10/30-9695①

LBL-38333
UC-419

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

Fringe Fields for the N Channel Permanent Magnet Array

E.P. Lee

April 1996

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED *kg*

MASTER

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor The Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or The Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof, or The Regents of the University of California.

Available to DOE and DOE Contractors
from the Office of Scientific and Technical Information
P.O. Box 62, Oak Ridge, TN 37831
Prices available from (615) 576-8401

Available to the public from the
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, VA 22161

Ernest Orlando Lawrence Berkeley National Laboratory
is an equal opportunity employer.

LBL-38333
UC-419
HIFAN 801

Fringe Fields for the N Channel Permanent Magnet Array

Edward P. Lee

Accelerator and Fusion Research Division
Ernest Orlando Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

April 1996

This work was supported by the Director, Office of Energy Research, Office of Fusion Energy, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Fringe Fields for the N Channel Permanent Magnet Array

Edward Lee
April, 1996

Analytical expressions are obtained for fringe field multipoles of an N channel permanent magnet quadrupole array [1]. See figure 1 below. It is assumed that the system of magnetic wedges starts at some transverse (x, y) plane located at $z=0$, and it continues to a magnet length $z=\ell$, where it stops. The iron yoke continues to $z = \pm \infty$, but it will be shown that only a small overhang is actually required to maintain the quadrupole and translational symmetries. Recall the 2-d solution for the magnetic potential $\left(\vec{H} = \nabla \phi \right)$:

$$\phi_2 = A \left[(x - x_i)^2 - (y - y_i)^2 \right],$$

where $A = -M_0/4b$, M_0 is the remnant field of the wedges, and (x_i, y_i) are the coordinates for the center of box (i). Boxes have dimensions $2b \times 2b$ and alternate between vacuum fill (for beams) and magnetic wedge fill. The 2-d system looks like a portion of an infinite transverse lattice with periodicity length $= 4b$ in both the x and y directions. For the magnetic potential ϕ , the periodicity length is $2b$.

3-d Magnetic potential for a semi-infinite system

More generally we have a 3-d situation where the potential ϕ is a function of z as well as (x, y) . It still satisfies

$$\nabla^2 \phi = - \nabla \cdot \underline{\underline{M}} ,$$

$$\nabla^2 = \nabla_{\perp}^2 + \frac{\partial^2}{\partial z^2} ,$$

where $\underline{\underline{M}} = \underline{\underline{B}} - \underline{\underline{H}}$ is the local density of magnetization.

For now we take $\underline{\underline{M}}$, which is still transverse, to turn on at $z=0$ and continue to $z=+\infty$.

Hence

$$\nabla \cdot \underline{\underline{M}} = \Theta(z) \nabla_{\perp} \cdot \underline{\underline{M}}_{\perp} .$$

Where $\Theta(z)$ is the unit step function and $\nabla_{\perp} \cdot \underline{\underline{M}}_{\perp}$ is the source of the 2-d potential (ϕ_2) given above. The truncated 2-d potential $\Theta(z) \phi_2$ is clearly not a valid 3-d solution.

However, we can add homogeneous 3-d potentials in the zones $z < 0$ and $z > 0$ to obtain the desired 3-d solution. Generally the homogeneous 3-d solutions consist of a sum of terms of the form

$$\left[\cos\left(\frac{n\pi x}{b}\right) \right] \cdot \left[\cos\left(\frac{m\pi y}{b}\right) \right] \cdot e^{\pm \sqrt{n^2 + m^2} \frac{\pi z}{b}} .$$

These terms preserve the periodicity in x and y exhibited by the 2-d solution. In fact, since ϕ_2 is a pure quadrupole potential in each $2b \times 2b$ box, we only use homogeneous combinations of the form

$$\left[\cos\left(\frac{n\pi x}{b}\right) - \cos\left(\frac{n\pi y}{b}\right) \right] e^{\pm \frac{n\pi z}{b}}.$$

Since M is transverse, $\partial\phi/\partial z$ must be continuous at $z = 0$, and we obtain

$$\underline{z < 0}: \quad \phi = - \sum_1^{\infty} F_n \left[\cos\left(\frac{n\pi x}{b}\right) - \cos\left(\frac{n\pi y}{b}\right) \right] e^{\frac{n\pi z}{b}},$$

$$\underline{z > 0}: \quad \phi = \phi_2 + \sum_1^{\infty} F_n \left[\cos\left(\frac{n\pi x}{b}\right) - \cos\left(\frac{n\pi y}{b}\right) \right] e^{-\frac{n\pi z}{b}}.$$

Here $\{F_n\}$ is a set of coefficients to be determined (below).

We must also have ϕ continuous at $z = 0$; this gives

$$\sum_1^{\infty} F_n \left[\cos\left(\frac{n\pi x}{b}\right) - \cos\left(\frac{n\pi y}{b}\right) \right] = -\frac{1}{2} \phi_2.$$

We may restrict attention to a single $2b \times 2b$ box centered at $x=y=0$; substituting for ϕ_2 we get (inside the box)

$$\sum_1^{\infty} F_n \left(\cos \frac{n\pi x}{b} - \cos \frac{n\pi y}{b} \right) = -\frac{A}{2} (x^2 - y^2).$$

To evaluate F_n we set $y = 0$; an elementary Fourier series on the interval

$-b < x < +b$ remains:

$$\sum_1^{\infty} F_n \left(\cos \frac{n\pi x}{b} - 1 \right) = -\frac{Ax^2}{2}.$$

It follows that

$$\sum_1^{\infty} F_n = \frac{Ab^2}{6},$$

and the individual coefficients are

$$F_n = \frac{1}{b} \int_{-b}^{+b} dx \left(-\frac{Ax^2}{2} \right) \cos \left(\frac{n\pi x}{b} \right) = \frac{-2b^2 A}{n^2 \pi^2} \cos(n\pi) = \frac{2b^2 A}{\pi^2} \cdot \left\{ \frac{1}{1^2}, \frac{-1}{2^2}, \frac{1}{3^2}, \frac{-1}{4^2}, \dots \right\}.$$

Evaluation of the quadrupole amplitude

Note that

$$\left(\cos \frac{n\pi x}{b} - \cos \frac{n\pi y}{b} \right) = \frac{-\pi^2}{2b^2} n^2 (x^2 - y^2) + \frac{\pi^4}{24b^4} n^4 (x^4 - y^4) - \frac{\pi^6}{720b^6} n^6 (x^6 - y^6) + \dots$$

So in the box centered at $x = y = 0$, the potential has the simple form

$$\phi = f(z) (x^2 - y^2) + g(z) (x^4 - y^4) + h(z) (x^6 - y^6) + \dots$$

This is true for $z > 0$ as well as $z < 0$ since $\phi_2 \sim \Theta(z) (x^2 - y^2)$ and can be absorbed into $f(z)$. For other boxes $(x-x_i)$ and $(y-y_i)$ must be inserted in this expression in place of x and y .

To obtain the quadrupole amplitude $f(z)$ we gather terms of ϕ proportional to $(x^2 - y^2)$; for $z < 0$

$$f(z) = \sum_1^{\infty} F_n \left(\frac{\pi^2 n^2}{2b^2} \right) e^{\frac{n\pi z}{b}} = \frac{A}{1 + e^{-\pi z/b}} = \frac{A}{2} \left(1 + \tanh \frac{\pi z}{2b} \right).$$

The same formula is found for $z > 0$ by direct summation (or by invoking analyticity).

To obtain the field for a magnet of finite length ℓ we add a displaced semi-infinite solution with reverse M to get

$$f = \frac{A}{2} \left[\tanh \frac{\pi z}{2b} - \tanh \frac{\pi(z-\ell)}{2b} \right].$$

Note that for the semi-infinite magnet the quadrupole amplitude has a “Fermi-Dirac” distribution form with amplitude $A/2$ at $z = 0$. We expect a profile of this general shape from elementary considerations. Outside the magnet ϕ falls off very rapidly with z , so it should actually be sufficient to continue the iron yoke to only about $\Delta z \approx 2b$ beyond the magnet ends; note the very low value of the quadrupole amplitude at this distance from the magnet:

$$f(z = -2b) \approx (.00186) A.$$

Other fringe field components

So far we have evaluated the lowest order quadrupole term:

$$\phi_{\text{quad}} = f(z) (x^2 - y^2) = f r^2 \cos 2\theta,$$

with the expressions for $f(z)$ derived in the previous section. For other fringe field features we will treat only the semi-infinite magnet, so

$$f(z) = \frac{A}{2} \left(1 + \tanh \frac{\pi z}{2b} \right).$$

A first result is a formula for H_z in lowest order:

$$\begin{aligned} H_z &= \frac{\partial \phi}{\partial z} \approx \frac{\partial f}{\partial z} r^2 \cos 2\theta \\ &= \left(\frac{A}{2} \right) \left(\frac{\pi}{2b} \right) \left(1 - \tanh^2 \frac{\pi z}{2b} \right) r^2 \cos 2\theta . \end{aligned}$$

This field component has magnitude similar to the transverse quadrupole components near $z = 0$. Higher order terms of ϕ can be extracted from the general expansion in the same way as was done for quadrupole component, however a short cut is available. We simply plug the expansion

$$\phi = f(x^2 - y^2) + g(x^4 - y^4) + h(x^6 - y^6) + \dots$$

directly into $\nabla^2 \phi = 0$ and equate coefficients of $(x^m - y^m)$:

$$\begin{aligned} 0 = \nabla^2 \phi &= f''(x^2 - y^2) + g''(x^4 - y^4) + h''(x^6 - y^6) + \dots \\ &+ 12g(x^2 - y^2) + 30h(x^4 - y^4) + \dots . \end{aligned}$$

We have immediately

$$g = \frac{-f''}{12},$$

$$h = \frac{-g''}{30} = \frac{f''''}{360},$$

and so forth. Hence

$$\phi = f(x^2 - y^2) - \frac{f''}{12}(x^4 - y^4) + \frac{f''''}{360}(x^6 - y^6) - \dots$$

After a little algebra this expansion may be cast in the form

$$\begin{aligned} \phi = & fr^2 \cos 2\theta - \frac{f''}{12} r^4 \cos 2\theta \\ & + \frac{f''''}{360} \left(\frac{15}{16} r^6 \cos 2\theta + \frac{1}{16} r^6 \cos 6\theta \right) + \dots, \end{aligned}$$

displaying directly the allowed terms through sixth order (quadrupole, pseudo-octopole, pseudo dodecapole, and dodecapole).

Returning to the semi-infinite quadrupole amplitude

$$f(z) = \frac{A}{2} \left(1 + \tanh \frac{\pi z}{2b} \right),$$

we find by successive differentiations

$$f' = \frac{A}{2} \left(\frac{\pi}{2b} \right) \left(1 - \tanh^2 \frac{\pi z}{2b} \right),$$

$$f'' = \frac{A}{2} \left(\frac{\pi}{2b} \right)^2 (-2 \tanh) \left(1 - \tanh^2 \right),$$

$$f''' = \frac{A}{2} \left(\frac{\pi}{2b} \right)^3 (-2 + 6 \tanh^2) \left(1 - \tanh^2 \right),$$

$$f'''' = \frac{A}{2} \left(\frac{\pi}{2b} \right)^4 (16 \tanh - 24 \tanh^3) \left(1 - \tanh^2 \right),$$

etc.

The pseudo octopole potential is

$$\begin{aligned} \phi_{PO} &= \frac{-f''}{12} r^4 \cos 2\Theta \\ &= \frac{A}{12} \left(\frac{\pi}{2b} \right)^2 \left(r^4 \cos 2\Theta \right) \left(\tanh \frac{\pi z}{2b} \right) \left(1 - \tanh^2 \frac{\pi z}{2b} \right). \end{aligned}$$

There is a very small dodecapole potential

$$\begin{aligned} \phi_{DOD} &= \frac{f''''}{(360) \cdot (16)} r^6 \cos 6\Theta \\ &= \frac{A}{11520} \left(\frac{\pi}{2b} \right)^4 \left(r^6 \cos 6\Theta \right) (16 \tanh - 24 \tanh^3) \left(1 - \tanh^2 \right). \end{aligned}$$

Comparison of Multipole Fields

We calculate the relative strengths of the various multipoles near $z = 0$.

Specifically, $B_r = \partial\phi/\partial r$ is computed and normalized to the 2-d maximum $= -M_0/2$:

$$\left(\frac{B_r}{-M_0/2}\right) = \left[\frac{1 + \tanh\left(\frac{\pi z}{2b}\right)}{2}\right] \left(\frac{r}{b}\right) \cos 2\theta \quad (\text{quadrupole})$$

$$+ \left[\frac{\pi^2}{24} \tanh(1 - \tanh^2)\right] \left(\frac{r}{b}\right)^3 \cos 2\theta \quad (\text{pseudo octopole})$$

$$+ \left[\frac{\pi^4}{240} \left(\tanh - \frac{3}{2} \tanh^3\right) (1 - \tanh^2)\right] \left(\frac{r}{b}\right)^5 \left[\frac{15}{16} \cos 2\theta + \frac{1}{16} \cos 6\theta\right] \quad \left(\begin{array}{c} \text{pseudo} \\ \text{dodecapole} \end{array} + \text{dodecapole}\right)$$

Values of the bracketed functions of z in the multipole field comparison are given in table 1. Note the antisymmetry of 3rd and 5th orders.

$\pi z/2b$	1st order $\frac{1 + \tanh\left(\frac{\pi z}{2b}\right)}{2}$	3rd order $\frac{\pi^2}{24} \tanh\left(1 - \tanh^2\right)$	5th order $\frac{\pi^4}{240} \left(\tanh - \frac{3}{2} \tanh^3\right) (1 - \tanh^2)$
-3.0	.002473	-.004037	+.001933
-2.5	.006693	-.010789	+.004900
-2.0	.017986	-.028009	+.010892
-1.5	.047426	-.067264	+.015199
-1.0	.119203	-.131533	-.016871
-.7	.197816	-.157756	-.070393
-.5	.268941	-.149455	-.100256
-.4	.310026	-.133692	-.103376
-.3	.354344	-.109631	-.094428
-.2	.401312	-.078005	-.072489
-.1	.450166	-.040580	-.039454
0.0	.500000	.000000	.000000
.1	.549834	.040580	.039454
.2	.598688	.078005	.072489
.3	.645656	.109631	.094428
.4	.689974	.133692	.103376
.5	.731059	.149455	.100256
.7	.802184	.157756	.070393
1.0	.880797	.131533	.016871
1.5	.957574	.067264	-.015199
2.0	.982014	.028009	-.010892
2.5	.993307	.010789	-.004900
3.0	.997527	.004037	-.001933

Table 1

Relative strength of components of B_r at $r = b$, $\theta = 0$.

Reference

1. E. Lee and M. Vella, LBL-38430, "Perfect 2-d Quadrupole Fields from Permanent Magnets".