AN EFFICIENT APPROXIMATE EXPRESSION FOR UNSTEADY PIPE FLOW WITH HIGH-VISCOSITY FLUID

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ABSTRACT

An approximate first-order expression for modeling frequency-dependent friction of unsteady pipe flow with high-viscosity fluid has been developed with the method of nonlinear square integral optimum in the frequency domain. This simple expression of first-order lag elements is more accurate and efficient than others in both the frequency and time domains and can be applied to calculations of both frequency and transient response of unsteady pipe flow for oil hydraulic systems.

INTRODUCTION

The effect of frequency dependence of wall shear stress on laminar flow must be taken into account in calculating unsteady flow with high-viscosity fluids, such as in oil hydraulic pipes. This makes the modeling of pipe flow highly complex and time-consuming (Brown 1962). Various approximation techniques have been developed to simplify the frequency-dependent friction term (Zielke 1968; Trikha 1975; Kagawa 1983; and Suzuki et al. 1991). Most of these efforts focused on the reduction of computer storage and computation time in simulating transient pipe flow by using the method of characteristics (Cai 1990; Suzuki et al. 1991).

This paper presents a new approximate expression of the frequency-dependent terms by using the method of nonlinear square integral optimum. This new model is optimized over a large frequency range and needs fewer terms of the first-order lag elements than previous models; moreover, it is more efficient and accurate in the calculations of both frequency and transient responses for hydraulic pipes.

EXPRESSIONS FOR FREQUENCY-DEPENDENT FRICTION

Based on one-dimensional wave theory, the momentum and continuity equations of pipe flow are

\[ \frac{\partial p}{\partial x} + p \frac{\partial v}{\partial t} + f(t) = 0 \]

\[ \frac{1}{K} \frac{\partial p}{\partial x} + \frac{\partial v}{\partial x} = 0. \]

where \( p \) is the fluid density, \( K \) is the fluid bulk modulus, \( p \) is pressure, \( v \) is uniform velocity over the pipe cross section, \( f(t) \) is pressure drop due to fluid friction per unit length, \( t \) is time, and \( x \) is axial coordinate.

If we consider the friction of laminar flow to be a frequency-dependent term, \( F(s) \) and \( V(s) \), which are Laplace transforms of \( f(t) \) and \( v(t) \), respectively, are related as (Brown 1962)

\[ F(s) = \frac{2p\lambda}{(\lambda J_0(\lambda) J_1(\lambda) - 2)} V(s), \]

where \( J_0 \) and \( J_1 \) are the zero and first-order, first-kind Bessel functions; \( \lambda = r \sqrt{s/v} \); \( r \) is the pipe radius; and \( v \) is the kinematic viscosity of the fluid.

Because of the complicated forms of Eq. 3, they are difficult to use for practical computation; thus, many researchers have tried to simplify the expression of frequency-dependent friction by approximation techniques.
Zielke (1968) introduced a weight function in the inverse Laplace transformation of $F(s)$ as

$$f(t) = \frac{8\nu v}{r^2} V(t) + \frac{4\nu v}{r^2} \int_{0}^{t} W(t-t_1) \frac{\partial V}{\partial t}(t_1) dt_1. \quad (4)$$

The weight function $W(t)$ shall be expressed as the function of Bessel function in Laplace domain

$$W(s) = \frac{2}{\left(j\lambda \frac{J_0(j\lambda)}{J_1(j\lambda)} - \frac{8}{\lambda^2}\right)}. \quad (5)$$

Zielke's model comes as

$$\tau = \frac{v}{r^2} t \quad (6)$$

$$W(t) = W(\tau) \quad (7)$$

$$W(t) = e^{-26.37\tau} + e^{-70.85\tau} + e^{-135.02\tau} + e^{-218.92\tau} + e^{-322.55\tau} + e^{-445.93\tau} + e^{-589.04\tau} + ... \quad \text{for } t \geq 0.02 \quad (8)$$

and

$$W(t) = 0.2821 t^{-1/2} - 1.2500 + 1.0579 t^{1/2} + 0.9375 t + 0.3967 t^{3/2} - 0.3516 t^{2} \quad \text{for } t < 0.02. \quad (9)$$

Zielke's model (Eqs. 8 and 9) is quite accurate in comparison with the exact model of Eq. 5, but requires excessive computer storage and computation time.

To reduce the required computer storage and simplify the computation procedure with the method of characteristics, Trikha (1975) developed an approximation expression for the weight function with a sum of first-order lag elements

$$W_1(\tau) = W_{\text{app}}(\tau) = \sum_{i=1}^{3} m_i e^{-n_i \tau} \quad (10)$$

and its Laplace domain expression is

$$W_1(s) = \frac{4v}{r^2} \sum_{i=1}^{3} \frac{m_i}{s + n_i} \frac{v}{r^2}. \quad (11)$$

Trikha's model (Eqs. 10 and 11) has large computation errors, so Kagawa (1983) applied the first-order lag elements to approximate Eqs. 8 and 9 in the time domain, and obtained a more accurate expression of the weight function of $W$ as

$$W_K(\tau) = \sum_{i=1}^{10} m_i e^{-n_i \tau} \quad (12)$$

and

$$W_K(s) = \frac{4v}{r^2} \sum_{i=1}^{10} \frac{m_i}{s + n_i} \frac{v}{r^2}. \quad (13)$$

where $m_i$ and $n_i$ are the coefficients shown in Table 1.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$n_i$</th>
<th>$m_i$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>26.3744</td>
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<td>2</td>
<td>72.8033</td>
<td>1.16725</td>
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<td>3</td>
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<td>5</td>
<td>1570.60</td>
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<tr>
<td>6</td>
<td>4618.13</td>
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<td>7</td>
<td>13601.1</td>
<td>20.0612</td>
</tr>
<tr>
<td>8</td>
<td>40082.5</td>
<td>34.4541</td>
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<tr>
<td>9</td>
<td>118153.</td>
<td>59.1642</td>
</tr>
<tr>
<td>10</td>
<td>348316.</td>
<td>101.590</td>
</tr>
</tbody>
</table>

The time constants in Eqs. 12 and 13 were obtained by approximating to exact $W(\tau)$ one element by one element in the time domain, so those constants converge very slowly. This means that many terms of the first-order lag elements are needed for sufficient accuracy (for example, $n = 10$) in calculations.

**OPTIMAL APPROXIMATION OF FREQUENCY-DEPENDENT FRICTION**

An approximate expression of frequency–dependent friction is presented in the frequency domain:

Substituting $s = j\omega$ in Eq. 5,

$$W(j\omega) = \frac{2}{[j\lambda(j\omega)] \frac{J_0[j\lambda(j\omega)]}{J_1[j\lambda(j\omega)]} - \frac{8}{[\lambda(j\omega)]^2}. \quad (14)$$

Again, we present the weight function as a sum of first-order lag elements:

$$W_c(j\omega) = \frac{4v}{r^2} \sum_{i=1}^{n} \frac{m_i}{j\omega + n_i} \frac{v}{r^2}. \quad (15)$$
The optimal approximation of Eq. 15 is determined by the solutions of \( m_1 \) and \( n_1 \), which are based on the solution of nonlinear square integral optimum problem as

\[
\begin{aligned}
\Omega(m_1, n_1, \ldots, m_n, n_n) &= \int C_1^2 |W(j\omega) - W_c(j\omega)|^2 d\omega \\
\Omega(m_1^*, n_1^*, \ldots, m_n^*, n_n^*) &= \inf(m_1, n_1, \ldots, m_n, n_n)
\end{aligned}
\]

\[m_1 \in (-\infty, +\infty), \quad n_1 \in (0, +\infty),\]

where \( C_1 \) and \( C_2 \) are integral constants, e.g., \( C_1 = 0.1 \omega_v \), \( C_2 = 1000 \omega_v^2 \), and \( \omega_v = 8\pi^2 \).

For \( n = 1, 2, 3, 4 \), the approximation results of \( m_1 \) and \( n_1 \) are shown in Table 2.

Table 3 shows the variance comparison of Eqs. 11, 13, and 15 in the frequency domain. It is obvious that the accuracy of Eq. 15 \((n = 3)\) is on the same level as Kagawa's model \((n = 10)\), while that of Eq. 15 \((n = 4)\) is better than Kagawa's model \((n = 10)\).

Figures 1 and 2 compare the approximation models (Eqs. 11, 13, and 15) to the exact model (Eq. 14) in magnitudes and phases, respectively. Kagawa's model is in good agreement with the exact model in magnitude and phase in the lower-frequency regions. But in the higher-frequency region, there are differences between Kagawa's model and the exact model, especially in phase. Trikha's model differs sharply from the exact model, especially in the higher-frequency region. Equation 15 \((n = 3)\), in Fig. 2(a) and Fig. 2(b), is much better than Trikha's model in both magnitude and phase, while Eq. 15 \((n = 4)\), in Fig. 2(a) and Fig. 2(b), is better than Kagawa's model \((n = 10)\).

Figures 3 and 4 compare the new approximation models, Trikha's model, and Kagawa's model with Zielke's weight function in the time domain. The new approximation model \((n = 4)\) in Fig. 4 agrees with Zielke's model, as well as with Kagawa's model \((n = 10)\) in Fig. 3.

### CONCLUSIONS

The new approximation for the frequency-dependent friction by the method of nonlinear square integral optimum can reduce the number of terms of the first-order lag elements in the weight function and presents sufficient accuracy in both frequency and time domains. The results of the approximation developed in this paper are obviously better than those of Trikha and Kagawa's models. Because this new model has fewer terms of first-order lag elements, the calculation procedure becomes simpler and faster and also requires less computer storage and computation time. The time efficiency of this model is also much better than that of Suzuki's model, which used a block-diagram representation of frequency-dependent terms (Suzuki et al. 1991).

### Table 2. Coefficients \( m_1 \) and \( n_1 \) of Eq. 15

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m_1 )</th>
<th>( m_2 )</th>
<th>( m_3 )</th>
<th>( m_4 )</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( n_3 )</th>
<th>( n_4 )</th>
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<td>4</td>
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### Table 3. Variance comparison of three models

<table>
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<th>model</th>
<th>variance</th>
<th>model</th>
<th>variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. 15: ( n = 1 )</td>
<td>0.062 \times 10^{-3}</td>
<td>Eq. 13: ( n = 1 )</td>
<td>0.203 \times 10^{-2}</td>
</tr>
<tr>
<td>( n = 2 )</td>
<td>0.800 \times 10^{-4}</td>
<td>( n = 2 )</td>
<td>0.164 \times 10^{-2}</td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>0.760 \times 10^{-5}</td>
<td>( n = 3 )</td>
<td>0.121 \times 10^{-2}</td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>0.527 \times 10^{-6}</td>
<td>( n = 4 )</td>
<td>0.788 \times 10^{-3}</td>
</tr>
<tr>
<td>( n = 5 )</td>
<td>0.434 \times 10^{-3}</td>
<td>( n = 5 )</td>
<td>0.434 \times 10^{-3}</td>
</tr>
<tr>
<td>( n = 6 )</td>
<td>0.187 \times 10^{-3}</td>
<td>( n = 6 )</td>
<td>0.187 \times 10^{-3}</td>
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<tr>
<td>( n = 7 )</td>
<td>0.672 \times 10^{-4}</td>
<td>( n = 7 )</td>
<td>0.672 \times 10^{-4}</td>
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<tr>
<td>( n = 8 )</td>
<td>0.229 \times 10^{-3}</td>
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<tr>
<td>( n = 9 )</td>
<td>0.778 \times 10^{-5}</td>
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<td>0.778 \times 10^{-5}</td>
</tr>
<tr>
<td>( n = 10 )</td>
<td>0.263 \times 10^{-5}</td>
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</tbody>
</table>
Fig. 1. Comparisons of approximation models Eqs. 11, 13, and 15 to exact model Eq. 14 in magnitudes and phases respectively.

Fig. 2. Comparisons of approximation models Eqs. 11, 13, and 15 to exact model Eq. 14 in magnitudes and phases respectively.

Fig. 3. Comparison of Trikha’s model and Kagawa’s model with Zielke’s weight function in time domain.

Fig. 4. Comparison of new approximation models with Zielke’s weight function in time domain.
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REFERENCES