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Detection and Track of a Stochastic Target using Multiple Measurements

Christopher T. Cunningham

Abstract

We present a statistical interpretation of the implications of measurements made on a stochastic system, one which makes random state transitions with known average rates. Knowledge of the system is represented as a statistical ensemble of instances which accord with measurements and prior information. The evolution of ratios of populations in this ensemble due to measurements and stochastic transitions may be calculated efficiently. Applied to target detection and tracking, this approach allows a rigorous definition of probability of detection and probability of false alarm and reveals a computationally useful functional relationship between the two. An example of a linear array of simple counters is considered in detail. For it, accurate analytic approximations are developed for detection and tracking statistics as functions of system parameters. A single measure of effectiveness for individual sensors is found which is a major determinant of system performance and which would be useful for initial sensor design.

I. Introduction

We are interested in search and tracking problems. In a search, the target might be located among a number of hiding places. Multiple measurements from various locations might be used to determine the likelihood that a particular hiding place is occupied. An obvious example would be a search for a weak radiation source in a building. Search teams might make many measurements with radiation detectors and analyze this data to determine likely areas for further searching.

The tracking problem is closely related: Here, however, the target locations are changing while the measurements are being made; so the estimates of likely locations must change accordingly. Additionally, there may be dynamic constraints on the motions of the target objects.

We may consider the target region to a system of many states, the target locations. Motions of the target may be described as transitions between states. Target dynamics determines the probabilities of such transitions.

A statistical approach to such a system is useful. Consider an ensemble of a large number similar multiple state systems. It is described by its populations, or numbers of systems that are in each of the possible states. Since transitions among states are assumed to exist, these populations evolve with time. If a measurement is made, it will have a number of possible outcomes. These partition the parent ensemble into daughters, each of which corresponds to a particular outcome.

Of interest is the ensemble corresponding to the observed results of all the measurements. Any inference which may be drawn from those measurements is a statement about the characteristics of this ensemble. Similarly, all such inferences must be implicit in the count of populations for this ensemble, which provides a complete description of it.

We find it convenient to introduce a measure of relative population counts termed the "likelihood" of a given state. This is proportional to the ratio of the population of a given state of the system to the population of a reference, or "null state", which, for a detection or tracking problem, corresponds to a target state "outside" the system. The equations that, describe the evolution of these likelihoods due to measurement or stochastic state transitions are quite simple: this is the great advantage of this approach.

A particular measurement changes a state likelihood by an amount related to the conditional probability of the measurement (assuming the system to be in that state). This quantity we term the "gain" for the state due to the measurement. These statistical notions are further developed in Section II. In Section III, we demonstrate the paradoxical result that mean gains and mean likelihoods in a network are independent of sensor characteristics as long as no target is present. This suggests that *medians*, rather than *means*, might be better measures of representative system performance.

In Section IV, we introduce a linear network in which a target is detected and tracked as it moves, haltingly, past a number of very poor quality detectors. We present the equations determining the development of the likelihood in the system and give results for mean and median system performance observed in Monte Carlo simulations. We introduce the logarithmic or "characteristic" average of gains and likelihoods as a surrogate for median performance.

In Section V we consider the statistical distribution of likelihoods with target present ("leakage" distribution) or not present ("false alarm" distribution) and derive the surprising, although very useful, result that either distribution determines the other. This technique is applied to our exemplary linear network in Section VI, where we demonstrate that a simple analysis of the likelihood in vicinity of the target accounts very accurately for the leakage and false alarm distributions.

In Section VII we present a similar analysis of likelihood near a target to discuss a network's tracking capability. This is related to the median or characteristic gain of its sensors. In Section VIII we review the major results of the paper and present some additional data from simulations to show that the characteristic gain is a good measure of performance of sensors in a network. We also show that it is a better determinant of performance than is the signal-to-noise ratio for the detectors.

Appendix A summarizes the mathematical symbols used in the paper. Appendix B gives details of the derivation of the approximations for leakage and false alarm distributions. Appendix C derives the relation between detector performance and target localization.

II. The basic statistical approach

Let s denote a possible state of the system and let T and $\emptyset$ be the sets

T : Target states : All s "inside" the system.
 $\emptyset$: No-target states : All s "outside" the system.

It is necessary to introduce the "outside" states $\emptyset$ to discuss, for example, the probability that a target is present in a region; there must a possibility that it is not in the region, or in $\emptyset$.

We shall consider two statistical ensembles of systems, the "general" ensemble $\mathcal{E}_0$, which contains all systems in accord with our *a priori* knowledge, and the "specific" ensemble $\mathcal{E}_{\mathcal{M}}$, which is the subset of $\mathcal{E}_0$ with systems whose measurements agree with the series of measurements $\mathcal{M}$. actually observed in the field. With no subscript, the ensemble $\mathcal{E}_{\mathcal{M}}$ is understood.

These ensembles are described by the populations of their states, denoted

$N_s, N_T, N_{\emptyset}$: Populations in $s, T, \emptyset$.

The magnitudes of these numbers are not meaningful, since the ensembles are arbitrarily large. Ratios of their magnitudes are significant, however. A particularly useful ratio compares "inside" and "outside" states:

$Q_s \equiv N_s / N_{\emptyset}$: Likelihood of s relative to $\emptyset$.

$Q_T \equiv N_T / N_{\emptyset} = \sum_{s \in T} Q_s$: Likelihood of T relative to $\emptyset$.

The current ensemble evolves for two reasons, state transitions and measurements. For transitions, define

$r_{s \rightarrow s'}$: Transition rate from s to s' (probability in unit time).

Thus, for the evolution of the ensemble populations from N to N' during the time interval

$$N'_{s'} = N_{s'} + \sum_{s \neq s'} (N_s r_{s \rightarrow s'} - N_{s'} r_{s' \rightarrow s}) \quad (1a)$$

It is useful to simplify this notation by admitting the "transition" $s' \rightarrow s'$, whose "rate" is the probability of remaining in state s' during the time step

$$\omega_{s'} \equiv r_{s' \rightarrow s'} = 1 - \sum_{s \neq s'} r_{s' \rightarrow s} \quad (1b)$$

Therefore,

$$N'_{s'} = \sum_s N_s r_{s \rightarrow s'} \quad (1c)$$

$$Q'_{s'} = \sum_s Q_s r_{s \rightarrow s'} / \sum_s Q_s r_{s \rightarrow \emptyset} \quad (1d)$$

Our system construct is a detection and tracking network that operates for a long time, waiting for a very rare event. In this case, the general ensemble must be approximately in a steady state (i.e. we have no *a priori* information that would distinguish one time from another for it. This steady-state condition is

$$Q_{s'0} = \sum_s Q_{s0} r_{s \rightarrow s'} / \sum_s Q_{s0} r_{s \rightarrow \emptyset} \quad (2)$$

(Note that we have dropped the ' which distinguishes one time step from another, since $\mathcal{E}_0$ is in a steady state.)

Now consider the evolution of the ensemble $\mathcal{E}_M$ due to all the measurements $M \in \mathcal{M}$ made in a time interval. The fraction of the elements of the ensemble making the transition $s \rightarrow s'$ for which M would be observed is

$$o_{M|s \rightarrow s'} : \text{Probability of observing } M \text{ for the transition } s \rightarrow s'.$$

The state $\emptyset$ represents an absent target, or one which has left the system. If the target is not present, it cannot influence any measurement, which is then taken against "background only". We require that the system be constructed so that this is also true for all transitions to $\emptyset$, i.e. their measurement probabilities also correspond to background measurements, denoted $o_{m|\emptyset}$:

$$o_{M|s \rightarrow \emptyset} = o_{M|\emptyset} : \text{all } m, s. \quad (3)$$

Therefore, the populations N' in the ensemble for which M is observed may be expressed as functions of populations N in the prior ensemble:

$$N'_{s'} = \sum_s N_s r_{s \rightarrow s'} o_{M|s \rightarrow s'} \quad (4a)$$

This implies

$$\begin{aligned} Q'_{s'} &= (\sum_s Q_s r_{s \rightarrow s'} o_{M|s \rightarrow s'}) / (\sum_s Q_s r_{s \rightarrow \emptyset} o_{M|\emptyset}) \\ &= \sum_s Q_s r_{s \rightarrow s'} g(s \rightarrow s', M) / \sum_s Q_s r_{s \rightarrow \emptyset} \end{aligned} \quad (4b)$$

$$g(s \rightarrow s', M) \equiv o_{M|s \rightarrow s'} / o_{M|\emptyset} \quad (4c)$$

The denominator $\sum_s Q_s r_{s \rightarrow \emptyset}$ in (4b) assures that $Q_{\emptyset} = 1$, but introduces a very undesirable nonlinearity. However, if we wish $\emptyset$ to be simply a reference state for detection, a consistent assumption is

$$\sum_s Q_s r_{s \rightarrow \emptyset} \sim Q_{\emptyset} = 1 \quad (5)$$

In this case, (4b) expresses the evolved state $Q'_{s'}$ as a linear combination of its antecedents Q_s . This linearity allows us to use the normalized quantities

$$q_T \equiv Q_T / Q_{T0}$$

$$q_T \equiv Q_T / Q_{T0}$$

These normalized quantities, which we will term "likelihoods", obey

$$q_{T0} = \sum_{s \in T} q_{s0} = 1 \quad (6)$$

Our motivation for this normalization is that Q_{T0} , the likelihood that a target is initially present, is an *a priori* characteristic of the ensemble which must be assumed, rather than measured. However, it is not a quantity that would be known with much confidence in practice. In this analysis, we try to determine what can be known about the system without evaluating Q_{T0} explicitly. The normalization isolates the influence of this evaluation.

Our final expression for the evolution of the ensemble $\mathcal{E}_{\mathcal{M}}$ is, in this case,

$$q'_{s'} = \sum_s q_s r_{s \rightarrow s'} g(s \rightarrow s', M) \quad (7)$$

The non-normalized quantities Q_s also obey this equation.

We denote $g(s \rightarrow s', M)$, the "gain" for the transition $s \rightarrow s'$ due to the measurements M . This important quantity is defined in (4c) as the ratio of conditional probabilities for observing M . If the measurements m which comprise M are independent, their cumulative probability is the product over the individual measurements. Therefore, the cumulative gain g may be written as a product of similarly defined gains for the individual measurements

$$g(s \rightarrow s', M) = \prod_{m \in M} g(s \rightarrow s', m) \quad (8a)$$

$$g(s \rightarrow s', m) \equiv o_{m|s \rightarrow s'} / o_{m|\emptyset} \quad (8b)$$

Suppose that the measurements being made are local in nature, i.e. they are influenced only by transitions involving a few nearby states. If a particular transition $s \rightarrow s'$ cannot affect the measurement of a particular quantity m (this will be the case for most transitions and measurements), then we know *a priori* $o_{m|s \rightarrow s'} \sim o_{m|\emptyset}$; so $g(s \rightarrow s', m) \sim 1$, and this term must drop out of the gain product (8a). Hence, these gain products are greatly simplified for local measurements: they will consist of only a few terms at most. This fact motivated our choice of the background state $\emptyset$ as a reference.

III. "Average" performance -- and a paradox

If the system is in the state $\emptyset$, the expected value for the gain is the mean of gains for all possible measurements, weighted by their likelihoods of occurrence:

$$\begin{aligned} \langle g(s \rightarrow s', M) \rangle_{|\emptyset} &\equiv \sum_M o_{M|\emptyset} g(s \rightarrow s', M) \\ &= \sum_M o_{M|\emptyset} (o_{M|s \rightarrow s'} / o_{M|\emptyset}) \\ &= \sum_M o_{M|s \rightarrow s'} = 1 \end{aligned} \quad (9)$$

The individual gains $g(s \rightarrow s', m)$ also have an expected value of unity for the state $\emptyset$.

The fact that $\langle g \rangle | \emptyset = 1$ has an important consequence: Consider a system for which q_s is being computed and let $\langle q_s \rangle | \emptyset$ denote the mean of q_s over those time steps in which no source is present. Then, by (7)

$$\begin{aligned} \langle q_{s'} \rangle | \emptyset &= \sum_s \langle q_s g(s \rightarrow s', M) \rangle | \emptyset r_{s \rightarrow s'} \\ &= \sum_s \langle q_s \rangle | \emptyset \langle g(s \rightarrow s', M) \rangle | \emptyset r_{s \rightarrow s'} \\ &= \sum_s \langle q_s \rangle | \emptyset r_{s \rightarrow s'} \end{aligned} \quad (10)$$

provided that the interval for which the mean was computed is long enough that $\langle q_s \rangle$ achieves a steady state, i.e. $\langle q'_{s'} \rangle | \emptyset = \langle q_s \rangle | \emptyset$. We also assumed that the outcomes of the measurements made in the simulations are independent of the prior measurements which determined q_s , i.e. they are influenced only by the fact that the source is not present¹.

Since both $\langle q_s \rangle | \emptyset$ and q_{s0} obey the same steady-state conditions (10),(2) and normalization condition (6), they are equivalent:

$$\langle q_s \rangle | \emptyset = q_{s0} \quad (11)$$

This says that the time-averaged values of q_s do not change until a target enters the system.

Note that the steady state determined by equation (10) is independent of the particular measurements being made, since $\langle g \rangle | \emptyset = 1$. The author finds this to be paradoxical, since it seems to imply that the sensors are giving no information when no target is present:

Suppose that we construct a barrier of very high quality detectors enclosing some central region. Then, if the network has been running for some time and no target is present:

- i) Won't there be a reduction in likelihood inside the barrier? The sensors would see no target crossing the barrier and the conclusion that there was no target inside it would seem inescapable.
- ii) Doesn't this imply that the steady-state q_{s0} is also reduced there, and thus depends upon details of the sensors (contradicting 10)?

In fact, q_s will be reduced inside the barrier *most of the time*. However, in rare cases, q_s may be sufficiently large inside the barrier, due to sensor false alarms, that its mean value $\langle q_s \rangle | \emptyset$ is similar to that outside. Good detectors may decrease the frequency of these rare events, but, of necessity, that must increase the interior values of q_s (since $\langle g \rangle | \emptyset = 1$) sufficiently that the time-averaged interior values are unchanged.

¹ $\langle A + B \rangle = \langle A \rangle + \langle B \rangle$; also, $\langle AB \rangle = \langle A \rangle \langle B \rangle$ if the probabilities of occurrence of values of A and B are independent.

This paradox seems to arise from a confusion between *mean* and *median* system performance. Thus, good sensors reduce *median* likelihoods within the barrier, even though they may have no effect on *mean* likelihoods.

So, it's not impossible that our contention (10) that sensors don't alter the steady-state likelihoods might be correct, but is there a simple justification for it? Suppose it were not the case and good sensors did indeed reduce the steady-state probability inside the barrier. This reduction could be calculated *a priori*, since it does not depend upon particular instantaneous measurements. Then, it would follow that the best strategy to apprehend the target would be to find it outside the network, where its likelihood was greater. *This would be the correct strategy in the absence of knowledge of the sensor measurements, since it may be determined a priori.* This leads to the contradictory result that the presence of instrumentation, the results of which are unknown to me and to the target, modifies the outcome of my searching for him.

In the next section we will illustrate the effects of measurement on likelihoods in a realistic situation.

IV. Example: a linear tracking network

Suppose that a number of sensors are positioned along a roadway to detect and track a target as it passes. The target may enter and leave the network, which may be represented as:

Thus, the detectors are spaced so that the target passes between two of them in a time step, traveling at maximum velocity. The parameter $\omega \equiv r_{S \rightarrow S}$ measures the likely dispersion in velocity.

If q_{ij} and g_{ij} are the likelihood and the sensor gain at position i and time step j , respectively:

$$q_{ij} = (1-\omega) q_{[i-1][j-1]} g_{ij} + \omega q_i[j-1] \quad (12a)$$

Boundary conditions are

$$q_{0j} = q_{i0} = q_0 = 1 \quad (12b)$$

Thus, (22) may be solved numerically if the gains g_{ij} are known.

For example, suppose the detectors are simple counters that see an average of B_d counts from the background and S_d counts from the source, if one is present.

The probability of observing c counts is, thus,

$$o(c) = \exp(-\langle c \rangle) \langle c \rangle^c / c! \quad (13a)$$

$$\begin{aligned} \langle c \rangle &= B_d && : \text{no source} \\ &= C_d \equiv B_d + S_d && : \text{source} \end{aligned} \quad (13b)$$

The gain corresponding to the observation of c counts is

$$g(c) = \exp(-S_d) (C_d/B_d)^c \quad (13c)$$

We ran a number of Monte Carlo simulations of a linear network. Some of our results for the likelihood in the network are shown in Figure 1:

Fig.1. Likelihoods observed in Monte Carlo simulations of a linear array. Upper curves give likelihoods at the target; lower curves give values with no target in the system. Example curves show two arbitrarily-chosen runs. Other curves are statistics from 10^4 time steps. We assume $C_d = 3$, $S_d = 2$, $B_d = 1$, $\omega = 0.2$.

The figure shows an exponential increase of likelihood at the target position (if one is present). With no target in the system, the median likelihood decreases exponentially with exposure to the sensors, while the mean likelihood remains constant. (Thus, we observe empirically the puzzling behavior remarked in the previous section.) The wait probability ω has great influence. It may seem surprising that the difference between $\omega = 0.2$ and $\omega = 0$ can make differences of orders of magnitude in likelihood values. Much of our remaining discussion will attempt to account for this effect.

Consider first the median likelihoods for $\omega = 0$. We introduce the logarithmic, or "characteristic", average

$$x_C \equiv \exp \langle \ln x \rangle \quad (14)$$

For simple counting sensors, the "characteristic gain" corresponding to (13) is

$$g_C = \exp(-S_d) (C_d/B_d)^{<c>} \quad (15a)$$

$$g_{C+} = \exp(-S_d) (C_d/B_d)^{C_d} : \text{target present} \quad (15b)$$

$$g_{C-} = \exp(-S_d) (C_d/B_d)^{B_d} : \text{target not present} \quad (15c)$$

The sensors considered in Figure 1 have characteristic gains of $g_{C+} = 3.65$ and $g_{C-} = 0.40$.

Since there is very little difference between the mean and median observed counts for a Poisson distribution (always less than one count, in fact), there is correspondingly little difference between the characteristic and median values of the sensor gain.

The characteristic average (14) has the important property that

$$(a * b)_C = a_C * b_C \quad (16)$$

This implies that in situations in which likelihoods are dominated by gain products, the characteristic (average of the) likelihood approximates the median likelihood.

Since for $\omega = 0$ the likelihood given by (12) is simply a gain product, $q_{NN} = \prod_i g_{ii}$, then to a very good approximation,

$$\text{median}(q_{NN}) = \prod_i g_{Cii} = g_{C\pm}^N \quad (+ : \text{target}; - : \text{no target}) \quad (17)$$

This generates the curves for $\omega = 0$ in Figure 1.

In general, if the sensors are fairly good, the value of the likelihood at the target may be approximated from (12) by the product of sensor gains along the target path with the factors $(1-\omega)$ or ω included for steps in which the target progresses or waits, respectively. Since the likelihood is a primarily a product, its characteristic value may be related to a product of characteristic gains. In Appendix B we derive an approximation for the characteristic likelihood measured at the target when it is at position N:

$$(qN)C+ \approx [gC+ (1-\omega)]^N [\omega (1+2gC-/gC+)]^{N\omega} \quad (18)$$

The term $(1+2gC-/gC+)$ arises from contributions to the likelihood along paths which differ slightly from that of the target. (This term is significant for our example, and so are such multiple path phenomena.)

Since the characteristic average does not distribute over addition, i.e. $(a+b)C \neq aC+bC$, there is not a simple relation between characteristic gains and likelihoods when sums of gains are important, as when no target is in the system with $\omega \neq 0$.²

However, we will find in the next section the surprising result that statistical properties of the likelihoods with no target present may be inferred from those when the target is present (which are simpler to estimate, as above). Ultimately, we will be able to present, in Section V, an accounting for the median likelihood, even when no target is in the system.

V. Detection statistics

The statistical issues raised in the previous sections involve more than a particular series of measurements. Hence, they must be addressed in the general ensemble $\mathcal{E}0$. Let us define two useful cumulative distributions: First, the probability of detection at s is the fraction of the elements in state s which have measurement series giving q_s greater than some threshold value q_s^* :

$$PD_s(q_s^*) \equiv N_{s0}(q_s > q_s^*) / N_{s0} \quad (19a)$$

The corresponding probability of leakage $PL_s = 1 - PD_s$ is also a useful quantity. Similarly, the probability of false alarm at s is the fraction of elements in which give $q_s > q_s^*$:

$$PFAs(q_s^*) \equiv N_{\emptyset 0}(q_s > q_s^*) / N_{\emptyset 0} \quad (19b)$$

These distributions have $PD_s(0) = PFAs(0) = 1$.

Now consider the effects of the final measurements M in a series $\mathcal{M}$ on the populations in $\mathcal{E}0$. Suppose that, due to M , the value of q_s^* increases by Δq_s^* , the corresponding changes in $\mathcal{E}0$ are

$$\begin{aligned} \Delta N_{s0}(q_s > q_s^*) &\equiv N_{s0}(q_s > q_s^*) - N_{s0}(q_s > q_s^* - \Delta q_s^*) \\ &= -N_{s\mathcal{M}} \end{aligned} \quad (20a)$$

$$\Delta N_{\emptyset 0}(q_s > q_s^*) = -N_{\emptyset \mathcal{M}} \quad (20b)$$

²Since the mean $\langle \rangle$ distributes over both "*" and "+", as in Note 1, there is a simple relation between mean likelihoods and mean gains, even with no target present.

Here, for clarity, we use " $N_{\mathcal{M}}$ " to refer explicitly to populations in $\mathcal{E}_{\mathcal{M}}$. These equations are perhaps more evident stated in words: How much larger is $N_{s0}(q_s > q_s^* - \Delta q_s^*)$ than $N_{s0}(q_s > q_s^*)$? Precisely by the amount $N_{s0}(q_s = q_s^*)$, which is $N_{s\mathcal{M}}$, etc.

Therefore,

$$\begin{aligned}\Delta P_{Ds} / \Delta P_{FAs} &= (\Delta N_{s0} / \Delta N_{\emptyset 0}) / (N_{s0} / N_{\emptyset 0}) \\ &= (N_{s\mathcal{M}} / N_{\emptyset \mathcal{M}}) / (N_{s0} / N_{\emptyset 0}) \\ &= q_s^* / q_{s0}\end{aligned}\tag{21}$$

This surprisingly simple relation is very useful because it allows either distribution to be derived from the other:

$$P_{FAs} = \int_0 (q_{s0} / q_s^*) dP_{Ds}\tag{22a}$$

$$P_{Ds} = \int_0 (q_s^* / q_{s0}) dP_{FAs}\tag{22b}$$

This gives a useful inequality

$$P_{Ds} / P_{FAs} > q_s^* / q_{s0}\tag{23}$$

It is straightforward to generalize these results to detection in a collection of states, in which case q_s^* and q_{s0} are replaced in the above by suitable sums $\sum_s q_s^*$ and $\sum_s q_{s0}$. For example, for detection of a target anywhere in the system,

$$P_{DT} = \int_0 q_{T^*} dP_{FAT}\tag{24}$$

The P_D , P_{FA} approach, above, is useful for deciding whether a target is present since it does not require a value for Q_{T0} , which we assume is poorly known.

VI. Detection in the linear network

We return to the results of Monte Carlo simulation of the linear network of Section III. In Figure 2 we compare the empirically observed leakage and false alarm distributions with those calculated from (22a) and (22b). The figure shows very good agreement (where there are enough observations) between the observed and calculated distributions. This provides some empirical validation of the derivations above. More importantly, the calculated distributions are much better defined for very low leakage and false alarm probabilities than are the empirical observations, and this is potentially very useful. For example, certain detection systems might well require such very low false alarm rates that determination of them by direct observation in a simulation is infeasible; however, they could be inferred by (22a).

Fig.2. The leakage and false alarm distributions observed in a large number of simulation runs; see text for parameters. The "likelihood" is the quantity q_N^* , measured at the tenth step. The false alarm distribution is the probability that $q_N > q_N^*$ with no source in the network. The leakage distribution is the probability that $q_N < q_N^*$ with the source at the tenth step. A threshold value of q_N^* could be set to give about 1% false alarm probability at 10% leakage. The "Calculated" curves demonstrate that either distribution may be calculated from the other by (22a,b).

Thus, if we can derive an analytic approximation for the leakage distribution, we can generate a corresponding false alarm distribution from it. If the target's path is well known *a priori*, i.e. $\omega \sim 0$, then it is easy to show that the leakage and false alarm distributions of the likelihood at step N will be

$$P_L(q_N^*) = \text{Poisson}(C_N, c_N^*) \quad (25a)$$

$$P_{FA}(q_N^*) = \text{Poisson}(B_N, c_N^*) \quad (25b)$$

$$q_N^* = \exp(-S_N) (C_N/B_N)^{c_N^*} \quad (25c)$$

$$\text{Poisson}(C, c) \equiv \exp(-C) \sum_{n=0}^c C^n / n! \quad (25d)$$

where, in this case, C_N , B_N , S_N are the sums of expected counts up to the N th step, i.e. NC_d , NB_d , NS_d .

For $\omega > 0$, we adopt eqs. (25) to give the approximate forms of the distributions, regarding C_N , B_N , S_N as fitting parameters. These analytic forms satisfy the interrelations (22a) and (22b) automatically, which is another reason for choosing them. We require the ratios of these "effective counts" to be the same as for the expected counts. Thus, there is one degree of freedom in the fit, described by the "fitting parameter" f :

$$C_N/NC_d = B_N/NB_d = S_N/NS_d = f \quad (26)$$

In (18) we gave an approximation for the median likelihood. Equating this to that corresponding to (25a) determines the fitting parameter:

$$(gC_+)^{f-1} = (1-\omega) [\omega (1 + 2gC_-/gC_+)]^\omega \quad (27)$$

For our example, this quantity, which is the ratio of the effective counts to the totals expected along the path, is 61%. Thus, the effect of $\omega \sim 0.2$ amounts to a 39% reduction in the effective counts.

Figure 3 shows very good agreement between the approximations (25) and the observed leakage and false alarm distributions:

Fig. 3. The observed leakage and false alarm distributions of Figs. 1,2. agree well with the approximations (25a,b) based on Poisson distributions.

Recall that we began the preceding discussion in an attempt to explain the median likelihoods presented in Figure 1. The characteristic gains for eqs. (26) are simply

$$(qN)_{C+} \approx (g_{C+})^f N \quad (28a)$$

$$(qN)_{C-} \approx (g_{C-})^f N \quad (28b)$$

where f is the fitting parameter (27); (0.61 for our example).

Since the likelihoods of eqs. (25) are distributed like the gains from a counting sensor, there is very little difference between characteristic and median likelihoods, as noted after eq. (15). Accordingly, in Figure 4, we plot the characteristic likelihoods (28a,b) with the median likelihoods of Figure 1. As with Figure 3, the agreement of approximations based on Poisson distributions with the simulation results is surprisingly good. It appears that the approximation slightly underestimates $(qN)_{C+}$. This may be due to the fact that the analysis of Appendix B only partially accounts for multiple path effects.

Fig. 4. The approximations (29a,b) for median likelihoods agree very well with the empirical values of Fig.1, which were observed in a simulation.

So, we can conclude this section with the heartening result that a simple analysis, based on Poisson distributions and characteristic averages, accounts very nicely for the performance of the network. It will be interesting to see if this approach applies as well to more complex networks, particularly 2-D networks in which the target may make turns at intersections.

VII. Tracking

If a target is determined to be present in the system, the (conditional) probability that it is in state s is simply

$$p_s = q_s / q_T \quad (29)$$

If it is judged to be in s , this is also the conditional probability that judgment is correct. As a measure of the target's localization, we could take the expected value of this quantity, assuming the target is present, i.e.

$$\langle p_s \rangle | T = \sum_s p_s^2 \quad (30)$$

This quantity has the desirable property (for a localization measure) that its value is $1/N$ if the probability distribution is spread uniformly over N states. If there is an associated size Δ_s which varies among states, then a good localization measure would be the "inverse size" of the distribution:

$$\lambda \equiv \langle p_s / \Delta_s \rangle | T = \sum_s p_s^2 / \Delta_s \quad (31)$$

For a tracking problem a good "size" would be the typical time a target remains in a state, since the time interval over which a target's distribution is spread will remain constant if it is moving without dispersion but with changing velocity.

If the distribution is highly localized to a single state " t ", then the probability of that state p_t may also be used as a measure of localization. If λ represents the inverse of the number of states populated and p_t is large,

$$\begin{aligned} \lambda &= p_t^2 + \sum_{s \neq t} (2 p_t p_s + p_s^2) \\ &\sim p_t (2 - p_t) \end{aligned} \quad (32)$$

In the previous section we evaluated the penalty, which may be significant, associated with the uncertainty of target behavior, in a network used for detection. Due to this penalty, a network configuration may be sub-optimal if detection alone is required -- a few large sensors at carefully chosen chokepoints may be more economical than an area monitoring system. However, an area monitor provides the additional capability to locate the target at all times, not just when it is at chokepoints, and this may be desirable.

Consider tracking in our one-dimensional network. A general analytic treatment of localization using the measure λ of eq. (31) would be even more difficult than the analysis of detection, since correlation of the values of the likelihood q at different positions is required and since λ is not a linear function of q . However, we noted in (32) that, for statistical distributions that are well-localized, the probability of the maximum state can be used as a measure of localization. In context, this amounts to the probability for the actual target location, which is not difficult to determine.

In Appendix C, we find that to maintain track,

$$g_{+avg} \sim g_{C+} \sim [(1-\omega)^{-1} - \langle p_t \rangle] / [(1+\omega)^{-1} - \langle p_t \rangle] \quad (33a)$$

$$\langle p_t \rangle \sim [g_{C+} (1+\omega)^{-1} - (1-\omega)^{-1}] / [g_{C+} - 1] \quad (33b)$$

where g_{+avg} is some suitable average value of the gain with the target present and $\langle p_t \rangle$ is the mean localization desired. We take this average to be the characteristic gain of eq. (15b). In this case, we expect the localization to improve about half the time.

Thus, for our exemplary parameters, $\omega = 0.2$, $C_d = 3$, $B_d = 1$, etc., we expect $p_t \sim 0.67$. If we want $p_t \sim 0.5$ and have detectors with $C_d/B_d = 3$, we would need $C_d \sim 2$.

Figure 5 plots the mean value and frequency distribution for p_t observed in a series of 10^4 measurements for detectors with $C_d = 2$ and $C_d = 3$.

Fig. 5. The distribution of values of the localization measure p_t observed in long simulation runs for two values of detector parameters. The mean values for these distributions observed (O) in the simulations and predicted (P) by (33b) compare nicely.

The figure shows that our estimate (33b) does quite a good job predicting the mean value of p_t . This provides some empirical justification for the use of the characteristic gain in problems of this sort (see Section VIII). The figure also shows that the localization may fluctuate widely: most of the time the target may be well localized with p_t near unity, but, even with good sensors, sometimes it will be poorly localized with p_t rather uniformly distributed between zero and unity. The mean value $\langle p_t \rangle$ indicates the relative strengths of these two distributions, rather than a particular value that is likely to be observed.

VIII. A detector performance measure

A simple figure of merit for detector performance in a network will be useful for making initial design tradeoffs.

The success of the (target present) characteristic gain g_{C+} of eq. (15b) in accounting for target detection and localization in Sections VI, VII leads us to propose this quantity as a general measure of detector performance in a network. So, we assert that detectors which have similar median gains should perform similarly, even though they differ in other respects; and detectors with different median gains should perform differently, even though they are similar in other respects.

To support this assertion, let us consider some of the results we have observed for our baseline detectors. The three network performance measures we will consider are, the false alarm probabilities for moderate leakage or equivalent false alarm and leakage probabilities from Section IV.B., and the localization measure $\langle p_t \rangle$ from Section IV.C. Table I gives results for these measures for our baseline detector design:

Table I. Baseline Network Performance
Test results for $S_d = 2$; $S_d/B_d = 2$.

Test:	$P_{FA PL=0.1}$	$P_{FA=PL}$	$\langle p_t \rangle$
Result:	7.5×10^{-3}	3.87×10^{-2}	0.644

Now consider some alternative detector parameters: We shall vary the source/background ratio, keeping some measure of detector performance fixed. Of course, we are primarily interested in the characteristic gain g_{C+} as a performance measure, but, for comparison, we shall also consider preserving another common measure of detector performance, the "signal-to-noise ratio"

$$SNR \equiv S_d / C_d^{1/2} \quad (34)$$

The resulting values of source counts S_d , as functions of S_d/B_d , are given in Table II:

Table II. Detector Excursions
Source counts S_d

Measure	Value	$S_d/B_d = 1$	$S_d/B_d = 4$
gC+	3.65	3.35	1.28
SNR	1.15	2.67	1.67

Thus, the effect of the background in driving source count requirements is weaker for the "signal-to-noise" scheme than for the "characteristic gain" scheme.

For these excursion designs, we calculated the performance of the network for the three measures of Table I. Our results are given in Table III, compared to the baseline values of Table I:

Table III. Results for Excursion Detectors
Ratio: Observed value / Baseline value

Test:	$P_{FA PL}=0.1$		$P_{FA}=PL$		$\langle p_t \rangle$	
S_d/B_d :	1	4	1	4	1	4
gC+	0.93	1.13	0.96	1.08	1.02	0.97
SNR	3.51	0.16	1.51	0.62	0.94	1.06

We see that with gC+ used as the measure of detector performance, there is typically less than 10% variation from baseline network performance for the detector excursions, but with SNR used as the measure, we observe quite large variations in network performance. This supports our claim that the characteristic gain gC+ is a good measure for detection or tracking performance, and is better than (some) other measures that might be used.

Appendix A. Symbols used

$\langle \rangle$	: Expected value (probability-weighted mean)
$\langle \rangle_T$	: Expectation with a target in the system.
$\langle \rangle_\emptyset$	: Expectation with no target in the system.
B_d, S_d, C_d	: Average background, source, total counts for detector d.
B_N, S_N, C_N	: "Effective" counts for a network.
c, C	: Observed and average counts for a simple detector.
$\mathcal{E}_0, \mathcal{E}_M$	: Ensembles for initial state, or for measurement series.
"0"	: Refers to $\mathcal{E}_0$.
no subscript	: Refers to $\mathcal{E}_M$.
time subscript	: Refers to $\mathcal{E}_M$, with M pertaining to a given time.
f	: Fit parameter : ratio of effective counts to total counts.
$g(s \rightarrow s', M)$	: "Gain", ratio of observation probabilities $\equiv \phi_M s \rightarrow s' / \phi_M \emptyset$
g_+, g_-	: Gains observed with target or no target.
g_{C+}, g_{C-}	: "Characteristic gains" $\equiv \exp \langle \ln g \rangle$ with target or no target
$N_s, N_T, N_\emptyset$	: Ensemble populations in given states.
$N_{s0}(q_s > q_s^*)$	: Partial population in $\mathcal{E}_0$ at s with the property $q_s > q_s^*$.
$\phi_M s \rightarrow s'$	: Probability of observing M for the transition $s \rightarrow s'$.
P_{Ds}, P_{Ls}, P_{FAs}	: Probabilities of detection, leakage, false alarm, at s .
p_s	: Conditional probability of state $s \equiv q_s / q_T$
p_t	: Conditional probability for actual target state.
$\langle p_t \rangle$	: Localization measure: expected probability of target state.
Q_s, Q_T	: Population ratios relative to $N_\emptyset$: $Q_s \equiv N_s / N_\emptyset$
Q_{s0}	: Initial, or steady-state, population ratios.
q_s, q_T	: "Likelihood", normalized population ratio : $q_s \equiv Q_s / Q_{T0}$
q_s^*	: Threshold in decision criterion q_s for target detection.
$r_{s \rightarrow s'}$	: Transition rate of from s to s' (probability in unit time).
$s, T, \emptyset$	: A system state, all "inside" states, all "outside" states.
SNR	: Detector "signal-to-noise" ratio $\equiv S_d / (S_d + B_d)^{1/2}$
λ	: Localization measure, inverse size of target distribution.
$\omega \equiv r_{s \rightarrow s}$	: "Wait", probability of remaining at s during a time step.

Appendix B. Derivation of leakage and false alarm approximations

In this section, we derive approximations for leakage and false alarm distributions for a linear sensor network, in which the likelihood propagates according to (12a).

This is a difficult issue because network effects are important, i.e. many paths through the network contribute significantly, particularly for the false alarm distribution. However, as demonstrated in Figure 2, the false alarm distribution may be derived from the leakage distribution by (22a), and the effects of multiple paths are not very important for the leakage distribution: Only paths in the region of high sensor gain in the vicinity of the target need to be considered.

Consider the propagation of characteristic likelihood at the target $(q_N)_{C+}$. As illustrated in Figure B1 there are two types of target behavior: The target either advances during a time step, or waits.

Fig.B1. Two types of target behavior. (a) The target proceeds without waiting. Only the target's path through the network develops significant likelihood. (b) The target waits during a time step. Likelihoods on two paths adjacent to that of the target may be significant.

If the target advances, the primary propagation of likelihood is directly along the target's path, since both the sensor gain g_+ and the probability of advance $(1-\omega)$ are large for this path. Propagation along other paths is inhibited by smaller gains g_- and the small probability of waiting ω . Thus, for this case

$$q_{N,t} = (1-\omega) g_{N,t} q_{N-1,t-1} \quad (B1a)$$

$$(q_N)_{C+} = (1-\omega) g_{C+} (q_{N-1})_{C+} \quad (B1b)$$

If the target waits, then the propagation of likelihood directly along the target's path is inhibited by the factor ω , and multiple paths must be considered, at least for the parameters of our example. Considering the two strongest alternate paths shown in Figure B1b, we have

$$q_{N,t} = (1-\omega)^2 \omega (g_{N-1,t-2} g_{N,t} + g_{N-1,t-2} g_{N,t-1} + g_{N-1,t-1} g_{N,t}) * \\ * q_{N-2,t-3} \quad (B2a)$$

Since the target is present for measurements $g_{N-1,t-2}$ and $g_{N,t}$ and not present for measurements $g_{N-1,t-1}$ and $g_{N,t-1}$ an approximation for the corresponding characteristic likelihood is

$$(q_N)_{C+} \approx (1-\omega)^2 \omega g_{C+} (g_{C+} + 2g_{C-}) (q_{N-2})_{C+} \quad (B2b)$$

This formula is an approximation, rather than an equation, since the characteristic average does not necessarily distribute over the sum of gain products in (B2a).

Since ω is the probability of waiting during a time step, our final approximation for the characteristic likelihood is

$$(q_N)_{C+} \approx [(1-\omega) g_{C+}]^N [\omega (1 + 2g_{C-}/g_{C+})]^{N\omega} \quad (B3)$$

It is convenient to describe the false alarm and leakage distributions as parametric functions of the decision variable threshold q_N^* . The Poisson function (25) provides a good shape for this task. Furthermore, such functions are the exact solutions for $\omega = 0$ and naturally satisfy the interrelationship between distributions (22). A minor problem is that the Poisson distributions are discontinuous in q_N^* , whereas Figure 2 shows that the distributions for $\omega=0.2$ should be approximated by continuous functions³. Therefore, we slightly modify the Poisson function to make it continuous:

$$\text{Poisson} \sim (C, c) \equiv \exp(-C) \left\{ \sum_{r=0}^{|c|} C^r / r! + (c - |c|) C^{|c|+1} / |c+1|! \right\} \quad (B4)$$

where c is a real variable and $| |$ is the greatest integer function.

Our approximations use this modification:

$$P_L(q_N^*) = \text{Poisson} \sim (C_N, c_N^*) \quad (B5a)$$

$$P_{FA}(q_N^*) = 1 - (1/2)[\text{Poisson} \sim (B_N, c_N^{*-1}) + \text{Poisson} \sim (B_N, c_N^*)] \quad (B5b)$$

$$c_N^* \equiv [\ln(q_N^*) + S_N] / \ln(C_N/B_N) \quad (B5c)$$

The false alarm distribution (B5b) follows from the leakage distribution (B5a), using the trapezoid rule for integration of (22a).

³The functional nature of these distributions is a fascinating subject, which will not be explored here.

As in Section V., S_N , B_N and $C_N \equiv S_N + B_N$ are fitting parameters, termed "effective counts", satisfying

$$C_N/NC_d = B_N/NB_d = S_N/NS_d = f \quad (B5)$$

with S_d and B_d the average counts for individual detectors.

The characteristic likelihood of the leakage distribution (B4a) is, therefore,

$$\begin{aligned} (q_N)_{C+} &= \exp(-S_N)(C_N/B_N)^{C_N} \\ &= (g_{C+})^{fN} \end{aligned} \quad (B6)$$

Equating this to (B3), which we derived by considering propagation in the network, gives the fit parameter f :

$$(g_{C+})^{f-1} = (1-\omega) [\omega (1 + 2g_{C-}/g_{C+})]^\omega \quad (B7)$$

With (B5), this completely determines our approximations for the leakage and false alarm distributions.

Appendix C. Detector requirements for tracking

In this Appendix, a target moves past a linear sensor array with a stochastic probability of waiting. We wish to relate p_t , the (conditional) probability for the actual target state, a measure of target localization, to properties of the sensors.

Suppose at time "1" the probability at the target is $p_{t1} \sim 1$, i.e. the target is well localized and probabilities elsewhere are negligible. The target may stay where it is, or it may move. If it stays where it is, then, at time "2",

$$(lg+)_{|wait} \sim \omega / (q_{S2} / q_{S1})_{|wait} \quad (C1a)$$

where, as in (6), q_S denotes the sum of likelihoods in the network. In this case, all the detectors have seen background only. Therefore, by eq. (11),

$$\langle q_{S2} / q_{S1} \rangle_{|wait} = 1 \quad (C1b)$$

$$\langle p_{t2} / p_{t1} \rangle_{|wait} = \omega \quad (C1c)$$

If the target moves

$$(p_{t2} / p_{t1})_{|move} = (1 - \omega) g_+ / (q_{S2} / q_{S1})_{|move} \quad (C2a)$$

where " g_+ " denotes the gain observed in the presence of the target. Similarly, " g_- " will denote gains observed away from the target. Since $\langle g_- \rangle = 1$,

$$\begin{aligned} \langle (q_{S2} / q_{S1}) \rangle_{lg+} &= (1 - \omega) g_+ p_{t1} + \langle g_- \rangle (1 - p_{t1}) \\ &= 1 + p_{t1} (1 - \omega) (g_+ - 1) \end{aligned} \quad (C2b)$$

where the expectation $\langle \rangle|_{g_+}$ is taken over situations in which the particular gain g_+ is observed at the target.

Since ω and $(1 - \omega)$ are the probabilities of waiting and moving, the mean value of the probability ratio is

$$\langle p_{t2} / p_{t1} \rangle = \omega^2 + (1 - \omega)^2 \langle g_+ / [1 + p_{t1} (1 - \omega) (g_+ - 1)] \rangle \quad (C3)$$

If the target is being successfully tracked $\langle p_{t2}/p_{t1} \rangle \sim 1$. Therefore, if particular localization $\langle p_t \rangle$ is required of the tracking system, the sensors must have

$$g_{+|avg} \sim [(1-\omega)^{-1} - \langle p_t \rangle] / [(1+\omega)^{-1} - \langle p_t \rangle] \quad (C4)$$

In the above, the average gain $g_{+|avg}$ is not the mean value $\langle g_+ \rangle$, since it arises from the expected value of a function of g_+ , not g_+ itself. For the characteristic gain $g(Cd)$ of eq. (23d), the probability ratio p_{t2}/p_{t1} should increase about 50% of the time, another reasonable condition for successful tracking.