DOE/ER/53207--T1

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JAN 16 1995

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Report for

Collisional and Chaotic Transport of Energetic Particles in Toroidal Plasma

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April, 1995

Abstract

We have made progress in two general areas of confinement plasma physics. (1) We studies a new loss mechanism of the toroidally trapped particles related to the up-down asymmetry of ripple in a tokamak. (2) We estimated the bootstrap current of the particles making transitions between the toroidally and locally states in non-axisymmetric tori, stellarators and tokamaks.

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I. Accomplishments

We carried out research in two general areas of confinement plasma physics. The main effort was to examine the mechanism for collisionless and collisional loss of toroidally trapped particles due to the vertically asymmetric magnetic ripple. Secondly, the longitudinal transport taking into account the so-called transition particles was studied, with the main emphasis on evaluating of the bootstrap current. These studies are useful for further calculations of neoclassical transport in non-axisymmetric toroidal devices, i.e. tokamaks with ripple and stellarators, in the low-collisional regime, relevant for the "near-fusion" temperatures.

1. The most important classical channel of the particle loss in toroidal magnetic systems is the ripple diffusion of the toroidally trapped particles¹⁻³. It arises due to distortion of the axial symmetry of the torus by the finite perturbation of the toroidal magnetic field. Hence, ripple diffusion has been a subject of intense studies for ideal configurations as well as for concrete devices.

The best solution to the problem is to diminish the toroidal asymmetry. But decreasing the ripple magnitude is quite expensive, because it requires the volume of the coils to be increased. That is why studies were aimed mostly to estimate the critical magnitude of ripple in order to find the balance between allowed losses and a reasonable design of the device.

In this regard, any new breaking of the symmetry of the magnetic field should be thoroughly investigated, as it might lead to an additional loss channel and affect the confining properties of the device. For example, the proposed ITER designs have sufficient magnetic ripple to form local magnetic wells, which can trap low-energy particles and provide a channel for their rapid escape from the machine. This effect may be

amplified in ITER by up-down asymmetry. Up-down asymmetry in combination with transitioning orbits allows particles to leave the machine convectively - i.e., much more rapidly. Given this new information we have modified this first project to developing an understanding of these two effects in combination.

In our studies we investigated the neoclassical transport associated with up-down asymmetry of the magnetic field in tokamaks and stellarators. In stellarators, the ripple magnitude is usually considered to be some function of the minor radius. In several cases, helical modulation dependent on the poloidal angle was studied⁴ but the up-down symmetry was always implied both for stellarators and tokamaks. This is not always the case in the real experiments. Substantial up-down asymmetry can arise due to even rather small vertical displacement of plasma with respect to the center of the chamber.^{5,6} Besides, all magnetic configurations with a single null divertor are naturally asymmetric from the very beginning. In such cases, the depth of the local magnetic wells, Δ , is different in the upper and lower parts of the magnetic surfaces. The value of this asymmetry can be of order unity, $(\Delta^+ - \Delta^-) / (\Delta^+ + \Delta^-) \approx 1$. In these magnetic configurations, the toroidally trapped particles - those with reflection points located in the region where the local magnetic wells exist - acquire a net radial velocity. This velocity is related to the toroidal drift velocity v_d and the toroidicity ϵ_t , $\dot{r}_t \approx v_d \sqrt{\frac{\Delta}{\epsilon_t}}$, and could lead to an additional transport, which we aim to study. We separate the collisionless and collisional cases, because at low collisionality, when the presence of the physical boundaries is essential and the losses due to new net drift motion are rapid. The situation should be different at higher collisionality, and this case requires solving rigorously the bounce-averaged drift kinetic equation both for the toroidally and locally trapped particles, because the distribution function should be continuous on the boundary between the corresponding regions of the velocity space. In Sec. II we present a calculation of the net radial velocity due in up-down asymmetric tori.

2. The second goal of the proposed research was to calculate the effects of collisionless trap-detrap processes on the longitudinal transport in non-axisymmetric tori. That is, we estimated the bootstrap current⁷ associated with transition particles. This problems were investigated before.⁸⁻¹⁰ The novelty of our approach is that we took into account the combined effects of the magnetic field modulations, in contrast to the approach where the toroidal modulation and ripple perturbation were considered in isolation which was useful for investigating the cases of an axisymmetric tokamak and a straight stellarator, but ignored such interesting and important phenomena as transitions of the toroidally trapped particles into the locally trapped particles and vice versa. The heuristic approach to estimate an change of the bootstrap current due to transitions is presented in Sec. III.

II. Transport in Non-axisymmetric Tori with Up-Down Asymmetry

As was stated in the previous section, any new breaking of the symmetry of the magnetic field leads to an additional loss channel and affect the confining properties of the device. One of new interesting transport phenomena is the neoclassical transport associated with up-down asymmetry of the magnetic field in tokamaks and stellarators. Substantial up-down asymmetry can arise due to even rather small vertical displacement of plasma with respect to the center of the chamber.^{9,10} Besides, all magnetic configurations with a single null divertor are naturally asymmetric from the very beginning. In such cases, the depth of the local magnetic wells, Δ , is different in the upper and lower parts of the magnetic surfaces. The value of this asymmetry can be of order unity, $(\Delta^+ - \Delta^-) / (\Delta^+ + \Delta^-) \approx 1$. In these magnetic configurations, the toroidally trapped particles - those with reflection points located in the region where the local magnetic wells exist - acquire a net radial velocity, which can be estimated from Liouville's theorem,

$$\dot{r}_t \approx v_d \sqrt{\frac{\epsilon_r}{\epsilon_t}}, \quad (1)$$

where v_d is the gradient drift velocity. This velocity could lead to an additional transport. This is the case at low collisionality, when the presence of the physical boundaries is essential and the losses due to new net drift motion are convective and quite rapid. The particles are lost at a rate

$$\frac{1}{\tau} \approx \min \left\{ \frac{a}{\dot{r}_t}, \frac{a}{v_d} \right\}. \quad (2)$$

where a is the minor radius of the torus.

The situation is different at higher collisionality, when the radial displacement of the particles between the collisions is small. In this case, we solve the bounce-averaged kinetic equation,

$$\langle \dot{r} \rangle^t \frac{\partial F^t}{\partial r} + \langle \dot{\beta} \rangle^t \frac{\partial F^t}{\partial \beta} = \frac{v}{B_0 \partial J^t / \partial E} \frac{\partial}{\partial \mu} \mu I^t \frac{\partial F^t}{\partial \mu}, \quad (3)$$

$$\langle \dot{r} \rangle^r \frac{\partial F^r}{\partial r} + \langle \dot{\beta} \rangle^r \frac{\partial F^r}{\partial \beta} = \frac{v}{B_0 \partial J^r / \partial E} \frac{\partial}{\partial \mu} \mu I^r \frac{\partial F^r}{\partial \mu}, \quad (4)$$

where

$$\langle \dot{r} \rangle^t = \frac{1}{r \Omega \partial J^t / \partial E} \left(\frac{\partial J^t}{\partial \beta} - \delta(Q) \frac{\partial Q}{\partial \beta} J_s^r \right) \equiv \dot{r}^t + \dot{r}_b^t, \quad (5)$$

$$\langle \dot{\beta} \rangle^t = -\frac{1}{r \Omega \partial J^t / \partial E} \left(\frac{\partial J^t}{\partial r} - \delta(Q) \frac{\partial Q}{\partial r} J_s^r \right) \equiv \dot{\beta}^t + \dot{\beta}_b^t, \quad (6)$$

are the average components of the drift velocity,

$$J = \oint R_0^2 \dot{\phi} d\phi = \oint R_0 \frac{u B_0}{B} d\phi, \quad (7)$$

$$I = \oint \frac{B R_0^2}{B_0} \dot{\phi} d\phi = R_0 \oint u d\phi, \quad (8)$$

$$T_b = \oint \frac{d\phi}{\dot{\phi}} = \frac{\partial J}{\partial E}. \quad (9)$$

where we omit the indices distinguishing the toroidally ("t") and locally trapped ("r") states and the function Q determines the boundary between these states, so that the separatrix is given by,

$$Q(r, \beta, E, \mu) = 0 \quad (10)$$

The rigorous analysis of the bounce-averaged kinetic equation (App. A) shows that the additional diffusive flux of the toroidally trapped particles, which is produced by the β -independent part of the distribution function, is totally compensated by the change in the flux of the locally trapped particles, so that the main transport remains unchanged. The remaining contribution to the diffusion in $1/v$ -regime of the toroidally trapped particles coming from $\tilde{F}^t$ is almost the same for the up-down symmetric and asymmetric tori. It has

been investigated before¹¹ and is related to 1/v-regime of the locally trapped particles by the relation, $D_t^{\text{sym}} / D_r^{\text{sym}} \approx \frac{1}{(qN)^2} \sqrt{\frac{\epsilon_t}{\epsilon_r}}$. Although, note that some new effects due to the

asymmetry, which do not change the order of the fluxes, can be expected in the results obtained before¹¹ since we should include the general (up-down asymmetric) dependence of ϵ_r on the poloidal angle. It will give new terms for $\frac{\partial J^{t,r}}{\partial \beta}$ with a relative order $\frac{\partial \epsilon_r / \partial \beta}{\epsilon_t}$

in comparison with the conventional ones.

III. Contribution of transitions into bootstrap current

The neoclassical theory^{4,7-10} predicted the bootstrap current, as the longitudinal current sustained without the longitudinal electric field. The most recent and rigorous description of the physics underlying the phenomenon can be found in Ref. 8. We will not dwell on the details of those explanations, except for some that are needed to estimate a contribution of the transition particles.

Transition particles are particles making transitions between toroidally and locally trapped states in tokamaks and stellarators. The standard model of the magnetic field strength in these devices has the form,

$$B = B_0(1 - \varepsilon_t \cos(\theta) - \varepsilon_r \cos(M\phi - l\theta)). \quad (11)$$

Here, ε_t is the inverse aspect ratio or toroidicity ($\varepsilon_t = r/R_0$ in the toroidal coordinates), ε_r is the ripple magnitude (or the helical modulation, for the stellarator); l and M (usually $M \gg l$) are the poloidal and toroidal mode numbers for a stellarator, while $l=0$ for a tokamak; θ and ϕ are the poloidal and toroidal angles, which are related to each other on the magnetic field lines via the rotational transform, $\frac{d\phi}{d\theta} = \frac{1}{q} = q$; r and $R = R_0(1 + \varepsilon_t \cos(\theta))$ are the minor and major radii. The axisymmetric configuration implies $\varepsilon_r=0$. The helically symmetric system can also be recovered by Eq. (1) by setting $\varepsilon_t = 0$, $\phi = L/R_0$ and $\lambda = M/R_0$: to give $B_h = B_0(1 - \varepsilon_r \cos(\lambda L - l\theta))$. Note, that in this section we use the toroidal coordinates for the sake of simplicity, although in the rest sections we shall use Boozer's coordinates to obtain the general answer.

The bootstrap current in an axisymmetric torus at low collisionality (banana regime), can be interpreted as an imbalance between the fluxes of the particles moving in parallel (+) and anti-parallel (-) directions along collisionless trajectories. This imbalance can be easily seen if we recognize that, due to the radial drift, + and - particles come from

different "home" magnetic surfaces with correspondingly different density and temperature. That is why the bootstrap current in the banana regime can be estimated as,

$$j_b \approx e \frac{\partial N}{\partial r} v_T (\Delta r)_b \quad (12)$$

where e is the species charge, v_T is the thermal velocity, N is the density,

$$(\Delta r)_b \approx \frac{\rho}{\tau \sqrt{\epsilon_t}}, \quad (13)$$

is the radial width of the toroidally trapped orbit (banana), with ρ the gyroradius.

In all the estimates made in this section we neglect the temperature gradient. Note also that there is no factor accounting for the fraction of the particles substituting the current, because the bootstrap current is produced mostly by passing particles (which constitute the majority at large aspect ratio) in spite that it is a width of toroidally trapped particle which is incorporated in Eq. (12). This happens due to collisions which maintain the friction between these two types of particles, although the value of this friction is irrelevant in the banana regime.

In terms of the drift kinetic equation, result described Eq. (12) can be roughly interpreted as follows. Due to the finite radial shift of toroidally trapped particles, their distribution function is perturbed by the amount,

$$\Delta F \approx \frac{\partial F_M}{\partial r} (\Delta r)_b, \quad (14)$$

where F_M is the Maxwellian. The distribution function is a solution to a second order differential equation, that is why it should be continuous on the boundary between toroidally trapped and passing particles, so that the distribution function of passing particles obtains the same order perturbation as given by Eq. (14). Since the number of passing particles is

larger than that of toroidally trapped particles, the longitudinal current is sustained mostly by them, which results in the estimate (12).

The helically symmetric case can be investigated in a similar way. There are two types of motion in the helically symmetric field: passing and trapped. The radial displacement of the trapped particle is easily evaluated,⁶

$$(\Delta r)_h \approx \frac{\rho}{M_t \sqrt{\epsilon_r}}, \quad (15)$$

which gives rise to the current,

$$j_h \approx e \frac{\partial N}{\partial r} v_T (\Delta r)_h. \quad (16)$$

Numerous papers about the bootstrap current in a stellarator⁷⁻⁸ always implied the straight stellarator (i.e. helically symmetric case), although this limitation was hardly pointed out.

Let us now take into account the finite value of toroidicity, $\epsilon_t \neq 0$, i.e. incorporate the full model (11). Then, instead of two types of motion, there are three: passing, toroidally and locally trapped. Besides, due to the drift, particle can make transitions between locally and toroidally trapped states. These transitions affect the low-collisional distribution function by the amount,⁹⁻¹¹

$$\Delta F \approx \frac{\partial F_M}{\partial r} \frac{(\Delta r)_r}{1 + \tau^t / \tau^r}, \quad (17)$$

where $(\Delta r)_r$ is the radial displacement of a particle in the locally trapped state, $(\Delta r)_r \approx \epsilon_t / \max \left\{ \epsilon_r, \epsilon_t, \frac{V}{E} \right\}$, $V = e\Phi / m$ is the electric potential per unit of mass, τ^t, τ^r is the relative time, during which transition particle stays in the toroidally ("t") or locally trapped ("r") states, respectively. One can show⁹⁻¹¹ that for small α ,

$$\alpha = \frac{t \epsilon_t}{N \epsilon_r} \ll 1, \quad (18)$$

the value of τ' / τ^r is negligibly small (order of ϵ_r). Then the numbers of transition particles with transition points separated by the radial distance Δr are different by $\Delta N = \frac{\partial}{\partial r} \left(\frac{\partial N}{\partial r} (\Delta r)_r \right) \Delta r \approx \frac{\partial N}{\partial r} \frac{\partial (\Delta r)_r}{\partial r} \Delta r$. In the case $\epsilon_i \gg \epsilon_r$, transition points of particles moving in + and - directions (Fig. 1) are separated by the distance approximately equal to the radial width of the toroidally trapped orbit, $(\Delta r)_b$. This gives rise to the additional bootstrap current,

$$j_T \approx e \frac{\partial N}{\partial r} \frac{\partial (\Delta r)_r}{\partial r} (\Delta r)_b v_T. \quad (19)$$

Note, that in the case $\epsilon_r \gg \epsilon_i$, instead of $(\Delta r)_b$ in Eq. (19) we should use $\Delta r \approx \frac{\rho}{\tau \sqrt{\epsilon_r}}$. At large values of α , the perturbation ΔF can be neglected, and there is no additional bootstrap current.

Fig. 1. Particles with opposite equal, but parallel velocities at the point A belong to different transition trajectories, separated by a distance ($B_1 B_2$ or $C_1 C_2$) approximately equal to the banana width. The points B_1, B_2 and C_1, C_2 are transition points.

IV.

References

- ¹ H.P. Furth, R.J. Goldston, S.J. Zweben, D.J. Sigmar, Nuclear Fusion **30**, 1799 (1990).
- ² R.B. White, H.E. Mynick, Phys. Fluids **B 1**, 980 (1989).
- ³ R.J. Goldston, R.B. White, A.H. Boozer, Phys. Rev. Let. **47**, 647 (1981).
- ⁴ L.M.Kovrizhnykh, Nuclear Fusion **24**, 851 (1984).
- ⁵ D.P. Schlissel, T.H. Osborne, J.C. DeBoo, J.R. Ferron, E.A. Lazarus, T.S. Taylor, Nuclear Fusion **32**, 689 (1992).
- ⁶ R.B. Bengston, A. Ouroua, A.G. Meigs, Bulletin of the American Physical Society, **34**, No 9, paper 8S3.
- ⁷ R.J. Bickerton, J.W. Connor, J.B. Taylor, Nature **229**, 110 (1971).
- ⁸ A.H. Boozer, H.J. Gardner, Phys. Fluids **B 2**, 2408 (1990).
- ⁹ A. Pytte, A.H. Boozer, Phys. Fluids **24**, 88 (1981).
- ¹⁰ K.C. Shaing, J.D. Callen, Phys. Fluids **26**, 3315 (1983).

Appendix A. Transport in Non-Axisymmetric Tori with Up-Down Asymmetry

Transport in Non-axisymmetric Tori with Up-Down Asymmetry

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Abstract

Transport phenomena arising due to up-down asymmetry of ripple and helical perturbations of the magnetic field in tokamaks and stellarators are investigated. Accounting for this asymmetry is important for the devices with vertically displaced magnetic axes or with a single-null divertor. The asymmetry manifests itself in different magnitudes of the ripple or helical perturbation in the up and down parts of the magnetic surface. Due to this difference, the toroidally trapped particles acquire a net radial drift. We study the collisionless dynamics of the particles in such configurations as well as solutions of the kinetic equation. We show that, in spite of the net radial drift, the up-down asymmetry does not lead to enhanced diffusive transport, compared to the conventional ripple transport, although the convective loss at low collisionality might be large.

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1. Introduction

The most important classical channel of the particle loss in toroidal magnetic systems is the ripple diffusion of the toroidally trapped particles [1]. It arises due to distortion of the axial symmetry of the torus by the finite perturbation of the toroidal magnetic field. Hence, the ripple diffusion has been the subject of intense studies for the ideal configurations [1-7] as well as for concrete devices [8-11].

The best solution to the problem is to diminish the toroidal asymmetry. But decreasing the ripple magnitude is quite expensive, because it requires that the volume of the coils be increased. That is why studies were aimed mostly at estimating the critical magnitude of ripple in order to find the balance between allowed losses and a reasonable design of the device. Not only is the loss value important, but also the distribution of the particle and energy flux on the chamber walls should be known. Here, the influence of the geometry and the features of the orbits come into play. Although the main loss of bulk plasma is expected to be governed by anomalous mechanisms, the classical loss of high energy particles (alpha-particles, NBI-ions, tail particles) is comparable with the anomalous one.

In this regard, any new breaking of the symmetry of the magnetic field should be thoroughly investigated, as it might lead to an additional loss channel and affect the confining properties of the device.

In the present paper we investigate the neoclassical transport associated with up-down asymmetry of the magnetic field in tokamaks and stellarators. In stellarators, the ripple magnitude is usually considered to be some function of the minor radius. In several cases, helical modulation dependent on the poloidal angle were studied (see [5,12]), but the up-down symmetry was always implied both for stellarators and tokamaks. But this is not always the case in real experiments. Substantial up-down asymmetry can arise due to even

rather small vertical displacement of plasma with respect to the center of the chamber [13,14]. Besides, all magnetic configurations with a single null divertor are naturally asymmetric from the very beginning. In such cases, the depth of the local magnetic wells, Δ , is different in the upper and lower parts of the magnetic surfaces. The value of this asymmetry, estimated in Secs .2 and 3, can be of order unity, $(\Delta^+ - \Delta^-) / (\Delta^+ + \Delta^-) \approx 1$. In these magnetic configurations, the toroidally trapped particles (those with reflection points located in the region where the local magnetic wells exist) acquire a net radial velocity. This velocity, evaluated in Sec. 4, is related to the toroidal drift velocity v_d and the toroidicity ϵ_t , $\dot{r}_t \approx v_d \sqrt{\frac{\Delta}{\epsilon_t}}$ and could lead to an additional transport. This is the case at low collisionality, when the presence of physical boundaries is essential and the losses due to new net drift motion are rapid. The situation is different at higher collisionality, when the radial displacement of particles between collisions is small. In this case, the rigorous analysis of the bounce-averaged kinetic equation (Sec. 5) shows that the additional diffusive flux of the toroidally trapped particles is totally compensated by the change in the flux of the locally trapped particles, so that the transport remains unchanged. Conclusions on the subject are presented in Sec. 6.

2. Up-down asymmetry of the ripple

Neoclassical transport in the toroidal magnetic confinement devices is determined mostly by the magnetic field strength,

$$B = B_0 [1 - \epsilon_t \cos(\theta) - \epsilon_r \cos(N\phi - l\theta)]. \quad (1)$$

Here, $\epsilon_t = r/R_0$ is the inverse aspect ratio (toroidicity), ϵ_r is the ripple magnitude (or the helical modulation, for the stellarator); l and N (usually $N \gg l$) are the poloidal and toroidal mode numbers for a stellarator, while $l=0$ for a tokamak; θ and ϕ are the poloidal and

toroidal angles, which are related to each other on the magnetic field lines via the rotational transform, $d\phi/d\theta = -1/q = -q$; r and $R=R_0(1+\varepsilon_r\cos(\theta))$ are the minor and major radii.

Let us estimate the influence of the vertical displacement of the plasma to the magnitude of the ripple. When the magnetic field in the tokamak is created by uniformly distributed circular coils, the contours of constant ripple strength ε_r are approximately circles with centers shifted towards the major axis of the torus. In this case, we have the following equation for the dependence of the ripple amplitude on coordinates [4].

$$\varepsilon_r = \varepsilon_{\min} I_0\left(\frac{Nb}{R_0 - L_0}\right), \quad (2)$$

where I_0 is the modified Bessel function, $\varepsilon_{\min}$ and L_0 are parameters determined by the shape of the coils, b is the radius of the circle $\varepsilon_r = \text{constant}$,

$$b = \left[(r \cos(\theta) + L_0)^2 + (r \sin(\theta) + \Delta z)^2 \right]^{1/2} \left(\frac{R_0 - L_0}{R_0 + r \cos(\theta)} \right)^{1/2}, \quad (3)$$

with Δz being the vertical displacement of plasma. The approximation $I_0(x) \approx \exp(x)$, $L_0 \ll a, R_0$, $\Delta z \ll a$, where a is the minor radius of the torus, gives then

$$\varepsilon_r \approx \varepsilon_r^{\text{sym}} \exp\left(\frac{N\Delta z \sin(\theta)}{R_0}\right), \quad (4)$$

$$\frac{\varepsilon_r(r, \pi/2)}{\varepsilon_r(r, -\pi/2)} \approx \exp\left(\frac{2N\Delta z}{R_0}\right). \quad (5)$$

For reasonable parameters, $N=16$ and $\Delta z / R_0 = 0.02$, the ripple varies by a factor of two over the plasma volume. Typical contours of the ripple amplitude for the shifted plasma are depicted in Fig.1.

The effective depth of the local magnetic well, Δ , depends not only on the magnitude of ripple. It also depends on parameter α (see [4,5]) via

$$\Delta = \varepsilon_r \left[\sqrt{1 - \alpha^2} - \alpha \left(\frac{\pi}{2} - \sin^{-1}(\alpha) \right) \right], \quad (6)$$

where

$$\alpha \equiv \frac{(\mathbf{B} \cdot \mathbf{e}_R)}{BN\varepsilon_r} = \frac{\varepsilon_t \sin(\theta)}{\varepsilon_r Nq}, \quad (7)$$

where $\mathbf{e}_R$ is the unit vector directed along the major radius. One can see that in configurations with a single null divertor, due to decrease of $(\mathbf{B} \cdot \mathbf{e}_R)/B$ in the vicinity of the X-point, the vertical asymmetry in the value of Δ can be of order of unity.

The magnetic field (1) has the same form for tokamaks and stellarators [5], although the ripple magnitudes are different functions of the coordinates. Since we do not use any particular form for these functions, all the results obtained below are applicable to both stellarator and tokamak, i.e. to the toroidally asymmetric tori with up-down asymmetry of the toroidal perturbation of the magnetic field.

3. Trajectories of the particles in the up-down asymmetric tori.

Let us consider the effect which the asymmetric ripple produces. There are essentially three types of motion in the rippled magnetic field: passing, toroidally trapped and locally trapped particles. As the magnetic wells are not generally constant, transitions between locally and toroidally trapped states occur, so that there exist transition particles which "live" part of the time in toroidally and locally trapped states [4,15,16]. In a vertically symmetric magnetic field, bananas (we use the term "banana" as short for "toroidally trapped") are closed on average, so they have no systematic radial displacement. Slight unclosing of the banana trajectories is related to the different toroidal phases of the reflection points of the toroidally trapped particles. Transport associated with this phase

uncompensation in radial displacement is studied in [3,4,5]. Averaging over this phase gives zero radial displacement. Note, that the phase averaging means an average over an ensemble of the toroidally trapped particles with the same energy and magnetic moment.

The situation is quite different when the magnetic wells confining the locally trapped particles are vertically asymmetric. Due to the asymmetry, the toroidally trapped particle reflects in the upper and lower parts of the torus at points which are not symmetric and lie on different magnetic surfaces (see Fig.2a,b). This difference persists after phase averaging. As a result, the banana acquires a net inward or outward drift. The sign of the net flux depends on the directions of the drift velocity and the gradient of the ripple amplitude. Let us assume, for example, that the ripple is bigger in the upper plane, and the drift velocity is directed downwards. Then the upper reflection point is closer to the midplane than the lower one, and the banana, as a whole, moves outwards. If we change the sign of the drift or make the ripple larger in the lower plane, the banana drifts inwards.

To balance the net banana flux and maintain Liouville's theorem, a counter-flux should exist. A counter-flux cannot be provided by the particles in the same state, because all of them have the same average velocity. The counter-flux can be provided only by particles in the locally trapped state which then collisionlessly detrap and become the toroidally trapped. This means that the new phenomena concern only the transition particles which drift inwards in one state and outwards in another. The resulting drift of the whole transition orbit is shown on Fig.2a,b. The non-transitional particles cannot experience non-zero average drift, because the invariant group of the particles cannot have a sink without having a source, and vice-versa. Only the transition orbits provide this kind of circulation. This means that, to obtain the new transport, the local wells should exist in the reflection points of the banana particles, i.e. the condition, $\alpha \leq 1$, must be fulfilled in some region of the cross-section, where α is given by Eq.(7).

According to this picture, the toroidally trapped region of the velocity space has sources and sinks on the boundary with the locally trapped region, meanwhile the locally trapped region has the corresponding sinks and sources. Note that in the vertically symmetric case, the sinks and sources of the transition orbits in the upper and lower parts compensate each other.

The balance between the incoming and outgoing fluxes actually determines the relationship between the drift velocities in toroidally and locally trapped states,

$$\dot{r}_t f_t \approx v_d f_r, \quad (8)$$

where $\dot{r}_t$ and v_d is the average drift velocity in the toroidally and locally trapped state,

$$v_d = \frac{\mu B_0}{R_0 \Omega}, \quad (9)$$

with Ω the gyro frequency, $\Omega = \frac{eB_0}{mc}$, and

$$f_t \approx \sqrt{\varepsilon_t} \quad (10)$$

and

$$f_r \approx \sqrt{\varepsilon_r} \quad (11)$$

are the relative phase space volumes corresponding to the toroidally and locally trapped states.

These transition particles can be lost either diffusively or convectively, as is determined by the characteristic time and length scales. If the drift radial displacement between collisions is much smaller than the minor radius, the loss is diffusive. This displacement is $\frac{\dot{r}_t}{v_t^{\text{eff}}} \approx \sqrt{\frac{\varepsilon_r}{\varepsilon_t}} v_d / v_t^{\text{eff}}$ in the toroidally trapped state, while the displacement of the locally trapped particles is v_d / v_r^{eff} . Since the transition particles have two branches

with different radial drift and different collisionality, the losses are diffusive, if the following two conditions are fulfilled simultaneously,

$$v_t^{\text{eff}} \gg \sqrt{\frac{\epsilon_r}{\epsilon_t}} \frac{v_d}{a} \quad (12)$$

$$v_r^{\text{eff}} \gg \frac{v_d}{a}, \quad (13)$$

with v_k^{eff} the effective collision frequency for scattering out of the state,

$$v_k^{\text{eff}} = \frac{v}{f_k^2}, \quad (13)$$

and f_k given in Eqs.(10) and (11). Eqs.(12) and (13) can be rewritten as

$$v \gg \frac{v_d}{a} \sqrt{\epsilon_r} \max\{\sqrt{\epsilon_r}, \sqrt{\epsilon_t}\}. \quad (14)$$

The diffusive regimes can be classified by a particular dependence of the diffusion coefficient, D , on the collision frequency [4,5]. At sufficiently low collisionality, when the drift frequencies are small,

$$v_k^{\text{eff}} \ll \frac{\dot{r}_k}{a}, \dot{\theta}_k, \quad (15)$$

we obtain the so-called v -regime ($D \propto v$). In the case opposite to (15), we get to $1/v$ -regime ($D \propto 1/v$). Since the drift frequency of the bananas is

$$\frac{\dot{r}_t}{a} \approx \frac{v_d}{a} \sqrt{\frac{\epsilon_t}{\epsilon_r}}, \quad (16)$$

the condition (12) for the onset of the regime of the diffusive losses coincides with the condition of the onset of the $1/v$ -regime. Hence, there is no v -regime for the toroidally trapped particles.

The situation might be different for the locally trapped particles. Their poloidal drift velocity can be different from $\frac{v_d}{a}$,

$$\dot{\theta}_r \approx \max\left\{\Omega_E, \frac{v_d}{a}, \frac{v_d}{\epsilon_t} \frac{\partial \epsilon_r}{\partial r}\right\}, \quad (17)$$

with

$$\Omega_E = \frac{e}{mr\Omega} \frac{\partial \Phi}{\partial r}. \quad (18)$$

Comparing Eqs.(13) and (15), we can conclude that the v-regime exists for the locally trapped particles, provided the condition,

$$\dot{\theta}_r > \frac{v_d}{a}, \quad (19)$$

is fulfilled. This can happen at a sufficiently large electric field ($e\Phi$ comparable with energy) or when the ripple is larger than the toroidicity.

If the collision frequency obeys the condition opposite to Eq.(14), then the loss is convective and quite rapid. The particles are lost at a rate

$$\frac{1}{\tau} \approx \min\left\{\frac{a}{\dot{r}_t}, \frac{a}{\dot{r}_r}\right\}. \quad (20)$$

Although the motion of the particles, in general, is very complicated, we can imagine the collisionless convective loss as the following: in one of the cases, when bananas drift outwards (Fig.2a), the transition particles start their motion in the locally trapped state and move downwards. As the ripple decreases, they become detrapped and start drifting directly outwards as bananas. In the opposite situation, as bananas drift inwards (Fig.2b), they become locally trapped and leave the system as locally trapped particles.

4. Radial displacement of the bananas

Here we evaluate the radial displacement of the toroidally trapped particle on one bounce - back and forth- between the reflection points without specifying the particular form of the ripple.

The radial displacement of the toroidally trapped particle on one circuit has been evaluated in many papers (see [3-5,17]). But the approach used in these papers implied that the ripple was vertically symmetric. In some papers the ripple was considered to be so small, that no wells existed in the reflection points of the banana. Here, we consider the case, when the ripple is strong enough to form the wells and when it is not not necessarily vertically symmetric.

To evaluate the radial displacement, it is convenient to introduce the adiabatic invariant of the banana, J^t (see [17]),

$$J^t = 2R_0 \int_{\phi_1}^{\phi_2} \frac{B_0}{B} \sqrt{2\{E - \mu B_0[1 - \varepsilon_i \cos(\beta - \phi' / q) - \varepsilon_r \cos(N\phi')]\}} d\phi', \quad (21)$$

where

$$\beta \equiv \theta + \phi / q = \text{constant} \quad (22)$$

denotes the field line; θ and ϕ being some reference angles through which the line goes. The integrand in Eq.(21) is just the absolute value of the parallel velocity, u , multiplied by the length factor B_0/B . The integration is performed between the reflection points (tips of the banana).

The radial displacement of the particle in one circuit is

$$d = \oint \frac{dr}{dt} dt,$$

where

$$\frac{dr}{dt} = \frac{1}{r\Omega} \frac{uB_0}{B} \frac{\partial}{\partial\beta} \left(\frac{uB_0}{B} \right), \quad (23a)$$

$$dt = \frac{d\phi}{\dot{\phi}}, \quad (23b)$$

$$\dot{\phi} = \frac{uB}{R_0 B_0}. \quad (24)$$

This yields,

$$d = \frac{2R_0}{r\Omega} \int_{\phi_1}^{\phi_2} \frac{\partial}{\partial\theta'} \left(\frac{uB_0}{B} \right) d\phi' = \frac{2R_0}{r\Omega} \int_{\phi_1}^{\phi_2} \frac{\partial}{\partial\beta} \left(\frac{uB_0}{B} \right) d\phi'. \quad (25)$$

Comparing Eqs.(21) and (25), one can see that d can be evaluated by differentiating J^t with respect to β [17]. But in our case, this should be done with caution, because the reflection points, ϕ_1 and ϕ_2 , are not smooth functions of β , but have steps at some β_1 and β_2 , corresponding to the situation when the reflection points touch the local maxima (see Fig.3). These steps give rise to delta-functions, when the limits of the integral (21) are being differentiated. In these points J^t has a step-like change equal to the separatrix value of the invariant of the locally trapped particle, J_s^r ,

$$J_s^r = R_0 \oint_{\text{well}} u \frac{B_0}{B} d\phi', \quad (26)$$

where the integration is performed over the local minima near the reflection point (between ϕ_0 and ϕ_2 on Fig.3). Bearing this in mind, for the derivative $\partial J^t / \partial \beta$ we obtain

$$\frac{\partial J^t}{\partial \beta} \equiv 2R_0 \int_{\phi_1}^{\phi_2} \frac{\partial}{\partial \beta} \left(\frac{uB_0}{B} \right) d\phi' - \delta(\beta - \beta_1) J_s^r(\tau) + \delta(\beta - \beta_2) J_s^r(-\tau). \quad (27)$$

Note, that in Eq.(27) we used the fact that, in accordance with Eq.(22), while one moves along the field line ($\beta=\text{constant}$) with ϕ increased, the poloidal angle, θ , decreases.

The adiabatic invariant J^t , with the energy and magnetic moment being fixed, is a periodic function of β , with a period of 2π . Thus, the average of $\partial J^t / \partial \beta$ over the period is zero. Averaging Eq.(27) and using Eq.(23) gives the following average radial displacement ,

$$\bar{d} = \frac{1}{2\pi} \int_0^{2\pi} d(\beta) d\beta = \frac{N}{2\pi r \Omega} (J_s^r(\tau) - J_s^r(-\tau)). \quad (28)$$

The factor N appears, because, while moving along β , we meet N local magnetic wells. Note, that the averaging in Eq.(28) is just the averaging over the toroidal phase of the reflection point of the banana. When the ripple does not exist in the upper or lower bounce points, the corresponding J_s^r in Eq.(28) should be set to zero.

The net radial drift of the banana particles, described by Eq.(28) and independent of β , i.e. constant on a magnetic surface, is a new phenomena. It exists only in the vertically asymmetric configurations, when the contributions from J_s^r in the upper and lower parts do not compensate each other. In the next section we study the effect of this net drift on transport.

5. Kinetic equation and diffusive transport

It is known that the kinetic equation can be written in the divergence form [18,19],

$$\text{div}_z(z|G|F) = |G|C(F) , \quad (29)$$

where z is $2*N$ ($N=3$) pseudo-canonical phase space variables with G being the Jacobian of the transformation and $C(F)$ the collision operator. The natural guiding center variables are $(y; E, \mu, \phi_g)$ with y being the guiding center location, E the energy, μ the magnetic moment and ϕ_g the gyro phase. The Jacobian of these variables is [18,19],

$$|G| = \frac{B_0^2}{B_u}. \quad (30)$$

To study the transport of thermal particles in the toroidal fusion devices, it is sufficient to use the Lorentz form of the collision operator,

$$|G|C(F) = v \left(\frac{B_0}{B} \right)^2 \frac{\partial}{\partial \mu} \left(\mu u \frac{\partial}{\partial \mu} F \right), \quad (31)$$

because the distribution function usually has steep changes in narrow regions of the pitch angle, so the slowing down and energy scattering terms can be neglected [4,5].

After the averaging over ϕ_g , Eq.(29) takes the following form in the toroidal coordinates, (r, β, ϕ) ,

$$\frac{\partial}{\partial r}(\dot{r}|G|F) + \frac{\partial}{\partial \beta}(\dot{\beta}|G|F) + \frac{\partial}{\partial \phi}(\dot{\phi}|G|F) = v \left(\frac{B_0}{B} \right)^2 \frac{\partial}{\partial \mu} \mu u \frac{\partial F}{\partial \mu}, \quad (32)$$

with the toroidal component of the drift velocity being proportional to the Jacobian (see Eq.(24)).

If the motion of the particles is almost periodic, Eq.(32) can be simplified by averaging over the fast parallel motion [5,19],

$$\frac{\partial}{\partial r}(T_b \langle \dot{r} \rangle \langle F \rangle) + \frac{\partial}{\partial \beta}(T_b \langle \dot{\beta} \rangle \langle F \rangle) = \frac{v}{B_0} \frac{\partial}{\partial \mu} \left(\mu I \frac{\partial}{\partial \mu} \langle F \rangle \right) + S. \quad (33)$$

Here,

$$\langle \dot{r} \rangle = \frac{1}{T_b r \Omega} \oint \frac{uB}{B_0} \frac{\partial}{\partial \beta} \left(\frac{uB_0}{B} \right) \frac{d\phi}{\dot{\phi}}, \quad (34)$$

$$\langle \dot{\beta} \rangle = -\frac{1}{T_b r \Omega} \oint \frac{uB}{B_0} \frac{\partial}{\partial r} \left(\frac{uB_0}{B} \right) \frac{d\phi}{\dot{\phi}}, \quad (35)$$

$$\langle F \rangle = \oint F d\phi, \quad (36)$$

$$J = \oint R_0 \frac{u B_0}{B} d\phi, \quad (37a)$$

$$I = R_0 \oint \left(\frac{B_0}{B} \right)^2 u d\phi, \quad (37b)$$

$$T_b = \oint \frac{d\phi}{\dot{\phi}} = \frac{\partial J}{\partial E}, \quad (38)$$

Here, integration is performed over the period of motion along the field line. We should bear in mind that Eqs.(33)-(38) are different for different types of motion, since the averaged quantities are different for the toroidally and locally trapped particles, although we omitted the distinguishing index for simplicity. Note that, to obtain Eq.(34), we assumed that $(F - \langle F \rangle)$ is small and, in what follows, we omit the brackets setting $F = \langle F \rangle$. We introduced the quantity I (Eq.(37b)) which is different from the adiabatic invariant, J , and coincides with J only if we neglect the ripple modulation in the length factor, which is usually implied in most of the papers [4,5,15].

In the right hand side of the kinetic equation (33) we inserted the source term S . The appearance of this term is due to the fact that, to perform the bounce averaging, we have to separate the groups of particles with one type of periodic motion from particles with the other. Since there exist collisionless transitions between these groups, we include the source term, S , describing these transitions.

Discussion about the radial displacement of the banana in the previous section led us to the following. Due to the fact that the adiabatic invariant of the toroidally trapped particles has steps on the boundary with the locally trapped particles (see Fig.4) the derivatives of J^t have additional terms proportional to delta-functions. That is why the averaged velocity of the toroidally trapped particles are not just given by the cross derivatives of the invariant, but also include δ -functions (compare with [19]):

$$\langle \dot{r} \rangle^t = \frac{1}{r\Omega \partial J^t / \partial E} \left(\frac{\partial J^t}{\partial \beta} - \delta(Q) \frac{\partial Q}{\partial \beta} J_s^r \right) \equiv \dot{r}^t + \dot{r}_b^t, \quad (39)$$

$$\langle \dot{\beta} \rangle^t = -\frac{1}{r\Omega \partial J^t / \partial E} \left(\frac{\partial J^t}{\partial r} - \delta(Q) \frac{\partial Q}{\partial r} J_s^r \right) \equiv \dot{\beta}^t + \dot{\beta}_b^t, \quad (40)$$

where the condition,

$$Q(r, \beta, E, \mu) = 0, \quad (41)$$

defines the separatrix between the locally and toroidally trapped particles. We take the convention that $Q > 0$ corresponds to the toroidally trapped particles and $Q < 0$ to the locally trapped particles. Eqs.(39) and (40) should be compared with Eq.(27). Note that the signs of $\partial Q / \partial \beta$ and $\partial Q / \partial r$ take care of the different signs of contribution which come from the solutions of Eq.(41) in the different midplanes. In Eqs.(39) and (40) and below, we split the velocities into the main part and the "boundary" part, marked by the subscript "b."

The adiabatic invariant of the locally trapped particle, J^r , is defined similarly to J^t , except that the integration over ϕ is performed inside the local magnetic well. We should also keep in mind that J^r exists only at $Q < 0$, i.e. it also has steps of the size of J^r itself. This gives rise to similar δ -functions for the derivatives of J^r . The averaged drift velocity of the locally trapped particle can then be presented in the form analogous to Eqs.(39) and (40),

$$\langle \dot{r} \rangle^r = \frac{1}{r\Omega \partial J^r / \partial E} \left(\frac{\partial J^r}{\partial \beta} + \delta(Q) \frac{\partial Q}{\partial \beta} J_s^r \right) \equiv \dot{r}^r + \dot{r}_b^r, \quad (42)$$

$$\langle \dot{\beta} \rangle^r = -\frac{1}{r\Omega \partial J^r / \partial E} \left(\frac{\partial J^r}{\partial r} + \delta(Q) \frac{\partial Q}{\partial r} J_s^r \right) \equiv \dot{\beta}^r + \dot{\beta}_b^r. \quad (43)$$

Substituting Eqs.(39),(40) into Eq.(33) yields

$$\langle \dot{r} \rangle^t \frac{\partial F^t}{\partial r} + \langle \dot{\beta} \rangle^t \frac{\partial F^t}{\partial \beta} + A^t = \frac{v}{B_0 T_b} \frac{\partial}{\partial \mu} \mu I^t \frac{\partial F^t}{\partial \mu} + S^t, \quad (44)$$

where we introduced the quantity A^t related to the non-divergence-free part of the velocity, $\dot{r}_b^t$ and $\dot{\beta}_b^t$,

$$\begin{aligned} A^t &= \frac{F^t \delta(Q)}{r \Omega \partial J^t / \partial E} \left(\frac{\partial Q}{\partial \beta} \frac{\partial J_s^r}{\partial r} - \frac{\partial Q}{\partial r} \frac{\partial J_s^r}{\partial \beta} \right) = \\ &= -F^t \delta(Q) \left(\frac{\partial Q}{\partial \beta} \dot{\beta}^r + \frac{\partial Q}{\partial r} \dot{r}^r \right) = -F^{t,r} \delta(Q) \frac{dQ}{dt} \Big| \end{aligned} \quad (45)$$

In obtaining Eq.(45) we used the fact that the distribution functions of the locally and toroidally trapped particles are equal on the separatrix.

For further transformation, let us notice that A^t turns out to be the velocity of change of the phase volume of the locally trapped region along the drift trajectory of the locally trapped particle. That is why, A^t is proportional to the number of collisionless transitions from the locally to toroidally trapped states i.e. is, in fact, equal to the change of the number of toroidally trapped particles per unit of time,

$$A^t = -\dot{n}^r = \dot{n}^t = S^t. \quad (46)$$

Thus, the part of the kinetic equation related to the divergent part of the velocity is equal to the source, which is not surprising. A similar procedure can be carried out for the kinetic equation of the locally trapped particles, to show that $A^r = S^r = -A^t$, so that the resulting equations take the familiar form [4,5],

$$\langle \dot{r} \rangle^t \frac{\partial F^t}{\partial r} + \langle \dot{\beta} \rangle^t \frac{\partial F^t}{\partial \beta} = \frac{v}{B_0 \partial J^t / \partial E} \frac{\partial}{\partial \mu} \mu I^t \frac{\partial F^t}{\partial \mu}, \quad (47)$$

$$\langle \dot{r} \rangle^r \frac{\partial F^r}{\partial r} + \langle \dot{\beta} \rangle^r \frac{\partial F^r}{\partial \beta} = \frac{v}{B_0 \partial J^r / \partial E} \frac{\partial}{\partial \mu} \mu I^r \frac{\partial F^r}{\partial \mu}, \quad (48)$$

where the velocities are given by Eqs.(42),(43),(45),(46) and the distribution functions obey the boundary condition,

$$\delta(Q)F^t = \delta(Q)F^r \equiv \delta(Q)F^b. \quad (49)$$

Let us split the the distribution function into two parts: the average part,

$$\bar{F}^{t,r} = \frac{1}{2\pi} \oint F^{t,r} d\beta, \quad (50)$$

and the part which varies on the magnetic surface,

$$\tilde{F}^{t,r} = F^{t,r} - \bar{F}^{t,r}. \quad (51)$$

We do this in order to separate the new effect described in the previous section (the β -independent drift of the toroidally trapped particles) which gives rise to the β -independent addition to the distribution function. The radial fluxes averaged over the magnetic surface,

$$\Gamma^{t,r} = \oint \frac{d\beta}{2\pi} \left(\oint \frac{d\phi}{2\pi} d^3v \right) \langle \hat{r} \rangle^{t,r} F^{t,r}, \quad (52)$$

where $\langle r \rangle^{t,r}$ are given by Eqs.(39) and (42), are split correspondingly,

$$\Gamma^{t,r} = \bar{\Gamma}^{t,r} + \tilde{\Gamma}^{t,r}, \quad (53)$$

so that $\bar{\Gamma}^{t,r}$ and $\tilde{\Gamma}^{t,r}$ come from $\bar{F}^{t,r}$ and $\tilde{F}^{t,r}$, respectively.

Since the adiabatic invariant is a periodic function of β , the β -average of $\partial J^{t,r}/\partial\beta$ is zero. That is why $\hat{r}^{t,r}$, proportional to $\partial J^{t,r}/\partial\beta$ (see Eqs.(39) and (42)), being averaged with $\bar{F}^{t,r}$, do not contribute to $\bar{\Gamma}^{t,r}$, so that only $\hat{r}_b^{t,r}$ are left there. The remaining parts of $\bar{\Gamma}^{t,r}$ turn out to be equal in the absolute value and have opposite signs for the toroidally and locally trapped particles,

$$\bar{\Gamma}^t = -\frac{1}{rR_0\Omega} \oint \frac{d\beta}{2\pi} \int_{\text{tor.trap}} d\mu B_0 \int dE \delta(Q) \frac{\partial Q}{\partial\beta} J_s^r \bar{F}^b = -\bar{\Gamma}^r. \quad (54)$$

To obtain Eq.(54) we used the equality of the distribution functions of the locally and toroidally trapped particles on the boundary (Eq.(49)), the relation

$$\oint \frac{d\phi}{2\pi} d^3v = \frac{\partial J^{t,r}}{\partial E} \frac{dE d\mu B_0}{R_0}, \quad (55)$$

and the fact that, due to the δ -function, the integral over μ in the toroidally trapped region gives the same as the one over the locally trapped region.

Eq.(54) shows that the fluxes produced by the constant part of the distribution function, $\bar{F}$, compensate each other, and only $\tilde{F}(\beta)$ contributes to transport. This means, in particular, that the net radial drift of bananas described in the previous section does not result in new transport, its flux being compensated by the corresponding flux of the locally trapped particles.

The contribution to fluxes $\tilde{\Gamma}^{t,r}$ coming from δ -terms also compensate each other. All that remains, is

$$\Gamma^{t,r} = \frac{1}{rR_0\Omega} \oint \frac{d\beta}{2\pi} \int_{\text{tor.,loc.trap.}} d\mu B_0 \int dE \frac{\partial J^{t,r}}{\partial \beta} \tilde{F}^{t,r}, \quad (56)$$

which is the diffusion produced by the irregular differences between the phases in the reflection points of the banana (which are zero on average) and the usual diffusion of the locally trapped particles.

The remaining contribution to the diffusion in the $1/v$ -regime of the toroidally trapped particles coming from $\tilde{F}^t$ is almost the same for the up-down symmetric and asymmetric tori. This has been investigated before (see Eq.A.I.4 of [5]) and is related to the $1/v$ -regime of the locally trapped particles by the relation, $D_t^{\text{sym}} / D_r^{\text{sym}} \approx \frac{1}{(qN)^2} \sqrt{\frac{\epsilon_t}{\epsilon_r}}$. Note however, that some new effects due to the asymmetry, which do not change the order of the fluxes, can be expected in the results obtained in [4,5] since we should include the general (up-down asymmetric) dependence of ϵ_r on the poloidal angle. It will give new terms for $\frac{\partial J^{t,r}}{\partial \beta}$ with a relative order $\frac{\partial \epsilon_r / \partial \beta}{\epsilon_r}$ in comparison with the conventional ones.

We find the effect of compensation of the diffusive fluxes quite curious. It is a consequence of Liouville's theorem, which turns out to work even for the collisional case, because the effects investigated here are independent on the magnetic field line of the magnetic surface. Substantial new effects due to the vertical asymmetry should occur in the convective transport, when the deviation of the distribution function from the Maxwellian can not be considered a perturbation. In this case the kinetic equations (50) and (51) are still valid, but need an introduction of a loss cone. This approach was developed in [19] and can be carried out analytically only for the simplest case of the vertically homogeneous ripple without electric field.

6. Explicit evaluation of $\bar{F}$

Although the flux of the toroidally trapped particles due to up-down asymmetry is compensated by the opposite flux of the locally trapped particles, it is interesting to evaluate it. We perform this procedure below, implying that $\varepsilon_r / \varepsilon_t \ll 1$, which allows us to set $I^{tr} = J^{tr}$ (given by Eqs.(37a) and (37b)) in Eqs.(47) and (48). The kinetic equation for $\bar{F}$ in the main order takes the form (compare with Eq.(47)),

$$\bar{d} \frac{\partial \bar{F}^t}{\partial r} = \frac{v}{B_0} \frac{\partial}{\partial \mu} \mu J^t \frac{\partial \bar{F}^t}{\partial \mu}, \quad (57)$$

where we took into account that the radial velocity of the banana $\langle \dot{r} \rangle^t$ (Eq.(39)) averaged over β is equal to $\frac{\bar{d}}{\partial J^t / \partial E}$ with $\bar{d}$ given by Eq.(28). For further convenience, let us transform from μ to the variable κ ,

$$\kappa = \cos(\tau), \quad (58)$$

where variable τ is defined to be the reflection point of the toroidally trapped particle in the upper part, when $\varepsilon_r = 0$,

$$1 - \varepsilon_t \cos(\tau) = \frac{E}{\mu B_0}, \quad (59)$$

$$0 \leq \tau \leq \pi.$$

Then Eq.(57) takes the form,

$$\bar{d} \frac{\partial F_m}{\partial r} = \frac{v}{\mu B_0 \varepsilon_t^2} \frac{\partial}{\partial \kappa} J^t \frac{\partial}{\partial \kappa} \Psi, \quad (60)$$

where we expanded the distribution function in the parameter $\frac{v_d}{av_t^{\text{eff}}}$, so that $\bar{F}^t = F_m + \Psi$ with F_m the Maxwellian and Ψ being the first order correction. The expansion parameter is small indeed, since in the diffusive regime condition (12) is fulfilled. The solution can be found straightforwardly,

$$\Psi = \frac{\partial F_m}{\partial r} \frac{\mu B_0 \varepsilon_t^2}{v} \int_{-1}^{\kappa} \frac{d\kappa'}{J^t(\kappa')} \int_1^{\kappa'} \bar{d}(\kappa'') d\kappa''. \quad (61)$$

To choose the integration limits in Eq.(61), we used the following boundary conditions:

$$\left| \frac{\partial \Psi}{\partial \kappa}(\kappa = 1) \right| < \infty, \quad (62)$$

$$\Psi(\kappa = -1) = 0, \quad (63)$$

at the boundary with the passing particles ($\kappa = -1$) and for the deeply trapped particles ($\kappa = 1$). The flux from Ψ is

$$\bar{\Gamma}^t = \oint \frac{d\beta}{2\pi} \left(\oint \frac{d\phi}{2\pi} d^3v \right) \frac{\bar{d}}{\partial J^t / \partial E} \Psi. \quad (64)$$

Let us simplify the calculation by assuming that the ripple exists only in one of the parts of the torus and does not depend on the poloidal angle, so that

$$|\bar{d}| = \frac{8R\sqrt{\mu B_0 \varepsilon_r}}{\pi r \Omega}, \quad (65)$$

where we used Eq.(28) and the lowest order (in α) formula,

$$J_s^i = \frac{16R}{M} \sqrt{\mu B_0 \epsilon_r}. \quad (66).$$

Substituting the lowest order (in ϵ_r/ϵ_t) expressions

$$J^i = 16qR\sqrt{\mu B_0 \epsilon_t} f(\tau) \quad (67)$$

$$f(\tau) = E(\sin^2(\tau/2)) - \cos^2(\tau/2)K(\sin^2(\tau/2)) \approx \frac{1-\kappa}{2},$$

into Eq.(61), then Eq.(65) into (64), yields

$$\bar{\Gamma}^i = -\frac{8\epsilon_t^{5/2}\epsilon_r}{(\pi\Omega r)^2} \int dE \frac{E^{5/2}}{v(E)} \frac{\partial F_m}{\partial r} I, \quad (68)$$

where

$$I = -\oint \frac{d\theta}{\pi} \int_{-1}^{\cos(\theta)} d\kappa \int_{-1}^{\kappa} \frac{d\kappa'}{(1-\kappa')} \int_1^{\kappa'} d\kappa'' = 2.$$

Note, that Eq.(68) is obtained in the limit $\alpha \rightarrow 0$, $\epsilon_r/\epsilon_t \rightarrow 0$. The diffusion coefficient corresponding to this flux, $D_t^{as} \propto \epsilon_r \sqrt{\epsilon_t} \frac{v_d^2}{v}$, could be found from the simple heuristic formula, $D_t^{as} \propto f_t \frac{\Delta r_t^2}{v_t^{eff}}$, with $\Delta r_t = \frac{\dot{r}_t}{v_t^{eff}}$, $\dot{r}_t \approx v_d \sqrt{\frac{\epsilon_r}{\epsilon_t}}$ (see Eq.(8)), v_t^{eff} given by Eq.(13), f_t - by Eq.(10). Using the same heuristic formula for the usual (up-down symmetric) $1/v$ -regime of the locally trapped particles, $D_r^{sym} \propto f_r \frac{\Delta r_r^2}{v_r^{eff}}$, with $\Delta r_r = \frac{v_d}{v_r^{eff}}$, f_r given by Eq.(11), we obtain the well known [5] result (correct for $\alpha < 1$), $D_r^{sym} \propto \epsilon_r^{3/2} \frac{v_d^2}{v}$. The value of the ratio of these diffusion coefficients, $D_t^{as} / D_r^{sym} \approx \sqrt{\frac{\epsilon_t}{\epsilon_r}}$, was what motivated us to investigate this diffusive regime. But, as we have shown above, the asymmetric flux of the toroidally trapped particles, originated from their net radial drift, independent on β , is compensated in the main order by the additional flux of the locally trapped particles. This can be explicitly shown, if we put Ψ given by Eq.(61) into the formula for the asymmetric flux of the locally trapped particles,

$$\bar{\Gamma}^r = \oint \frac{d\beta}{2\pi} \left(\oint \frac{d\phi}{2\pi} d^3v \right) \frac{\partial J^r / \partial \beta}{\partial J^r / \partial E} \Psi_b = -\bar{\Gamma}^t, \quad (68)$$

where Ψ_b is the distribution function Ψ on the boundary with the locally trapped particles ($\kappa = \cos(\theta)$).

Note, that explicit investigation of the case $\varepsilon_r / \varepsilon_t \approx 1$ is more complicated. We do not perform that, because the general result of compensation of the fluxes arising from $\bar{F}$ is valid for the arbitrary relationship between ε_t and ε_r .

6. Discussion of the results

The phenomena investigated above are a new kind of transport process which arise due to the ripple asymmetry in upper and lower parts of the torus. This asymmetry can occur due to the vertical displacement of the plasma or the vertical asymmetry of the magnetic configuration (like in the case of the single null divertor). The change of the ripple magnitude can be as high as unity for the tokamak. Similar phenomena can occur in the stellarator. The resulting equations for the radial displacement and the radial flux are applicable both for the stellarator and tokamak.

It is shown that the vertical asymmetry gives rise to a net radial drift of the toroidally trapped particles with the characteristic velocity $\frac{v_d}{\sqrt{\varepsilon_t}} (\sqrt{\varepsilon_r^+} - \sqrt{\varepsilon_r^-})$, where ε_r^+ and ε_r^- are the ripple magnitudes in the upper and lower parts of the torus. At low collisionality, the losses of the particles have a convective character. The particles rapidly leave the device either in the toroidally trapped state, when they acquire the net radial outward drift, or in the locally trapped state after experiencing entrapment in the unfavorable part of the system. The new mechanism presents a new channel of the particle

loss, so that for every device there exists a critical value of allowed vertical asymmetry, which needs to be estimated in each particular case.

In the high collisionality regime, the diffusion of the toroidally trapped particles associated with the asymmetry is shown to be compensated by the additional diffusive flux of the locally trapped particles. This is related to the fact that a distribution function constant on a magnetic surface does not contribute to transport. That is the reason why the net radial drift of bananas (the dominant part of which does not vary on a magnetic surface) does not change the transport. The remaining part of the radial drift velocity of the bananas is a rapid function of the poloidal angle (or β); its contribution to the fluxes has been investigated in [4,5].

Figure captions:

- Fig.1. Contour lines, $\varepsilon_r = \text{constant}$, for the vertically shifted plasma. The picture is obtained numerically (courtesy of Azita Valinia) for TEXT-U, plasma has been moved down by 2.5 cm.
- Fig.2a. The bounce-averaged transition orbit for the case when the ripple magnitude is higher in the upper part of the torus: "r" corresponds to the locally trapped part, "t" - to the toroidally trapped of the orbit. The particle tends to detrap in the lower part and drifts outwards in the toroidally trapped state due to asymmetry of the reflection points.
- Fig.2b. The bounce-averaged transition orbit for the case when the ripple magnitude is higher in the lower part of the torus: "r" corresponds to the locally trapped part, "t" - to the toroidally trapped of the orbit. The particle tends to entrap in the lower part and drifts inwards in the toroidally trapped state due to asymmetry of the reflection points.
- Fig.3. Illustration to Eqs.(23), (24). When the toroidal phase of reflection corresponds to "touching" of the magnetic maxima, the adiabatic invariant J^l increases by amount J^r , equal to the separatrix value of the invariant of the locally trapped particle.
- Fig.4. The schematic depiction of the longitudinal motion of the particles with the same magnetic moment, μ , on the plane (ϕ, θ) . Straight lines correspond to different values of β , marking different field lines. At some particular values of β , β_i and β_j , the adiabatic invariant J^l changes abruptly by amount J^r . The "total" invariant J , obtained from all the pieces along the field line, is free from these abrupt changes.

References:

1. H.P. Furth, R.J. Goldston, S.J. Zweben, D.J. Sigmar, *Nuclear Fusion* **30**, 1799 (1990).
2. R.B. White, H.E. Mynick, *Phys. Fluids B* **1**, 980 (1989).
3. R.J. Goldston, R.B. White, A.H. Boozer, *Phys. Rev. Let.* **47**, 647 (1981).
4. P.N. Yushmanov, in *Reviews of Plasma Physics*, edited by B.B.Kadomtsev (Consultants Bureau, New York, 1991), Vol.16, p.117.
5. L.M.Kovrizhnykh, *Nuclear Fusion* **24**, 851 (1984).
6. K. Tani, T. Takizuka, M. Azumi, H. Kishimoto, *Nuclear Fusion* **23**, 657 (1983).
7. L.M. Hively, *Nuclear Fusion* **24**, 779 (1984).
8. F.S.Zajtsev, A.P.Smirnov, P.N.Yushmanov, *Nuclear Fusion* **26**, 1311 (1986).
9. R.B. White, A.H. Boozer, R.J. Goldston, R. Hay, J. Albert, C.F.F. Karney, in *Plasma Physics and Controlled Nuclear Fusion Research, 1982*, Proc. of the 9th International Conference (IAEA, Vienna, 1983), Vol.III, p.391.
10. K. Tani, T. Takizuka, M. Azumi, in *Proceedings of the Workshop on 'Statistical Plasma Physics'*, US-Japan Joint Institute for Fusion Theory Program (Institute of Plasma Physics), Nagoya **16**, 199 (1976).
11. G. Sadler, P. Barabaschi, E. Bertolini e.a., in *Proceedings of the 19th International Conference on Plasma Physics*, Innsbruck **16C**, Part I, 167 (1992).

12. C.D. Beidler, W.N.G. Hitchon, J.L. Shohet, *J.Comput.Phys.* **72**, 220 (1987).
13. D.P. Schlissel, T.H. Osborne, J.C. DeBoo, J.R. Ferron, E.A. Lazarus, T.S. Taylor, *Nuclear Fusion* **32**, 689 (1992).
14. R.B. Bengston, A. Ouroua, A.G. Meigs, *Bulletin of the American Physical Society*, **34**, No 9, paper 8S3.
15. L.M. Kovrizhnykh, S.G. Shasharina, *Nuclear Fusion* **30**, 453 (1990).
16. John R. Cary, S.G. Shasharina, submitted to *Phys.Fluids*.
17. B.B. Kadomtsev, in *Plasma Phys. and the Problem of Controlled Thermonuclear Reactors*, ed. MA. Leontovich (Izd. Akad.Nauk SSSR,1958) v.3, p.285-299.
18. R.G. Littlejohn, *J.Plasma Physics* **29**, 111 (1983).
19. A.V. Gurevich, Ya.S. Dimant, in *Reviews of Plasma Physics*, edited by B.B.Kadomtsev (Consultants Bureau, New York, 1991), Vol.16, p.1.

Fig.1

Fig. 2a

Fig.3

Fig. 2b

Fig.4

Appendix B. Bootstrap Cuurent of Transition Particles in Non-Axisymmetric Tori

Bootstrap Current of Transition Particles in Non-Axisymmetric Tori

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A specific group of particles exists in 3-D confinement systems, which make transitions back and forth between the toroidally and locally trapped states. The low-collisional distribution function of such particles is found with the appropriate boundary condition on the separatrix derived. This solution is then used to evaluate the bootstrap current. The physical origin of the current is the dependence of the distribution function of transition particles on the radial location of the transition point. Since the transition particles in the toroidally trapped states moving in the parallel and anti-parallel directions undergo the transition on different magnetic surfaces (separated by the distance of the order of the banana width), the number of particles moving in the parallel and anti-parallel directions in any particular point is also different, which gives rise to the additional longitudinal current. This current is found significant for the stellarator configuration.

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1. Introduction

Bootstrap, or "diffusion-driven", current is an interesting phenomenon predicted by the neoclassical transport theory in its early age.¹⁻⁴ The nature of this current is well explained in Ref. 1 and recent Refs. 5, 6. Basically, any parallel current in the toroidal confinement devices, sustained without the parallel electric field, is the bootstrap current. It exists because of a certain asymmetry of the distribution function for the particles moving in parallel and anti-parallel direction. This asymmetry is caused by radial displacement of these particles, due to which parallel and anti-parallel particles come from different radii, so that they "feel" different conditions due to non-uniformity of the temperature and density across the magnetic field.

A precise calculation of the bootstrap current in axisymmetric tori was carried out in Refs. 2, 3, 6. Studies performed for non-axisymmetric configurations reduced the problem to the 2-D case⁶⁻⁸ with corresponding changes in the parameters and were quite limited, indeed. Thus, in particular, the net drift of trapped particles was not taken into account, which implied a hidden symmetry and oversaw specific 3-D features of these systems. Being applied to studies of perpendicular transport, such approach would lead to the loss of contribution of the locally trapped particles into transport.^{2,9} Besides, it neglects the drift change of the parameters of the magnetic wells along trajectories, which, in real 3-D systems, leads to the possibility of collisionless transitions of toroidally trapped particles into locally trapped and vice versa. This process was found to be very important for the low-collisional diffusion.^{9,10} Its influence on the longitudinal transport has never been studied before.

Thus, in the present paper, we are going to study the effect produced by the trap-detrap processes on the bootstrap current, i.e. we evaluate the parallel current of the transition particles in a truly 3-D system. The paper is organized as follows. In Sec. 2 we

briefly explain the physics of the bootstrap current with the emphasis on the details needed to understand new effects arising from transitions of toroidally trapped particles into locally trapped and vice-versa. In Sec. 3 we remind necessary facts on the averaging procedure of the drift kinetic equation, study the transition orbits and introduce the transition probabilities. Then we solve the bounce-averaged kinetic equation for transition particles. In Sec 4, calculate the additional bootstrap current. In Sec. 5 we discuss the results and summarize the paper.

2. The origin of the bootstrap current

The neoclassical theory¹⁻⁵ predicted the bootstrap current, as the longitudinal current sustained without the longitudinal electric field. The palpable explanation, which might appeal to the experimentally oriented reader, is proposed in Ref. 5. More rigorous description of the physics underlying the phenomenon can be found in Ref. 6. Although, being instructive, Ref. 6 does not propose anything really new compared to the original calculation of the bootstrap current carried out in Ref. 2, where the drift kinetic equation was solved in the banana and plateau regimes for the axisymmetric devices.

We will not dwell on the details of those explanations, except for some that are needed. First of all, let us introduce the magnetic configuration we are working with. The standard model of the magnetic field strength has the form,

$$B = B_0(1 - \varepsilon_t \cos(\theta) - \varepsilon_r \cos(M\phi - l\theta)). \quad (1)$$

Here, ε_t is the inverse aspect ratio or toroidicity ($\varepsilon_t = r/R_0$ in the toroidal coordinates), ε_r is the ripple magnitude (or the helical modulation, for the stellarator); l and M (usually $M \gg l$) are the poloidal and toroidal mode numbers for a stellarator, while $l=0$ for a tokamak; θ and ϕ are the poloidal and toroidal angles, which are related to each other on the magnetic field lines via the rotational transform, $\frac{d\phi}{d\theta} = \frac{1}{q} = q$; r and $R = R_0(1 + \varepsilon_t \cos(\theta))$ are the minor

and major radii. The axisymmetric configuration implies $\varepsilon_r=0$. The helically symmetric system can also be recovered by Eq. (1) by setting $\varepsilon_t = 0$, $\phi = L / R_0$ and $\lambda = M / R_0$: to give $B_h = B_0(1 - \varepsilon_r \cos(\lambda L - l\theta))$. Note, that in this section we use the toroidal coordinates for the sake of simplicity, although in the rest sections we shall use Boozer's coordinates to obtain the general answer.

The bootstrap current in an axisymmetric torus at low collisionality (banana regime), can be interpreted as an imbalance between the fluxes of the particles moving in parallel (+) and anti-parallel (-) directions along collisionless trajectories. This imbalance can be easily seen if we recognize that, due to the radial drift, + and - particles come from different "home" magnetic surfaces with correspondingly different density and temperature. That is why the bootstrap current in the banana regime can be estimated as,

$$j_b \approx e \frac{\partial N}{\partial r} v_T (\Delta r)_b \quad (2)$$

where e is the species charge, v_T is the thermal velocity, N is the density,

$$(\Delta r)_b \approx \frac{\rho}{\tau \sqrt{\varepsilon_t}}, \quad (3)$$

is the radial width of the toroidally trapped orbit (banana), with ρ the gyroradius.

In all the estimates made in this section we neglect the temperature gradient. Note also that there is no factor accounting for the fraction of the particles substituting the current, because the bootstrap current is produced mostly by passing particles (which constitute the majority at large aspect ratio) in spite that it is a width of toroidally trapped particle which is incorporated in Eq. (2). This happens due to collisions which maintain the friction between these two types of particles, although the value of this friction is irrelevant in the banana regime.

In terms of the drift kinetic equation, result described Eq. (2) can be roughly interpreted as follows. Due to the finite radial shift of toroidally trapped particles, their distribution function is perturbed by the amount,

$$\Delta F \approx \frac{\partial F_M}{\partial r} (\Delta r)_b, \quad (4)$$

where F_M is the Maxwellian. The distribution function is a solution to a second order differential equation, that is why it should be continuous on the boundary between toroidally trapped and passing particles, so that the distribution function of passing particles obtains the same order perturbation as given by Eq. (4). Since the number of passing particles is bigger than that of toroidally trapped particles, the longitudinal current is sustained mostly by them, which results in the estimate (2).

The helically symmetric case can be investigated in a similar way. There are two types of motion in the helically symmetric field: passing and trapped. The radial displacement of the trapped particle is easily evaluated,⁶

$$(\Delta r)_h \approx \frac{\rho}{Mt\sqrt{\epsilon_r}}, \quad (5)$$

which gives rise to the current,

$$j_h \approx e \frac{\partial N}{\partial r} v_T (\Delta r)_h. \quad (6)$$

Numerous papers about the bootstrap current in a stellarator⁷⁻⁸ always implied the straight stellarator (i.e. helically symmetric case), although this limitation was hardly pointed out.

Let us now take into account the finite value of toroidicity, $\epsilon_t \neq 0$, i.e. incorporate the full model (1). Then, instead of two types of motion, there are three: passing, toroidally and locally trapped. Besides, due to the drift, particle can make transitions

between locally and toroidally trapped states (Fig. 1). These transition affect the low-collisional distribution function by the ammount,⁹⁻¹¹

$$\Delta F \approx \frac{\partial F_M}{\partial r} \frac{(\Delta r)_r}{1 + \tau^t / \tau^r}, \quad (7)$$

where $(\Delta r)_r$ is the radial displacement of a particle in the locally trapped state, $(\Delta r)_r \approx \varepsilon_t / \max \frac{\partial}{\partial r} \left\{ \varepsilon_r, \varepsilon_t, \frac{V}{E} \right\}$, $V = e\Phi / m$ is the electric potential per unit of mass, $\tau^{t,r}$ is the relative time, during which transition particle stays in the toroidally ("t") or locally trapped ("r") states, respectively. One can show⁹⁻¹¹ that for small α ,

$$\alpha = \frac{\varepsilon_t}{N\varepsilon_r} \ll 1, \quad (8)$$

the value of τ^t / τ^r is negligibly small (order of ε_r). Then the numbers of transition particles with transition points separated by the radial distance Δr are different by $\Delta N = \frac{\partial}{\partial r} \left(\frac{\partial N}{\partial r} (\Delta r)_r \right) \Delta r \approx \frac{\partial N}{\partial r} \frac{\partial (\Delta r)_r}{\partial r} \Delta r$. In the case $\varepsilon_t \gg \varepsilon_r$, transition points of particles moving in + and - directions (see Fig. 2) are separated by the distance approximately equal to the radial width of the toroidally trapped orbit, $(\Delta r)_b$. This gives rise to the additional bootstrap current,

$$j_T \approx e \frac{\partial N}{\partial r} \frac{\partial (\Delta r)_r}{\partial r} (\Delta r)_b v_T. \quad (9)$$

Note, that in the case $\varepsilon_r \gg \varepsilon_t$, instead of $(\Delta r)_b$ in Eq. (9) we should use $\Delta r \approx \frac{\rho}{\varepsilon_r \sqrt{\varepsilon_r}}$ (see Sec.5). At large values of α , the perturbation ΔF can be neglected, and there is no additional bootstrap current.

3. Kinetic equation and boundary conditions for transition trajectories

(a) Averaged kinetic equation

From now on we use Boozer coordinates¹² (ψ, θ, ϕ) : ψ is the flux variable, θ is the poloidal angle, ϕ is the toroidal angle, so that the magnetic field is

$$\mathbf{B} = B_\psi \nabla \psi + B_\theta \nabla \theta + B_\phi \nabla \phi, \quad (10)$$

with the magnetic field strength given by Eq. (1). The contravariant components of the magnetic field are related via rotational transform,

$$\tau = \frac{1}{q} = \frac{R_0 B^\theta}{r B^\phi}, \quad (11)$$

which, in turn, is related to the toroidal component of the vector potential A_ϕ ,

$$\tau = -\frac{dA_\phi(\psi)}{d\psi}. \quad (12)$$

The guiding center Lagrangian in these coordinates takes the following form¹³:

$$L = \frac{e}{c} \psi \dot{\theta} + p_\phi \dot{\phi} + \frac{mc}{e} \mu \dot{\phi}_g - h, \quad (13)$$

where

$$p_\phi = m u \frac{B_\phi}{B} + \frac{e}{c} A_\phi, \quad (14)$$

$$h = \frac{m u^2}{2} + U, \quad (15)$$

$$U = \mu B + e \Phi, \quad (16)$$

with ϕ_g being the gyro phase, Φ the electrostatic potential,

$$u = \pm \sqrt{2(E - U) / m}, \quad (17)$$

the parallel velocity, E the full energy and $\mu = \frac{mv_{\perp}^2}{2B}$ the magnetic moment.

In Eq. (13) we neglected the terms $\dot{\theta}\mu\frac{B_{\theta}}{B}$ and $\dot{\phi}\mu\frac{B_{\phi}}{B}$, because they are relatively small.¹³ Note, that B is given by Eq. (1) and in the system with the vacuum field with the circular magnetic surfaces one has $B_{\phi} = B_0 R_0$, where B_0 is the magnetic field on the magnetic axis.

Introducing the canonical momenta,

$$p_{\theta} = \frac{e}{c}\psi, \quad (18)$$

and

$$p_g = \frac{mc}{e}\mu,$$

we transform the Lagrangian to the canonical form,¹³

$$L = p_{\theta}\dot{\theta} + p_{\phi}\dot{\phi} + p_g\dot{g} - h, \quad (19)$$

$$h = \frac{B^2}{2mB_{\phi}^2} \left(p_{\phi} - \frac{eA_{\phi}}{c} \right)^2 + U. \quad (20)$$

Hamilton's equations then give the guiding equations in our coordinate system,

$$\dot{\psi} = \frac{c}{e}\dot{p}_{\theta} = -\frac{c}{e}\frac{\partial h}{\partial \theta} = -\frac{c}{e} \left(\frac{\mu u^2}{B} \frac{\partial B}{\partial \theta} - \frac{\partial U}{\partial \theta} \right), \quad (21)$$

$$\dot{\theta} = \frac{\partial h}{\partial p_{\theta}} = \frac{c}{e} \frac{\partial h}{\partial \psi} = \frac{uB}{B_{\phi}} + \frac{c}{e} \left(\frac{\mu u^2}{B} \frac{\partial B}{\partial \psi} + \frac{\partial U}{\partial \psi} \right), \quad (22)$$

$$\dot{\phi} = \frac{\partial h}{\partial p_{\phi}} = \frac{B}{B_{\phi}} u, \quad (23)$$

$$\dot{p}_{\phi} = -\frac{\partial h}{\partial \phi} = -\frac{\mu u^2}{B} \frac{\partial B}{\partial \phi} - \frac{\partial U}{\partial \phi}, \quad (24)$$

$$\dot{p}_g = \frac{\partial h}{\partial p_g} = \frac{e}{mc} \frac{\partial h}{\partial \mu} = \frac{eB}{mc}, \quad (25)$$

$$\dot{\mu} = \frac{e}{mc} \dot{p}_g = \frac{e}{mc} \frac{\partial h}{\partial \phi_g} = 0. \quad (26)$$

Since the coordinates, which we use, are canonical, the Jacobian of transformation is unity:

$$d^3x d^3p = m^3 d^3x d^3v = dp_\theta d\theta dp_\phi d\phi dp_g d\phi_g, \quad (27)$$

or

$$d^3x d^3v = \frac{1}{m^2} d\psi d\theta dp_\phi d\phi d\mu d\phi_g. \quad (28)$$

It is more convenient to use the full energy E rather than p_ϕ , because E is constant. This gives rise to the Jacobian G , defined by

$$d^3x d^3v = G d\psi d\theta d\phi dE d\mu d\phi_g, \quad (28)$$

so that

$$G = \frac{1}{\frac{\partial E}{\partial p_\phi} m^2} = \frac{1}{\frac{\partial h}{\partial p_\phi} m^2} = \frac{B_\phi}{m^2 B u}, \quad (29)$$

where we used Eqs. (14) and (17).

The drift kinetic equation^{14,15} describing transport in our coordinates has the form:

$$\frac{\partial}{\partial \psi}(\psi GF) + \frac{\partial}{\partial \theta}(\dot{\theta} GF) + \frac{\partial}{\partial \phi}(\dot{\phi} GF) = GC(F), \quad (30)$$

where all the derivatives are taken at constant W . The collision operator in Eq. (30) is the pitch angle scattering operator,¹⁻³

$$GC(F) = Gv \frac{\mu u}{B} \frac{\partial}{\partial \mu} \mu u \frac{\partial F}{\partial \mu} = v \frac{B_\phi}{m B^2} \frac{\partial}{\partial \mu} \mu u \frac{\partial F}{\partial \mu}. \quad (31)$$

Note that in Eq. (31) we neglect the term providing the momentum conservation.³ This term is odd in the sign of the parallel velocity and averages out for the locally trapped particles. It does not for the toroidally trapped particles, but in what follows, we will show that toroidally trapped branch of transition orbits can be neglected in the solvability condition for the kinetic equation at low collisionality.

Using Eqs. (21)-(23) and (29), we obtain

$$\dot{\psi}G = -\frac{c}{em^2} \frac{\partial h / \partial \theta}{\partial h / \partial p_\phi} = -\frac{c}{em^2} \frac{\partial p_\phi}{\partial \theta} \Big|_{E=\text{const}}, \quad (32)$$

$$\dot{\theta}G = \frac{c}{em^2} \frac{\partial h / \partial \psi}{\partial h / \partial p_\phi} = \frac{c}{em^2} \frac{\partial p_\phi}{\partial \psi} \Big|_{E=\text{const}}, \quad (33)$$

$$\dot{\phi}G = \frac{1}{m^2} \frac{\partial h / \partial p_\phi}{\partial h / \partial p_\phi} = \frac{1}{m^2}. \quad (34)$$

With the help of Eqs. (32)-(34) we can transform Eq. (30),

$$\frac{c}{e} \left(\frac{\partial}{\partial \psi} \left(\frac{\partial p_\phi}{\partial \theta} F \right) - \frac{\partial}{\partial \theta} \left(\frac{\partial p_\phi}{\partial \psi} F \right) \right) + \frac{\partial F}{\partial \phi} = \frac{\partial p_\phi}{\partial E} C(F). \quad (35)$$

For a large class of the machines, the rotational transform per period, t/M , is small, so that we can also assume that condition (8) is fulfilled. In such a case,^{11,16} the motion of all particles is nearly periodic in the toroidal angle ϕ , which can be considered a "fast variable" compared to ψ and θ . The period of the toroidally trapped particles is also related to the local magnetic well and is equal $\frac{2\pi}{M}$. Besides, we are working with sufficiently low collision frequency (but higher than the typical bounce frequency in the local magnetic well) so that the biggest operator in Eq. (35) is $\frac{\partial}{\partial \phi}$. Splitting the distribution function into two parts,

$$F = \bar{F} + \tilde{F}(\phi), \quad (36)$$

independent and dependent on ϕ , we obtain the zeroth and first order equations,

$$\frac{\partial \bar{F}}{\partial \phi} = 0, \quad (37)$$

$$\frac{c}{e} \left(\frac{\partial}{\partial \psi} \left(\frac{\partial p_\phi}{\partial \theta} \bar{F} \right) - \frac{\partial}{\partial \theta} \left(\frac{\partial p_\phi}{\partial \psi} \bar{F} \right) \right) + \frac{\partial \tilde{F}}{\partial \phi} = \frac{\partial p_\phi}{\partial W} C(\bar{F}). \quad (38)$$

Now we can average Eq. (38) over ϕ . This averaging is different for different kind of particles. For the locally trapped particle, whose unperturbed orbit is closed, it is

performed over the period of motion : back and forth over the whole bounce inside the well. For the toroidally trapped particle, whose orbit is not closed, it is performed over the period from ϕ_k to $\phi_k \pm \frac{2\pi}{M}$ (the sign here is determined by the sign of the parallel velocity). The integration of Eq. (38) is performed at constant energy E and eliminates the last term in the left hand side. The resulting averaged kinetic equation for the distribution function in the i -state ($i=+, -$ or r) takes the form,³

$$\frac{\partial J^i}{\partial \theta} \frac{\partial \bar{F}^i}{\partial \psi} - \frac{\partial J^i}{\partial \psi} \frac{\partial \bar{F}^i}{\partial \theta} = v \frac{eR_0}{cB_\phi} \frac{\partial}{\partial \mu} \mu \tilde{J}^i \frac{\partial \bar{F}^i}{\partial \mu} = v \frac{e}{cB_0} \frac{\partial}{\partial \mu} \mu \tilde{J}^i \frac{\partial \bar{F}^i}{\partial \mu}, \quad (39)$$

where we introduced the adiabatic invariant,

$$J^i = \int_i p_\phi d\phi, \quad (40)$$

with

$$\int_i = \begin{cases} \oint & \text{for the locally trapped particles} \\ \int_{\phi_k}^{\phi_k \pm \frac{2\pi}{M}} & \text{for the } \pm \text{toroidally trapped particles} \end{cases} \quad (41)$$

The adiabatic invariant is different for different kind of particles,

$$J^\pm = \int_{\phi_k}^{\phi_k \pm \frac{2\pi}{M}} d\phi \left(\frac{u m B_\phi}{B} + \frac{e A_\phi}{c} \right) = X \pm \frac{2\pi e A_\phi}{M c}, \quad (42)$$

$$X \equiv \int_{\phi_k}^{\phi_k \pm \frac{2\pi}{M}} d\phi \left(\frac{|u| m B_\phi}{B} \right), \quad (43)$$

$$J^r = \oint \frac{\mu u B_\phi}{B} d\phi, \quad (44)$$

The quantity $\tilde{J}^i$ is different from the adiabatic invariant,

$$\tilde{J}^i = \int \frac{um}{R_0} \left(\frac{B_\phi}{B} \right)^2 d\phi, \quad (45)$$

but if the condition (8) is fulfilled, we can put

$$\tilde{J}^\pm = X, \quad (46)$$

and

$$\tilde{J}^r = J^r. \quad (47)$$

In any case, if this condition is violated, the motion of the toroidally trapped particle is not periodic on the period $(\phi_k, \phi_k \pm \frac{2\pi}{M})$. That is why, in what follows, we neglect the effects of the α -order and higher and put $\tilde{J}^i = J^i$. It is also convenient to omit the bar over the distribution function in Eq. (39) and introduce the Poisson bracket,

$$\{a, b\} \equiv \frac{\partial a}{\partial \psi} \frac{\partial b}{\partial \theta} - \frac{\partial a}{\partial \theta} \frac{\partial b}{\partial \psi}, \quad (48)$$

so that the kinetic equation (39) for the distribution function averaged over ϕ takes the following form,

$$\{F^i, J^i\} = \frac{ve}{cB} \frac{\partial}{\partial \mu} \mu \tilde{J}^i \frac{\partial F}{\partial \mu}, \quad (49)$$

with J^i and $\tilde{J}^i$ related via Eqs. (46) and (47).

(b) Transition orbits

As a particle moves, the position of the particle in the potential μB changes. This change occurs due to the drift motion, which is similar to the motion in a slowly varying potential. The value of the adiabatic invariant and the energy of the particle remain constant in this motion, while the volumes of the phase space corresponding to different types of motion changes together with the energy of the separatrix, E_x , and, when the separatrix

energy becomes equal to the energy of the particle, $E_x = E$, the particle crosses the separatrix, and the orbit is transformed.

These transformations impose certain boundary conditions on the distribution functions F^i , following from the conservation of the particles. In order to obtain them, we need to evaluate the elementary flux, S^i , through the surface $E_x = E$, between the points ϕ and $\phi_k \pm \frac{2\pi}{M}$ for + and - particles and over the period for the locally trapped particle. Since the only different quantity of different types of particles at the separatrix $E_x = E$ is the adiabatic invariant J^i , it is convenient to transform from ψ, θ to E_x and $J^i(E_x = E) \equiv Y^i$,

$$dx^3 dv^3 = G dE_x dY^i d\phi dE d\mu d\phi_g \frac{\partial(\psi, \theta)}{\partial(E_x, Y^i)} = - \frac{G}{\{E_x, Y^i\}} dE_x dY^i d\phi dE d\mu d\phi_g. \quad (50)$$

The flux S^i is then given by

$$S^i = - \frac{F}{\{E_x, Y^i\}} dY^i dE d\mu d\phi_g \int_i \dot{E}_x G d\phi \Big|_{E_x=E}, \quad (51)$$

where the integral over the period is determined by Eq. (41). Using Eqs. (21), (22) and (29), we obtain,

$$\begin{aligned} \int_i \dot{E}_x G d\phi \Big|_{E_x=E} &= \frac{1}{m^2} \int_i \left(\frac{\partial E_x}{\partial \theta} \dot{\theta} + \frac{\partial E_x}{\partial \psi} \dot{\psi} \right) d\phi \Big|_{E_x=E} = \\ &= \frac{e}{cm^2} \left(\frac{\partial E_x}{\partial \psi} \int_i \frac{\partial p_\phi}{\partial \theta} d\phi - \frac{\partial E_x}{\partial \theta} \int_i \frac{\partial p_\phi}{\partial \psi} d\phi \right) \Big|_{E_x=E} = - \frac{e}{cm^2} \{E_x, Y^i\} \end{aligned} \quad (52)$$

Substituting Eq. (52) in Eq. (51) gives,

$$S^i = \frac{e F^i}{cm^2} dY^i dE d\mu d\phi_g \propto F^i dY^i, \quad (53)$$

i. e. we proved that the fluxes are just proportional to dY^i .

The differentials dY^i are defined on the surface $E_x = \text{const.}$ That is why,

$$\frac{\partial E_x}{\partial \theta} d\theta + \frac{\partial E_x}{\partial \psi} d\psi = 0. \quad (54)$$

This permits us to relate dY^i to the Poisson brackets by the following relations,

$$dY^i = \frac{\partial Y^i}{\partial \theta} d\theta + \frac{\partial Y^i}{\partial \psi} d\psi = \frac{1}{\frac{\partial E_x}{\partial \psi}} \{E_x, Y^i\}. \quad (55)$$

The collisionless transition orbit can look as depicted at Figs. 3a-d. All the orbits have two symmetric transition points (we consider up-down symmetric case, so that orbits are closed in (ψ, θ) -plane). These transition points are different. For one, two species of particles flow in, one out. The final state is obvious for such a transition. We call this point a confluence (C) point. For another point, two species flow out, one flows in. Here the "branching" occurs. We call this transformation point a branching (B) point and the species that flows in here is called a majority state, since all the particles pass through it. The majority state is the one which loses the particle in the branching point. Figs. 3(a,b) depict the situation when + and - states are respectively the majority states. Figs. 3(c,d) show one of the orbits with the majority state being the locally trapped state. This situation is quite rare and corresponds to the case of such a strong drift (large radial electric field) that both + and - states move in the direction of the drift in (r, θ) -plane. One of the quantities relevant to the branching is the transformation probability,^{9-11,16-19} which defines the relative numbers of transformations from the majority state to two other states in the branching point. To simplify the consideration, we shall refer to the case when + particles constitute the majority state (Fig. 3(a)). Then the transition probabilities are

$$P_{+ \rightarrow r} = \frac{\{E_x, Y^r\}}{\{E_x, Y^+\}} \Big|_B, \quad (56)$$

$$P_{+ \rightarrow -} = -\frac{\{E_x, Y^-\}}{\{E_x, Y^+\}} \Big|_B. \quad (57)$$

Note, that transition probability is defined in the branching point B. The sum of two probabilities of transformation is unity, which follows from

$$Y^+ + Y^- = Y^r, \quad (58)$$

$$\{Y^+, E_x\} + \{Y^-, E_x\} = \{Y^r, E_x\}. \quad (59)$$

Eqs. (55)-(57) give interesting relations,

$$\frac{dY^-}{dY^+} = \frac{\{E_+, Y^-\}}{\{E_+, Y^+\}}, \quad (60)$$

$$\frac{dY^r}{dY^+} = \frac{\{E_x, Y^r\}}{\{E_x, Y^+\}}. \quad (61)$$

At the branching point the conservation of particles demands that fluxes were related via transition probabilities,

$$F^+(B)dY^+P_{+\rightarrow-} + F^-(B)dY^- = 0, \quad (62)$$

$$F^+(B)dY^+P_{+\rightarrow r} = F^r(B)dY^r. \quad (63)$$

The difference between Eq. (62) and (63) is due to the fact that in the first case there is no total flux of + and - particles through the separatrix, meanwhile in the second case there is flux, crossing the separatrix. Substituting Eqs. (60) and (61) into Eqs. (62) and (63), we obtain the boundary condition in the branching point:

$$F^+(B) = F^-(B) = F^r(B). \quad (64)$$

The situation is different at the C point, since two fluxes (- and r) confluence together to make the majority flux. That is why we demand

$$F^+(C)dY^+ + F^-(C)dY^- = F^r(C)dY^r, \quad (65)$$

which, together with Eqs. (61) and (62), give the boundary condition at the confluence point,

$$F^+(C) = F^-(C)P_{+\rightarrow-} + F^r(C)P_{+\rightarrow r}, \quad (66)$$

which involves trapping probabilities referring to the confluence point C. To obtain Eq. (66), we used the equalities $\{E_x, Y^i\}|_B = -\{E_x, Y^i\}|_C$.

(c) Solution of kinetic equation at low collisionality

As was shown in Refs. 3 and 11, the equation for the transition particles is solved more easily, if we transform from the variables (ψ, θ, μ) to the variables $(\psi_0, \theta_0, \theta)$, where ψ_0, θ_0 are the coordinates of the transition point, i.e. can be found from the following relations:

$$J^i(\psi, \theta, \mu, E) = J^i(\psi_0, \theta_0, \mu, E) \equiv Y^i, \quad (67)$$

$$E_x(\psi_0, \theta_0, \mu) = E. \quad (68)$$

Taking derivative $\frac{\partial}{\partial \mu} + \frac{\partial \theta_0^i}{\partial \mu} \frac{\partial}{\partial \mu} + \frac{\partial \psi_0^i}{\partial \mu} \frac{\partial}{\partial \mu}$ of these equations, we easily obtain (compare with Refs. 3 and 11)

$$\frac{\partial \theta_0^i}{\partial \mu} = \frac{\left(\left(\frac{\partial J^i}{\partial \mu} - \frac{\partial Y^i}{\partial \mu} \right) \frac{\partial E_x}{\partial \psi_0} + \frac{\partial Y^i}{\partial \psi_0} \frac{\partial E_x}{\partial \mu} \right)}{\{Y^i, E_x\}}, \quad (69)$$

$$\frac{\partial \psi_0^i}{\partial \mu} = - \frac{\left\{ \left(\frac{\partial J^i}{\partial \mu} - \frac{\partial Y^i}{\partial \mu} \right) \frac{\partial E_x}{\partial \theta_0} + \frac{\partial Y^i}{\partial \theta_0} \frac{\partial E_x}{\partial \mu} \right\}}{\{Y^i, E_x\}}, \quad (70)$$

The use of coordinates $(\psi_0, \theta_0, \theta)$ transforms Eq. (49) into,

$$\frac{\partial F^i}{\partial \theta} = - \frac{ve}{cB} \frac{\frac{\partial J^i}{\partial \mu}}{\frac{\partial J^i}{\partial \psi}} \frac{d}{d\mu} \mu \tilde{J}^i \frac{dF^i}{d\mu} \equiv \hat{C}^i(F^i), \quad (71)$$

where all the quantities are expressed in new variables, in particular,

$$\frac{d}{d\mu} = \frac{\partial\theta_0^i}{\partial\mu} \frac{\partial}{\partial\theta_0} + \frac{\partial\psi_0^i}{\partial\mu} \frac{\partial}{\partial\psi_0}, \quad (72)$$

with $\frac{\partial\theta_0^i}{\partial\mu}$ and $\frac{\partial\psi_0^i}{\partial\mu}$ given by Eqs. (69) and (70).

To solve the kinetic equation (71), let us integrate over the branches of a transition orbit. To make everything specific, we use the example considered in the previous section, when the + state is the majority state (see Fig. 3a). This leads us to the following relations:

$$F^r(B) - F^r(C) = \int_{-\theta_0}^{\theta_0} \hat{C}^r(F^r) d\theta, \quad (73)$$

$$F^-(B) - F^-(C) = \int_{-\theta_0}^{\theta_0} \hat{C}^-(F^-) d\theta, \quad (74)$$

$$F^+(C) - F^+(B) = \int_{\theta_0}^{-\theta_0} \hat{C}^+(F^+) d\theta. \quad (75)$$

Using the boundary conditions given by Eq. (66) transforms Eq. (75) into

$$F^+(B) = F^+(C) - \int_{\theta_0}^{-\theta_0} \hat{C}^+(F^+) d\theta =$$

$$-\frac{\{Y^-, E_x\}}{\{Y^+, E_x\}} \Big|_C F^-(C) + \frac{\{Y^r, E_x\}}{\{Y^+, E_x\}} \Big|_C F^r(C) - \int_{\theta_0}^{-\theta_0} \hat{C}^+(F^+) d\theta \quad (76)$$

Substituting Eqs. (73), (75) together with Eq. (64) into Eq. (76) yields

$$F^+(B) = -\frac{\{Y^-, E_x\}}{\{Y^+, E_x\}} \Big|_C \left(F^-(B) - \int_{-\theta_0}^{\theta_0} \hat{C}^-(F^-) d\theta \right) +$$

$$\frac{\{Y^r, E_x\}}{\{Y^+, E_x\}} \Big|_C \left(F^r(B) - \int_{-\theta_0}^{\theta_0} \hat{C}^r(F^r) d\theta \right) - \int_{\theta_0}^{-\theta_0} \hat{C}^+(F^+) d\theta \quad (77)$$

which, due to Eq. (66), gives the following solvability condition:

$$\begin{aligned} & -\frac{\{Y^-, E_x\}}{\{Y^+, E_x\}} \bigg|_{C=-\theta_0} \int_{-\theta_0}^{\theta_0} \hat{C}^-(F^-) d\theta + \frac{\{Y^r, E_x\}}{\{Y^+, E_x\}} \bigg|_{C=-\theta_0} \int_{-\theta_0}^{\theta_0} \hat{C}^r(F^r) d\theta + \int_{\theta_0}^{-\theta_0} \hat{C}^+(F^+) d\theta = \\ & P_{+ \rightarrow -} \int_{-\theta_0}^{\theta_0} \hat{C}^-(F^-) d\theta + P_{+ \rightarrow r} \int_{-\theta_0}^{\theta_0} \hat{C}^r(F^r) d\theta + \int_{\theta_0}^{-\theta_0} \hat{C}^+(F^+) d\theta = 0 \end{aligned} \quad (78)$$

Eq. (78) was first obtained in Refs. 9-11 and gives the generalization of the ordinary solvability condition² to the case of transition trajectories.

Let us look for the low collisional solution of Eq. (71) in the following form,

$$F^i(\psi_0, \theta_0, \theta) = F_0(\psi_0, \theta_0) + \tilde{\Psi}^i(\psi_0, \theta_0, \theta), \quad (79)$$

$$F_0(\psi_0, \theta_0) = F_M(\psi_0) + \bar{\Psi}(\psi_0, \theta_0), \quad (80)$$

where $F_M(\psi)$ is the Maxwellian function, $\bar{\Psi}$ is independent on θ and, being the function of the integrals of motion, does not depend on the state. Function $\tilde{\Psi}^i$ varies on the magnetic surface and is different for on different branches of the trajectory. Using the well-known technique^{2,9-11}, we find that the solution in the first order in the collision frequency obeys

$$\frac{\partial \tilde{\Psi}^i}{\partial \theta} = \hat{C}^i(F_0^i) = -\frac{\nu \Omega}{B_0^2 \frac{\partial J^i}{\partial \psi}} \frac{d}{d\mu} \mu \tilde{J}^i \frac{dF_0^i}{d\mu}, \quad (81)$$

Assuming that the radial displacements are small,

$$\frac{\partial \psi_0^i}{\partial \mu} \ll \psi_0 \frac{\partial \theta_0^i}{\partial \mu}, \quad (82)$$

we can transform the operator in the right hand side of Eq. (81) to

$$\hat{C}^i(F_0^i) = -\frac{\nu \Omega}{B_0^2 \frac{\partial J^i}{\partial \psi}} \frac{\partial \theta_0^i}{\partial \mu} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^i \left(\frac{\partial \psi_0^i}{\partial \mu} \frac{\partial F_M}{\partial \psi_0} + \frac{\partial \theta_0^i}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right), \quad (83)$$

which allows to find the lowest order solution, $\bar{\Psi}$, from the solvability condition (78):

$$\begin{aligned}
 & \left\{ Y^+, E_x \right\} \Big|_c \int_{\theta_0}^{-\theta_0} \frac{d\theta}{\partial J^+} \frac{\partial \theta_0^+}{\partial \mu} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^+ \left(\frac{\partial \psi_0^+}{\partial \mu} \frac{\partial F_M}{\partial \psi_0} + \frac{\partial \theta_0^+}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right) + \\
 & \left\{ Y^r, E_x \right\} \Big|_c \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\partial J^r} \frac{\partial \theta_0^r}{\partial \mu} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^r \left(\frac{\partial \psi_0^r}{\partial \mu} \frac{\partial F_M}{\partial \psi_0} + \frac{\partial \theta_0^r}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right) - \\
 & \left\{ Y^-, E_x \right\} \Big|_c \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\partial J^-} \frac{\partial \theta_0^-}{\partial \mu} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^- \left(\frac{\partial \psi_0^-}{\partial \mu} \frac{\partial F_M}{\partial \psi_0} + \frac{\partial \theta_0^-}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right) = 0
 \end{aligned} \tag{84}$$

To simplify this equation, we need to compare its terms. We use the lowest order expressions in ε_r , ε_t for the invariants^{11,16} for this purpose:

$$J^\pm = \frac{Y^r}{2} \hat{f}(k^2) \pm I^r, \tag{85}$$

$$\hat{f}(k^2) = kE\left(\frac{1}{k}\right), \tag{86}$$

$$I^r = \frac{2\pi}{M} \frac{\Omega}{B_0} A_\phi, \tag{87}$$

$$J^r = Y^r f(k^2), \tag{88}$$

$$f(k^2) = E(k) - (1 - k^2)K(k), \tag{89}$$

$$Y^r = \frac{16R\sqrt{\mu B_0 \varepsilon_r}}{MR_0}, \tag{90}$$

$$\tilde{J}^\pm = \pm \frac{R}{2R_0} Y^r \hat{f}(k^2), \tag{91}$$

$$\tilde{J}^r = \frac{R}{R_0} Y^r f(k^2), \tag{92}$$

where $R = R_0 + \varepsilon_t \cos(\theta)$, $E(k)$ and $K(k)$ are the complete elliptic integrals of the first and second order, $\pm$ is the sign of the parallel velocity of the toroidally trapped particles.

Parameter k^2 ,

$$k^2 = \frac{E - e\Phi - \mu B_0(1 - \varepsilon_t \cos(\theta) - \varepsilon_r)}{2\mu B_0 \varepsilon_r}, \quad (93)$$

defines the type of motion: $k^2 \leq 1$ corresponds to the locally trapped motion, $k^2 \geq 1$ corresponds to the toroidally trapped (+ or -) motion. Here, E is the full energy, Φ is the electrostatic potential. Instead of using k^2 , it is rather convenient to use the energy E . Say, the critical value $k^2=1$ marking the separatrix, corresponds to

$$E_x \equiv e\Phi + \mu B_0(1 - \varepsilon_t \cos(\theta) + \varepsilon_r) = E. \quad (94)$$

Leaving only the main terms in ε_r and $\frac{1}{\Omega}$, we can simplify Eqs. (69) and (70):

$$\frac{\partial \theta_0^r}{\partial \mu} = \frac{\frac{\partial J^r}{\partial \mu} \frac{\partial E_x}{\partial \psi_0}}{\{Y^r, E_x\}}, \quad (95)$$

$$\frac{\partial \psi_0^r}{\partial \mu} = -\frac{\frac{\partial J^r}{\partial \mu} \frac{\partial E_x}{\partial \theta_0}}{\{Y^r, E_x\}}, \quad (96)$$

$$\frac{\partial \theta_0^\pm}{\partial \mu} = \frac{\frac{\partial Y^\pm}{\partial \psi_0} \frac{\partial E_x}{\partial \mu}}{\{Y^\pm, E_x\}}, \quad (97)$$

$$\frac{\partial \psi_0^\pm}{\partial \mu} = -\frac{\frac{\partial J^\pm}{\partial \mu} \frac{\partial E_x}{\partial \theta_0}}{\{Y^\pm, E_x\}}, \quad (98)$$

where the transition point is the C-point. Substituting Eqs. (95)-(98) into Eq. (84), we get rid of the Poisson brackets at all:

$$\begin{aligned}
& \int_{\theta_0}^{-\theta_0} \frac{d\theta}{\partial J^+} \frac{\partial Y^+}{\partial \psi_0} \frac{\partial E_x}{\partial \mu} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^+ \left(\frac{\partial \psi_0^+}{\partial \mu} \frac{\partial F_M}{\partial \psi_0} + \frac{\partial \theta_0^+}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right) + \\
& \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\partial J^r} \frac{\partial J^r}{\partial \mu} \frac{\partial E_x}{\partial \psi_0} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^r \left(\frac{\partial \psi_0^r}{\partial \mu} \frac{\partial F_M}{\partial \psi_0} + \frac{\partial \theta_0^r}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right) - \\
& \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\partial J^-} \frac{\partial Y^-}{\partial \psi_0} \frac{\partial E_x}{\partial \mu} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^- \left(\frac{\partial \psi_0^-}{\partial \mu} \frac{\partial F_M}{\partial \psi_0} + \frac{\partial \theta_0^-}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right) = 0
\end{aligned} \tag{99}$$

Comparing the terms containing $\frac{\partial \psi_0^i}{\partial \mu} \frac{\partial F_M}{\partial \psi_0}$, with the help of Eqs. (85)-(94), we can obtain have the following ordering:

$$\frac{\frac{\partial J^r}{\partial \psi} \frac{\partial Y^\pm}{\partial \psi_0} \frac{\partial E_x}{\partial \mu} \frac{\partial \psi_0^\pm}{\partial \mu} \frac{\partial F_M}{\partial \psi_0}}{\frac{\partial J^\pm}{\partial \psi} \frac{\partial J^r}{\partial \mu} \frac{\partial E_x}{\partial \psi_0} \frac{\partial \psi_0^r}{\partial \mu} \frac{\partial F_M}{\partial \psi_0}} \approx \frac{\frac{\partial \psi_0^\pm}{\partial \mu}}{\frac{\partial \psi_0^r}{\partial \mu}} \approx \frac{(\Delta r)_b}{(\Delta r)_r} \ll 1, \tag{100}$$

i.e. has an order of the ratio of the radial displacements in the toroidally and locally trapped particles. This allows us to neglect such terms on the toroidally trapped parts of the trajectories in Eq. (99). which yields

$$\begin{aligned}
& \int_{\theta_0}^{-\theta_0} \frac{d\theta}{\partial J^+} \frac{\partial Y^+}{\partial \psi_0} \frac{\partial E_x}{\partial \mu} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^+ \left(\frac{\partial \theta_0^+}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right) + \\
& \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\partial J^r} \frac{\partial J^r}{\partial \mu} \frac{\partial E_x}{\partial \psi_0} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^r \left(\frac{\partial \psi_0^r}{\partial \mu} \frac{\partial F_M}{\partial \psi_0} + \frac{\partial \theta_0^r}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right) - \\
& \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\partial J^-} \frac{\partial Y^-}{\partial \psi_0} \frac{\partial E_x}{\partial \mu} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^- \left(\frac{\partial \theta_0^-}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right) = 0
\end{aligned} \tag{101}$$

Comparing the terms in Eq. (101) containing $\bar{\Psi}$, we obtain:

$$A = \frac{\tilde{J}^\pm}{\tilde{J}^r} \frac{\frac{\partial Y^\pm}{\partial \psi_0} \frac{\partial E_x}{\partial \mu} \frac{\partial \theta_0^\pm}{\partial \mu} \frac{\partial J^r}{\partial \psi}}{\frac{\partial J^r}{\partial \mu} \frac{\partial E_x}{\partial \psi_0} \frac{\partial \theta_0^r}{\partial \mu} \frac{\partial J^\pm}{\partial \psi}} \approx \frac{\tilde{J}^\pm}{\tilde{J}^r} \frac{\frac{\partial \theta_0^\pm}{\partial \mu}}{\frac{\partial \theta_0^r}{\partial \mu}} \approx \frac{\tilde{J}^\pm}{\tilde{J}^r} \frac{\frac{\partial E_x}{\partial \mu}}{\frac{\partial E_x}{\partial \theta} \frac{\partial \theta_0^r}{\partial \mu}} \approx \frac{\tilde{J}^\pm}{\tilde{J}^r} \frac{1}{\epsilon_{t\mu} \frac{\partial \theta_0^r}{\partial \mu}}. \quad (102)$$

Substituting

$$\{Y^r, E_x\} \approx Y^r \frac{\partial E_x}{\partial \psi_0} \epsilon_t, \quad (103)$$

into Eq. (85), we obtain

$$\frac{\partial \theta_0^r}{\partial \mu} \approx \frac{1}{\epsilon_t \epsilon_r \mu}. \quad (104)$$

Then using

$$\frac{\tilde{J}^\pm}{\tilde{J}^r} \approx \max \left\{ 1, \sqrt{\frac{\epsilon_t}{\epsilon_r}} \right\}, \quad (105)$$

together with Eqs. (103) and (104) gives the following estimate for the value of A:

$$A \approx \epsilon_r \max \left\{ 1, \sqrt{\frac{\epsilon_t}{\epsilon_r}} \right\}. \quad (106)$$

Thus, we can neglect the terms in Eq. (89) coming from the toroidally trapped part of the transition orbit at all, which ends up with a simple equation:

$$\int_{-\theta_0}^{\theta_0} \frac{d\theta}{\frac{\partial J^r}{\partial \psi}} \frac{\partial J^r}{\partial \mu} \frac{\partial E_x}{\partial \psi_0} \frac{\partial}{\partial \theta_0} \mu \tilde{J}^r \left(\frac{\partial \psi_0^r}{\partial \mu} \frac{\partial F_M}{\partial \psi_0} + \frac{\partial \theta_0^r}{\partial \mu} \frac{\partial \bar{\Psi}}{\partial \theta_0} \right) = 0. \quad (107)$$

The quantity A can be shown (see Ref. 11) to be roughly equal to the ratio of times τ^t / τ^r . Its small value means that a transition particle spends most of the time in the toroidally trapped state, and the deviation of the distribution function from Maxwellian is

"formed" in the locally trapped state. It can be understood from the following considerations. Since the collision integral is the second order differential operator and inversely proportional to squared phase volume corresponding to the particular state,^{9,11}

$$C^i(F_0) \propto \left(\frac{1}{\tau^i}\right)^2. \quad (108)$$

On the other hand, the solvability condition could be roughly represented in the following form

$$\int_r C_r(F_0)dt + \int_+ C_+(F_0)dt + \int_- C_-(F_0)dt = 0, \quad (109)$$

where each has the order $\tau^i \left(\frac{1}{\tau^i}\right)^2 = \frac{1}{\tau^i}$. The ratio of the times τ^i can be found from the following calculations:

$$\frac{\tau^t}{\tau^r} \approx P_{+ \rightarrow -} \left| \frac{\dot{\theta}^r}{\dot{\theta}^t} \right| = P_{+ \rightarrow -} \left| \frac{\frac{\partial J^r}{\partial \psi}}{\frac{\partial J^+}{\partial \psi}} \right| \approx A. \quad (110)$$

To obtain Eq. (110) we used that

$$\frac{\partial J^\pm}{\partial \psi} = \pm \frac{\partial I^t}{\partial \psi} = - \left(\pm \frac{2\pi\Omega}{MB_0} \right), \quad (111)$$

$$\frac{\partial J^r}{\partial \psi} = Y^r \frac{\partial \hat{f}(k^2)}{\partial k^2} \frac{\partial k^2}{\partial E_x} \frac{\partial E_x}{\partial \psi} = Y^r \frac{K(k^2)}{4\mu B_0 \epsilon_r} \frac{\partial E_x}{\partial \psi} = Y^r \frac{K(k^2)}{4\mu B_0 \epsilon_r} \frac{\partial e\Phi}{\partial \psi}, \quad (112)$$

and the lowest order expression for the trapping probability (see Refs. 11, 16):

$$P_{+ \rightarrow r} = \frac{8B_0 \sqrt{\mu B_0 \epsilon_r}}{\pi \Omega} \left(\frac{e \partial \Phi / \partial \psi}{\mu B_0} + \frac{\partial \epsilon_r / \partial \psi}{2\epsilon_r} \right). \quad (113)$$

Using the estimate given by Eq. (110) we can reduce Eq. (109) to

$$\int_r C^r(F_0)dt = 0, \quad (114)$$

which basically coincides with Eq. (107).

We have already used the assumption about small radial displacement given by Eq. (82). It implies that the radial electric field should be relatively large,

$$\frac{e \frac{\partial \Phi}{\partial \psi}}{\mu B_0} \ll \frac{\partial \varepsilon_r}{\partial \psi}, \frac{\partial \varepsilon_t}{\partial \psi}. \quad (115)$$

and, in the lowest order, we can assume

$$\frac{\frac{\partial J^r}{\partial \mu} \frac{\partial E_x}{\partial \psi_0}}{\frac{\partial J^r}{\partial \psi}} = B_0 \frac{\frac{\partial E_x}{\partial \psi_0}}{\frac{\partial E_x}{\partial \psi}} = B_0 \frac{\frac{\partial \Phi}{\partial \psi_0}}{\frac{\partial \Phi}{\partial \psi}} \approx B_0 = \text{constant}. \quad (116)$$

Relation (116) allows to find the solution of Eq. (107) immediately:

$$\frac{\partial \bar{\Psi}}{\partial \theta_0} = - \frac{\frac{\partial F_M}{\partial \psi_0} \int_{-\theta_0}^{\theta_0} \frac{\partial \psi_0^r}{\partial \mu} d\theta}{\int_{-\theta_0}^{\theta_0} \frac{\partial \theta_0^r}{\partial \mu} d\theta} \approx \frac{\frac{\partial F_M}{\partial \psi_0} \frac{\partial E_x}{\partial \theta_0}}{\frac{\partial E_x}{\partial \psi_0}}. \quad (117)$$

Inequality (115) provided, Eq. (94) gives:

$$\frac{\partial E_x}{\partial \psi_0} = e \frac{\partial \Phi}{\partial \psi_0}, \quad (118)$$

$$\frac{\partial E_x}{\partial \theta_0} = \mu B_0 \varepsilon_t(\psi_0) \sin(\theta_0). \quad (119)$$

Substituting Eqs. (118) and (119) into Eq. (117) and integrating over θ_0 , we finally obtain

$$\overline{\Psi}(\psi_0, \theta_0) = -\frac{\partial F_M}{\partial \psi_0} \frac{\mu B_0 \varepsilon_t(\psi_0)}{e \frac{\partial \Phi}{\partial \psi_0}} (1 + \cos(\theta_0)). \quad (120)$$

Here we imposed the boundary condition, $\overline{\Psi}(\theta_0 = 0) = 0$, since in the absence of the locally trapped part of the trajectory, there are no transitions and the perturbation of the distribution function is zero.

4. Bootstrap current

The distribution function is a solution to the second order differential equation (10). Hence, it is continuous through the boundary between toroidally trapped and passing particles. That is why the solution found in the previous section, $F_0(\psi_0, \theta_0) = F_M(\psi_0) + \overline{\Psi}(\psi_0, \theta_0)$ with $\overline{\Psi}$ given by Eq. (99), should be continued into the region corresponding to the passing motion. This means that the distribution function of passing particles gets an addition related to collisionless transitions. This addition can be found by setting $\theta_0 = \pi$ in F_0 , because, in this case, the reflection point of a banana moves to $\theta = \pi$, which corresponds to the boundary with the passing region. Let us denote this boundary value of the distribution function $F^* \equiv F_0(\psi_0, \theta = \pi)$. Due to the finite radial width of the orbits, F^* is different for + and - states. This can be easily seen, if we follow the + and - trajectories which start at the same point (see Fig. 4): the radial coordinates of the transition points ψ_0^+ and ψ_0^- are different and so are the corresponding values of F^* , since F^* is a function of $\psi_0^\pm$. This difference explains the origin of the additional bootstrap current.

Since the number of the passing particles is much larger (at $\varepsilon_t \ll 1$) than that of toroidally trapped, the bootstrap current is mostly sustained by the passing particles. In the lowest order in ε_t , the bootstrap current arising from transitions is then given by,

$$j_T = e \oint \frac{d\theta}{2\pi} \int_0^{2\pi/M} \frac{d\phi}{2\pi/M} \int d^3v \frac{u B_0}{B} \overline{\Psi}^*, \quad (100)$$

where

$$d^3v = \sum_{\sigma=+1,-1} 2\pi dE d\mu B \frac{1}{u}, \quad (101)$$

is the element in the velocity space, $\bar{\Psi}^* = \bar{\Psi}|_{\theta_0=\pi}$ is the boundary value of $\bar{\Psi}$. Note, that we did not include $F_M(r_0)$ into Eq. (100) since this term partly produces the main bootstrap current which was evaluated before.^{2,3,6} Using the fact that $\bar{\Psi}^*$ does not depend on μ , and performing the integration, we transform Eq.(100) into

$$j_T = e \oint \frac{d\theta}{2\pi} \int dE (\bar{\Psi}^*(\sigma=+1) - \bar{\Psi}^*(\sigma=-1)) E. \quad (102)$$

Expanding the distribution function in the radial displacement,

$$\bar{\Psi}(\psi_0^{+-}, \theta_0^{+-} = \pi) = \bar{\Psi}(\psi, \pi) + \frac{\partial \bar{\Psi}(\psi, \pi)}{\partial \psi} (\psi_0^{+-} - \psi), \quad (103)$$

we finally obtain,

$$j_T = 2\pi e \int E \frac{\partial \bar{\Psi}(\psi, \pi)}{\partial \psi} \langle \psi_0^+ - \psi_0^- \rangle dE, \quad (104)$$

where

$$\langle \psi_0^+ - \psi_0^- \rangle = \frac{1}{\pi} \int_0^\pi d\theta (\psi_0^+ - \psi_0^-), \quad (105)$$

and, according to Eq. (99),

$$\frac{\partial \bar{\Psi}(\psi, \pi)}{\partial \psi} = 2 \frac{\partial F_M}{\partial \psi} \frac{\partial}{\partial \psi} \left(\frac{\epsilon_i E}{e \partial \Phi / \partial \psi} \right). \quad (106)$$

We find $(\psi_0^+ - \psi_0^-)$ using the adiabatic invariant J^t given by Eq. (64):

$J_\pm^t(\psi, \theta, k_\pm^2) = J_\pm^t(\psi_0^\pm, \pi, 1)$, or

$$\frac{Y^t(\psi, \theta)}{2} \hat{f}(k_+^2) + I^t(\psi) = \frac{Y^t(\psi_0^+, \pi)}{2} \hat{f}(k_-^2) + I^t(\psi_0^+), \quad (107)$$

$$\frac{Y^r(\psi, \theta)}{2} \hat{f}(k_{\pm}^2) - I^r(\psi) = \frac{Y^r(\psi_0^{\pm}, \pi)}{2} \hat{f}(k_{\pm}^2) - I^r(\psi_0^{\pm}). \quad (108)$$

Here, $k_{\pm}^2$ is such, that + or - trajectory, being started at (ψ, θ) (with $k^2 = k_{\pm}^2$ move to have the reflection point at $(\psi_0^{\pm}, \pi)$). In other words, $k_{\pm}^2$ corresponds to the boundary value $\mu_{\pm}$:

$$k_{\pm}^2 = \frac{W - e\Phi(\psi) - \mu_{\pm} B_0 (1 - \varepsilon_t(\psi) \cos(\theta) - \varepsilon_r(\psi))}{2\mu_{\pm} B_0 \varepsilon_r(\psi)}, \quad (109)$$

with $\mu_{\pm}$ found from

$$1 = \frac{W - e\Phi(\psi_0^{\pm}) - \mu_{\pm} B_0 (1 + \varepsilon_t(\psi_0^{\pm}) - \varepsilon_r(\psi_0^{\pm}))}{2\mu_{\pm} B_0 \varepsilon_r(\psi_0^{\pm})}. \quad (110)$$

Eq. (110) is just a statement that $(\psi_0^{\pm}, \pi)$ is the transition point, which is simultaneously the reflection point of a banana (see Fig. 4, showing the relationship between the kinetic energy and $\mu^{\pm}$). It is easily shown that, in the lowest order in the gyroradius, we can put $\psi_0^{\pm} = \psi$ in Eq. (110), because $k_{\pm}^2$ is incorporated in the higher order term in Eqs. (107) and (108). Then Eqs. (109) and (110) yield,

$$k_+^2 = k_-^2 = k_*^2, \quad (111)$$

$$k_*^2 \equiv 1 + \frac{\varepsilon_t}{2\varepsilon_r} (1 + \varepsilon_t \cos(\theta)). \quad (112)$$

Using Eq. (69), we can represent Y^r in the form,

$$Y^r(\psi, \theta) = a(\psi) (1 + \varepsilon_t \cos(\theta)), \quad (113)$$

$$a(\psi) = \frac{16R_0 \sqrt{\mu B_0 \varepsilon_r}}{M}. \quad (114)$$

Inserting Eqs. (111) and (113) into Eqs. (107) and (108), then expanding in $\Delta\psi^{\pm} = |\psi_0^{\pm} - \psi|$, we obtain,

$$\psi_0^+ - \psi_0^- = -(\Delta\psi^+ + \Delta\psi^-), \quad (115)$$

with

$$\Delta\psi^\pm = \frac{a}{2\frac{\partial I^t}{\partial \psi}} \left(1 - \hat{f}(k_*^2) + \varepsilon_t [1 + \cos(\theta) \hat{f}(k_*^2)] \right), \quad (116)$$

$$\frac{a}{\frac{\partial I^t}{\partial \psi}} = -\frac{4B_0\sqrt{E\varepsilon_r}}{4\pi\Omega}. \quad (117)$$

We evaluated $(\psi_0^+ - \psi_0^-)$, which is the difference in the radial positions of the reflection points of + and - trajectories as a function of the current coordinates (ψ, θ) . These trajectories were the boundary trajectories: with the reflection points at $\theta=\pi$. Now we need the average of this difference given by Eq. (105), which can be easily calculated in two limits: $\varepsilon_t \gg \varepsilon_r$ and $\varepsilon_r \gg \varepsilon_t$.

(a) Let us first assume $\varepsilon_t \gg \varepsilon_r$. Then, according to Eq. (112), the average value of k_*^2 is much larger than unity and $\hat{f}(k_*^2) \approx k_* \frac{\pi}{2}$. Then the average radial displacement becomes,

$$\langle \psi_0^+ - \psi_0^- \rangle = -\frac{8B_0\sqrt{E\varepsilon_r}}{4\pi\Omega} \int_0^\pi d\theta \sqrt{1 + \frac{\varepsilon_t}{2\varepsilon_r} (1 + \cos(\theta))} = -\frac{8B_0\sqrt{E\varepsilon_t}}{4\pi\Omega}. \quad (118)$$

(b) Now let us consider the opposite case, $\varepsilon_r \gg \varepsilon_t$, which is not very frequent, though. In this case we can neglect the term proportional to ε_t in Eq. (116) and use the asymptotic expression for $\hat{f}(k_*^2)$ at $k_*^2 \approx 1$,

$$\hat{f}(k_*^2) \approx 1 + \frac{k_*^2 - 1}{4} \left(1 + \ln \left(\frac{16}{|k_*^2 - 1|} \right) \right). \quad (119)$$

The integration over θ is then easily performed to lead to the following expression for the radial displacement,

$$\langle \psi_0^+ - \psi_0^- \rangle = -\frac{B_0\varepsilon_t\sqrt{E}}{24\pi\Omega\sqrt{\varepsilon_r}} \left(\ln \left(\frac{16\varepsilon_r}{\varepsilon_t} \right) - 1 \right). \quad (120)$$

Combining Eqs. (118) and (120), we can represent the average radial displacement in the unified form,

$$\langle \psi_0^+ - \psi_0^- \rangle = -\frac{B_0 \varepsilon_t \sqrt{E}}{4\pi\Omega} G, \quad (121)$$

where the geometrical factor is different for the cases (a) and (b),

$$G = \begin{cases} \frac{8}{\sqrt{\varepsilon_t}}, & \text{for } \varepsilon_t \gg \varepsilon_r \\ \frac{1}{2\sqrt{\varepsilon_r}} \left(\ln \left(\frac{16\varepsilon_r}{\varepsilon_t} \right) - 1 \right), & \text{for } \varepsilon_r \gg \varepsilon_t \end{cases} \quad (122)$$

Substituting Eqs. (106) and (121) into Eq. (104), we finally obtain,

$$\begin{aligned} j_T = & -\frac{4eB_0\varepsilon_t G}{4\pi\Omega} \int E^{3/2} \frac{\partial}{\partial \psi} \left(\frac{\varepsilon_t}{\frac{\partial}{\partial \psi} \left(\frac{e\Phi}{E} \right)} \right) \frac{\partial F_M}{\partial \psi} dE = \\ & -\frac{15}{4\sqrt{2}} \frac{eB_0\varepsilon_t G}{4\pi\Omega} v_T^2 \frac{T \frac{\partial \varepsilon_t}{\partial \psi}}{\frac{\partial e\Phi}{\partial \psi}} \left(\frac{\partial N}{\partial \psi} - 5N \frac{\partial T}{T} \right). \end{aligned} \quad (123)$$

5. Conclusions

The bootstrap current found in our paper is sustained by transitions of toroidally trapped particles into locally trapped and vice versa. These transitions change the distribution function in the toroidally trapped region and, due to smoothness of the distribution function, in the passing region as well. The perturbation of the distribution function is significant for the magnetic configurations with $\alpha \ll 1$, which is typical for stellarators and less typical for tokamaks. The additional bootstrap current for configurations with $\alpha \ll 1$ has the order $j_T \approx e \frac{\partial N}{\partial r} (\Delta r)_r v_T \frac{\rho}{4\pi} \frac{1}{\max\{\sqrt{\varepsilon_t}, \sqrt{\varepsilon_r}\}}$, where $(\Delta r)_r$ is the radial width of the locally trapped orbit.

Strictly speaking, all the formulas obtained in the paper (as well as those in Ref.), are correct when the radial displacements of the locally trapped particles are small. But they correctly describe the tendency of the perturbation of the distribution function to become of order unity as the radial width of the locally trapped orbit becomes comparable with the minor radius. In such a case, the additional bootstrap current has the same order as the usual bootstrap current (see Eqs. (2) and (6)) in the banana regime.

References

1. A.A. Galeev, R.Z. Sagdeev, in Reviews of Plasma Physics, M.A. Leontovich, ed., v. 7, ? (Consultants Bureau, 1979).
2. M.N. Rosenbluth, R.H. Haseltine, F.L. Hinton, Phys. Fluids **15**, 112 (1972).
3. L.M. Kovrizhnykh, Nuclear Fusion **24**, 851 (1984).
4. R.J. Bickerton, J.W. Connor, J.B. Taylor, Nature Phys. Sci. **229**, 110 (1971).
5. J.W. Connor, J.B. Taylor, Comments Plasma Phys. Controlled Fusion **11**, 37 (1987).
6. A.H. Boozer, H.J. Gardner, Phys. Fluids **B2**, 2408 (1990).
7. A. Pytte, A.H. Boozer, Phys. Fluids **24**, 88 (1981).
8. K.C. Shaing, J.D. Callen, Phys. Fluids **26**, 3315 (1983).
9. P.N. Yushmanov, Dokl. Akad. Nauk SSSR **283**, 581 (1985).
10. P. N. Yushmanov, in Reviews of Plasma Physics, B. B. Kadomtsev, ed., v. 16 102 (Consultants Bureau, 1987).
11. L.M. Kovrizhnykh, S.G. Shasharina, Nuclear Fusion **30**, 453 (1990).
12. A. H. Boozer, Phys. Fluids **24**, 1999 (1981).
13. R. G. Littlejohn, J. Plasma Phys. **29**, 111 (1983).
14. R. Balescu, *Transport Processes in Plasmas* (Elsevier Science Publisher, North Holland Amsterdam-Oxford-New York-Tokyo, 1988).

15. A.V. Gurevich, Ya.S. Dimant, in Reviews of Plasma Physics, B. B. Kadomtsev, ed., v. 16, 1 (Consultants Bureau, 1987).
16. J. R. Cary, C. L. Hedrick, J.S. Tolliver, Phys. Fluids **31**, 1586 (1988).
17. John R. Cary, S.G. Shasharina, Phys. Fluids **B5**, 2098 (1993).
18. R. J. Goldston, H.H. Towner, J. Plasma Phys. **26**, 283 (1981).
19. D. Dobrott, J. M. Greene, Phys. Fluids **14**, 1525 (1971).

Figure Captions

Fig. 1 The process of detrapment: a locally trapped particle drifts into the region with lower local modulation of the magnetic field and becomes toroidally trapped.

Fig. 2 Particles with opposite equal, but parallel velocities at the point A belong to different transition trajectories, separated by a distance (B_1B_2 or C_1C_2) approximately equal to the banana width. The points B_1, B_2 and C_1, C_2 are transition points.

Fig. 3 Typical transition trajectories with the strong electric field. Point B is the branching point, C is the confluence point.

- (a) + state is the majority state,
- (b) - state is the majority state,
- (c) r state is the majority state: the drift is so strong that both + and - states drift in the same direction in the poloidal cross section,
- (d) the same trajectory as on (c) as it looks on the torus.

Fig. 4 The magnetic field strength along the field line versus the poloidal angle θ . The straight lines correspond to the critical value $B_* = B_0(1 + \epsilon_i + \epsilon_r)$ which defines the value of the magnetic moment, $\mu_* = \frac{E}{B_*}$, on the boundary between toroidally trapped and passing particles.

- (a) $\epsilon_r \ll \epsilon_i$: the average value of k^2 of the toroidally trapped particles is much larger than unity,
- (b) $\epsilon_i \ll \epsilon_r$: the average value of k^2 of the toroidally trapped particles is close to unity.

Fig. 1

Fig. 2

Fig. 3a

Fig. 3b

Fig. 3c

Fig. 3d

B

θ

B

θ

Fig. 4 a, b