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**Computing Spacetime Curvature via
Differential-Algebraic Equations**

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MANAGED AND OPERATED BY
LOCKHEED MARTIN ENERGY RESEARCH CORPORATION
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Computer Science and Mathematics Division

**COMPUTING SPACETIME CURVATURE VIA
DIFFERENTIAL-ALGEBRAIC EQUATIONS**

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Abstract

The equations that govern the behavior of physical systems can often be solved numerically using a method of lines approach and differential-algebraic equation (DAE) solvers. For example, such an approach can be used to solve the Einstein field equations of general relativity, and thereby simulate significant astrophysical events. In this paper, we describe some preliminary work in which two model problems in general relativity are formulated, spatially discretized, and then numerically solved as a DAE. In particular, we seek to reproduce the solution to the spherically symmetric Schwarzschild spacetime. This is an important testbed calculation in numerical relativity since the solution is the steady-state for the collision of two (or more) non-rotating black holes. Moreover, analytic late-time properties of the Schwarzschild spacetime are well known and can be used to verify the accuracy of the simulation.

1. Introduction

Einstein's theory of general relativity is the foundation of our current understanding of the large-scale universe. This revolutionary theory asserts that the force of gravity is really a manifestation of the spacetime curvature produced by massive objects. The bending of light rays, the redshifting of spectral lines and the precession of Mercury's orbit are observed phenomena that can each be explained in general relativistic terms. To gain deeper knowledge, however, a numerical investigation of the Einstein equations must be pursued. Numerical solutions reveal the finer details of gravitational behavior, and such information can greatly improve our understanding of the physical theory.

Numerical solutions also guide the experimental work of physicists in search of new phenomena in the universe. General relativity predicts that massive objects in nonspherical motion will generate waves (that is, gravitational radiation) in the fabric of spacetime. The weakly interacting nature of gravitational waves allows their signal of remote, cataclysmic events to travel great distances with little distortion. Thus, gravitational waves are an ideal source of information concerning significant astrophysical events—if they can be detected and precisely measured. It is anticipated that detailed measurements of gravitational radiation will be detected by gravitational wave observatories (GWOs) by the end of this century [1]. Sensitive detection devices such as GWOs are being designed and calibrated based, in part, on what we learn from various solutions to the Einstein equations. The interpretation of gravitational waveforms is another challenging component of research involving GWOs. In this way, numerical solutions to the Einstein equations will play an important role in helping to identify and interpret the various waveforms recorded by these new observatories of the universe.

In this paper, we investigate the numerical solution of the Einstein field equations using a method of lines (MOL) approach and a differential-algebraic equation (DAE) solver. The Einstein equations are coupled nonlinear, hyperbolic-elliptic partial differential equations (PDEs). A DAE is obtained by spatially discretizing these PDEs yielding a system of ordinary differential equations (ODEs) whose solution satisfies a system of algebraic equations (a discretized elliptic PDE) at each time step. DASPK is a FORTRAN code for solving these types of DAEs and, as a first step in applying DAEs to numerical relativity, we solve some simple model problems involving black holes and spacetime curvature. By treating the Einstein equations within the framework of DAEs, other difficult

problems (for example, colliding black holes) can also be solved using the best available numerical methods and DAE software.

An outline of this paper is as follows. In §2, we describe two model problems for computing the spacetime that surrounds a black hole, and give some well known properties of the solution. In §3, we show how the model problems can be formulated, spatially discretized, and then numerically solved using the DAE solver, DASPK. In §4, we present some numerical results from solving the model problems using DASPK. In §5, we summarize our findings and indicate the need for future work in this area.

2. Spherically symmetric black hole spacetimes

The theory of general relativity unifies space and time into the four-dimensional entity called *spacetime*, and further describes gravity as the manifestation of spacetime curvature. In weak gravitational fields, spacetime is nearly flat and Newton’s law of gravitation gives adequate approximations. However, Newton’s laws are not valid in the intense gravitational fields and highly warped spacetimes near massive objects. In this section, we model the region of spacetime surrounding the extremely compact object that forms when a massive star dies and then collapses upon itself. These compact objects are known as *black holes*.

In general relativity, the geometry of 4D spacetime can be fully described using a 4×4 symmetric metric tensor, $g_{\mu\nu}$. For the spacetime surrounding a spherically symmetric, non-rotating black hole, $g_{\mu\nu}$ can be simplified to a diagonal tensor whose entries are functions of 3 variables: a , b , and α . The variables a and b are metric components and α determines the “slicing” of the spacetime. We also define the time derivatives H_a , H_b for a , b so that the Einstein equations become first-order in time. A standard approach for computing the metric components involves splitting the spacetime into “space plus time”, and then evolving the Einstein equations as an initial-value problem (for suitable initial values and boundary conditions, of course). Some earlier work on solving black hole spacetimes using this approach is described in [4], for example. We take a similar approach, but will integrate the model problems using a DAE solver. We will also compare the computed DAE results with some of the analytic results known for the black hole spacetimes.

For this spherically symmetric, one (space) dimensional problem, the evolu-

tion begins at a time and in a region where spacetime is flat; that is, at $t = 0$ we set

$$a = 1, \quad b = 1, \quad H_a = 0, \quad H_b = 0 \quad (1)$$

at each point along the gridline. The evolution equations in §2.1–§2.2 below assume a black hole of two solar masses. The gridline $0 \leq \eta \leq \eta_{\max} = 10$ is chosen large enough so that spacetime remains flat at all coordinates $(t, \eta \geq \eta_{\max})$ that we consider. The variable α is initialized to one at each gridpoint. For $t > 0$, subsequent values for α are determined by the slicing condition used. In geodesic slicing, the metric components (and their time derivatives) are computed along gridlines that march parallel to each other in time. However, it is not possible to evolve beyond $t = 2\pi$ since a singularity due to infinite curvature exists at $(2\pi, 0)$. To avoid the singularity, maximal slicing defines α in a way that allows the evolution to proceed normally at the outermost gridpoints, but slows down and eventually halts the evolution at the innermost gridpoints. The result is that successive gridlines are not parallel to each other in time. Instead, the gridlines become sharply curved as the time lag between the innermost and outermost gridpoints increases. This singularity-avoiding behavior is useful because it allows us to explore a larger portion of the black hole spacetime. For more details on the slicing conditions, evolution equations and constraint equations (in §2.3 below), see [4, Appendix].

2.1. Geodesic slicing conditions

For geodesic slicing conditions, the first-order evolution of initial data occurs on a gridline $0 \leq \eta \leq \eta_{\max}$ with four unknowns (a, b, H_a, H_b) per gridpoint. The variable α has constant value one throughout the evolution. (In §4 below, we consider numerical experiments in which the gridline spacing $\Delta\eta$ is on the order of 0.01.) The black hole equations for geodesic slicing conditions simplify to

$$\frac{\partial a}{\partial t} = -2H_a, \quad (2)$$

$$\frac{\partial b}{\partial t} = -2H_b, \quad (3)$$

$$\frac{\partial H_a}{\partial t} = \frac{R_{\eta\eta}}{\psi^4} + \frac{2H_a H_b}{b} - \frac{H_a^2}{a}, \quad (4)$$

$$\frac{\partial H_b}{\partial t} = \frac{R_{\xi\xi}}{\psi^4} + \frac{H_a H_b}{a} \quad (5)$$

with the auxiliary variables

$$R_{\eta\eta} = -\frac{4\frac{\partial^2\psi}{\partial\eta^2}}{\psi} + \frac{4\left(\frac{\partial\psi}{\partial\eta}\right)^2}{\psi^2} - \frac{2\frac{\partial b}{\partial\eta}\frac{\partial\psi}{\partial\eta}}{b\psi} + \frac{2\frac{\partial a}{\partial\eta}\frac{\partial\psi}{\partial\eta}}{a\psi} - \frac{\frac{\partial^2 b}{\partial\eta^2}}{b} + \frac{\left(\frac{\partial b}{\partial\eta}\right)^2}{2b^2} + \frac{\frac{\partial a}{\partial\eta}\frac{\partial b}{\partial\eta}}{2ab}, \quad (6)$$

$$R_{\xi\xi} = -\frac{2b\frac{\partial^2\psi}{\partial\eta^2}}{a\psi} - \frac{2b\left(\frac{\partial\psi}{\partial\eta}\right)^2}{a\psi^2} - \frac{3\frac{\partial b}{\partial\eta}\frac{\partial\psi}{\partial\eta}}{a\psi} + \frac{b\frac{\partial a}{\partial\eta}\frac{\partial\psi}{\partial\eta}}{a^2\psi} - \frac{\frac{\partial^2 b}{\partial\eta^2}}{2a} + \frac{\frac{\partial a}{\partial\eta}\frac{\partial b}{\partial\eta}}{4a^2} + 1, \quad (7)$$

$$\psi = 2 \cosh(\eta/2). \quad (8)$$

In (6) and (7), partial derivatives with respect to η are replaced by second-order centered difference approximations. For boundary conditions, we assume spatial symmetry across $\eta = 0$, and we approximate values at $\eta_{\max}$ by extrapolating the values at the nearby interior gridpoints.

The analytic values at $\eta = 0$ are [3, pg. 45]

$$a(t, 0) = (-1 + \cos(\omega)) \times \frac{\left((-3 + \cos(\omega)) \sin^2 \omega + (1 + \cos(\omega)) (8 + 3\omega \sin(\omega))\right)^2}{-16(1 + \cos(\omega))^3 \sin^2 \omega}, \quad (9)$$

$$b(t, 0) = \frac{1}{4}(1 + \cos \omega)^2, \quad (10)$$

$$H_a(t, 0) = \frac{a(t, 0)}{2(1 + \cos(\omega))} \times \frac{-(1 + \cos(\omega)) \sin(\omega) - 3(\omega + \sin(\omega))}{2(1 + \cos(\omega))^2 + 3 \sin(\omega)(\omega + \sin(\omega))}, \quad (11)$$

$$H_b(t, 0) = \frac{\sin \omega}{8}, \quad (12)$$

where $\omega = \omega(t)$ is determined by solving

$$t = 2(\omega + \sin \omega). \quad (13)$$

It is well known that a singularity develops at $\eta = 0$ as $t \rightarrow 2\pi$. In terms of the evolution, we find that $b \rightarrow 0$ as $t \rightarrow 2\pi$ at $\eta = 0$, which causes an overflow condition for the term $2H_a H_b / b$ in (4). Thus, a “good” black hole solver will halt just prior to $t = 2\pi$ due to this overflow difficulty. For $t < 2\pi$, the analytic solutions given by (9)–(12) are helpful in verifying the accuracy of the evolution.

2.2. Maximal slicing conditions

The second study of spherically symmetric, non-rotating black hole spacetimes requires the evolution of a more complicated set of equations. For maximal slicing conditions,

$$\frac{\partial a}{\partial t} = -2\alpha H_a, \quad \partial \quad (14)$$

$$\frac{\partial b}{\partial t} = -2\alpha H_b, \quad (15)$$

$$\frac{\partial H_a}{\partial t} = \frac{\alpha R_{\eta\eta}}{\psi^4} + \frac{2\frac{\partial\alpha}{\partial\eta}\frac{\partial\psi}{\partial\eta}}{\psi^5} - \frac{\frac{\partial^2\alpha}{\partial\eta^2}}{\psi^4} + \frac{\frac{\partial a}{\partial\eta}\frac{\partial\alpha}{\partial\eta}}{2a\psi^4} + \frac{2\alpha H_a H_b}{b} - \frac{\alpha H_a^2}{a}, \quad (16)$$

$$\frac{\partial H_b}{\partial t} = \frac{\alpha R_{\xi\xi}}{\psi^4} - \frac{2b\frac{\partial\alpha}{\partial\eta}\frac{\partial\psi}{\partial\eta}}{a\psi^5} - \frac{\frac{\partial\alpha}{\partial\eta}\frac{\partial b}{\partial\eta}}{2a\psi^4} + \frac{\alpha H_a H_b}{a}, \quad (17)$$

where $R_{\xi\xi}$, $R_{\eta\eta}$, and ψ are given in (6)–(8). The initial values and boundary conditions for a , b , H_a , H_b are the same as before. At time $t = 0$, we set $\alpha = 1$ at each gridpoint. At later times, however, the lapse $\alpha(t, \eta)$ is determined by solving the linear two-point ODE boundary value problem

$$\frac{\partial^2 \alpha}{\partial \eta^2} + \left(\frac{2\frac{\partial\psi}{\partial\eta}}{\psi} + \frac{\frac{\partial b}{\partial\eta}}{b} - \frac{\frac{\partial a}{\partial\eta}}{2a} \right) \frac{\partial \alpha}{\partial \eta} - \left(R_{\eta\eta} + \frac{2a}{b} R_{\xi\xi} \right) \alpha = 0, \quad (18)$$

where $\partial\alpha/\partial\eta = 0$ at $\eta = 0$ and $\alpha = 1$ at $\eta_{\max}$. For higher-dimensional black hole spacetimes, we must solve a linear elliptic PDE to determine α .

We can formulate the lapse equation (18) and its boundary conditions as a linear system by replacing the spatial derivatives with second-order centered differences, and thereby obtain

$$M\alpha = d. \quad (19)$$

M is a time-dependent, $m \times m$ tridiagonal matrix that contains the coefficients resulting from the finite differencing of the lapse equation. The vector α contains the values of the lapse at each grid point. The tridiagonal system (19) is easily solved using Gaussian elimination with partial pivoting.

For maximal slicing conditions, the limiting values at $\eta = 0$ are [3, pg. 67]

$$\lim_{t \rightarrow \infty} a(t, 0) = \frac{256}{81} \approx 3.16049, \quad (20)$$

$$\lim_{t \rightarrow \infty} b(t, 0) = \frac{9}{16} = 0.5625, \quad (21)$$

$$\lim_{t \rightarrow \infty} H_a(t, 0) = \frac{-1024}{486\sqrt{3}} \approx -1.21647, \quad (22)$$

$$\lim_{t \rightarrow \infty} H_b(t, 0) = \frac{\sqrt{3}}{16} \approx 0.10825. \quad (23)$$

In practice, these limiting values are achieved fairly early in the evolution (that is, at $t \approx 15$). It is not possible to evolve this problem indefinitely on a fixed grid because the solution curves eventually develop gradients too steep to be resolved. For our preliminary studies, it is sufficient to evolve the problem to $t = 200$.

2.3. Constraints

The numerical solution of these initial-value problems can be attempted using various ODE and DAE methods. As another partial check of numerical accuracy and solution correctness, it is useful to monitor the extent to which the numerical solution satisfies the Hamiltonian constraint

$$k_1(a, b, H_a, H_b) = \frac{2R_{\xi\xi}}{b\psi^4} + \frac{R_{\eta\eta}}{a\psi^4} + \frac{2H_b^2}{b^2} + \frac{4H_aH_b}{ab} = 0, \quad (24)$$

and the momentum constraint

$$k_2(a, b, H_a, H_b) = \frac{4H_b}{b\psi} \frac{\partial\psi}{\partial\eta} - \frac{4H_a}{a\psi} \frac{\partial\psi}{\partial\eta} + \frac{2}{b} \frac{\partial H_b}{\partial\eta} - \frac{\partial b}{\partial\eta} \frac{H_b}{b^2} - \frac{\partial b}{\partial\eta} \frac{H_a}{ab} = 0, \quad (25)$$

since the continuous, analytic solution is known to satisfy them exactly. It is particularly important to satisfy (24) since a violation of the Hamiltonian constraint implies the introduction of energy (hence, mass) into the vacuum black hole spacetime.

3. Black hole spacetimes as differential-algebraic equations

Now we consider modeling the two, black hole spacetimes as DAEs. Roughly speaking, a DAE is a system of ODEs coupled with algebraic equations [5]. For example, the evolution equations for a , b , H_a , H_b with maximal slicing conditions (14)–(17) are ODEs coupled with the algebraic equation (19) for the

lapse α . In contrast, the evolution for geodesic slicing is not coupled with an algebraic equation, and therefore it is simply an ODE initial-value problem.

In fully implicit form, the above DAEs may be written as the general, first-order nonlinear system

$$F(t, y, y', \alpha) = 0, \quad (26)$$

where y and α are vectors containing the differential and algebraic variables, respectively. For the maximal slicing problem, the general form (26) separates into the following differential and algebraic subsystems:

$$F_1(t, y, y', \alpha) = y' - f(t, y, \alpha) = 0, \quad (27)$$

$$F_2(t, y, \alpha) = M(y)\alpha - d = 0. \quad (28)$$

In terms of (26), t is a scalar, y and y' are vectors of length $4m$ for the unknown metric components and their time derivatives at each of the m gridpoints, α is a vector of length m for the unknown algebraic variable at each gridpoint. Hence $F(t, y, y', \alpha)$ is a vector-valued function of length $5m$. The geodesic slicing problem is a typical ODE initial-value problem and the general form (26) reduces to

$$y' - f(t, y) = 0. \quad (29)$$

The vector-valued function $F(t, y, y')$ described by the left-side of (29) has length $4m$ since $4m$ ODEs (and no algebraic equations) are present.

For numerically solving these DAEs, we will use the FORTRAN code DASPK developed by Brown, Hindmarsh and Petzold [6]. The code is based on an idea due to Gear [7], in which backward differentiation formula (BDF) methods of orders 1 through 5 are used to approximate the y' term in (27) and (29). In particular, DASPK uses the order k approximation

$$y'_n \approx \frac{\rho y_n}{h_n}, \quad (30)$$

where $\rho y_n = \sum_{i=0}^k \gamma_i y_{n-i}$, γ_i are the coefficients for the BDF method, and h_n is the step size $h_n = t_n - t_{n-1}$. By substituting (30) into (26), we obtain the nonlinear equation

$$F\left(t_n, y_n, \frac{\rho y_n}{h_n}, \alpha_n\right) = 0, \quad (31)$$

that must be solved for y_n and α_n at each time step t_n .

In DASPK, this nonlinear equation is solved using an Inexact Newton method. That is, at each Newton iteration, we approximately solve the linear system

$$J_n \delta_n = -F\left(t_n, y_n, \frac{\rho y_n}{h_n}, \alpha_n\right) \quad (32)$$

using a preconditioned Krylov method (for example, GMRES [9]). The matrix J_n is the Jacobian matrix of the nonlinear function (31), and δ_n is the correction to the current Newton iterate. At each time step t_n , DASPK obtains the initial guess for the Inexact Newton iteration by extrapolating from the DAE solutions computed at previous time steps.

If the maximal slicing problem is formulated as in (27)-(28), we must approximately solve the following linear system at each Newton iteration:

$$\begin{pmatrix} \frac{\gamma_0}{h_n} I - \frac{\partial f}{\partial y_n} & -\frac{\partial f}{\partial \alpha_n} \\ \frac{\partial F_2}{\partial y_n} & M(y_n) \end{pmatrix} \begin{pmatrix} \Delta y_n \\ \Delta \alpha_n \end{pmatrix} = - \begin{pmatrix} \frac{\rho y_n}{h_n} - f(t_n, y_n, \alpha_n) \\ M(y_n) \alpha_n - d_n \end{pmatrix}, \quad (33)$$

where Δy_n and $\Delta \alpha_n$ are corrections to the differential and algebraic variables, respectively. For the geodesic slicing example of (29), each Newton iteration gives the linear system

$$\left(\frac{\gamma_0}{h_n} I - \frac{\partial f}{\partial y_n} \right) \Delta y_n = - \left(\frac{\rho y_n}{h_n} - f(t_n, y_n) \right) \quad (34)$$

where Δy_n is the correction to the differential variables. We hasten to remark that DAE problems can often be formulated in several different, but mathematically equivalent, ways. This observation is important since certain DAE formulations can lead to linear systems (32) for which GMRES converges quickly. An alternate DAE formulation for maximal slicing, and preconditioners for both problems, are discussed later in this section. We also note that DASPK does not explicitly compute the Jacobian J_n since a matrix-free method can be used to approximate the matrix-vector products $J_n v$ that GMRES requires in solving (32). This matrix-vector product is obtained at the cost of evaluating the DAE residual (31).

A more detailed description of DASPK is given in [6], and the prologue to the actual code. In particular, an in-depth discussion of critical issues such as error estimates, convergence tests, diagnostics, weighted norms, and so on, are provided there.

In general, DASPK chooses the variable step size and the BDF order based on

the behavior of the computed DAE solution. Ideally, DASPK will vary its step size and order selection so that the physical system is evolved as efficiently as possible. Implicit integration has the advantage of (possibly) large time-stepping, and the drawback that large, linear systems need to be solved at each Newton iteration. Unfortunately, the Inexact Newton method may fail to converge if the step sizes are too large since the initial Newton iterate may be poor and/or the preconditioned linear system may be too difficult for GMRES to solve in a few iterations. The optimal BDF order and step size selection for a given problem and preconditioner is difficult, if not impossible, to determine. At best, we hope for simple preconditioners that are relatively effective for the range of step sizes that DASPK tends to select.

3.1. Geodesic slicing conditions

The geodesic slicing problem, as formulated in (29), is straightforward to precondition since it is really an ODE initial-value problem. In particular, the diagonal preconditioner

$$P^{-1} = \frac{h_n}{\gamma_0} I \quad (35)$$

is a natural choice, and yields the preconditioned linear system

$$\left(I - \frac{h_n}{\gamma_0} \frac{\partial f}{\partial y_n} \right) \Delta y_n = -\frac{1}{\gamma_0} (\rho y_n - h_n f(t_n, y_n)) \quad (36)$$

at each Newton iteration. Note that the preconditioned Jacobian in (36) approaches the identity matrix as $h_n \rightarrow 0$. This is a highly desirable property since GMRES should converge quickly as the eigenvalues become more tightly clustered around 1. The main objective, however, is to keep low the overall cost of evolving the problem—so, taking too many small time steps may be counterproductive. We also remark that a good preconditioner enhances the reliability of the convergence tests in the Inexact Newton method [6, §2.2]. In our experience with the geodesic slicing problem, the preconditioner (35) has worked extremely well.

3.2. Maximal slicing conditions

For the maximal slicing problem, a DAE formulation other than (27)–(28) is needed. This is because the ODE preconditioner (35) applied to the linear sys-

tem (33) gives the preconditioned system

$$\begin{pmatrix} I - \frac{h_n}{\gamma_0} \frac{\partial f}{\partial y_n} & -\frac{h_n}{\gamma_0} \frac{\partial f}{\partial \alpha_n} \\ \frac{h_n}{\gamma_0} \frac{\partial \hat{F}_2}{\partial y_n} & \frac{h_n}{\gamma_0} M(y_n) \end{pmatrix} \begin{pmatrix} \Delta y_n \\ \Delta \alpha_n \end{pmatrix} = -\frac{1}{\gamma_0} \begin{pmatrix} \rho y_n - h_n f(t_n, y_n, \alpha_n) \\ h_n M(y_n) \alpha_n - h_n d_n \end{pmatrix}. \quad (37)$$

Unfortunately, the preconditioned Jacobian becomes highly ill-conditioned (and, ultimately, singular) as $h_n \rightarrow 0$. Thus, the properties of the Jacobian become less favorable for GMRES as the step size is reduced. A complete explanation of this ill-conditioning problem, and the recommended scaling to avoid it, is described in [5, §5.4.2].

A mathematically equivalent (and more favorable) DAE formulation can be obtained by premultiplying the algebraic equations (28) by $\frac{\gamma_0}{h_n} M^{-1}(y)$ so that

$$F_1(t, y, y', \alpha) = y' - f(t, y, \alpha) = 0, \quad (38)$$

$$\hat{F}_2(t, y, \alpha) = \frac{\gamma_0}{h_n} (\alpha - M^{-1}(y) d) = 0. \quad (39)$$

The Jacobian for the DAE (38)–(39) is

$$\begin{pmatrix} \frac{\gamma_0}{h_n} I - \frac{\partial f}{\partial y_n} & -\frac{\partial f}{\partial \alpha_n} \\ \frac{\partial \hat{F}_2}{\partial y_n} & \frac{\gamma_0}{h_n} I \end{pmatrix} \quad (40)$$

and the preconditioned linear system is

$$\begin{pmatrix} I - \frac{h_n}{\gamma_0} \frac{\partial f}{\partial y_n} & -\frac{h_n}{\gamma_0} \frac{\partial f}{\partial \alpha_n} \\ \frac{h_n}{\gamma_0} \frac{\partial \hat{F}_2}{\partial y_n} & I \end{pmatrix} \begin{pmatrix} \Delta y_n \\ \Delta \alpha_n \end{pmatrix} = -\frac{1}{\gamma_0} \begin{pmatrix} \rho y_n - h_n f(t_n, y_n, \alpha_n) \\ \gamma_0 \alpha_n - \gamma_0 M^{-1}(y_n) d_n \end{pmatrix}, \quad (41)$$

where the preconditioner P^{-1} is given in (35). As with the geodesic slicing problem, we now have a preconditioned Jacobian whose eigenvalues become more tightly clustered around 1 as $h_n \rightarrow 0$. Of course, in implementing this formulation in DASPK, the Jacobian matrix (40) and the inverse of the tridiagonal matrix $M(y)$ are never explicitly formed. Instead, $M^{-1}(y)$ is applied to the vector d in (39) by directly solving the linear system $M(y)z = d$ for z , and DASPK uses a matrix-free method to approximate the Jacobian-vector products $J_n v$ needed by GMRES. For the maximal slicing problems that we studied, GMRES converged rapidly using this formulation and preconditioning.

4. Numerical results

In this section, we present numerical results obtained from evolving the black hole problems described in §2 using DASPK. The evolution occurs on the 1D gridline, and we consider gridlines containing $m = 200, 400, \dots, 1000$ uniformly spaced gridpoints for $0 \leq \eta \leq \eta_{\max} = 10$. The absolute and relative integration error tolerances are $\text{ATOL} = 10^{-6}$ and $\text{RTOL} = 0$, respectively. To obtain our results, we found that DASPK performed best when the maximum BDF order was restricted to 2. This is not unexpected since the black hole equations are hyperbolic in nature and often the DAE Jacobian (40) may have eigenvalues near the imaginary axis. If some of the eigenvalues are too large and too close to the imaginary axis, then these eigenvalues may lie outside the stability region for BDF methods of order $k > 2$. Diagrams of the absolute stability regions for order k BDF methods can be found in [8, p. 100], for example.

4.1. Geodesic slicing conditions

For geodesic slicing conditions, equations (9)–(12) are evolved to $t = 6.2 \approx 2\pi$. In Table 1, we compare the exact values of the unknown metric components and their time derivatives versus the values computed using $m = 200, \dots, 1000$ gridpoints. The accuracy of the computed solutions for a and H_a steadily improves as more gridpoints are used. On the other hand, the small values for b and H_b vary slightly, but they have essentially converged with 3 or 4 digits of accuracy.

Table 1: Geodesic slicing: comparison of exact vs. computed values.

Geodesic slicing: $t = 6.2, \eta = 0, m$ gridpoints.				
	a	b	H_a	H_b
exact	51.6433	9.43115e-3	-224.189	7.40283e-2
$m = 200$	51.8427	9.37444e-3	-226.070	7.39272e-2
$m = 400$	51.7112	9.42722e-3	-224.570	7.40113e-2
$m = 600$	51.6884	9.43649e-3	-224.309	7.40259e-2
$m = 800$	51.6805	9.43968e-3	-224.219	7.40310e-2
$m = 1000$	51.6771	9.44105e-3	-224.180	7.40331e-2

Figure 1 gives the solutions for a and b for $m = 1000$ and various values of t . This figure also shows that, as $t \rightarrow 2\pi^-$, we have $b \rightarrow 0^+$ at $\eta = 0$; this is the

Figure 1: Geodesic slicing: a and b for $t = 6.2$, $m = 1000$ gridpoints.

feature of geodesic slicing that causes an overflow condition due to division by zero in (4).

In Table 2, the maximum absolute error in the Hamiltonian and momentum constraints (24)–(25) are given. For the momentum constraint, the error seems to decrease by four each time the number of gridpoints is doubled. In contrast to this quadratic convergence of momentum error, the Hamiltonian error increases slightly as more gridpoints are used. Overall, the Hamiltonian error seems to have converged, and the momentum error falls quadratically with finer grids. The physical significance of these trends may be a matter for further investigation.

Table 2: Geodesic slicing: maximum error in Hamiltonian and momentum constraints.

Geodesic slicing: $t = 6.2$, $0 \leq \eta \leq 10$, m gridpoints					
constraint	$m = 200$	$m = 400$	$m = 600$	$m = 800$	$m = 1000$
Hamiltonian	0.02190	0.02625	0.02720	0.02749	0.02765
momentum	0.73977	0.19726	0.08289	0.04245	0.02397

Finally, we note that DASPK performed well on this problem. In each of the 5 cases, DASPK used approximately 950 time steps and averaged 1–2 Newton iterations per time step and 1–2 GMRES iterations per Newton iteration.

DASPK also reported no difficulties or failures in the DAE local error test or Inexact Newton convergence tests.

4.2. Maximal slicing conditions

For maximal slicing conditions, equations (14)–(18) are evolved to $t = 200$. In Table 3, we compare the limit values at $\eta = 0$ with the computed values. The accuracy of the computed solutions steadily improves or remains converged as more gridpoints are used.

Table 3: Maximal slicing: comparison of exact vs. computed values.

Maximal slicing: $t = 200$, $\eta = 0$, m gridpoints				
	a	b	H_a	H_b
exact	3.16049	0.56250	-1.21647	0.10825
$m = 200$	3.16084	0.56246	-1.21670	0.10826
$m = 400$	3.16060	0.56248	-1.21653	0.10826
$m = 600$	3.16056	0.56249	-1.21650	0.10826
$m = 800$	3.16054	0.56249	-1.21649	0.10826
$m = 1000$	3.16053	0.56249	-1.21649	0.10826

Table 4 shows the maximum absolute error in the Hamiltonian and momentum constraints. For the Hamiltonian constraint, the error decreases linearly as more gridpoints are used. However, the momentum error seems to have converged at approximately 0.44. (Conversely, in geodesic slicing, the Hamiltonian error was the one that converged.) Ideally, both of these nonlinear constraints should be satisfied exactly.

Table 4: Maximal slicing: maximum error in Hamiltonian and momentum constraints.

Maximal slicing: $t = 200$, m gridpoints					
constraint	$m = 200$	$m = 400$	$m = 600$	$m = 800$	$m = 1000$
Hamiltonian	0.1475e-1	0.9974e-2	0.7855e-2	0.6586e-2	0.5586e-2
momentum	0.4412	0.4508	0.4525	0.4344	0.3976

Figure 2 shows the solutions of a , b , and their time derivatives for $m = 1000$ and various values of t . An undesirable feature of a and H_a is the unbounded

Figure 2: Maximal slicing: a , b , H_a , H_b for $t = 200$, $m = 1000$ gridpoints.

growth of their peaks as t increases. Special techniques for helping to overcome this difficulty are described in [2] and the references therein.

Figure 3 shows the error in the constraints for $m = 1000$ and various values of t . Note that these errors also grow rapidly as t increases.

Table 5 gives DASPK performance results for $m = 200, \dots, 1000$. Column 5, "func evals", indicates the number of times the DAE residual (38)–(39) was evaluated. Recall that the DAE residual is evaluated at each Newton iteration and at each GMRES iteration; thus, column 5 is the sum of column 3 and column 4 if no error test failures occurred in DASPK. The latter two columns also indicate that, on average, less than 1 GMRES iteration was executed per Newton iteration. (No GMRES iterations are performed if the initial preconditioned GMRES

Figure 3: Maximal slicing: error in constraints for $t = 200, m = 1000$ gridpoints.

residual is sufficiently small.) We can also determine that, on average, about 2 Newton iterations were executed per time step. Note that the total number of time steps does not grow rapidly as more gridpoints are used. In fact, the overall cost of DASPK, as measured by “func evals”, only increases modestly as m becomes large. Finally, Table 5 indicates that the average step size is always much larger than the gridspacing, and this makes the DAE approach competitive with explicit time-stepping schemes for this problem. DASPK performed extremely well for maximal slicing, and no failures in the DAE local error test or Inexact Newton tests were reported.

Table 5: Maximal slicing: DASPK performance.

Maximal slicing: $t = 200, m$ gridpoints				
	time steps	Newton iters	GMRES iters	func evals
$m = 200$	2415	4826	3446	8272
$m = 400$	2639	5274	3849	9123
$m = 600$	2824	5644	3702	9346
$m = 800$	2825	5645	3765	9410
$m = 1000$	2985	5966	4141	10107

5. Summary and future work

To summarize, we have described two model problems for spherically symmetric, non-rotating black hole spacetimes, and solved them numerically using an MOL approach and the DAE solver, DASPK. By evolving these DAE initial-value problems in space and time, we are able to obtain the metric components a , b (and the lapse α , in maximal slicing) at various spacetime coordinates. These components, and the slicing condition used, determine a metric tensor $g_{\mu\nu}$ that completely describes the geometry of the spacetime. In this context, we focused mainly on using DASPK to compute the metric components and their time derivatives accurately in comparison with their analytic properties, such as their limiting values and constraint errors.

In §4, we found that DASPK was fairly accurate for the examples studied. The DAE approach was successful for these problems due to a DAE formulation and preconditioner for which DASPK worked well; see §3. The computed solutions were obviously not perfect, and the constraint errors grew as t increased. Such inaccuracies are expected as DASPK (and other methods) attempt to compute values at spacetime coordinates near the black hole—where curvature becomes infinite.

The proper role of the constraints is an important, open question in numerical relativity. It was somewhat surprising that the momentum error remained almost constant for maximal slicing at $t = 200$ and for various values of m ; see Table 4. Given that the constraints are not naturally satisfied by the numerical integrator, it is an important issue whether the constraints should be ignored, monitored, or actually enforced in some way. Another advantage of the DAE approach is that such constraints can be included as algebraic equations to be satisfied at each time step. By simultaneously enforcing such constraints, we hope to improve other aspects of the simulation of the solution of the physical problem (for example, conserving the mass of the black hole). An investigation of this issue is the subject of future DAE numerical relativity research.

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