I. INTRODUCTION

An achromat is a transport system that carries a beam without distorting its transverse phase space distribution. In this study, we apply the Lie algebraic technique [1-6] to a repetitive FODO array to make it either a second-order or a third-order achromat. (Achromats based on reflection symmetries [7,8] are not studied here.) We consider third-order achromats whose unit FODO cell layout is shown in Fig. 1. The second-order achromat layout is the same, except the octupoles are absent.

For the second-order achromats, correction terms (due to the finite bending of the dipole) to the well-known formulae for the sextupole strengths are derived. For the third-order achromats, analytic expressions for the fifth octupole strengths are given. The quadrupole, sextupole and octupole magnets are assumed to be thin-lens elements. The dipole are assumed to be sector magnets filling the drift spaces. More details of the analysis have been reported elsewhere.[9] We thank Y. Yan, H. Ye, J. Irwin and A. Dragt for their help.

II. ANALYSIS

We first calculate the Lie maps of each of the magnet elements. The map for a magnet element of length $L$ is given by $e^{-LH}$, where $H$ is the Hamiltonian of the element. For a particle with $\delta = \Delta P/P_0$, we use (we ignore the path-length dynamics)

thin quadrupole: $HL = \frac{1}{2F_k}(x^2-y^2)(1-\delta+\delta^2)$

thin sextupole: $HL = \frac{S_k}{3}(x^3-3xy^2)(1-\delta)$

thin octupole: $HL = \frac{O_k}{4}(x^4-6x^2y^2+y^4)$

sector dipole: $H = \frac{P^2+P_z^2}{2} + \frac{x^2}{2R^2} - \frac{x\delta}{R} + \frac{x(P_z+2P_y)}{2R}$

and obtained

thin quadrupole:

$$f_3 = \frac{1}{2F_k} (x^2-y^2)\delta$$

thin sextupole:

$$f_4 = \frac{1}{2F_k} (x^2-y^2)\delta$$

thin octupole:

$$f_3 = \frac{S_k}{3} (x^3-3xy^2)\delta$$

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sector dipole:

$$f_3 = -\frac{1}{6R^2}\sin^2 L R \left( \frac{x^3}{4R} - \frac{1}{4R} \sin L R \frac{2L}{R} x^2 P_x \right)$$

$$+ \frac{1}{4} \cos L R \sin L R \frac{2L}{R} x^2 P_x + \frac{R}{6} (1 - \cos^2 L R) P_z^3$$

$$f_4 = \frac{1}{2} \sin L R \frac{x^2 P_z + x^2 \delta}{2R} \sin L R \left( \cos L R + \sin^2 L R \right)$$

$$+ R \sin^2 L R \frac{x^2 P_z}{2R} \sin^2 L R \frac{x^2 L R}{2R} x^2 P_x \delta$$

$$\frac{R}{2} \sin^2 L R \frac{x^2 P_z}{2R} \sin^2 L R \frac{x^2 P_z}{2R}$$

$$- \frac{x^2}{2} \left( \sin^2 L R + \sin^2 L R \right) + \frac{R}{2} \left( \cos L R \sin^2 L R \right) \sin^2 L R \frac{P_z^2}{2R} \delta$$

$$+ \frac{1}{12} (-6L + R \sin^3 L R + 3R \sin \frac{2L}{R} \delta) \delta^3,$$
\[ M_{\text{cell}} = \prod_{i=1}^{N}(e^{i \phi_i}; e^{i \phi_i}; e^{i \psi_i}; e^{i \Phi_i}) = e^{i \phi_1 e^{i \phi_2} e^{i \phi_3} e^{i \Phi_4} e^{i \Phi_5} e^{i \Phi_6} e^{i \Phi_7} e^{i \Phi_8}}; \]

where

\[ R = e^{i \phi_2}; \quad h_3 = \sum_{i=1}^{N} \tilde{f}_3; \]

\[ h_4 = \sum_{i=1}^{N} \tilde{f}_4 + \frac{1}{2} \sum_{j=1}^{N} [\tilde{f}_3, \tilde{f}_3]. \]

In Eq. (5), \( f_i \) means \( f_i(X) = f_i(R_{N-i}X) \) with \( R_{N-i} \) the linear map from the last element to the \( i \)-th element. The map of the \( N \)-cell achromat is \( M = M_{\text{cell}}^N \). The number of cells \( N \) is so that (the total phase advances in \( x \) are \( y \)) are both multiples of \( 2\pi \), but avoid resonances.

We now make a canonical coordinate transformation from \( (x, P_x, y, P_y) \) to \( (\tilde{x}, P_x, \tilde{y}, P_y) \) by \( x = \sqrt{2} A_x \beta_x \sin \phi_x + \eta \delta \), \( P_x = \sqrt{2} A_x \beta_x (\cos \phi_x - \alpha_x \sin \phi_x) + \eta \delta \), and similarly for \( y \) and \( P_y \) without the \( \eta \) and \( \eta' \) terms, where \( \beta_{x,y}, \alpha_{x,y} \) and \( \eta, \eta' \) are the Courant-Snyder and the dispersion functions [10]. The linear map generator \( h_2 \) becomes \( h_2 = -\mu_x A_x - \mu_y A_y - \frac{1}{2} \alpha_x \beta_x \delta^2 \) where \( \alpha_x \) is the momentum compaction factor. We then decompose \( h_n \) in terms of the eigenmodes of \( h_2 \); as [5]

\[ h_n = \sum_{a+c+d+e=m} C_{abcd, e}^{m} (\alpha_0, e), \]

\[ |\alpha_0, e\rangle \equiv A_x^{a+b/b} A_y^{c+d/2} e^{i(a-b)\phi_x} e^{i(c-d)\phi_y} e^{i\phi_x} \tag{6} \]

To reduce a nonlinear map to its normal form, it can be shown [11] that (in the absence of resonances) all the non-secular terms can be transformed away via a symplectic similarity transformation leaving only terms with \( a = b \) and \( c = d \); i.e., terms depending on \( A_x, A_y \) and \( \delta \) only. In particular, we have

\[ h_3 = C_{1100,1} A_x \delta + C_{0011,1} A_y \delta + C_{0000,3} \delta^3, \]

\[ h_4 = C_{2200,0} A_x^2 + C_{0222,0} A_y^2 + C_{1111,0} A_x A_y + \]

\[ + C_{1100,2} A_x^2 \delta^2 + C_{0111,2} A_y^2 \delta^2 + C_{0000,4} \delta^4. \tag{7} \]

III. SECOND-ORDER ACHROMATS

For a second-order achromat, we follow Eqs. (6-7) and find the normal form of the unit cell is given by \( h_3 \) of Eq. (7) with

\[ C_{1100,1} = \sum_{k=1}^{\text{quads}} \left[ \frac{1}{2} \frac{1}{2} \lambda_k \eta(k) \right] \beta_x(k) + \omega_x, \]

\[ C_{0011,1} = - \sum_{k=1}^{\text{quads}} \left[ \frac{1}{2} \frac{1}{2} \lambda_k \eta(k) \right] \beta_y(k) + \omega_y, \tag{8} \]

and

\[ w_x = \sum_{k=1,2} \frac{1}{2} \sin^2 \left( \frac{L}{R} \right) \left\{ \frac{\beta_x(k)}{R} \left[ \sin \frac{L}{R} + \cot \frac{L}{R} \right] \right. \]

\[ - \frac{\eta(k)}{R} \sin \frac{L}{R} + \frac{\eta'(k) \cos \frac{L}{R}}{R} + 2 \alpha_x(k) \left[ 1 - \cos \frac{L}{R} \right] \]

\[ + \frac{\eta(k)}{R} \cos \frac{L}{R} + \eta'(s) \cdot \cos \frac{L}{R} \cot \frac{L}{R} \]

\[ + \frac{\eta_x(k)}{R} \left[ - \cos \frac{L}{R} \tan \frac{L}{2R} - \right] \left[ \eta(k) \cos \frac{L}{R} \cot \frac{L}{R} \right] \]

\[ + \frac{\eta'(k)}{R} \left[ 1 - \cos \frac{L}{R} \right]. \tag{9} \]

The lattice functions are evaluated at the two quadrupoles in Eq. (8) and at the ends of the two dipoles in Eq. (9). In the limit of weak bending with \( \epsilon_1 = \frac{L}{R} \ll 1 \), we have

\[ w_x \approx \epsilon_1 \sum_{s} \frac{1}{2} \alpha_x(s) \eta'(s) + \frac{1}{4} \frac{\gamma_x(s)}{s} (3 \lambda s' - 2 \eta(s)), \]

\[ w_y \approx \epsilon_1 \sum_{s} \frac{1}{2} \alpha_y(s) \eta'(s) - (3 \lambda s' - 2 \eta(s)). \tag{10} \]

To form a second-order achromat, we set the two \( C \)-coefficients to zero, and obtain

\[ S_1 = \frac{1}{2 \eta_1(1) F_{\frac{1}{2}}} + \frac{\beta_2(2) w_x + \beta_2(2) w_y}{\eta_1(1) [\beta_2(1) \beta_2(2) - \beta_2(2) \beta_2(1)]}, \]

\[ S_2 = \frac{1}{2 \eta_2(2) F_{\frac{1}{2}}} - \frac{\beta_1(1) w_x + \beta_1(1) w_y}{\eta_2(2) [\beta_1(1) \beta_2(2) - \beta_2(1) \beta_2(1)]}. \tag{11} \]

The first terms usually dominate and give the well known results. The correction terms with \( w_x \) and \( w_y \) are normally, but not always, small.

IV. THIRD-ORDER ACHROMATS

We also studied the case of a third-order achromat. An algebraic program using Mathematica was developed to do the analysis. Here, we only report our results. The normal form of the third-order generator for a unit cell is given by Eq. (9) with

\[ C_{2200,0} = - \frac{3}{8} \sum_{k=1}^{5} \beta_x(k)^2 O_k + w_{xx}, \]

\[ C_{1111,0} = \frac{3}{2} \sum_{k=1}^{5} \beta_x(k) \beta_y(k) O_k + w_{xy}, \]

\[ C_{0022,0} = - \frac{3}{8} \sum_{k=1}^{5} \beta_y(k)^2 O_k + w_{yy}, \]

\[ C_{1100,2} = \frac{3}{2} \sum_{k=1}^{5} \beta_x(k) \eta(k) O_k + w_{xd}, \]

\[ C_{0011,2} = \frac{3}{2} \sum_{k=1}^{5} \beta_y(k) \eta(k) O_k + w_{yd}, \tag{12} \]

and (when \( \epsilon_1 = \frac{L}{R} \ll 1 \))

\[ w_{xx} \approx \frac{3}{2} \mu_x \left[ 2 + 3 \cos \mu_x \left( \sum_{s} \frac{S_s}{4} \beta_x(s) \right)^2 - \frac{3}{16} \sum_{s} \gamma_x(s)^2 \right] \]

\[ - \frac{3}{16} \sum_{s} \gamma_x(s)^2 \]

\[ - \frac{3}{16} \sum_{s} \gamma_x(s)^2 \]
\[ w_{xy} \simeq -\frac{1}{4} \sum_{s} \gamma_x(s) \gamma_y(s) - \frac{1}{2} \cot \frac{\mu_z}{2} \sum_{s} S_p^2 \beta_x(s)^2 \beta_y(s) \]
\[ w_{yy} \simeq -\frac{L}{16} \sum_{s} \gamma_y(s)^2 + \frac{1}{16} \sum_{s} S_p^2 \beta_x(s) \beta_y(s)^2 - \frac{1}{8} \left[ 4 \cot \frac{\mu_z}{2} + \sin \mu_z \csc \left( \frac{\mu_z}{2} + \mu_y \right) \csc \left( \frac{\mu_z}{2} - \mu_y \right) \right] \]
\[ w_{yd} \simeq \frac{3L}{4} \sum_{s} \gamma_y(s) \eta'(s)^2 - \sum_{s} \beta_x(s) \left( \frac{1}{F_p} - S_p \beta_x(s) \right) \]
\[ w_{yd} \simeq \frac{1}{2} \cot \mu_y \sum_{s} \left[ \beta_y(s) \left( \frac{1}{F_p} - S_p \beta_x(s) \right) \right]^2 \]
\[ w_{yd} \simeq -\frac{L}{4} \sum_{s} \gamma_y(s) \eta'(s)^2 + \sum_{s} \beta_y(s) \left( \frac{1}{F_p} - S_p \beta_x(s) \right) \]
\[ w_{yd} \simeq -\frac{1}{2} \cot \mu_y \sum_{s} \left[ \beta_y(s) \left( \frac{1}{F_p} - S_p \beta_x(s) \right) \right]^2 \]

Figure 1. Unit cell of an achromat layout.

The required octupole strengths are such that the five C-coefficients in Eq. (12) are equal to zero. For the case when two of the octupoles are located next to the two sextupoles and the other three are at the \( \frac{1}{3} \), \( \frac{2}{3} \), and \( \frac{1}{2} \) locations of the two bending magnets, we find

\[ a = 2f(1360 - 22846 f^2 - 74476 f^4 + 695809 f^6 - 1438146 f^8 + 1200096 f^{10} - 326592 f^{12}) \]
\[ b = -352 - 3360 f^2 + 233290 f^4 - 1070910 f^6 + 1917603 f^8 - 1364850 f^{10} + 361584 f^{12} \]
\[ c = 6f(-42 + 1076 f^2 + 16306 f^4 - 14368 f^6 + 4032 f^{10}) \]
\[ d = 8 - 394 f^2 + 5322 f^4 + 16907 f^6 + 14866 f^8 - 4464 f^{10} \]
\[ e = -368 + 10536 f^2 - 92342 f^4 + 307222 f^6 - 470547 f^8 + 330642 f^{10} - 81648 f^{12} \]
\[ D = (4f^2 - 1)^3(3f^2 - 4)(10 - 173 f^2 - 261 f^4 + 324 f^6)L^3 \]

We have defined the dimensionless parameter \( f = \frac{2F_1}{L} \) and have assumed that \( \epsilon_1 = \frac{1}{L} \ll 1 \) and \( \left| \frac{F_1 + F_2}{L} \right| \ll 1 \).

References

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