock Techniques*, John Wiley & Sons, 1979.
- ¹¹ J. W. Goodman, *Statistical Optics*, John Wiley & Sons, 1985.