CONF-960927-1

Title:

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MAR 13 1996

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Submitted to:

Fifth IEEE International Conference on Fuzzy Systems
New Orleans, Louisiana
September 8-11, 1996

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Effect of Noise on Chaotic Fuzzy Mappings

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Abstract— Chaotic mappings in the space of fuzzy sets induced by mappings of the underlying reference set are investigated. Different fuzzification schemes are considered and their impact on the resultant iterated fuzzy set, under a quadratic mapping, is studied numerically. The fuzzy set mapping is described in terms of the mapping of level cuts, resulting from the resolution theorem for fuzzy sets. In the two-dimensional case, a generalized notion, given as a fuzzy set, of the Hausdorff dimension is formulated. An example, based on the Henon Mapping, is provided.

I. INTRODUCTION

To a large extent, chaos in fuzzy dynamical systems has been investigated by Kloeden,¹ Diamond,² and Diamond and co-workers.³ The mathematical definitions of chaos rely on positive topological entropy, sensitive dependence on initial conditions, and positive Liapunov exponents. When transcribed to dynamical fuzzy systems, the definition of chaos invokes the notion of topological entropy; roughly speaking, a system is chaotic if the trajectories are mixing.

In the simplest formulation, based on the classic extension principle,⁴ a mapping $f: \Omega \rightarrow \Omega$ induces a fuzzification $\bar{f}: D(\Omega) \rightarrow D(\Omega)$, defined on the space $D(\Omega)$ of fuzzy sets on a set Ω , as

$$(\bar{f}u)(y) = \sup_{x \in f^{-1}(y)} \{u(x)\} \quad (1)$$

In general, however, different fuzzification schemes based on the notion of s - and t -norms are possible. Whereas the classic extension principle leads to a mapping of each level set to a level set with the same level value, the level values as well are mapped by the generalized extension principle.

The goal of this paper is twofold. First, we study the chaotic behavior of a simple quadratic one-dimensional map corresponding to different fuzzification schemes. Second, we turn to two-dimensional chaotic systems, exemplified by Henon's attractor, with the attempt to introduce a measure of chaos, analogous to the Hausdorff dimension that characterizes strange attractors. If we preserve the notion of level sets, we arrive at a novel description of the Hausdorff dimension, given in terms of a fuzzy set. The membership function of this fuzzy set is a constant when the chaotic system is perturbed by an additive noise; on the other hand, in the absence of noise, the fuzzy set provides a quantitative measure allowing one to distinguish chaotic from noisy mappings.

II. CHAOS UNDER DIFFERENT FUZZIFICATION SCHEMES

To discuss different fuzzification schemes, we recall the definition of level sets associated with the fuzzy set u , viewed as a mapping $u: \Omega \rightarrow [0, 1]$, which is required to be upper semicontinuous. For $0 \leq \alpha \leq 1$, the level set is a crisp set $[u]^\alpha = \{x \in \Omega: u(x) \geq \alpha\}$, where the level set $[u]^0$ can be identified with the support of u .

By virtue of the resolution (decomposition) theorem, any fuzzy set, $X = \sum u(x)/x$, can be represented as

$$X = \bigcup_{\alpha \in [0,1]} \alpha [u]^\alpha. \quad (2)$$

In Eq. (2), $[u]^\alpha$ is interpreted as a fuzzy set with a membership function whose value is unity. As the standard usage does not reserve a special term for the summand in Eq. (2), we refer to $\alpha [u]^\alpha$ as an α -cut. The resolution theorem reduces the transformations of fuzzy sets to interval arithmetic.⁵

The t -norms generalize the minimum operation, whereas the s -norms generalize the maximum. Thus, given x and y , a t -norm T is a binary operation $T(x, y)$, which is commutative and satisfies $T(x, y) \leq T(x', y')$ for all $x \leq x'$ and $y \leq y'$; furthermore, for all x , $T(x, 1) = x$. The s -norm can be regarded as a dual to t -norm; indeed, to each t -norm T , one can associate an s -norm $S(x, y) = 1 - T(1-x, 1-y)$. The monograph of Pedrycz contains a useful table listing various t - and s -norms.⁶ The fuzzification scheme defined by Eq. (1) implies that the level set of the transformed mapping is given as the transform of the level set; that is,

$$[\bar{f}u]^\alpha = f([u]^\alpha) \quad (3)$$

If Γ is any t - or s -norm, the diagonal Δ of Γ is defined by

$$\Delta_\Gamma(x) = \Gamma(x, x). \quad (4)$$

In terms of the diagonal Δ , a Γ -fuzzification is defined as

$$(\bar{f}_\Gamma u)(y) = \sup_{x \in f^{-1}(y)} \{\Delta_\Gamma(u(x))\}, \quad (5)$$

and an α -level set with respect to G is

$$[u]_\Gamma^\alpha = \{x \in \Omega : \Delta_\Gamma(u(x)) \geq \alpha\}, \quad (6)$$

which replaces the former definition. As has been shown in Ref. 2, Eq. (3) holds for a Γ -fuzzification provided that for every element y , of the support of f , $\sup_{\xi \in f^{-1}(y)} \{\Delta_\Gamma(u(\xi))\}$ is attained.

The consequence of the Γ -fuzzification is that the parameter α that defines the level sets is also transformed. For example, for the t -norm $T(x, y) = \max(x + y - 1, 0)$, the iterated values of the parameter α are $\alpha_n = 1 - (1 - \alpha)/2^n$. In this example, the fuzzy set is known in terms of the α -cuts approaching the vicinity of unity.³

To show the impact of different fuzzification schemes on fuzzy chaos, we provide numerical examples of the transformation of an initial triangular fuzzy number, induced by the quadratic mapping

$$y = 3.9x(1 - x) \quad (7)$$

of the unit interval $[0, 1]$.

When iterating fuzzy sets, two conditions are imposed on the level sets. First, let α^l and α^r denote the left and right bounds of the level set. If $f(\alpha^l) > f(\alpha^r)$, we interchange $f(\alpha^l)$ and $f(\alpha^r)$. Second, to ensure that the resulting fuzzy set is convex, we impose a condition that, for $\alpha_1 < \alpha_2$, the bounds satisfy $\alpha_1^l \leq \alpha_2^l$ and $\alpha_1^r \geq \alpha_2^r$.

In the numerical experiments described here we always start with a triangular fuzzy set centered around 0.25 with the base width (support) of 0.1. As we iterate the set, a convenient way to describe the loss of information is in terms of the nonspecificity measure,⁷ defined as

$$U = \prod_{i=1}^n (\alpha_i - \alpha_{i+1}) \log \left| \frac{\alpha_i}{[u]_i} \right|, \quad (8)$$

where, for the sake of simplicity, we do not indicate the Γ -dependence. The nonspecificity measure, termed also U -uncertainty, will evolve in a manner dependent on the fuzzification scheme. The U -uncertainty of the initial triangular fuzzy set is -4.734. In the presence of white noise, the boundary values of the level sets are perturbed additively with random numbers from the interval $[0, 1]$, scaled by a factor of 0.05.

A. Standard Fuzzification

As the first example, we consider the standard fuzzification scheme, in which $\Delta(u) = u$. The form of the initial fuzzy set after 5 and 1000 iterations is shown in Fig. 1.

FIGURE 1. Triangular fuzzy set after 5 (left) and 1000 (right) iterations induced by the quadratic mapping.

When the additive random noise is applied, the results of successive application of quadratic mapping change, as shown in Fig. 2, for 5 and 1000 iterations.

FIGURE 2. The results of iterative quadratic mapping in the presence of white noise after 5 (left) and 1000 (right) iterations.

In the absence of noise, the U -uncertainty increases from -1.46 to -0.44, as we iterate the initial fuzzy set 5 and 1000 times, respectively. When noise is present, after five iterations, the corresponding U -uncertainty is -0.42; the U -uncertainty increases to -0.15 after 1000 iterations. These numbers indicate that, as is well known, some measures do not distinguish between chaos and noise. In this case, the U -uncertainty measure does not distinguish between many iterations without additive noise and a few iterations in the presence of noise.

B. Fuzzification Induced by T -norm $xy / (x + y - xy)$

The diagonal, corresponding to the norm $t(x,y) = xy / (x + y - xy)$, is $D(u) = u^2 / (2u - u^2)$. As the number of iterations increases, the α -levels accumulate around the value of 1.0. Figure 3 shows the results of iterations for this fuzzification corresponding to 5 and 10 iterations.

FIGURE 3. Transform of the triangular fuzzy set after 5 (left) and 10 (right) iterations. Fuzzification induced by the t -norm $xy / (x + y - xy)$.

In this example, the U -uncertainty increases from -0.44 to -0.19 as the number of iterations increases from 5 to 10. We notice that the fuzzification produces the α -cuts accumulating near 1.0 from below.

C. Fuzzification Induced by S -norm $x + y - xy$

In contrast to the t -norm $xy / (x + y - xy)$, the iterations induced by s -norm $x + y - xy$ have the α -cuts focused in the vicinity of the zero level. Figure 4 illustrates the situation for four and six iterations.

FIGURE 4. Transform of the triangular fuzzy set after four (left) and eight (right) iterations. Fuzzification induced by the s -norm $x + y - xy$.

The U -uncertainty increases from -1.44 to -0.27 as the number of iterations increases from 4 to 8.