SPIN TRACKING IN RHIC (CODE SPINK)*

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ABSTRACT

The evolution of the spin during acceleration in the Relativistic Heavy Ion Collider (RHIC) has been studied with a new numerical code, Spink. Results of spin tracking through resonances are shown.

1. Introduction

In the acceleration of polarized protons in a synchrotron, many depolarizing resonances are encountered. 1 2, of two species: intrinsic resonances that depend on the lattice structure of the ring and arise from the coupling of betatron oscillations with horizontal magnetic fields, and imperfection resonances caused by orbit distortions.

The spectrum of resonances vs. spin tune $G\gamma$ ($G = 1.7928$, proton gyromagnetic anomaly, and $\gamma$, proton relativistic energy ratio) for a given betatron tune $\nu$, or vs. $\nu$ for a given $G\gamma$, contains many lines with various resonance strengths. The polarization crossing the resonances can be studied by numerical tracking of protons with spin in a model accelerator. Tracking will allow one to check the strength of the resonances, to study the effect of devices like Siberian snakes 3, to find safe lattice tune regions, and finally to study the operation of special devices like spin flippers. 4 5

A few computer codes exist that calculate resonance strengths $\epsilon_k$ and perform tracking. Most relevant to our work are two programs: Depol, originally written by E.D. Courant 6 that calculates the $\epsilon_k$'s by Fourier analysis, and Snake, written by J. Buon, and modified by Courant, S.Y. Lee and others, that does the tracking on a synthetic machine. Depol reads the output of an optical code, such as Synch or Mad, containing all the details of the lattice. The machine for Snake contains a number of periods, FODO cells, Siberian snakes, etc.

Since the complexities of machines like RHIC could not be adequately represented by Snake, we decided to write a new code, Spink, that reads the output of Mad like Depol and tracks like Snake. In addition, Spink does many more things (and it is still growing). The code and examples for RHIC are described in the following.

2. Spink Formalism

Spink tracks a number of protons randomly generated in phase space, through the machine lattice. A proton is characterized by four transverse coordinates, $x$, $x'$, $y$, $y'$.
y', by two longitudinal coordinates \( dp = p - p_s \), \( d\phi \), and by three spin coordinates, \( S_x, S_y, S_z \) (where \( S_x^2 + S_y^2 + S_z^2 = 1 \)). Matrices are used to transform orbit and spin coordinates. Orbit matrices are built from the output of Mad. We only retain the relevant (“active”) elements for orbit and spin tracking. Everything else in the lattice is lumped in a drift space. Typically, for RHIC the number of Spin matrices is 981, each active element being surrounded by two drifts, keyword (D) in the code. At present, active elements are: bends (B), quadrupoles (Q), snakes (S), RF cavities (R), spin flippers, (F).

The spin is transformed by rotation, using matrices. In a bend, the rotation is around a vertical \( y \)-axis, in a quad, around a radial axis, in a snake, around an axis
of orientation gives as input, and in a spin flipper around a horizontal rotating or oscillating axis.

Tracking, say, 25 particles for a number of revolutions of the order of 100,000, through a thousand matrices takes a computer time of the order of a few hours for a typical fast workstation. A “reasonable” turnaround time is obtained with some shortcuts. The results presented in this paper were obtained with the following limitations: (i) only the vertical motion was considered, (ii) the longitudinal synchrotron motion was decoupled from the other degrees of freedom, (iii) there was no attempt to consider in detail the spin precession within each element (i.e. for the purpose of representing spin motion, the machine elements are thin).

3. Orbit

With these limitations, the orbit matrices are \( 2 \times 2 \) and act only on \( y, y' \). They are computed from the Twiss file using twiss function values at the entrance and exit of each element, as follows

\[
T \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_1} (\cos \Delta \phi + \alpha_1 \sin \Delta \phi) & \sqrt{\beta_2 \beta_1} \sin \Delta \phi \\ \frac{\Delta p - 1}{\Delta p} & \sqrt{\beta_2} (\cos \Delta \phi - \alpha_2 \sin \Delta \phi) \end{pmatrix}
\]

with \( \Delta \phi = \phi_2 - \phi_1 \).

Orbit matrices are unitary, except than in a RF (thin) cavity, where the D element in the matrix is decreased by a quantity proportional to the momentum gain per turn \( dp_n \), corresponding to an instantaneous change of vertical orbit angle (together with the change of momentum)

\[
D = 1 - \frac{dp_n}{\beta \gamma}, \quad dp_n = V_{RF} \sin \phi, \quad G \gamma = G \sqrt{1 + (\beta \gamma)^2}.
\]

Since an important study is to track polarized protons through resonances at constant energy (storage mode) as a function of lattice tune, we needed a means to vary the tune. It would be very cumbersome to run Mad and read its output for every tune in the range. Besides, one might want to study acceleration with a programmed
tune change. To accomplish this, two thin lenses of strength $\delta$ are added to each quadrupole in the machine, immediately up- and down-stream. The new quadrupole matrix and the quadrupole gradient become (see Appendix A)

\[
\begin{pmatrix}
A + B\delta & B \\
C + 2A\delta + B\delta^2 & A + B\delta
\end{pmatrix}, \quad \begin{cases}
K_1 := K_1 - \frac{2A}{B}\delta - \delta^2 \quad \text{(focusing quad)} \\
K_1 := K_1 + \frac{2A}{B}\delta + \delta^2 \quad \text{(defocusing quad)}
\end{cases}
\] (3)

4. Spin

Spin rotation is represented by the equation

\[
\begin{pmatrix}
S_x \\
S_y \\
S_z
\end{pmatrix} := R \begin{pmatrix}
S_x \\
S_y \\
S_z
\end{pmatrix}.
\] (4)

In a bend, the spin vector precesses around the y-axis by an angle $\psi$ proportional to the bend angle $\theta$: $\psi = G\gamma\theta$, with spin matrix

\[
R_B = \begin{pmatrix}
\cos \psi & 0 & -\sin \psi \\
0 & 1 & 0 \\
\sin \psi & 0 & \cos \psi
\end{pmatrix}.
\] (5)

In a quadrupole, the spin vector precesses around a radial axis. The angle of precession is proportional to the quadrupole integrated gradient $K_1L$ and to the orbit (vertical) displacement, $y_\beta$ due to the betatron motion, and $y_{er}$ due to lattice errors: $\psi = -K_1L(1 + G\gamma)(y_\beta + y_{er})$.

The spin rotation matrix for a quadrupole is

\[
R_Q = \begin{pmatrix}
0 & 1 & 0 \\
-\cos \psi & 0 & \sin \psi \\
\sin \psi & 0 & \cos \psi
\end{pmatrix}.
\] (6)

In a Siberian snake, two angles are given as input to the program: the angle $\phi$ of the precession axis with the longitudinal $z$-axis (the precession axis is in the horizontal plane) and the snake spin rotation $\psi$. The spin rotation matrix for a snake is (see Appendix B)

\[
R_S = \begin{pmatrix}
1 - \cos^2 \phi(1 - \cos \psi) & \cos \phi \sin \psi & \sin \phi \cos \phi(1 - \cos \psi) \\
-\cos \phi \sin \psi & 1 - \sin^2 \phi(1 - \cos \psi) & \sin \phi \sin \psi \\
\sin \phi \cos \phi(1 - \cos \psi) & -\sin \phi \sin \psi & 1 - \sin^2 \phi(1 - \cos \psi)
\end{pmatrix}.
\] (7)

In an RF spin flipper, the spin precesses around an axis that rotates (or oscillates) in a horizontal plane with a frequency $\omega$. This frequency varies. When $\omega$ equals half of the revolution frequency of the protons in the machine (with two snakes on), the
spin flips. There are variations of this scheme that will be described later. The spin flipping angle is proportional to the strength of the SF integrated field

$$\psi = \frac{\int B \, dl}{B} \gamma.$$  

(8)

The spin rotation matrix for a rotating field spin flipper is

$$R_F = \begin{pmatrix}
1 - 2 \sin^2 \frac{1}{2} \psi \sin^2 \omega t & \sin \psi \sin \omega t & \sin^2 \frac{1}{2} \psi \sin 2\omega t \\
- \sin \psi \sin \omega t & 1 - 2 \sin^2 \frac{1}{2} \psi & \sin \psi \cos \omega t \\
\sin^2 \frac{1}{2} \psi \sin 2\omega t & - \sin \psi \cos \omega t & 1 - 2 \cos^2 \frac{1}{2} \psi \cos^2 \omega t
\end{pmatrix}. \quad (9)$$

5. Froissart-Stora

Spink can be used to check the strength of resonances calculated by Depol using the Froissart-Stora formula. 7 Conditions are that the resonances being studied are reasonably separated (say, ten times their width) and are not too strong. The first condition insures that the vertical polarization, at the beginning of the run, is stable around the value of one. The second should be met in order that the Froissart-Stora formula would not saturate.

Assume that after crossing an isolated resonance the polarization is reduced to some final value $< \rho >$. The resonance strength is then calculated with

$$\epsilon_k = \sqrt{-4 \alpha \ln \frac{1+< \rho >}{2}}, \quad \alpha = \frac{\Delta \gamma_n}{2\pi}, \quad (10)$$

with $\alpha$ the rate of resonance crossing ($\Delta \gamma_n$ is the change of $\gamma$ per turn). Since after the crossing the polarization performs some oscillations, $< \rho >$ is calculated as the average over a few turns, starting after a value of $G \gamma$ given as input.

6. Example. Strength of an isolated intrinsic resonance

RHIC contains two full Siberian snakes with axis at $\pm 45^\circ$ respectively. $G \gamma$ ranges from 45 to 500. 8 Intrinsic resonances are located where $G \gamma = \text{Integer} \pm \nu$, $\nu$ being the (vertical) tune of the lattice. Integeres that are multiple of some characteristic periodicity of the lattice are particularly important. Fig. 1 shows the spectrum of intrinsic resonances for RHIC, calculated with Depol for a particle on the contour of a vertical emittance of $10\pi$ mm-mrad for a tune $\nu = 29.18$.

One of the strongest resonance is at

$G \gamma = 381.82 = 5 \times 81 - (\nu - 6)$, with strength $\epsilon_k = 0.41288$.

With Spink, one particle was tracked through this resonance. The Froissart-Stora formula was used to calculate the strength and compare with Depol. In the run, the emittance was reduced by a factor of 100, since the strength of the resonance would
have saturated the formula. With this reduced emittance, a predicted value for the strength is 10 times smaller. A series of acceleration rates were used. The results are shown in Fig. 2 and in Table 1

<table>
<thead>
<tr>
<th>dp [(GeV/c)/turn]</th>
<th>(\langle P \rangle) in Spin&amp;k</th>
<th>(\epsilon_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>-0.865</td>
<td>0.0395</td>
</tr>
<tr>
<td>0.04</td>
<td>-0.752</td>
<td>0.0400</td>
</tr>
<tr>
<td>0.05</td>
<td>-0.752</td>
<td>0.0408</td>
</tr>
</tbody>
</table>

We consider the agreement with Depol very good, also recalling that in this example only one test particle was used.

Tracking through the resonance at \(G\gamma = 260.18\) with both snakes on, as shown in Fig. 3, and snakes off, Fig. 4 and emittance at the (design) value of 10\(\pi\) mm-mrad shows how snakes are an effective means to avoid depolarization.


For experiments with polarized protons in RHIC, the spin must be reversed periodically in the machine. This can be obtained with a spin flipper. In an idealization of this device, a magnetic field is established in some position around the lattice. The field rotates in the horizontal plane, or simply oscillates along the radial or longitudinal direction. The frequency of rotation of the field is varied. With two Siberian snakes active (with axis at ±45\(^\circ\)), when the frequency of the SF field becomes equal
Fig. 2. Spin tracking through a resonance. Froissart-Stora formula is used to calculate the strength.

Fig. 3. Spin tracking through a resonance. Snakes on. $\epsilon = 10\pi$ mm-mrad.

Fig. 4. Spin tracking through the same resonance(s) of 3. Snakes off.
to half the frequency of revolution of the protons in the ring, and the field is strong enough, the polarization is inverted.

We have simulated with Spink a rotating spin flipper by tracking at constant proton energy (storage mode) $G\gamma = 378$, for various SF field strengths and frequency sweep rates. At that energy, the frequency of revolution in RHIC is 78.25042 KHz. The SF frequency range was varied between 38 and 41 KHz. The results are shown in Fig. 5. A good flipping was found at the correct frequency of 39.125 KHz with an integrated field of 0.006 Tesla-m and a sweep rate of 0.05 Hz/turn.

An oscillating field spin flipper will produce the desired results when the snakes precession axis angles are detuned from their standard value at $\pm 45^0$. The required field of a transverse spin flipper is about $\sqrt{2}$ times larger than in a rotating field SF. However, an oscillating magnetic field is much easier to build than a rotating field. We studied with Spink a transverse field SF with the precession axis angles detuned by $\pm 50^0$. The results are given in Fig. 6, that shows how a transverse field SF flips the polarization back and forth at two symmetrically placed frequencies.

8. Appendix A. Modified quadrupole.  

Add two thin quadrupoles at both ends of a regular quadrupole

$$M = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} = \begin{pmatrix} A + B\delta & B \\ C + 2A\delta + B\delta^2 & A + B\delta \end{pmatrix}. \tag{11}$$

For a focusing quad it is

$$A = \cos \phi, \quad B = \frac{1}{\sqrt{k_0}}\sin \phi, \quad C = -\sqrt{k_0}\sin \phi. \tag{12}$$
From the identities

\[
\begin{align*}
A + B\delta &= \cos\phi_0 + \frac{\delta}{\sqrt{k_0}}\sin\phi_0 = \cos \phi_0 \\
B &= \frac{1}{\sqrt{k_0}}\sin\phi_0 = \frac{1}{\sqrt{k_0}}\sin\phi \Rightarrow k = k_0 \frac{\sin^2 \phi}{\sin \phi_0} \\
C + 2A\delta + B\delta^2 &= \frac{k_0 - \delta^2}{\sqrt{k_0}} + 2\delta \cos \phi_0 = -\sqrt{k} \sin \phi
\end{align*}
\] (13)

it is obtained

\[
\begin{align*}
k &= k_0 - 2\frac{A}{B}\delta - \delta^2 & \text{focusing quad} \\
k &= k_0 + 2\frac{A}{B}\delta + \delta^2 & \text{defocusing quad}
\end{align*}
\] (14)

9. Appendix B. Spin rotation around a given axis.

\(X = (x, y, z)\) are the Lab coordinates, \(U = (u, v, w)\) are the proton coordinates. \(R\) is a rotation of \(U\) with respect to \(X\).

\(S_u\) are the spin coordinates in \(U\). Assume that in \(U\) the spin precesses around the axis \(\hat{u}\) by an angle \(\theta\)

\[
S'_u = P S_u, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}, \quad \begin{bmatrix} c = \cos \theta \\ s = \sin \theta \end{bmatrix} \] (15)

If \(S_x\) are the spin coordinates in the Lab, it is: \(S'_x = R^{-1} P R S_x\), with

\[
R = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{pmatrix} \] (16)
and $\lambda, \mu, \nu$ direction cosines of $\hat{x}, \hat{y}, \hat{z}$ in $U$. We obtain

$$R^{-1}PR = \begin{pmatrix}
\lambda_1^2 + A_{11c} & \lambda_1 \lambda_2 + A_{12c} + B_{12s} & \lambda_1 \lambda_3 + A_{13c} + B_{13s} \\
\lambda_2 \lambda_1 + A_{21c} - B_{12s} & \lambda_2^2 + A_{22c} & \lambda_2 \lambda_3 + A_{23c} + B_{23s} \\
\lambda_3 \lambda_1 + A_{31c} - B_{13s} & \lambda_3 \lambda_2 + A_{32c} - B_{23s} & \lambda_3^2 + A_{33c}
\end{pmatrix}$$

(17)

with

$$A_{ij} = \mu_i \mu_j + \nu_i \nu_j$$

$$B_{ij} = \mu_i \nu_j - \nu_i \mu_j$$

(18)

All spin rotation matrices in Sec. 2 are a special case of the preceding equations.

We obtain

- A spin rotation around $\hat{y}$ (bend) with $u = y$ and $v = x$ (arbitrary).
- Around $\hat{x}$ (quadrupole) is obtained by putting $u = x$.
- Around $\hat{z}$, with $v = y$ (arbitrary).
- Around an axis in the $xz$ plane (snake and spin flipper), with $w = y$.

10. Acknowledgments

Discussions with Thomas Roser and Ernest Courant have been very important

11. References

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