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Derivation of the Coupled Equations of Motion for a Circular Ring Rotating About an Axis in the Plane of the Ring

G. A. Benedetti

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DERIVATION OF THE COUPLED EQUATIONS OF MOTION FOR A CIRCULAR RING ROTATING ABOUT AN AXIS IN THE PLANE OF THE RING

G. A. Benedetti
Solid Mechanics Department
Sandia National Laboratories/California

ABSTRACT

The coupled equations of motion for a circular ring or circular ring segment are developed for the case where the ring is rotating about an axis in its plane and subjected to an angular velocity as well as an angular acceleration. Coupling results from bending in and out of the plane of the ring as well as from extension and torsion of the ring. These equations are then applied to special cases to determine the coupled equations of motion for a ring, beam and cable rotating at a constant angular speed.

Coupled equations of motion for a non-rotating circular ring or circular ring segment are developed for the cases of extensional motion and inextensional motion. These equations are subsequently linearized and uncoupled for extensional and inextensional motion in the plane of the ring as well as for uncoupled motion out of the plane of the ring.

The critical angular speed for lateral dynamic instability is determined for a rotating circular shaft which supports several rotating circular ring segments.

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NOMENCLATURE

a	radius of ring in body fixed x-y plane
$\bar{a}$	absolute acceleration vector
A	cross-sectional area of ring
$\bar{d}$	position vector
$\bar{D}$	displacement vector
e	offset distance
E	modulus of elasticity
$\bar{f}(\phi)$	force vector per unit arc length
$\bar{F}(\phi, t)$	force vector per unit arc length
$\bar{F}$	force vector
g	local acceleration of gravity
G	shear modulus of elasticity
$\bar{h}_o$	absolute angular momentum vector about displaced mass center of ring cross-section per unit arc length of ring
$\bar{i}, \bar{j}, \bar{k}$	unit vectors along the x y z axes
$\bar{I}, \bar{J}, \bar{K}$	unit vectors along the X Y Z axes
I	area moments of inertia of ring circular cross-section about y or z axes originating at centroid of ring cross-section
I_p	polar mass moment of inertia of ring circular cross-section per unit arc length
J	torsion constant; also, number of symmetrical circular ring segments, $J = 2, 3, \dots, k$
J_1	area moment of inertia of ring circular cross-section about x axis passing through centroid of cross-section or polar area moment of inertia of ring circular cross-section. For a straight beam with a circular cross-section, $T = GJ_1\theta_x$ where $J_1 = \frac{\pi}{2}r_o^4$ and $J_1 = I_y + I_z = 2I$. For a straight beam with a non-circular cross section, J_1 is replaced with the torsion constant J .
ℓ	beam length
m	ring mass per unit arc length; $m = \rho A$
m_b	beam mass per unit length

$\bar{M}$	moment vector
M_y, M_z	bending moments in plane of deformed ring cross section and normal to P
P	axial force normal to deformed ring cross-section
Q_y, Q_z	shear forces in plane of deformed ring cross-section and normal to P
r	radial coordinate
r_o	radius of ring circular cross section
$\bar{r}_1, \bar{r}_2$	position vectors
$\bar{r}$	position or displacement vector
$\bar{R}_o$	position vector
t	time
$\underline{T}$	transformation matrix
T	torque or twisting moment normal to deformed ring cross-section
u, v, w	displacements (centroidal) of mass center of ring cross-section along the directions of the x,y,z axes
$\left. \begin{array}{l} x', y', z' \\ x'', y'', z'' \\ x, y, z \end{array} \right\}$	body fixed coordinate systems
X, Y, Z	fixed coordinate system
ρ	mass density
$\bar{\rho}$	displacement vector
ξ	position along arc length of ring
$\theta_x = \phi'_x = \frac{\partial \phi_x}{a \partial \phi}$	unit angle of twist of ring cross - section about x axis or angle of twist per unit of arc length
α	specified angular location
ξ, ϕ	spatial coordinates
β, ψ	angular rotations
$\omega_x, \omega_y, \omega_z$	specified rotational speeds about the X,Y,Z axes
$\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z$	specified rotational accelerations about the X,Y,Z axes
$\gamma_1, \gamma_2, \gamma_3$	direction cosine angles; $\cos \gamma_i =$ direction cosines, $i = 1, 2, 3$
$\bar{\omega}, \bar{\Omega}$	angular velocity vectors
$\dot{\bar{\omega}}, \dot{\bar{\Omega}}$	angular acceleration vectors

$$\left. \begin{array}{l} \phi_x \\ \phi_y = -\frac{\partial w}{a\partial\phi} \\ \phi_z = \frac{\partial v}{a\partial\phi} \end{array} \right\} \begin{array}{l} \text{centroidal rotations of ring cross-section about x,y,z axes, respectively.} \\ \phi_x \text{ is angle of twist} \end{array}$$

$(\dot{})$ indicates differentiation with respect to time

$()'$ indicates differentiation with respect to space coordinate ϕ

Introduction

This report develops the coupled partial differential equations of motion for a rotating ring or ring segment. Before proceeding with the derivation, it is necessary to list a few assumptions.

1. The ring material is homogeneous and isotropic. In addition, the constitutive relationship between stress and strain is linear.
2. Damping of the ring material is neglected.
3. The ring is circular and lies in a plane. Its cross-section is symmetrical (e.g., a solid ellipse) with one principal axis lying in the plane of the ring. The other principal axis for the cross-section is therefore perpendicular to the plane of the ring.
4. The ring rotates about an axis in the plane of the ring.
5. All points on a plane of the ring cross-section, which is perpendicular to the ring center line, before deformation remain on that plane after deformation; i.e., plane sections remain plane before and after deformation. Therefore, deformations due to the transverse shear forces Q_y and Q_z are neglected. Also, warping of the ring's non-circular cross-section due to torsion is neglected.
6. The extensional stresses or strains in the y and z directions are negligible compared to the stress or strain in the ϕ direction. Therefore, the ring is thin compared to its radius of curvature.
7. All ring displacements are small compared with the radius of curvature of the ring's middle surface. That is, after deformation the ring still appears circular. Also, the first spatial derivatives of the displacements (slopes) are negligible compared to unity.
8. The ring's radius of curvature is large compared to the ring thickness. For example, $2r_o/a < 0.1$. Consequently, the centroid of the ring cross-section lies on the bending neutral axes. For $2r_o/a < 0.25$, the error is still small.
9. Warping of the ring's non-circular cross-section due to torsion is neglected in determining the angular momentum of the cross-section. Also, the center of twist of the cross-section coincides with the centroid of the cross-section.
10. The ring's rotational angular speed is small compared to the frequency of the speed of sound traveling around the ring circumference.

The coupled equations of motion for a circular ring or circular ring segment are developed for the case where the ring is rotating about an axis in its plane and subjected to an angular velocity as well as an angular acceleration. Coupling results from bending in and out of the plane of the ring as well as from extension and torsion of the ring. These equations are then applied to special cases to determine the coupled equations of motion for a ring, beam and cable rotating at a constant angular speed.

Coupled equations of motion for a non-rotating circular ring or circular ring segment are developed for the cases of extensional motion and inextensional motion. These equations are subsequently applied to special cases and uncoupled for extensional and inextensional motion in the plane of the ring as well as for uncoupled motion out of the plane of the ring.

Boundary conditions for a circular ring segment are briefly discussed.

Appendix A determines the critical angular speed for lateral dynamic instability of a rotating circular shaft which supports several rotating circular ring segments.

Force-Strain and Moment-Curvature Relations

Before proceeding with the derivation of the force-strain and moment-curvature relations it is required to study Figures 1 through 10. Referring to Figure 6, the displacement vector for a point in the deformed plane of the ring segment (or curved beam) cross-section relative to the same point in the undeformed cross-section is expressed by Eqs. (1) and (2), provided higher order terms are neglected. Note that the thickness of the ring $2r_o$ is small relative to its radius of curvature a ; i.e., $2r_o/a \leq 0.1$. Consequently, the centroid of the ring cross-section practically lies on the intersection of the bending neutral axes.

$$\bar{D} = \bar{r}_2 - \bar{r}_1 - \bar{d} + \bar{\rho} \quad (1)$$

where

$$\begin{aligned} \bar{r}_2 - \bar{r}_1 &= u\bar{i} + v\bar{j} + w\bar{k} \\ \bar{\rho} &= \bar{d} + \left(\frac{y}{a}u - y\phi_z + z\phi_y \right)\bar{i} + (-z\phi_x)\bar{j} + y\phi_x\bar{k} \\ \bar{d} &= y\bar{j} + z\bar{k} \end{aligned} \quad (2)$$

Utilizing Eqs. (2) in Eq. (1) yields Eq. (3) for the displacement vector $\bar{D}$ for the i th point in the plane of the deformed ring segment cross-section relative to the i th point in the plane of the undeformed cross-section.

$$\bar{D} = U\bar{i} + V\bar{j} + W\bar{k} \quad (3)$$

where

$$\begin{aligned} U &= u\left(1 + \frac{y}{a}\right) - y\phi_z + z\phi_y \\ V &= v - z\phi_x \\ W &= w + y\phi_x \end{aligned} \quad (4)$$

Note that point O is the centroid of the cross-section and that the cross-section is symmetrical (not necessarily circular) with respect to the y-z axis. That is, the y-z axes are principal centroidal axes. For a Bernoulli-Euler ring segment, the small centroidal displacements (u , v , and w) and the small rotations about the centroid (ϕ_x , ϕ_y , and ϕ_z) are functions of only one spatial coordinate, ϕ .

Figure 1. Symmetrically offset circular ring segments rotating about centerline axis. Center shaft is supported in bearings. Two links support each ring segment.

Figure 2. Offset circular ring segment rotating about X axis.

Figure 3. Offset circular ring segment.

Figure 4. First coordinate rotation.

Figure 5. Second coordinate rotation.

Figure 6. Displaced cross-section. O and O' are original and displaced positions of mass center, respectively. xyz axes are not necessarily parallel to XYZ axes.

Figure 7. Portion of circular ring segment and its cross-section.

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Figure 9. Deformed element in $x-z$ plane.

Figure 10. Angles for direction cosines. The angles denoted by $\gamma_i (i = 1, 2, 3)$ exist in the planes formed by the gravity vector and each axis (X, Y, and Z).

The strains associated with the displacements given by Eqs. (3) and (4) are expressed by Eqs. (5). Note that $\bar{D} = \bar{D}(\phi, y, z, t)$.

$$\begin{aligned}
\varepsilon_\phi &= \frac{V}{r} + \frac{1}{r} \frac{\partial U}{\partial \phi} = \frac{1}{a+y} (\bar{D} \cdot \bar{j}) + \frac{1}{a+y} \frac{\partial}{\partial \phi} (\bar{D} \cdot \bar{i}) \\
\varepsilon_y &= \frac{\partial V}{\partial r} = \frac{\partial}{\partial y} (\bar{D} \cdot \bar{j}) \\
\varepsilon_z &= \frac{\partial W}{\partial z} = \frac{\partial}{\partial z} (\bar{D} \cdot \bar{k}) \\
\gamma_{\phi y} &= \frac{1}{r} \frac{\partial V}{\partial \phi} + \frac{\partial U}{\partial r} - \frac{U}{r} = \frac{1}{a+y} \frac{\partial}{\partial \phi} (\bar{D} \cdot \bar{j}) + \frac{\partial}{\partial y} (\bar{D} \cdot \bar{i}) - \frac{1}{a+y} (\bar{D} \cdot \bar{i}) \\
\gamma_{\phi z} &= \frac{1}{r} \frac{\partial W}{\partial \phi} + \frac{\partial U}{\partial z} = \frac{1}{a+y} \frac{\partial}{\partial \phi} (\bar{D} \cdot \bar{k}) + \frac{\partial}{\partial z} (\bar{D} \cdot \bar{i}) \\
\gamma_{zy} &= \frac{\partial V}{\partial z} + \frac{\partial W}{\partial r} = \frac{\partial}{\partial z} (\bar{D} \cdot \bar{j}) + \frac{\partial}{\partial y} (\bar{D} \cdot \bar{k})
\end{aligned} \tag{5}$$

Note that $r = a + y$ and $\varepsilon_y = \varepsilon_z = \gamma_{zy} = 0$. Since $\varepsilon_\phi \neq 0$ and $\varepsilon_y = \varepsilon_z = 0$, this implies, along with assumption 6, that Poisson's ratio is negligible; i.e., $\varepsilon_y = \varepsilon_z \neq -\nu \varepsilon_\phi$. Also, $\phi_y = -\frac{\partial w}{a \partial \phi}$ and $\phi_z = \frac{\partial v}{a \partial \phi}$.

Utilizing Eqs. (3) and (4) in Eqs. (5) results in Eqs. (6).

$$\begin{aligned}
\varepsilon_\phi &= \frac{1}{a} \frac{\partial u}{\partial \phi} + \frac{1}{(a+y)} \left[v - \frac{y}{a} \frac{\partial^2 v}{\partial \phi^2} \right] - \frac{z}{a+y} \phi_x - \frac{z}{a(a+y)} \frac{\partial^2 w}{\partial \phi^2} \\
\gamma_{\phi y} &= \frac{\partial v}{\partial \phi} \left[\frac{1}{a+y} - \frac{1}{a} + \frac{y}{a(a+y)} \right] - \frac{z}{a+y} \frac{\partial \phi_x}{\partial \phi} + \frac{z}{a(a+y)} \frac{\partial w}{\partial \phi} \\
\gamma_{\phi z} &= \frac{\partial w}{\partial \phi} \left[\frac{1}{a+y} - \frac{1}{a} \right] + \frac{y}{(a+y)} \frac{\partial \phi_x}{\partial \phi}
\end{aligned} \tag{6}$$

Note that

$$\frac{1}{(a+y)} = \frac{1}{a} \frac{1}{\left(1 + \frac{y}{a}\right)} = \frac{1}{a} \left[1 - \frac{y}{a} + \left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)^3 \dots \right]$$

Since $y/a \ll 1$,

$$\frac{1}{a+y} \cong \frac{1}{a} \left[1 - \frac{y}{a} + \left(\frac{y}{a} \right)^2 \right] \quad (7)$$

Using Eq. (7) in Eqs. (6) results in Eqs. (8). Some higher order terms are neglected.

$$\begin{aligned} \varepsilon_\phi &= \frac{1}{a} \frac{\partial u}{\partial \phi} + \frac{1}{a} \left[1 - \frac{y}{a} + \left(\frac{y}{a} \right)^2 \right] \left[v - \frac{y}{a} \frac{\partial^2 v}{\partial \phi^2} \right] - \frac{1}{a} \left[1 - \frac{y}{a} + \left(\frac{y}{a} \right)^2 \right] z \phi_x - \frac{1}{a^2} \left[1 - \frac{y}{a} + \left(\frac{y}{a} \right)^2 \right] z \frac{\partial^2 w}{\partial \phi^2} \\ \gamma_{\phi y} &= \frac{\partial v}{\partial \phi} \left[\frac{1}{a} \left(1 - \frac{y}{a} \right) - \frac{1}{a} + \frac{y}{a^2} \left(1 - \frac{y}{a} \right) \right] - \frac{1}{a} \left(1 - \frac{y}{a} \right) z \frac{\partial \phi_x}{\partial \phi} + \frac{1}{a^2} \left(1 - \frac{y}{a} \right) z \frac{\partial w}{\partial \phi} \\ \gamma_{\phi z} &= \frac{\partial w}{\partial \phi} \left[\frac{1}{a} \left(1 - \frac{y}{a} \right) - \frac{1}{a} \right] + \frac{1}{a} \left(1 - \frac{y}{a} \right) y \frac{\partial \phi_x}{\partial \phi} \end{aligned} \quad (8)$$

Eqs. (9) result by neglecting several higher order geometrical terms which make the strain non-linear over the cross section. Also neglected are geometrical terms which result in area product of inertia terms since the cross-section is symmetrical with respect to the principal centroidal axes, y and z . Note that $y/a \ll 1$ and $z/a \ll 1$.

$$\begin{aligned} \varepsilon_\phi &= \frac{1}{a} \frac{\partial u}{\partial \phi} + \frac{v}{a} - \frac{y}{a^2} v + \frac{y^2}{a^3} v - \frac{y}{a^2} \frac{\partial^2 v}{\partial \phi^2} + \frac{y^2}{a^3} \frac{\partial^2 v}{\partial \phi^2} - \frac{z}{a} \phi_x - \frac{z}{a^2} \frac{\partial^2 w}{\partial \phi^2} \\ \gamma_{\phi y} &= -\frac{z}{a} \frac{\partial \phi_x}{\partial \phi} + \frac{z}{a^2} \frac{\partial w}{\partial \phi} \\ \gamma_{\phi z} &= -\frac{y}{a^2} \frac{\partial w}{\partial \phi} + \frac{y}{a} \frac{\partial \phi_x}{\partial \phi} \end{aligned} \quad (9)$$

Eqs. (9) represent the strain-displacement relations for the ring segment shown in Figures 2 and 6.

The stresses in terms of the displacements can be obtained from Hooke's law by substituting Eqs. (9) into Eqs. (10). (Recall that Poisson's ratio is neglected. Refer to Eqs. (5).)

$$\begin{aligned}
\sigma_\phi &= E\varepsilon_\phi \\
\tau_{\phi y} &= G\gamma_{\phi y} \\
\tau_{\phi z} &= G\gamma_{\phi z}
\end{aligned} \tag{10}$$

The resultant forces and moments, associated with the displacements given by Eq. (3), are expressed by Eqs. (11). Refer to Figures 8 and 9.

$$\begin{aligned}
P &= \int_A \sigma_\phi dA \\
Q_y &= \int_A \tau_{\phi y} dA \\
Q_z &= \int_A \tau_{\phi z} dA \\
T &= \int_A (\tau_{\phi z} y - \tau_{\phi y} z) dA \\
M_y &= \int_A \sigma_\phi z dA \\
M_z &= - \int_A \sigma_\phi y dA
\end{aligned} \tag{11}$$

Substituting Eqs. (9) into Eqs. (10) and those results into Eqs. (11), yields Eqs. (12). Recall the ring segment cross-section is uniform with respect to ϕ and the moduli are constant. Since the cross-section is symmetrical with respect to the principal centroidal axes, y and z , the area products of inertia are zero. Also, recall that P is normal to the deformed cross-section and passes through the centroid. Refer to Figures 8 and 9.

$$\begin{aligned}
P &= \frac{EA}{a} \left[\frac{\partial u}{\partial \phi} + v \right] + \frac{EI_z}{a^3} \left[\frac{\partial^2 v}{\partial \phi^2} + v \right] \\
Q_y &= 0 \\
Q_z &= 0 \\
T &= \frac{G}{a} (I_y + I_z) \left[\frac{\partial \phi_x}{\partial \phi} - \frac{1}{a} \frac{\partial w}{\partial \phi} \right] \\
M_y &= - \frac{EI_y}{a} \left[\frac{1}{a} \frac{\partial^2 w}{\partial \phi^2} + \phi_x \right] \\
M_z &= \frac{EI_z}{a^2} \left[\frac{\partial^2 v}{\partial \phi^2} + v \right]
\end{aligned} \tag{12}$$

Note that

$$\int_A y dA = \int_A z dA = 0 \quad (13)$$

and that

$$\begin{aligned} I_y &= \int_A z^2 dA \\ I_z &= \int_A y^2 dA \\ \int_A yz dA &= \int_A zy dA = 0 \end{aligned} \quad (14)$$

Eqs. (12) represent the force-strain and moment-curvature relations for the ring segment under consideration. These relations will be used in the equilibrium equations to obtain the four equations of motion for the ring segment in terms of the dependent variables u , v , w , and ϕ_x . Note that for thin circular rings of large radius

$$(u' + v) \gg \frac{I_z}{Aa^2}(v'' + v)$$

The second term in the second bracket in the first equation and the second term in the sixth equation of Eqs. (12) occur due to a change in curvature of the ring. When the radius of curvature of the ring increases from a to $a + v$, the increase in curvature is $\sim -\frac{v}{a^2}$. For thin rings of large radius and small radial displacement, this change in curvature is a small quantity.

The fourth equation of Eqs. (12) is valid for torsion of a circular cross-section where $I_y + I_z = 2I$. For torsion of a non-circular cross-section, $(I_y + I_z)$ can be replaced by the torsion constant J where J depends upon the shape of the cross-section. The shear stress distribution over the cross-section as well as the maximum shear stress for torsion of a non-circular cross-section is not the same as for a circular cross-section.

Also, torsion of a non-circular cross-section causes warping to occur. That is, the cross-section displaces in and out of its plane. Consequently, this warping affects the absolute angular momentum about the mass center of a unit length of cross-section. Also, warping of the cross-section probably has a small effect for the lower modes of torsional vibration as well as for the lower modes of vibration out of the plane of the ring. This warping effect on the angular momentum of the cross-section is neglected herein.

Absolute Acceleration Vector

Before developing an expression for the absolute acceleration of the mass center, O' , of a unit length of cross-section, it is necessary to determine an expression between the unit vectors in the fixed XYZ system and the unit vectors in the body fixed xyz system.

Referring to Figures 3 and 4, the unit vectors in the $x'y'z'$ system are related to the unit vectors in the $x''y''z''$ system by Eq. (15).

$$\begin{Bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix} \begin{Bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{Bmatrix} \quad (15)$$

Note that the $x''y''z''$ system is parallel to the XYZ system. Referring to Figure 5, the unit vectors in the xyz system are related to the unit vectors in the $x'y'z'$ by Eq. (16).

$$\begin{Bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{Bmatrix} = \begin{bmatrix} \sin \phi & \cos \phi & 0 \\ -\cos \phi & \sin \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{Bmatrix} \quad (16)$$

Note that $\psi = \frac{\pi}{2} - \phi$ and that $\cos \psi = \sin \phi$ and $\sin \psi = \cos \phi$.

Substituting Eq. (15) into Eq. (16) results in Eq. (17).

$$\begin{Bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{Bmatrix} = \begin{bmatrix} \sin \phi & \cos \phi \cos \beta & \cos \phi \sin \beta \\ -\cos \phi & \sin \phi \cos \beta & \sin \phi \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix} \begin{Bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{Bmatrix} \quad (17)$$

Since the coordinate transformations are orthogonal, Eq. (18) follows.

$$\begin{Bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{Bmatrix} = \begin{bmatrix} \sin \phi & -\cos \phi & 0 \\ \cos \phi \cos \beta & \sin \phi \cos \beta & -\sin \beta \\ \cos \phi \sin \beta & \sin \phi \sin \beta & \cos \beta \end{bmatrix} \begin{Bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{Bmatrix} \quad (18)$$

The absolute acceleration of the mass center, O' , of a unit length of ring segment cross-section, shown in Figure 6 (originally positioned at $\bar{r}_1$ and displaced to position $\bar{r}_2$) is expressed by Eq. (19).

$$\bar{a}_{cm} = \ddot{\bar{R}}_o + \ddot{\bar{r}}_r + 2\bar{\omega} \times \dot{\bar{r}}_r + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) \quad (19)$$

where

$$\begin{aligned} \bar{a}_{cm} &= a_x \bar{i} + a_y \bar{j} + a_z \bar{k} \\ \ddot{\bar{R}}_o &= \dot{\bar{\omega}} \times \bar{r}_1 + \bar{\omega} \times (\bar{\omega} \times \bar{r}_1) \\ \bar{r}_1 &= r_x \bar{i} + r_y \bar{j} + r_z \bar{k} \\ \bar{r} &= u \bar{i} + v \bar{j} + w \bar{k} \\ \dot{\bar{r}}_r &= \dot{u} \bar{i} + \dot{v} \bar{j} + \dot{w} \bar{k} \\ \ddot{\bar{r}}_r &= \ddot{u} \bar{i} + \ddot{v} \bar{j} + \ddot{w} \bar{k} \\ \bar{\omega} &= \Omega_x \bar{i} + \Omega_y \bar{j} + \Omega_z \bar{k} \\ \dot{\bar{\omega}} &= \dot{\Omega}_x \bar{i} + \dot{\Omega}_y \bar{j} + \dot{\Omega}_z \bar{k} \\ (\dot{}) &= \frac{\partial()}{\partial t} \end{aligned} \quad (20)$$

The first two terms in the expression for $\ddot{\bar{R}}_o$ result because the center of mass of the cross-section (point O) is subjected to a specified angular acceleration, $\dot{\bar{\omega}}$, and a specified angular velocity, $\bar{\omega}$. Since $\bar{\omega} = \omega_x \bar{i} + 0\bar{j} + 0\bar{k}$ and $\dot{\bar{\omega}} = \dot{\omega}_x \bar{i}$, it follows from Eq. (17) that

$$\begin{aligned} \Omega_x &= \omega_x \sin \phi \\ \Omega_y &= -\omega_x \cos \phi \\ \Omega_z &= 0 \end{aligned} \quad (21)$$

and

$$\begin{aligned} \dot{\Omega}_x &= \dot{\omega}_x \sin \phi \\ \dot{\Omega}_y &= -\dot{\omega}_x \cos \phi \\ \dot{\Omega}_z &= 0 \end{aligned} \quad (22)$$

Note that ω_x and $\dot{\omega}_x$ are specified.

Observe from Figures 2, 3, and 4 that

$$\bar{r}_1 = X\bar{I} + Y\bar{J} + Z\bar{K} \quad (23)$$

where

$$\begin{aligned} X &= a(\cos \alpha - \cos \phi) \\ Y &= [a(\sin \phi - \sin \alpha) + e] \cos \beta \\ Z &= [a(\sin \phi - \sin \alpha) + e] \sin \beta \end{aligned} \quad (24)$$

Substituting Eqs. (24) into Eq. (23) and utilizing Eqs. (18) results in Eqs. (25) and (26). Note that $\bar{r}_1$ is expressed in terms of the unit vectors in the body fixed xyz system.

$$\bar{r}_1 = r_x \bar{i} + r_y \bar{j} + r_z \bar{k} \quad (25)$$

where, after some simplification,

$$\begin{aligned} r_x &= a \sin(\phi - \alpha) + e \cos \phi \\ r_y &= a[1 - \cos(\phi - \alpha)] + e \sin \phi \\ r_z &= 0 \end{aligned} \quad (26)$$

Note that a , α , and e are specified constants and ϕ is an independent variable.

Substitution of Eqs. (20) into Eq. (19) results in Eqs. (27a), (27b), and (27c) for a_x , a_y , and a_z , respectively.

$$a_x = [\Omega_x \Omega_y r_y - \Omega_y^2 r_x + \ddot{u} + 2\Omega_y \dot{w} + \dot{\Omega}_y w + \Omega_x \Omega_y v - \Omega_y^2 u] \quad (27a)$$

$$a_y = [-\Omega_x^2 r_y + \Omega_x \Omega_y r_x + \ddot{v} - 2\Omega_x \dot{w} - \dot{\Omega}_x w - \Omega_x^2 v + \Omega_x \Omega_y u] \quad (27b)$$

$$a_z = [\dot{\Omega}_x r_y - \dot{\Omega}_y r_x + \ddot{w} + 2\Omega_x \dot{v} - 2\Omega_y \dot{u} + \dot{\Omega}_x v - \dot{\Omega}_y u - \Omega_x^2 w - \Omega_y^2 w] \quad (27c)$$

where

$$\begin{aligned} \Omega_x &= \omega_x \sin \phi \\ \Omega_y &= -\omega_x \cos \phi \end{aligned} \quad (28)$$

$$\begin{aligned}\dot{\Omega}_x &= \dot{\omega}_x \sin \phi \\ \dot{\Omega}_y &= -\dot{\omega}_x \cos \phi\end{aligned}\tag{29}$$

$$\begin{aligned}r_x &= a \sin(\phi - \alpha) + e \cos \phi \\ r_y &= a[1 - \cos(\phi - \alpha)] + e \sin \phi\end{aligned}\tag{30}$$

Eqs. (27a) through (27c) represent the absolute accelerations of the displaced mass center, O' , of a unit length of ring segment cross-section, along the xyz axes, respectively, as viewed by an observer in the fixed XYZ system. These relations will be used in the equilibrium equations.

Absolute Angular Momentum Vector and Its Time Rate of Change

Referring to Figures 2 and 6, the moment of the absolute linear momentum for all particles in a unit length of the displaced cross-section of the ring segment about its mass center, O' , is expressed by Eq. (31).

$$\bar{h}_{o'} = \sum_i (\bar{\rho}_i \times m_i \bar{V}_i)\tag{31}$$

$\bar{\rho}_i$ is the vector from O' to the i th particle and is given by Eq. (32), provided higher order terms are neglected.

$$\bar{\rho}_i = \left(\frac{y_i}{a} u - y_i \phi_z + z_i \phi_y \right) \bar{i} + (y_i - z_i \phi_x) \bar{j} + (z_i + y_i \phi_x) \bar{k}\tag{32}$$

m_i is the mass of the i th particle. $\bar{V}_i$ is the absolute velocity of the i th particle and is given by Eq. (33).

$$\bar{V}_i = \dot{\bar{r}}_2 + \dot{\bar{\rho}}_i\tag{33}$$

where

$$\dot{\bar{\rho}}_i = \bar{\omega} \times \bar{\rho}_i + \dot{\bar{\rho}}_{i_r}\tag{34}$$

and

$$\dot{\vec{p}}_{i_r} = \left(\frac{y_i}{a} \dot{u} - y_i \dot{\phi}_z + z_i \dot{\phi}_y \right) \bar{i} + \left(-z_i \dot{\phi}_x \right) \bar{j} + \left(y_i \dot{\phi}_x \right) \bar{k} \quad (35)$$

Substituting Eq. (34) into Eq. (33) and that result into Eq. (31) yields Eq. (36).

$$\bar{h}_{o'} = -\dot{\vec{r}}_2 \times \sum_i m_i \bar{\rho}_i + \sum_i \bar{\rho}_i \times m_i (\bar{\omega} \times \bar{\rho}_i) + \sum_i \bar{\rho}_i \times m_i \dot{\vec{p}}_{i_r} \quad (36)$$

Since the mass moments are taken about the mass center, O' , the first term on the right hand side of Eq. (36) is zero. Substituting $\bar{\rho}$ for $\bar{\rho}_i$, $\dot{\vec{p}}_r$ for $\dot{\vec{p}}_{i_r}$ and dm for m_i (recall $dm = \rho dA$ where ρ is the mass density), Eq. (36), in the limit, becomes Eq. (37).

$$\bar{h}_{o'} = \int_A [\bar{\rho} \times (\bar{\omega} \times \bar{\rho})] \rho dA + \int_A [\bar{\rho} \times \dot{\vec{p}}_r] \rho dA \quad (37)$$

where

$$\begin{aligned} \bar{h}_{o'} &= h_{o'_x} \bar{i} + h_{o'_y} \bar{j} + h_{o'_z} \bar{k} \\ \bar{\rho} &= \left(\frac{y}{a} u - y \phi_z + z \phi_y \right) \bar{i} + (y - z \phi_x) \bar{j} + (z + y \phi_x) \bar{k} \\ \dot{\vec{p}}_r &= \left(\frac{y}{a} \dot{u} - y \dot{\phi}_z + z \dot{\phi}_y \right) \bar{i} + (-z \dot{\phi}_x) \bar{j} + (y \dot{\phi}_x) \bar{k} \\ \bar{\omega} &= \Omega_x \bar{i} + \Omega_y \bar{j} + \Omega_z \bar{k} \\ \Omega_x &= \omega_x \sin \phi \\ \Omega_y &= -\omega_x \cos \phi \end{aligned} \quad (38)$$

$\bar{h}_{o'}$ is the absolute angular momentum, of a unit length of cross section, about its displaced mass center, O' . Note the velocities $\bar{\omega} \times \bar{\rho}$ and $\dot{\vec{p}}_r$ are relative to the displaced mass center O' .

The first term on the right hand side of the equal sign in Eq. (37) represents the absolute angular momentum, of a unit length of cross-section, about its mass center, O' , for the case where the particles in the unit length of cross-section are rigidly connected (e.g., a rigid thin disk). The second term in Eq. (37) arises because the particles in the deformed plane of the cross-section are moving, in a prescribed manner, relative to each other.

Substitution of Eqs. (38) into Eq. (37) results in Eqs. (39), (40), and (41), after some lengthy algebra. Recall that the cross-section (not necessarily circular) is symmetrical with respect to the principal centroidal axes y and z , and the mass products of inertia are zero; e.g., refer to Eqs. (13) and (14). For the case under consideration, the mass center for a unit length

of cross-section projected onto the centroidal y-z axes is concurrent with the centroid of the cross-section. That is, the mass density of the ring segment is constant over its cross-section.

$$\begin{aligned} & \int_A [\bar{\rho} \times (\bar{\omega} \times \bar{\rho}) + \bar{\rho} \times \dot{\bar{\rho}}_r] \rho dA \bullet \bar{i} \\ &= \rho \left\{ (\Omega_x + \dot{\phi}_x)(I_y + I_z) + \Omega_y I_z \left(\phi_z - \frac{u}{a} \right) + \Omega_y I_y \phi_x \phi_y + \Omega_x I_z \phi_x^2 \right\} \end{aligned} \quad (39a)$$

Neglecting higher order terms

$$h_{o'_x} = \rho \left\{ (\Omega_x + \dot{\phi}_x)(I_y + I_z) + \Omega_y I_z \left(\phi_z - \frac{u}{a} \right) \right\} \quad (39b)$$

$$\begin{aligned} & \int_A [\bar{\rho} \times (\bar{\omega} \times \bar{\rho}) + \bar{\rho} \times \dot{\bar{\rho}}_r] \rho dA \bullet \bar{j} \\ &= \rho \left\{ -I_z (\Omega_x + \dot{\phi}_x) \frac{u}{a} + I_z (\Omega_x + \dot{\phi}_x) \phi_z + \Omega_x I_y \phi_x \phi_y \right. \\ & \quad \left. + \Omega_y I_z \left(\frac{u^2}{a^2} - 2 \frac{u}{a} \phi_z + \phi_z^2 + \phi_x^2 \right) + \Omega_y I_y (\phi_y^2 + 1) \right. \\ & \quad \left. + I_y \dot{\phi}_y + I_z \phi_x \left(\frac{\dot{u}}{a} - \dot{\phi}_z \right) \right\} \end{aligned} \quad (40a)$$

Neglecting higher order terms

$$h_{o'_y} = \rho \left\{ \Omega_x \left(\phi_z - \frac{u}{a} \right) I_z + (\Omega_y + \dot{\phi}_y) I_y \right\} \quad (40b)$$

$$\begin{aligned} & \int_A [\bar{\rho} \times (\bar{\omega} \times \bar{\rho}) + \bar{\rho} \times \dot{\bar{\rho}}_r] \rho dA \bullet \bar{k} \\ &= \rho \left\{ -(\Omega_x + \dot{\phi}_x) I_y \phi_y + \Omega_x I_z \phi_x \left(\phi_y - \frac{u}{a} \right) + \Omega_y (I_y - I_z) \phi_x \right. \\ & \quad \left. + I_z \left(\dot{\phi}_z - \frac{\dot{u}}{a} \right) + I_y \phi_x \dot{\phi}_y \right\} \end{aligned} \quad (41a)$$

Neglecting higher order terms

$$h_{o'_z} = \rho \left\{ -\Omega_x I_y \phi_y + \Omega_y (I_y - I_z) \phi_x + I_z \left(\dot{\phi}_z - \frac{\dot{u}}{a} \right) \right\} \quad (41b)$$

where

$$\begin{aligned}\Omega_x &= \omega_x \sin \phi \\ \Omega_y &= -\omega_x \cos \phi\end{aligned}\quad (42)$$

Note that

$$I_x = \int_A (y^2 + z^2) dA = I_z + I_y \quad (43)$$

and that if the cross-section is circular, $I_y + I_z = 2I$. For a non-circular cross-section $I_y + I_z$ can be replaced by the torsion constant J where J depends upon the shape of the cross-section.

An important and well known relationship between the resultant moment vector and the time rate of change of the angular momentum vector is given by Eq. (44) (refer to Reference 1).

$$\overline{M}_{o'} = \dot{\overline{h}}_{o'} \quad (44)$$

Since the xyz axes are rotating with an angular velocity $\overline{\omega}$ relative to the fixed system XYZ ,

$$\dot{\overline{h}}_{o'} = \dot{\overline{h}}_{o'r} + \overline{\omega} \times \overline{h}_{o'} \quad (45)$$

where

$$\overline{h}_{o'} = h_{o'_x} \bar{i} + h_{o'_y} \bar{j} + h_{o'_z} \bar{k} \quad (46)$$

and

$$\dot{\overline{h}}_{o'r} = \dot{h}_{o'_x} \bar{i} + \dot{h}_{o'_y} \bar{j} + \dot{h}_{o'_z} \bar{k} \quad (47)$$

$\dot{h}_{o'_x}$, $\dot{h}_{o'_y}$, and $\dot{h}_{o'_z}$ are obtained from Eqs. (39b), (40b), and (41b), respectively, by differentiating these equations with respect to time. Substituting Eq. (45) into Eq. (44) results in Eq. 48.

$$\overline{M}_{o'} = \dot{\overline{h}}_{o'r} + \overline{\omega} \times \overline{h}_{o'} \quad (48)$$

where

$$\begin{aligned}
\dot{\bar{h}}_{o'_r} = & \rho \left\{ \left(\dot{\Omega}_x + \ddot{\phi}_x \right) (I_y + I_z) + \dot{\Omega}_y I_z \left(\phi_z - \frac{u}{a} \right) + \Omega_y I_z \left(\dot{\phi}_z - \frac{\dot{u}}{a} \right) \right\} \bar{i} \\
& + \rho \left\{ \dot{\Omega}_x \left(\phi_z - \frac{u}{a} \right) I_z + \Omega_x \left(\dot{\phi}_z - \frac{\dot{u}}{a} \right) I_z + \left(\dot{\Omega}_y + \ddot{\phi}_y \right) I_y \right\} \bar{j} \\
& + \rho \left\{ -\dot{\Omega}_x I_y \phi_y - \Omega_x I_y \dot{\phi}_y + \dot{\Omega}_y (I_y - I_z) \phi_x + \Omega_y (I_y - I_z) \dot{\phi}_x \right. \\
& \left. + I_z \left(\ddot{\phi}_z - \frac{\ddot{u}}{a} \right) \right\} \bar{k}
\end{aligned} \tag{49}$$

Substitution of Eq. (49) and Eqs. (39b), (40b), and (41b) into Eq. (45) results in Eq. (50).

$$\begin{aligned}
\dot{\bar{h}}_{o'} = \dot{\bar{h}}_{o'_r} + \bar{\omega} \times \bar{h}_{o'} = & \rho \left[\left\{ \left(\dot{\Omega}_x + \ddot{\phi}_x \right) (I_y + I_z) + \dot{\Omega}_y I_z \left(\phi_z - \frac{u}{a} \right) + \Omega_y I_z \left(\dot{\phi}_z - \frac{\dot{u}}{a} \right) \right. \right. \\
& \left. \left. - \Omega_x \Omega_y I_y \phi_y + \Omega_y^2 (I_y - I_z) \phi_x + \Omega_y I_z \left(\dot{\phi}_z - \frac{\dot{u}}{a} \right) \right\} \bar{i} \right. \\
& + \left[\left[\dot{\Omega}_x \left(\phi_z - \frac{u}{a} \right) I_z + \Omega_x \left(\dot{\phi}_z - \frac{\dot{u}}{a} \right) I_z + \left(\dot{\Omega}_y + \ddot{\phi}_y \right) I_y \right] \right. \\
& \left. + \Omega_x^2 I_y \phi_y - \Omega_x \Omega_y (I_y - I_z) \phi_x - \Omega_x I_z \left(\dot{\phi}_z - \frac{\dot{u}}{a} \right) \right\} \bar{j} \\
& + \left[\left[-\dot{\Omega}_x I_y \phi_y - \Omega_x I_y \dot{\phi}_y + \dot{\Omega}_y (I_y - I_z) \phi_x \right. \right. \\
& \left. \left. + \Omega_y (I_y - I_z) \dot{\phi}_x + I_z \left(\ddot{\phi}_z - \frac{\ddot{u}}{a} \right) \right] \right. \\
& \left. + \Omega_x^2 \left(\phi_z - \frac{u}{a} \right) I_z + \Omega_x (\Omega_y + \dot{\phi}_y) I_y - \Omega_y (\Omega_x + \dot{\phi}_x) (I_y + I_z) \right. \\
& \left. \left. - \Omega_y^2 I_z \left(\phi_z - \frac{u}{a} \right) \right\} \bar{k} \right]
\end{aligned} \tag{50}$$

where

$$\begin{aligned}\Omega_x &= \omega_x \sin \phi \\ \Omega_y &= -\omega_x \cos \phi \\ \phi_y &= -\frac{\partial w}{a \partial \phi} \\ \phi_z &= +\frac{\partial v}{a \partial \phi}\end{aligned}$$

Eq. (50) represents the time rate of change of the absolute angular momentum vector of a unit length of cross-section about its displaced mass center O' . Note that higher order terms are neglected.

At this point it is convenient to determine $\frac{\partial \dot{\vec{h}}_{o'}}{\partial \phi} \cdot \bar{j}$ and $\frac{\partial \dot{\vec{h}}_{o'}}{\partial \phi} \cdot \bar{k}$ from Eq. (50). This results in Eqs. (51) and (52).

$$\begin{aligned}\frac{\partial \dot{\vec{h}}_{o'}}{\partial \phi} \cdot \bar{j} &= \rho \left\{ \left[\dot{\Omega}_x' \left(\phi_z - \frac{u}{a} \right) I_z + \dot{\Omega}_x' \left(\phi_z' - \frac{u'}{a} \right) I_z + \Omega_x' \left(\dot{\phi}_z - \frac{\dot{u}}{a} \right) I_z \right. \right. \\ &\quad \left. \left. + \Omega_x' \left(\dot{\phi}_z' - \frac{\dot{u}'}{a} \right) I_z + (\dot{\Omega}_y' + \ddot{\phi}_y') I_y \right] \right. \\ &\quad \left. + 2\Omega_x \Omega_x' I_y \phi_y + \Omega_x^2 I_y \phi_y' - \Omega_x' \Omega_y (I_y - I_z) \phi_x - \Omega_x \Omega_y' (I_y - I_z) \phi_x \right. \\ &\quad \left. - \Omega_x \Omega_y (I_y - I_z) \phi_x' - \Omega_x' I_z \left(\dot{\phi}_z - \frac{\dot{u}}{a} \right) - \Omega_x I_z \left(\dot{\phi}_z' - \frac{\dot{u}'}{a} \right) \right\} \quad (51)\end{aligned}$$

$$\begin{aligned}\frac{\partial \dot{\vec{h}}_{o'}}{\partial \phi} \cdot \bar{k} &= \rho \left\{ \left[-\dot{\Omega}_x' I_y \phi_y - \dot{\Omega}_x I_y \phi_y' - \Omega_x' I_y \dot{\phi}_y - \Omega_x I_y \dot{\phi}_y' + \dot{\Omega}_y' (I_y - I_z) \phi_x \right. \right. \\ &\quad \left. \left. + \dot{\Omega}_y (I_y - I_z) \phi_x' + \Omega_y' (I_y - I_z) \dot{\phi}_x + \Omega_y (I_y - I_z) \dot{\phi}_x' + I_z \left(\ddot{\phi}_z' - \frac{\ddot{u}'}{a} \right) \right] \right. \\ &\quad \left. + 2\Omega_x \Omega_x' \left(\phi_z - \frac{u}{a} \right) I_z + \Omega_x^2 \left(\phi_z' - \frac{u'}{a} \right) I_z + \Omega_x' (\Omega_y + \dot{\phi}_y) I_y + \Omega_x (\Omega_y' + \dot{\phi}_y') I_y \right. \\ &\quad \left. - \Omega_y' (\Omega_x + \dot{\phi}_x) (I_y + I_z) - \Omega_y (\Omega_x' + \dot{\phi}_x') (I_y + I_z) - 2\Omega_y \Omega_y' I_z \left(\phi_z - \frac{u}{a} \right) \right. \\ &\quad \left. - \Omega_y^2 I_z \left(\phi_z' - \frac{u'}{a} \right) \right\} \quad (52)\end{aligned}$$

where

$$\begin{aligned}
 ()' &= \frac{\partial ()}{\partial \phi} \\
 \Omega_x &= \omega_x \sin \phi \\
 \Omega_y &= -\omega_x \cos \phi \\
 \dot{\Omega}_x &= \dot{\omega}_x \sin \phi \\
 \dot{\Omega}_y &= -\dot{\omega}_x \cos \phi \\
 \Omega'_x &= \omega_x \cos \phi \\
 \Omega'_y &= \omega_x \sin \phi
 \end{aligned} \tag{53}$$

Since the circular ring segment is uniform and of constant cross-section, ρ , I_y , and I_z are not functions of ϕ .

Eqs. (50), (51), and (52) will be incorporated into the equilibrium equations to obtain the equations of motion for the curved beam or circular ring segment.

Equilibrium Equations

Figures 8 and 9 show the forces and moments acting on a displaced and deformed infinitesimal element of the ring segment cross-section in the x-y and x-z planes, respectively.

Applying Newton's second law for translational motion to the mass center of the element results in Eq. (54).

$$-\bar{F} + \bar{F} + \Delta \bar{F} + a \Delta \phi [\bar{f}(\phi) + \bar{F}(\phi, t)] = a \Delta \phi m \bar{a}_{CM} \tag{54}$$

Eq. (55) is obtained as $\Delta \phi \rightarrow 0$.

$$\frac{\partial \bar{F}}{\partial \phi} + a [\bar{f}(\phi) + \bar{F}(\phi, t)] = a m \bar{a}_{CM} \tag{55}$$

where $\bar{f}(\phi)$ is a specified static force per unit of arc length (e.g., a gravity load) and $\bar{F}(\phi, t)$ is a specified externally applied dynamic force per unit of arc length. The line of action of the applied forces passes through the shear center, sometimes called the center of twist. For

simplicity, the cross-section is assumed to have two axes of symmetry in which case the shear center coincides with the centroid of the cross-sectional area.

Applying Newton's second law for rotational motion about the mass center of the element results in Eq. (56).

$$-\bar{M} + \bar{M} + \Delta\bar{M} + \left[a \left(1 + \frac{v}{a} \right) \Delta\phi \bar{i} + \Delta\bar{D} \Big|_{y=z=0} \right] \times (\bar{F} + \Delta\bar{F}) = \dot{\bar{h}}_o a \Delta\phi \quad (56)$$

where $\Delta\bar{D} \Big|_{y=z=0} = \Delta u \bar{i} + \Delta v \bar{j} + \Delta w \bar{k}$. Eq. (57) is obtained as $\Delta\phi \rightarrow 0$, provided higher order terms are neglected.

$$\frac{\partial \bar{M}}{\partial \phi} + a \bar{i} \times \bar{F} + \frac{\partial \bar{D}}{\partial \phi} \Big|_{y=z=0} \times \bar{F} = a \dot{\bar{h}}_o \quad (57)$$

Note that the specified static and dynamic forces $\bar{f}(\phi)$ and $\bar{F}(\phi, t)$, do not contribute to the moment about the mass center of the element.

Figures 8 and 9 show the forces (P , Q_y , and Q_z) and moments (T , M_y , and M_z) acting on (parallel or normal to) the displaced and deformed cross-sections of the element. It is necessary to express these forces and moments in terms of the unit vectors $\bar{i}\bar{j}\bar{k}$ in the xyz system; Eqs. (59) through (64) can be used to do this.

$$\bar{F} = \underline{T}_1^T [P, Q_y, Q_z]^T \quad (59)$$

$$\bar{M} = \underline{T}_1^T [T, M_y, M_z]^T \quad (60)$$

$$\bar{F} + d\bar{F} = \underline{T}_2^T [P + dP, Q_y + dQ_y, Q_z + dQ_z]^T \quad (61)$$

$$\bar{M} + d\bar{M} = \underline{T}_2^T [T + dT, M_y + dM_y, M_z + dM_z]^T \quad (62)$$

$$\underline{T}_1^T = \begin{bmatrix} 1 & -\phi_z & \phi_y \\ \phi_z & 1 & -\phi_x \\ -\phi_y & \phi_x & 1 \end{bmatrix} \quad (63)$$

$$\underline{T}_2^T = \begin{bmatrix} 1 & -(\phi_z + d\phi_z - d\phi) & (\phi_y + d\phi_y) \\ (\phi_z + d\phi_z - d\phi) & 1 & -(\phi_x + d\phi_x) \\ -(\phi_y + d\phi_y) & (\phi_x + d\phi_x) & 1 \end{bmatrix} \quad (64)$$

where $[\dots]^T$ is a column matrix.

Using Eqs. (59) through (64) results in Eqs. (65) and (67)

$$\begin{aligned}\bar{F} + \frac{\partial \bar{F}}{\partial \phi} - \bar{F} = \frac{\partial \bar{F}}{\partial \phi} = & \left[\frac{\partial P}{\partial \phi} + Q_y - \frac{\partial}{\partial \phi}(Q_y \phi_z) + \frac{\partial}{\partial \phi}(Q_z \phi_y) \right] \bar{i} \\ & + \left[\frac{\partial}{\partial \phi}(P \phi_z) - P + \frac{\partial Q_y}{\partial \phi} - \frac{\partial}{\partial \phi}(Q_z \phi_x) \right] \bar{j} \\ & + \left[-\frac{\partial}{\partial \phi}(P \phi_y) + \frac{\partial Q_z}{\partial \phi} + \frac{\partial}{\partial \phi}(Q_y \phi_x) \right] \bar{k}\end{aligned}\quad (65)$$

Neglecting higher order terms in Eq. (65) results in Eq. (66).

$$\frac{\partial \bar{F}}{\partial \phi} = \left[\frac{\partial P}{\partial \phi} + Q_y \right] \bar{i} + \left[\frac{\partial}{\partial \phi}(P \phi_z) - P + \frac{\partial Q_y}{\partial \phi} \right] \bar{j} + \left[-\frac{\partial}{\partial \phi}(P \phi_y) + \frac{\partial Q_z}{\partial \phi} \right] \bar{k} \quad (66)$$

$$\begin{aligned}\bar{M} + \frac{\partial \bar{M}}{\partial \phi} - \bar{M} = \frac{\partial \bar{M}}{\partial \phi} = & \left[\frac{\partial T}{\partial \phi} + M_y - \frac{\partial}{\partial \phi}(M_y \phi_z) + \frac{\partial}{\partial \phi}(M_z \phi_y) \right] \bar{i} \\ & + \left[-T + \frac{\partial M_y}{\partial \phi} + \frac{\partial}{\partial \phi}(T \phi_z) - \frac{\partial}{\partial \phi}(M_z \phi_x) \right] \bar{j} \\ & + \left[\frac{\partial M_z}{\partial \phi} - \frac{\partial}{\partial \phi}(T \phi_y) + \frac{\partial}{\partial \phi}(M_y \phi_x) \right] \bar{k}\end{aligned}\quad (67)$$

Eq. (68) results from Eqs. (3) and (4).

$$\left. \frac{\partial \bar{D}}{\partial \phi} \right|_{y=z=0} = \frac{\partial u}{\partial \phi} \bar{i} + \frac{\partial v}{\partial \phi} \bar{j} + \frac{\partial w}{\partial \phi} \bar{k} \quad (68)$$

Eq. (69) results from Eq. (59).

$$\begin{aligned}\bar{F} = & [P - Q_y \phi_z + Q_z \phi_y] \bar{i} + [P \phi_z + Q_y - Q_z \phi_x] \bar{j} \\ & + [-P \phi_y + Q_y \phi_x + Q_z] \bar{k}\end{aligned}\quad (69)$$

Eq. (70) is obtained by utilizing Eq. (69).

$$a\bar{i} \times \bar{F} = 0\bar{i} + a[P\phi_y - Q_y\phi_x - Q_z]\bar{j} + a[P\phi_z + Q_y - Q_z\phi_x]\bar{k} \quad (70)$$

Eq. (71) is obtained by utilizing Eqs. (68) and (69).

$$\begin{aligned} \left. \frac{\partial \bar{D}}{\partial \phi} \right|_{y=z=0} \times \bar{F} = & \left\{ [-P\phi_y + Q_y\phi_x + Q_z] \frac{\partial v}{\partial \phi} - [P\phi_z + Q_y - Q_z\phi_x] \frac{\partial w}{\partial \phi} \right\} \bar{i} \\ & + \left\{ [-P\phi_y + Q_y\phi_x + Q_z] \frac{\partial u}{\partial \phi} + [P - Q_y\phi_z + Q_z\phi_y] \frac{\partial w}{\partial \phi} \right\} \bar{j} \\ & + \left\{ [P\phi_z + Q_y - Q_z\phi_x] \frac{\partial u}{\partial \phi} - [P - Q_y\phi_z + Q_z\phi_y] \frac{\partial v}{\partial \phi} \right\} \bar{k} \end{aligned} \quad (71)$$

Neglecting some of the non-linear terms results in Eq. (73). Recall that

$$\begin{aligned} \phi_y &= -\frac{1}{a} \frac{\partial w}{\partial \phi} \\ \phi_z &= \frac{1}{a} \frac{\partial v}{\partial \phi} \end{aligned} \quad (72)$$

$$\left. \frac{1}{a} \frac{\partial \bar{D}}{\partial \phi} \right|_{y=z=0} \times \bar{F} = [Q_z\phi_z + Q_y\phi_y]\bar{i} + \left[-Q_z \frac{1}{a} \frac{\partial u}{\partial \phi} - P\phi_y \right] \bar{j} + \left[Q_y \frac{1}{a} \frac{\partial u}{\partial \phi} - P\phi_z \right] \bar{k} \quad (73)$$

Substitution of Eq. (66) into Eq. (55) results in Eq. (74).

$$\begin{aligned} & \left[\frac{\partial P}{\partial \phi} + Q_y \right] \bar{i} + \left[\frac{\partial}{\partial \phi} (P\phi_z) - P + \frac{\partial Q_y}{\partial \phi} \right] \bar{j} + \left[-\frac{\partial}{\partial \phi} (P\phi_y) + \frac{\partial Q_z}{\partial \phi} \right] \bar{k} \\ & + a[\bar{f}(\phi) + \bar{F}(\phi, t)] = a m \bar{a}_{CM} \end{aligned} \quad (74)$$

Eq. (74) represents the equilibrium equation for translational motion of the mass center of the element.

Substitution of Eqs. (67), (70), and (73) into Eq. (57) results in Eq. (75).

$$\begin{aligned}
& \left\{ \frac{\partial T}{\partial \phi} - \frac{\partial}{\partial \phi} (M_y \phi_z) + M_y + \frac{\partial}{\partial \phi} (M_z \phi_y) + a[Q_z \phi_z + Q_y \phi_y] \right\} \bar{i} \\
& + \left\{ \frac{\partial}{\partial \phi} (T \phi_z) - T + \frac{\partial M_y}{\partial \phi} - \frac{\partial}{\partial \phi} (M_z \phi_x) + a[P \phi_y - Q_y \phi_x - Q_z] \right. \\
& \quad \left. + a \left[-Q_z \frac{1}{a} \frac{\partial u}{\partial \phi} - P \phi_y \right] \right\} \bar{j} \\
& + \left\{ -\frac{\partial}{\partial \phi} (T \phi_y) + \frac{\partial}{\partial \phi} (M_y \phi_x) + \frac{\partial M_z}{\partial \phi} + a[P \phi_z + Q_y - Q_z \phi_x] \right. \\
& \quad \left. + a \left[Q_y \frac{1}{a} \frac{\partial u}{\partial \phi} - P \phi_z \right] \right\} \bar{k} = a \dot{\bar{h}}_o.
\end{aligned} \tag{75}$$

Neglecting higher order terms results in Eq. (76).

$$\left[\frac{\partial T}{\partial \phi} + M_y \right] \bar{i} + \left[-T + \frac{\partial M_y}{\partial \phi} - a Q_z \right] \bar{j} + \left[\frac{\partial M_z}{\partial \phi} + a Q_y \right] \bar{k} = a \dot{\bar{h}}_o. \tag{76}$$

Eq. (76) represents the equilibrium equation for rotational motion about the mass center of the element.

It is convenient to reduce the six scalar equilibrium equations, Eqs. (74) and (76), to four scalar equations where the unknowns are P , T , M_y , and M_z .

Determine Q_y from the coefficient of $\bar{k}$ in Eq. (76) and substitute that result into the coefficient of $\bar{i}$ in Eq. (74). This results in Eq. (77).

$$\frac{\partial P}{\partial \phi} - \frac{\partial M_z}{a \partial \phi} = a m \bar{a}_{CM} \bullet \bar{i} - \dot{\bar{h}}_o \bullet \bar{k} - a \bar{f}(\phi) \bullet \bar{i} - a \bar{F}(\phi, t) \bullet \bar{i} \tag{77}$$

Eq. (78) results from the coefficient of $\bar{i}$ in Eq. (76).

$$\frac{\partial T}{\partial \phi} + M_y = a \dot{\bar{h}}_o \bullet \bar{i} \tag{78}$$

Determine $\frac{\partial Q_y}{\partial \phi}$ and $\frac{\partial Q_z}{\partial \phi}$ from the coefficients of $\bar{k}$ and $\bar{j}$, respectively, in Eq. (76) and substitute those results into the coefficients of $\bar{j}$ and $\bar{k}$, respectively, in Eq. (74). This results in Eqs. (79) and (80).

$$\begin{aligned} \frac{\partial}{\partial \phi} \left[\frac{-\partial M_z}{a \partial \phi} \right] - P + \frac{\partial}{\partial \phi} (P \phi_z) = a m \bar{a}_{CM} \cdot \bar{j} - \frac{\partial \dot{\bar{h}}_{e'}}{\partial \phi} \cdot \bar{k} \\ - a \bar{f}(\phi) \cdot \bar{j} - a \bar{F}(\phi, t) \cdot \bar{j} \end{aligned} \quad (79)$$

$$\begin{aligned} \frac{\partial}{\partial \phi} \left[\frac{\partial M_y}{a \partial \phi} \right] - \frac{\partial T}{a \partial \phi} - \frac{\partial}{\partial \phi} (P \phi_y) = a m \bar{a}_{CM} \cdot \bar{k} + \frac{\partial \dot{\bar{h}}_{e'}}{\partial \phi} \cdot \bar{j} \\ - a \bar{f}(\phi) \cdot \bar{k} - a \bar{F}(\phi, t) \cdot \bar{k} \end{aligned} \quad (80)$$

where $\bar{a}_{CM}$ is given by Eqs. (27a), (27b), (27c) and the first of Eqs. (20). $\frac{\partial \dot{\bar{h}}_{e'}}{\partial \phi} \cdot \bar{j}$ and $\frac{\partial \dot{\bar{h}}_{e'}}{\partial \phi} \cdot \bar{k}$ are given by Eqs. (51) and (52), respectively. Consequently, the four scalar equilibrium equations of interest are Eqs. (77), (78), (79), and (80).

The specified dynamic force $\bar{F}(\phi, t)$ is expressed by Eq. (81).

$$\bar{F}(\phi, t) = F_x(\phi, t) \bar{i} + F_y(\phi, t) \bar{j} + F_z(\phi, t) \bar{k} \quad (81)$$

For the case when $\bar{f}(\phi)$ is a gravity force, $\bar{f}(\phi)$ is given by Eq. (82)

$$\bar{f}(\phi) = \underline{T}_3 \begin{Bmatrix} mg_x \\ mg_y \\ mg_z \end{Bmatrix} \quad (82)$$

where $\underline{T}_3$ is given in Eq. (17). $f_x(\phi)$, $f_y(\phi)$, and $f_z(\phi)$ are the components of $\bar{f}(\phi)$ in the $\bar{i}$, $\bar{j}$, and $\bar{k}$ directions, respectively.

$$\begin{aligned} mg_x &= mg \cos \gamma_1 \\ mg_y &= mg \cos \gamma_2 \\ mg_z &= mg \cos \gamma_3 \end{aligned} \quad (83)$$

where γ_1 , γ_2 , and γ_3 are the direction cosine angles shown in Figure 10. When the gravity vector is parallel to the X axis and points in a direction opposite to the direction of the X axis shown in Figure 10 (i.e., $\gamma_1 = \pi, \gamma_2 = \gamma_3 = -\pi/2$), then Eq. (82) reduces to Eq. (84).

$$\bar{f}(\phi) = -mg \sin \phi \bar{i} + mg \cos \phi \bar{j} \quad (84)$$

Note that when the gravity force is not parallel to the X axis (i.e., $\cos \gamma_2 \neq 0$ and $\cos \gamma_3 \neq 0$), the components of the gravity force in the xyz system become functions of time. This can be observed from Eq. (17) and noting that $\beta = \omega_x t + \beta_o$. When $\beta_o = 0$, the y' and Y axes in Figure 4 coincide at $t = 0$.

Equations of Motion for a Circular Ring

Equations of Motion for a Rotating Ring

Equations (77), (78), (79), and (80) are four scalar equilibrium equations with four unknown generalized forces P , T , M_y , and M_z and four unknown generalized displacements u , ϕ_x , w , and v . Recall that $\phi_y = -\frac{1}{a} \frac{\partial w}{\partial \phi}$ and $\phi_z = \frac{1}{a} \frac{\partial v}{\partial \phi}$.

Utilizing the force-strain and moment-curvature relations, Eqs. (12), the equations for the absolute acceleration, Eqs. (27a), (27b), and (27c), and the equations for the absolute angular momentum, Eqs. (50), (51), and (52), in Eqs. (77) through (80) results in Eqs. (85), (86), (87) and (88) for the motion of a ring, after some lengthy algebra. Note that in Eqs. (85) through (88), the unknowns are the four generalized displacements. Recall that E , A , I_y , I_z , and ρ are not functions of ϕ .

Equation for extensional motion for the ring in the ϕ direction (extensional motion):

$$\begin{aligned} & P' + a m \Omega_y^2 u + \rho I_z (\Omega_y^2 - \Omega_x^2) \frac{u}{a} - \left(a m + \frac{\rho I_z}{a} \right) \ddot{u} \\ & - \frac{EI_z}{a^3} v''' + \left(-\frac{EI_z}{a^3} + \rho \frac{\Omega_x^2}{a} I_z - \rho \Omega_y^2 \frac{I_z}{a} \right) v' - a m \Omega_x \Omega_y v \\ & + \rho \dot{\Omega}_x I_y \frac{1}{a} w' - a m \dot{\Omega}_y w + \rho \dot{\Omega}_y (I_y - I_z) \phi_x - 2 a m \Omega_y \dot{w} \\ & + \rho \Omega_y (I_y - I_z) \dot{\phi}_x - \rho \Omega_y (I_y + I_z) \dot{\phi}_x \\ & + \rho I_z \frac{1}{a} \ddot{v} = a m \Omega_x \Omega_y r_y - a m \Omega_y^2 r_x + \rho \Omega_x \Omega_y I_z \\ & - a f_x(\phi) - a F_x(\phi, t) \end{aligned} \quad (85)$$

Equation for twisting motion for the ring about the x axis (torsional motion):

$$\begin{aligned}
& \frac{G}{a}(I_y + I_z)\phi_x'' + \left[-\frac{EI_y}{a} - a\rho\Omega_y^2(I_y - I_z) \right] \phi_x \\
& - \left[\frac{G}{a^2}(I_y + I_z) + \frac{EI_y}{a^2} \right] w'' - \rho\Omega_x\Omega_y I_y w' - \rho\dot{\Omega}_y I_z v' \\
& + \rho\dot{\Omega}_y I_z u - a\rho(I_y + I_z)\ddot{\phi}_x - 2\rho\Omega_y I_z \dot{v}' + 2\rho\Omega_y I_z \dot{u} = \\
& a\rho\dot{\Omega}_x(I_y + I_z)
\end{aligned} \tag{86}$$

For a circular cross-section, $I_y + I_z = 2I$ and $I_y - I_z = 0$.

Equation for transverse motion for the ring in the y direction (bending motion about the z axis):

$$\begin{aligned}
& \frac{-EI_z}{a^3} v'''' + \left[\frac{P}{a} + \rho I_z \Omega_x^2 \frac{1}{a} - \rho I_z \Omega_y^2 \frac{1}{a} - \frac{EI_z}{a^3} \right] v'' \\
& + \left[\frac{P'}{a} + 2\rho I_z \Omega_x \Omega_x' \frac{1}{a} - 2\rho I_z \Omega_y \Omega_y' \frac{1}{a} \right] v' + [am\Omega_x^2] v \\
& + \rho I_y \dot{\Omega}_x \frac{1}{a} w'' + \rho I_y \dot{\Omega}_x' \frac{1}{a} w' + am\dot{\Omega}_x w + \rho\dot{\Omega}_y (I_y - I_z) \phi_x' \\
& + \rho\dot{\Omega}_y' (I_y - I_z) \phi_x + [-\rho I_z \Omega_x^2 + \rho I_z \Omega_y^2] \frac{u'}{a} \\
& + [-a^2 m \Omega_x \Omega_y - 2\Omega_x \Omega_x' \rho I_z + 2\Omega_y \Omega_y' \rho I_z] \frac{u}{a} \\
& - am\ddot{v} - \rho I_z \frac{1}{a} \ddot{v}'' + am2\Omega_x \dot{w} \\
& - 2\Omega_y' \rho I_z \dot{\phi}_x - 2\Omega_y \rho I_z \dot{\phi}_x' - \rho I_z \frac{1}{a} \ddot{u}' = P \\
& - am\Omega_x^2 r_y + am\Omega_x \Omega_y r_x - \rho I_y \Omega_x' \Omega_y - \rho I_y \Omega_x \Omega_y' \\
& + \rho(I_y + I_z) \Omega_x \Omega_y' + \rho(I_y + I_z) \Omega_x' \Omega_y - af_y(\phi) - aF_y(\phi, t)
\end{aligned} \tag{87}$$

Equation for transverse motion for the ring in the z direction (bending motion about the y axis):

$$\begin{aligned}
& \frac{-EI_y}{a^3} w'''' + \left[\frac{P}{a} + \frac{G}{a^3} (I_y + I_z) + \rho I_y \frac{\Omega_x^2}{a} \right] w'' \\
& + \left[\frac{P'}{a} + 2\rho I_y \frac{\Omega_x \Omega'_x}{a} \right] w' + am [\Omega_x^2 + \Omega_y^2] w - \rho I_z \frac{\dot{\Omega}_x}{a} v'' \\
& - \rho I_z \frac{\dot{\Omega}'_x}{a} v' - am \dot{\Omega}_x v + \left[\frac{-EI_y}{a^2} - \frac{G}{a^2} (I_y + I_z) \right] \phi'' \\
& + \rho \Omega_x \Omega_y (I_y - I_z) \phi'_x + \rho (I_y - I_z) [\Omega'_x \Omega_y + \Omega_x \Omega'_y] \phi_x \\
& + \rho I_z \dot{\Omega}_x \frac{u'}{a} + \left[a^2 m \dot{\Omega}_y + \rho I_z \dot{\Omega}'_x \right] \frac{u}{a} - am \ddot{w} + \rho I_y \frac{1}{a} \ddot{w}'' \\
& - 2am \Omega_x \dot{v} + 2a^2 m \Omega_y \frac{\dot{u}}{a} \\
& = am \dot{\Omega}_x r_y - am \dot{\Omega}_y r_x + \rho I_y \dot{\Omega}'_y - af_z(\phi) - aF_z(\phi, t)
\end{aligned} \tag{88}$$

where

$$\begin{aligned}
P &= \frac{EA}{a} \left[\frac{\partial u}{\partial \phi} + v \right] + \frac{EI_z}{a^3} \left[\frac{\partial^2 v}{\partial \phi^2} + v \right] \\
P' &= \frac{EA}{a} \left[\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial v}{\partial \phi} \right] + \frac{EI_z}{a^3} \left[\frac{\partial^3 v}{\partial \phi^3} + \frac{\partial v}{\partial \phi} \right] \\
\Omega_x &= \omega_x \sin \phi \\
\Omega_y &= -\omega_x \cos \phi \\
r_x &= a \sin(\phi - \alpha) + e \cos \phi \\
r_y &= a [1 - \cos(\phi - \alpha)] + e \sin \phi \\
(\dot{}) &= \frac{\partial()}{\partial t} \\
()' &= \frac{\partial()}{\partial \phi}
\end{aligned} \tag{89}$$

Equations (85) through (88), along with their associated boundary and initial conditions, describe the extensional, torsional, and transverse (in and out of the plane of the ring) motions of the ring segment. These equations of motion are coupled and many terms contain variable coefficients; i.e., coefficients of the dependent variables which depend on the space coordinate ϕ . Eqs. (85) and (86) are linear. However, in general, Eqs. (87) and

(88) are non-linear because the force P acting on the ring cross-section is not zero (i.e., $P \neq 0$) nor is P quasi-static and a predetermined function of ϕ .

Equations of Motion for a Ring Rotating at Constant Speed

Eqs. (85) through (88) can be simplified by letting the angular accelerations $\dot{\Omega}_x$ and $\dot{\Omega}_y$ equal zero. This results in Eqs. (90) through (93).

Equation for extensional motion for the ring in the ϕ direction (extensional motion) for $\dot{\Omega}_x = \dot{\Omega}_y = 0$:

$$\begin{aligned} & \frac{P'}{a} + m\Omega_y^2 u + \frac{\rho I_z}{a^2} (\Omega_y^2 - \Omega_x^2) u - \left(m + \frac{\rho I_z}{a^2} \right) \ddot{u} \\ & - \frac{EI_z}{a^4} v''' + \left[-\frac{EI_z}{a^4} + \frac{\rho I_z}{a^2} (\Omega_x^2 - \Omega_y^2) \right] v' - m\Omega_x \Omega_y v \\ & - 2m\Omega_y \dot{w} - 2\rho I_z \frac{\Omega_y}{a} \dot{\phi}_x + \rho I_z \frac{1}{a^2} \ddot{v}' = m\Omega_x \Omega_y r_y - m\Omega_y^2 r_x \\ & + \frac{\rho I_z}{a} \Omega_x \Omega_y - f_x(\phi) - F_x(\phi, t) \end{aligned} \quad (90)$$

Equation for twisting motion for the ring about the x axis (torsional motion) for $\dot{\Omega}_x = \dot{\Omega}_y = 0$:

$$\begin{aligned} & \frac{G}{a^2} (I_y + I_z) \phi_x'' + \left[\frac{-EI_y}{a^2} - \rho\Omega_y^2 (I_y - I_z) \right] \phi_x - \left[\frac{G}{a^3} (I_y + I_z) + \frac{EI_y}{a^3} \right] w'' - \frac{\rho}{a} I_y \Omega_x \Omega_y w' \\ & - \rho (I_y + I_z) \ddot{\phi}_x - 2\frac{\rho}{a} \Omega_y I_z \dot{v}' + 2\frac{\rho}{a} \Omega_y I_z \dot{u} = 0 \end{aligned} \quad (91)$$

Equation for transverse motion for the ring in the y direction (bending motion about the z axis) for $\dot{\Omega}_x = \dot{\Omega}_y = 0$:

$$\begin{aligned}
& -\frac{EI_z}{a^4}v'''' + \left[\frac{P}{a^2} + \frac{\rho I_z}{a^2}(\Omega_x^2 - \Omega_y^2) - \frac{EI_z}{a^4} \right] v'' \\
& + \left[\frac{P'}{a^2} + 2\frac{\rho I_z}{a^2}(\Omega_x\Omega'_x - \Omega_y\Omega'_y) \right] v' + m\Omega_x^2 v + \frac{\rho I_z}{a^2}[-\Omega_x^2 + \Omega_y^2] u' \\
& + \left[-ma\Omega_x\Omega_y + \frac{2\rho I_z}{a}(-\Omega_x\Omega'_x + \Omega_y\Omega'_y) \right] \frac{u}{a} - m\ddot{v} - \frac{\rho I_z}{a^2}\ddot{v}'' \\
& + 2m\Omega_x\dot{w} - 2\Omega'_y\frac{\rho}{a}I_z\dot{\phi}_x - 2\Omega_y\frac{\rho}{a}I_z\dot{\phi}'_x - \frac{\rho I_z}{a^2}\ddot{u}' = \\
& \frac{P}{a} - m\Omega_x^2 r_y + m\Omega_x\Omega_y r_x - \frac{\rho I_y}{a}[\Omega'_x\Omega_y + \Omega_x\Omega'_y] \\
& + \frac{\rho}{a}(I_y + I_z)[\Omega_x\Omega'_y + \Omega'_x\Omega_y] - f_y(\phi) - F_y(\phi, t)
\end{aligned} \tag{92}$$

Equation for transverse motion for the ring in the z direction (bending about the y axis) for $\dot{\Omega}_x = \dot{\Omega}_y = 0$:

$$\begin{aligned}
& -\frac{EI_y}{a^4}w'''' + \left[\frac{P}{a^2} + \frac{G}{a^4}(I_y + I_z) + \rho I_y \frac{\Omega_x^2}{a^2} \right] w'' + \left[\frac{P'}{a^2} + \frac{2\rho I_y}{a^2}\Omega_x\Omega'_x \right] w' \\
& + m[\Omega_x^2 + \Omega_y^2]w + \left[-\frac{EI_y}{a^3} - \frac{G}{a^3}(I_y + I_z) \right] \phi''_x + \frac{\rho}{a}\Omega_x\Omega_y(I_y - I_z)\phi'_x \\
& + \frac{\rho}{a}(I_y - I_z)[\Omega'_x\Omega_y + \Omega_x\Omega'_y]\phi_x \\
& - m\ddot{w} + \frac{\rho I_y}{a^2}\ddot{w}'' - 2m\Omega_x\dot{v} + 2m\Omega_y\dot{u} = -f_z(\phi) - F_z(\phi, t)
\end{aligned} \tag{93}$$

where:

$$\begin{aligned}
P &= \frac{EA}{a} \left[\frac{\partial u}{\partial \phi} + v \right] + \frac{EI_z}{a^3} \left[\frac{\partial^2 v}{\partial \phi^2} + v \right] \\
P' &= \frac{EA}{a} \left[\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial v}{\partial \phi} \right] + \frac{EI_z}{a^3} \left[\frac{\partial^3 v}{\partial \phi^3} + \frac{\partial v}{\partial \phi} \right] \\
\Omega_x &= \omega_x \sin \phi \\
\Omega_y &= -\omega_x \cos \phi \\
r_x &= a \sin(\phi - \alpha) + e \cos \phi \\
r_y &= a [1 - \cos(\phi - \alpha)] + e \sin \phi \\
(\dot{}) &= \frac{\partial()}{\partial t} \\
()' &= \frac{\partial()}{\partial \phi}
\end{aligned} \tag{94}$$

Eqs. (90) through (93) are still coupled and have many terms which still contain variable coefficients. Eqs. (92) and (93) are, in general, non-linear due to the extensional motion; i.e., $P \neq 0$.

Equations of Motion for a Beam Rotating at Constant Speed

Eqs. (90) through (93) are simplified further by considering a straight beam. For this case,

$$\begin{aligned}
\lim a\gamma &= \ell \\
a &\rightarrow \infty \\
\gamma &\rightarrow 0
\end{aligned} \tag{95}$$

where ℓ is the beam length and

$$a d\phi = a d\xi = dx \tag{96}$$

Note that $\xi = \phi - \alpha$ and as $a \rightarrow \infty$ and $\gamma \rightarrow 0$, $\alpha \rightarrow \frac{\pi}{2}$, $\phi \rightarrow \frac{\pi}{2}$ and that

$$\Omega_x \rightarrow \omega_x$$

$$\Omega_y \rightarrow 0$$

$$r_x \rightarrow 0$$

$$r_y \rightarrow e$$

Therefore, Eqs. (90) through (93) become Eqs. (97) through (100), respectively, for the motions of a straight beam rotating at a constant angular speed about the X (or x) axis and offset from the X axis by e .

For extensional or longitudinal motion for the beam:

$$EAu'' - m\ddot{u} = -f_x(x) - F_x(x, t) \quad (97)$$

For torsional motion for the beam:

$$G(I_y + I_z)\phi_x'' - \rho(I_y + I_z)\ddot{\phi}_x = 0 \quad (98)$$

For transverse motion for the beam in the y direction or bending about the z axis:

$$EI_z v'''' - [P + \rho I_z \omega_x^2] v'' - P' v' - m\omega_x^2 v + m\ddot{v} + \rho I_z \ddot{v}'' - 2m\omega_x \dot{v} = m\omega_x^2 e - f_y(x) - F_y(x, t) \quad (99)$$

For transverse motion for the beam in the z direction or bending about the y axis:

$$EI_y w'''' - [P + \rho I_y \omega_x^2] w'' - P' w' - m\omega_x^2 w + m\ddot{w} - \rho I_y \ddot{w}'' + 2m\omega_x \dot{w} = -f_z(x) - F_z(x, t) \quad (100)$$

where $(\)' = \frac{\partial(\)}{\partial x}$, $(\)^\bullet = \frac{\partial(\)}{\partial t}$, $P = EA \frac{\partial u}{\partial x}$, and $P' = EA \frac{\partial^2 u}{\partial x^2}$. If P (or u) is quasi-static, then P can be predetermined as a function of x . In this case, Eqs. (99) and (100) are linear. The first extensional and torsional natural frequencies for a beam are usually much higher than the first bending natural frequencies. Consequently, load application times which result in significant bending vibration may not result in significant extensional or torsional vibration. When $P \neq P(x)$, $P' = 0$ in Eqs. (99) and (100). When the angular speed ω_x is zero, Eqs. (99) and (100) uncouple. Note that Eqs. (97) through (100) are consistent with Eqs. (56) through (59) in Reference 1 when most of the non-linear terms are neglected.

Equations of Motion for a Cable Rotating at Constant Speed

Consider the case where the uniform beam is very flexible (i.e., $I_y \rightarrow 0$ and $I_z \rightarrow 0$) but subjected to a large constant axial tensile force P applied at its ends. In this case the beam can be considered as a uniform cable and Eqs. (97) through (100) simplify to Eqs. (101) and (102) for the coupled transverse vibration of a cable rotating at a constant angular speed about the X (or x) axis and offset from the X (or x) axis by e . When the angular speed ω_x is zero, Eqs. (101) and (102) uncouple.

$$-Pv'' - m\omega_x^2 v + m\ddot{v} - 2m\omega_x \dot{w} = m\omega_x^2 e - f_y(x) - F_y(x, t) \quad (101)$$

$$-Pw'' - m\omega_x^2 w + m\ddot{w} + 2m\omega_x \dot{v} = -f_z(x) - F_z(x, t) \quad (102)$$

Equations of Motion for a Non-Rotating Circular Ring

Extensional Equations of Motion

Referring to the extensional equations of motion for a rotating ring, Eqs. (90) through (93) are simplified by considering a non-rotating ring. That is, for the case where $\Omega_x = \Omega_y = 0$, Eqs. (90) through (93) reduce to Eqs. (103) through (106). The extensional equations of motion are:

$$\frac{P'}{a} - \left(m + \frac{\rho I_z}{a^2} \right) \ddot{u} - \frac{EI_z}{a^4} v''' - \frac{EI_z}{a^4} v' + \frac{\rho I_z}{a^2} \ddot{v}' = -f_x(\phi) - F_x(\phi, t) \quad (103)$$

$$\frac{G}{a^2} (I_y + I_z) \phi_x'' - \frac{EI_y}{a^2} \phi_x - \left[\frac{G}{a^3} (I_y + I_z) + \frac{EI_y}{a^3} \right] w'' - \rho (I_y + I_z) \ddot{\phi}_x = 0 \quad (104)$$

$$-\frac{EI_z}{a^4} v'''' + \left[\frac{P}{a^2} - \frac{EI_z}{a^4} \right] v'' + \frac{P'}{a^2} v' - m\ddot{v} - \frac{\rho I_z}{a^2} \ddot{v}'' - \frac{\rho I_z}{a^2} \ddot{u}' = \frac{P}{a} - f_y(\phi) - F_y(\phi, t) \quad (105)$$

$$\begin{aligned} & -\frac{EI_y}{a^4} w'''' + \left[\frac{P}{a^2} + \frac{G}{a^4} (I_y + I_z) \right] w'' + \frac{P'}{a^2} w' + \left[-\frac{EI_y}{a^3} - \frac{G}{a^3} (I_y + I_z) \right] \phi_x'' \\ & - m\ddot{w} + \frac{\rho I_y}{a^2} \ddot{w}'' = -f_z(\phi) - F_z(\phi, t) \end{aligned} \quad (106)$$

where

$$P = \frac{EA}{a} \left[(u' + v) + \frac{I_z}{Aa^2} (v'' + v) \right] \quad (107)$$

Note that Eqs. (105) and (106) are, in general, non-linear.

Inextensional Equations of Motion

Note that for thin rings of large radius

$$(u' + v) \gg \frac{I_z}{Aa^2} (v'' + v) \quad (108)$$

and Eq. (107) simplifies to Eq. (109).

$$P = \frac{EA}{a} (u' + v) \quad (109)$$

Further simplification to Eqs. (103) through (106) result for inextensional motion. That is, $\varepsilon_\phi = 0$ at the centroid of the cross-section (refer to Equation (6) or (8)). This results in Eq. (110). Also note that the force P acting on the ring cross-section is practically zero for thin rings of large radius.

$$\frac{1}{a} \frac{\partial u}{\partial \phi} + \frac{v}{a} = 0 \quad (110)$$

or

$$u' = -v \quad (111)$$

Substituting Eq. (107) in Eqs. (103), (105), and (106), neglecting non-linear terms, and utilizing Eq. (111), results in Eqs. (112) through (115). The inextensional equations of motion are:

$$-\left(m + \frac{\rho I_z}{a^2}\right) \ddot{u} - \frac{\rho I_z}{a^2} \ddot{u}'' = -f_x(\phi) - F_x(\phi, t) \quad (112)$$

$$\frac{G}{a^2} (I_y + I_z) \phi_x'' - \frac{EI_y}{a^2} \phi_x - \rho (I_y + I_z) \ddot{\phi}_x - \left[\frac{G}{a^3} (I_y + I_z) + \frac{EI_y}{a^3} \right] w'' = 0 \quad (113)$$

$$-\frac{EI_z}{a^4}v'''' - \frac{2EI_z}{a^4}v'' - \frac{EI_z}{a^4}v + \left[-m + \frac{\rho I_z}{a^2}\right]\ddot{v} - \frac{\rho I_z}{a^2}\ddot{v}'' = -f_y(\phi) - F_y(\phi, t) \quad (114)$$

$$-\frac{EI_y}{a^4}w'''' + \frac{G}{a^4}(I_y + I_z)w'' + \left[-\frac{EI_y}{a^3} - \frac{G}{a^3}(I_y + I_z)\right]\phi_x'' - m\ddot{w} + \frac{\rho I_y}{a^2}\ddot{w}'' = -f_z(\phi) - F_z(\phi, t) \quad (115)$$

Note that for $\bar{\Omega} = 0$ and for inextensional motion (i.e., $u' = -v$) Eqs. (103) and (105) uncouple and Eq. (106) simplifies. Also note that, for inextensional motion, Eqs. (114) and (115) are linear.

Linearized Equations of Motion for a Non-Rotating Circular Ring

Extensional Equations of Motion

When non-linear terms are neglected in Eqs. (103) through (106), which describe the extensional motions for a non-rotating ring, these equations simplify to Eqs. (116) through (119).

$$\frac{EA}{a^2}u'' - \left(m + \frac{\rho I_z}{a^2}\right)\ddot{u} + \frac{EA}{a^2}v' + \frac{\rho I_z}{a^2}\ddot{v}' = -f_x(\phi) - F_x(\phi, t) \quad (116)$$

$$\frac{G}{a^2}(I_y + I_z)\phi_x'' - \frac{EI_y}{a^2}\phi_x - \left[\frac{G}{a^3}(I_y + I_z) + \frac{EI_y}{a^3}\right]w'' - \rho(I_y + I_z)\ddot{\phi}_x = 0 \quad (117)$$

$$-\frac{EI_z}{a^4}v'''' - \frac{EI_z}{a^4}v'' - m\ddot{v} - \frac{\rho I_z}{a^2}\ddot{v}'' - \frac{\rho I_z}{a^2}\ddot{u}' = \frac{P}{a} - f_y(\phi) - F_y(\phi, t) \quad (118)$$

$$-\frac{EI_y}{a^4}w'''' + \frac{G}{a^4}(I_y + I_z)w'' + \left[-\frac{EI_y}{a^3} - \frac{G}{a^3}(I_y + I_z)\right]\phi_x'' - m\ddot{w} + \frac{\rho I_y}{a^2}\ddot{w}'' = -f_z(\phi) - F_z(\phi, t) \quad (119)$$

where

$$P = \frac{EA}{a} \left[(u' + v) + \frac{I_z}{Aa^2}(v'' + v) \right] \quad (120)$$

Inextensional Equations of Motion

Note that for inextensional motion (i.e., $u' + v = 0$), Eqs. (116) through (119) reduce to Eqs. (112) through (115). That is, $\epsilon_\phi = 0$ at the centroid ($y = z = 0$) of the cross-section.

Linear Homogeneous Equations of Motion for a Non-Rotating Circular Ring

Since the general solution for the linear dynamic response of a ring can be represented by a superposition of the ring normal modes, it is convenient to examine the linear homogeneous equations of motion for a ring.

Extensional Equations for In-Plane Motion

The linearized homogenous equations for extensional motion of a non-rotating ring are obtained from Eqs. (116) through (120) and are expressed by Eqs. (121) through (124).

$$\frac{EA}{a^2}u'' - \left(m + \frac{\rho I_z}{a^2}\right)\ddot{u} + \frac{EA}{a^2}v' + \frac{\rho I_z}{a^2}\ddot{v}' = 0 \quad (121)$$

$$\frac{G}{a^2}(I_y + I_z)\phi_x'' - \frac{EI_y}{a^2}\phi_x - \left[\frac{G}{a^3}(I_y + I_z) + \frac{EI_y}{a^3}\right]w'' - \rho(I_y + I_z)\ddot{\phi}_x = 0 \quad (122)$$

$$-\frac{EI_z}{a^4}v'''' - \frac{2EI_z}{a^4}v'' - \frac{EI_z}{a^4}v - m\ddot{v} - \frac{\rho I_z}{a^2}\ddot{v}'' - \frac{\rho I_z}{a^2}\ddot{u}' - \frac{EA}{a^2}u' - \frac{EA}{a^2}v = 0 \quad (123)$$

$$-\frac{EI_y}{a^4}w'''' + \frac{G}{a^4}(I_y + I_z)w'' + \left[-\frac{EI_y}{a^3} - \frac{G}{a^3}(I_y + I_z)\right]\phi_x'' - m\ddot{w} + \frac{\rho I_y}{a^2}\ddot{w}'' = 0 \quad (124)$$

The coupled linearized extensional equations for in-plane motion of a non-rotating ring are given by equations (125) and (126). Note that the linearized equations for out-of-plane motion for a non-rotating ring are the same for either extensional or inextensional motion. For example, compare Eqs. (122) and (124) to Eqs. (113) and (115), respectively.

$$\frac{EA}{ma^2}u'' - \left(1 + \frac{\rho I_z}{ma^2}\right)\ddot{u} + \frac{EA}{ma^2}v' + \frac{\rho I_z}{ma^2}\ddot{v}' = 0 \quad (125)$$

$$\frac{EI_z}{ma^4}v'''' + \frac{2EI_z}{ma^4}v'' + \left[\frac{EA}{ma^2} + \frac{EI_z}{ma^4}\right]v + \ddot{v} + \frac{\rho I_z}{ma^2}\ddot{v}'' + \frac{\rho I_z}{ma^2}\ddot{u}' + \frac{EA}{ma^2}u' = 0 \quad (126)$$

Define the following parameters.

$$p_1 = \frac{I_z}{Aa^2}$$

$$\omega_o^2 = \frac{E}{\rho a^2} \quad (127)$$

where ω_o is the circular frequency associated with an extensional wave with a speed of $\sqrt{\frac{E}{\rho}}$ to travel one revolution (2π radians) around a complete circular ring. That is, the period of the extensional wave is $T_o = \frac{2\pi a}{\sqrt{\frac{E}{\rho}}}$. Since $m = \rho A$, $\frac{EI_z}{ma^4} = p_1 \omega_o^2$, $\frac{\rho I_z}{ma^2} = p_1$, and $\frac{EA}{ma^2} = \omega_o^2$,

Eqs. (125) and (126) are expressed as coupled Eqs. (128) and (129), for in-plane extensional motion of the ring.

$$u'' - \frac{1}{\omega_o^2}(1 + p_1)\ddot{u} + v' + \frac{p_1}{\omega_o^2}\ddot{v}' = 0 \quad (128)$$

$$p_1 v'''' + 2p_1 v'' + (1 + p_1)v + \frac{1}{\omega_o^2}\ddot{v} + \frac{p_1}{\omega_o^2}\ddot{v}'' + \frac{p_1}{\omega_o^2}\ddot{u}' + u' = 0 \quad (129)$$

For thin rings of large radius $p_1 \ll 1$ and $\omega_o^2 \gg 1$, and Eqs. (128) and (129) reduce to Eqs. (130) and (131).

$$u'' - \frac{1}{\omega_o^2}\ddot{u} + v' + \frac{p_1}{\omega_o^2}\ddot{v}' = 0 \quad (130)$$

$$p_1 v'''' + 2p_1 v'' + v + \frac{1}{\omega_o^2}\ddot{v} + \frac{p_1}{\omega_o^2}\ddot{v}'' + \frac{p_1}{\omega_o^2}\ddot{u}' + u' = 0 \quad (131)$$

and the terms involving p_1/ω_o^2 can be neglected.

Inextensional Equations for In-Plane Motion

The homogeneous equations for inextensional uncoupled motion of a non-rotating circular ring are obtained from Eqs. (112) through (115). The in-plane inextensional equations of motion are obtained from Eqs. (112) and (114). Using the definitions given by Eq. (127), Eqs. (112) and (114) are expressed by Eqs. (132) and (133), for in-plane inextensional motion of a ring.

$$\frac{1}{\omega_o^2}(1+p_1)\ddot{u} + \frac{p_1}{\omega_o^2}\ddot{u}'' = 0 \quad (132)$$

$$p_1 v'''' + 2p_1 v'' + p_1 v + \frac{1}{\omega_o^2}(1-p_1)\ddot{v} + \frac{p_1}{\omega_o^2}\ddot{v}'' = 0 \quad (133)$$

For thin rings of large radius $p_1 \ll 1$ and $\omega_o^2 \gg 1$, Eqs. (132) and (133) for inextensional in-plane motion, reduce to Eqs. (134) and (135).

$$\frac{1}{\omega_o^2}\ddot{u} + \frac{p_1}{\omega_o^2}\ddot{u}'' = 0 \quad (134)$$

$$p_1 v'''' + 2p_1 v'' + p_1 v + \frac{1}{\omega_o^2}\ddot{v} + \frac{p_1}{\omega_o^2}\ddot{v}'' = 0 \quad (135)$$

and the terms involving $\ddot{u}''$ and $\ddot{v}''$ can be neglected. Note that $(1-p_1) \cong 1$.

Note that the inextensional equations, Eqs. (132) and (133), can be obtained from the linearized extensional equations, Eqs. (128) and (129), by using $u' = -v$ in Eqs. (128) and (129). Also note that Eqs. (132) and (133) are uncoupled whereas Eqs. (128) and (129) are coupled.

Equations for Out-of-Plane Motion

Note that inextensional Eqs. (113) and (115) or the linearized extensional Eqs. (122) and (124) are identical for out-of-plane motion for the ring. Eqs. (136) and (137) result from rewriting Eqs. (113) and (115) or Eqs. (122) and (124).

$$\frac{G}{a^2 \rho A}(I_y + I_z)\phi_x'' - \frac{EI_y}{ma^4}a^2\phi_x - \frac{1}{A}(I_y + I_z)\ddot{\phi}_x - \left[\frac{EI_y}{ma^3} + \frac{G(I_y + I_z)}{ma^3} \right] w'' = 0 \quad (136)$$

$$\frac{EI_y}{ma^4}w'''' - \frac{G}{ma^4}(I_y + I_z)w'' + \ddot{w} - \frac{\rho I_y}{ma^2}\ddot{w}'' + \left[\frac{EI_y}{ma^3} + \frac{G}{ma^3}(I_y + I_z) \right] \phi_x'' = 0 \quad (137)$$

Define the following parameters.

$$\begin{aligned}
p_2 &= \frac{I_y}{Aa^2} \\
\omega_s^2 &= \frac{G}{\rho a^2}
\end{aligned} \tag{138}$$

where ω_s is the circular frequency associated with a shear wave with a speed $\sqrt{\frac{G}{\rho}}$ to travel one revolution (2π radians) around a complete ring. That is, the period of the shear wave is $T_s = \frac{2\pi a}{\sqrt{\frac{G}{\rho}}}$. Since $m = \rho a$, $\frac{EI_y}{ma^4} = p_2\omega_o^2$, $\frac{\rho I_y}{ma^2} = p_2$, $\frac{EA}{ma^2} = \omega_o^2$, $\frac{GA}{ma^2} = \omega_s^2$, $\frac{GI_y}{ma^4} = p_2\omega_s^2$,

Eqs. (136) and (137) are expressed as Eqs. (139) and (140), for out-of-plane motion of the ring.

$$\omega_s^2 \frac{I_y}{A} \left(1 + \frac{I_z}{I_y}\right) \phi_x'' - p_2 \omega_o^2 a^2 \phi_x - \frac{I_y}{A} \left(1 + \frac{I_z}{I_y}\right) \ddot{\phi}_x - ap_2 \omega_o^2 \left[1 + \frac{\omega_s^2}{\omega_o^2} \left(1 + \frac{I_z}{I_y}\right)\right] w'' = 0 \tag{139}$$

$$\omega_o^2 p_2 w'''' - \omega_s^2 p_2 \left(1 + \frac{I_z}{I_y}\right) w'' + \ddot{w} - p_2 \ddot{w}'' + a \omega_o^2 p_2 \left[1 + \frac{\omega_s^2}{\omega_o^2} \left(1 + \frac{I_z}{I_y}\right)\right] \phi_x'' = 0 \tag{140}$$

or

$$\frac{\omega_s^2}{\omega_o^2} \frac{I_y}{A} \left(1 + \frac{I_z}{I_y}\right) \phi_x'' - p_2 a^2 \phi_x - \frac{I_y}{A} \left(1 + \frac{I_z}{I_y}\right) \frac{\ddot{\phi}_x}{\omega_o^2} - ap_2 \left[1 + \frac{\omega_s^2}{\omega_o^2} \left(1 + \frac{I_z}{I_y}\right)\right] w'' = 0 \tag{141}$$

$$p_2 w'''' - p_2 \frac{\omega_s^2}{\omega_o^2} \left(1 + \frac{I_z}{I_y}\right) w'' + \frac{1}{\omega_o^2} \ddot{w} - \frac{p_2}{\omega_o^2} \ddot{w}'' + ap_2 \left[1 + \frac{\omega_s^2}{\omega_o^2} \left(1 + \frac{I_z}{I_y}\right)\right] \phi_x'' = 0 \tag{142}$$

For thin rings of large radius $p_2 \ll 1$, $\omega_o^2 \gg 1$, $\omega_s^2 \gg 1$, $\omega_o^2 > \omega_s^2$, and $\frac{\omega_o^2}{\omega_s^2} > 1$. The rotary inertia term $\ddot{w}''$ can be neglected. Note that Eqs. (141) and (142) are coupled.

Uncoupled Equations for Out-of-Plane Motion

Eqs. (141) and (142) for motion out-of-the-plane of the ring can be uncoupled. Define the following linear differential operators.

$$L_1 = \frac{\omega_s^2 I_y}{\omega_o^2 A} \left(1 + \frac{I_z}{I_y} \right) \frac{\partial^2}{\partial \phi^2} - p_2 a^2 - \frac{I_y}{A} \left(1 + \frac{I_z}{I_y} \right) \frac{1}{\omega_o^2} \frac{\partial^2}{\partial t^2} \quad (143)$$

$$L_2 = ap_2 \left[1 + \frac{\omega_s^2}{\omega_o^2} \left(1 + \frac{I_z}{I_y} \right) \right] \frac{\partial^2}{\partial \phi^2} \quad (144)$$

$$L_3 = p_2 \frac{\partial^4}{\partial \phi^4} - p_2 \frac{\omega_s^2}{\omega_o^2} \left(1 + \frac{I_z}{I_y} \right) \frac{\partial^2}{\partial \phi^2} + \frac{1}{\omega_o^2} \frac{\partial^2}{\partial t^2} - \frac{p_2}{\omega_o^2} \frac{\partial^4}{\partial \phi^2 \partial t^2} \quad (145)$$

Rewriting Eqs. (141) and (142) in operator form yields Eqs. (146) and (147).

$$L_1 \phi_x - L_2 w = 0 \quad (146)$$

$$L_2 \phi_x - L_3 w = 0 \quad (147)$$

The uncoupled linear homogeneous equations for motion out-of-the-plane of the ring are expressed by Eqs. (148) and (149).

$$[L_1 L_3 + L_2 L_2] \phi_x = 0 \quad (149)$$

$$[L_1 L_3 + L_2 L_2] w = 0 \quad (149)$$

Note that the uncoupled equations are sixth order.

Uncoupled Equations for Extensional In-Plane Motion

The equations for extensional motion in the plane of the ring for, Eqs. (128) and (129), can be uncoupled. Define the following linear differential operators.

$$L_4 = \frac{\partial^2}{\partial \phi^2} - \frac{(1 + p_1)}{\omega_o^2} \frac{\partial^2}{\partial t^2} \quad (150)$$

$$L_5 = \frac{\partial}{\partial \phi} + \frac{p_1}{\omega_o^2} \frac{\partial^3}{\partial \phi \partial t^2} \quad (151)$$

$$L_6 = p_1 \frac{\partial^4}{\partial \phi^4} + 2p_1 \frac{\partial^2}{\partial \phi^2} + (1 + p_1) + \frac{1}{\omega_o^2} \frac{\partial^2}{\partial t^2} + \frac{p_1}{\omega_o^2} \frac{\partial^4}{\partial \phi^2 \partial t^2} \quad (152)$$

Rewriting Eqs. (128) and (129) in operator form yields Eqs. (153) and (154).

$$L_4 u + L_5 v = 0 \quad (153)$$

$$L_5 u + L_6 v = 0 \quad (154)$$

The uncoupled linear homogeneous equations for extensional motion in the plane of the ring are expressed by Eqs. (155) and (156).

$$[L_4 L_6 - L_5 L_5] u = 0 \quad (155)$$

$$[L_4 L_6 - L_5 L_5] v = 0 \quad (156)$$

Note that the uncoupled equations are sixth order.

The inextensional equations for in-plane motion of the ring are uncoupled. Refer to Eqs. (132) and (133).

Using Eqs. (150) through (152) in Eq. (156) results in Eq. (157) for the uncoupled homogeneous equations for extensional in-plane motion of a circular ring.

$$\begin{aligned} \frac{\partial^6 v}{\partial \phi^6} + 2 \frac{\partial^4 v}{\partial \phi^4} + \frac{\partial^2 v}{\partial \phi^2} + \frac{1}{\omega_o^2} \left(\frac{1}{p_1} - 4 \right) \frac{\partial^4 v}{\partial \phi^2 \partial t^2} - \frac{(1+p_1)}{p_1 \omega_o^2} \frac{\partial^2 v}{\partial t^2} \\ - \frac{1}{p_1 \omega_o^4} \frac{\partial^4 v}{\partial t^4} - \frac{1}{\omega_o^4} (1+p_1) \frac{\partial^6 v}{\partial \phi^2 \partial t^4} = 0 \end{aligned} \quad (157)$$

For thin rings of large radius, $p_1 \ll 1$ and $\omega_o^2 \gg 1$, and Eq. (157) reduces to Eq. (158).

$$\begin{aligned} \frac{\partial^6 v}{\partial \phi^6} + 2 \frac{\partial^4 v}{\partial \phi^4} + \frac{\partial^2 v}{\partial \phi^2} + \frac{1}{p_1 \omega_o^2} \frac{\partial^4 v}{\partial \phi^2 \partial t^2} - \frac{1}{p_1 \omega_o^2} \frac{\partial^2 v}{\partial t^2} \\ - \frac{1}{p_1 \omega_o^4} \frac{\partial^4 v}{\partial t^4} - \frac{1}{\omega_o^4} \frac{\partial^6 v}{\partial \phi^2 \partial t^4} = 0 \end{aligned} \quad (158)$$

Based on Eq. (155), u may be substituted for v in Eqs. (157) and (158).

The uncoupled homogeneous equations for the out-of-plane motion of a circular ring can be obtained by utilizing Eqs. (143) through (145) in Eqs. (146) or (147).

Boundary Conditions

Returning to the equations of motion for a rotating circular ring, Eqs. (85) through (88), boundary conditions need to be specified for some common conditions. Equations (159) through (164) are obtained from Eqs. (12) and (76) where $\dot{\vec{h}}_o$ is given by Eq. (50).

$$P = \frac{EA}{a} \left[\left(\frac{\partial u}{\partial \phi} + v \right) + \frac{I_z}{Aa^2} \left(\frac{\partial^2 v}{\partial \phi^2} + v \right) \right] \quad (159)$$

$$T = \frac{G}{a} (I_y + I_z) \left[\frac{\partial \phi_x}{\partial \phi} - \frac{1}{a} \frac{\partial w}{\partial \phi} \right] \quad (160)$$

$$M_y = -\frac{EI_y}{a} \left[\frac{1}{a} \frac{\partial^2 w}{\partial \phi^2} + \phi_x \right] \quad (161)$$

$$M_z = \frac{EI_z}{a^2} \left[\frac{\partial^2 v}{\partial \phi^2} + v \right] \quad (162)$$

$$Q_y = \frac{1}{a} \left[a \dot{\vec{h}}_o \cdot \vec{k} - \frac{\partial M_z}{\partial \phi} \right] \quad (163)$$

$$Q_z = \frac{1}{a} \left[-a \dot{\vec{h}}_o \cdot \vec{j} + \frac{\partial M_y}{\partial \phi} - T \right] \quad (164)$$

Simply Supported End (i.e., at $\phi = \alpha$, the ring segment end is pinned in the y and z directions; this support can transmit P and T).

For $\phi = \alpha$:

$$u = 0$$

$$\phi_x = 0$$

$$v = 0$$

$$w = 0$$

$$M_y = 0; \text{ therefore, } \frac{\partial^2 w}{\partial \phi^2} = 0 \quad (165 \text{ a-f})$$

$$M_z = 0; \text{ therefore, } \frac{\partial^2 v}{\partial \phi^2} = 0$$

Ball and Socket End (i.e., at $\phi = \alpha$, the ring segment end is supported in a ball and socket; this support can transmit P but not T).

For $\phi = \alpha$:

$$u = 0$$

$$v = 0$$

$$w = 0$$

$$T = 0; \text{ therefore, } \frac{\partial \phi_x}{\partial \phi} - \frac{1}{a} \frac{\partial w}{\partial \phi} = 0 \quad (166 \text{ a-f})$$

$$M_y = 0; \text{ therefore, } \frac{1}{a} \frac{\partial^2 w}{\partial \phi^2} + \phi_x = 0$$

$$M_z = 0; \text{ therefore, } \frac{\partial^2 v}{\partial \phi^2} = 0$$

Built in End (i.e., at $\phi = \alpha$, the ring segment end is fixed or prevented from displacing or rotating; this support can transmit P , T , M_y , and M_z).

For $\phi = \alpha$:

$$u = 0$$

$$v = 0$$

$$w = 0$$

$$\phi_x = 0$$

(167 a-f)

$$\frac{\partial v}{\partial \phi} = 0$$

$$\frac{\partial w}{\partial \phi} = 0$$

Free End (i.e., at $\phi = \alpha$, the ring segment end is free from any forces or moments; this support cannot transmit P , T , M_y , M_z , Q_y , or Q_z).

For $\phi = \alpha$:

$$P = 0 \text{ or}$$

$$\left(\frac{\partial v}{\partial \phi} + v \right) + \frac{I_z}{Aa^2} \left(\frac{\partial^2 v}{\partial \phi^2} + v \right) = 0$$

$$T = 0 \text{ or}$$

$$\frac{\partial \phi_x}{\partial \phi} - \frac{1}{a} \frac{\partial w}{\partial \phi} = 0$$

$$M_y = 0 \text{ or}$$

$$\frac{1}{a} \frac{\partial^2 w}{\partial \phi^2} + \phi_x = 0$$

$$M_z = 0 \text{ or}$$

$$\frac{\partial^2 v}{\partial \phi^2} + v = 0$$

(168 a-f)

$$Q_y = 0 \text{ or}$$

$$\dot{\bar{h}}_0 \cdot \bar{k} - \frac{1}{a} \frac{\partial M_z}{\partial \phi} = 0$$

$$Q_z = 0 \text{ or}$$

$$\dot{\bar{h}}_0 \cdot \bar{j} - \frac{1}{a} \frac{\partial M_y}{\partial \phi} + T = 0$$

where

$$\frac{\partial M_z}{\partial \phi} = \frac{EI_z}{a^2} \left[\frac{\partial^3 v}{\partial \phi^3} + \frac{\partial v}{\partial \phi} \right] \quad (169a)$$

$$\frac{\partial M_y}{\partial \phi} = -\frac{EI_y}{a} \left[\frac{1}{a} \frac{\partial^3 w}{\partial \phi^3} + \frac{\partial \phi_x}{\partial \phi} \right] \quad (169b)$$

and $\ddot{h}_o$ is given by Eq. (50).

Note that the second term in Eq. (168a) and the last term in Eq. (168f) are zero due to Eq. (168d) and Eq. (168b), respectively.

Pinned End and Circumferentially Guided (i.e., at $\phi = \alpha$, the ring segment end is pinned in the y and z directions but guided on frictionless rollers in the circumferential direction; this support can transmit T but not P).

For $\phi = \alpha$:

$$\begin{aligned} \phi_x &= 0 \\ v &= 0 \\ w &= 0 \\ M_y &= 0; \text{ therefore, } \frac{\partial^2 w}{\partial \phi^2} = 0 \\ M_z &= 0; \text{ therefore, } \frac{\partial^2 v}{\partial \phi^2} = 0 \\ P &= 0; \text{ therefore, } \frac{\partial u}{\partial \phi} = 0 \end{aligned} \quad (170 \text{ a-f})$$

Examples of other geometric and natural boundary conditions can be determined by utilizing Eqs. (159) through (164).

Initial Conditions

The initial conditions associated with Eqs. (85) through (88) are expressed by Eqs. (171) through (172).

At $t = 0$:

$$\begin{aligned}u &= g_1(\phi) \\ \phi_x &= g_2(\phi) \\ v &= g_3(\phi) \\ w &= g_4(\phi)\end{aligned}\tag{171 a-d}$$

At $t = 0$:

$$\begin{aligned}\dot{u} &= h_1(\phi) \\ \dot{\phi}_x &= h_2(\phi) \\ \dot{v} &= h_3(\phi) \\ \dot{w} &= h_4(\phi)\end{aligned}\tag{172 a-d}$$

where $(\dot{}) = \frac{\partial()}{\partial t}$ and $g_i(\phi)$ and $h_i(\phi)$, $i = 1, 2, 3, 4$, are specified functions of ϕ .

Discussion

Eqs. (85) through (88) represent the coupled partial differential equations of motion for a circular ring or circular ring segment rotating about an axis in its plane and subjected to an angular velocity as well as an angular acceleration.

Eqs. (90) through (93) are for the case where the circular ring's angular velocity is constant; that is, the angular acceleration is zero.

Eqs. (97) through (100) and Eqs. (101) and (102) represent the coupled equations of motion for a straight beam and cable, respectively, rotating at a constant angular speed about an axis parallel to their longitudinal axes.

Eqs. (103) through (106) and Eqs. (112) through (115) represent the coupled extensional and inextensional equations of motion, respectively, for a non-rotating circular ring.

Eqs. (116) through (119) represent the coupled linearized equations for extensional motion of a non-rotating circular ring. The linearized equations for inextensional motion are given by Eqs. (112) through (115).

Linear coupled homogeneous equations of motion for a non-rotating circular ring are determined for extensional and inextensional motion. These equations of motion are subsequently uncoupled.

The equations of motion for a circular ring rotating about an axis in the plane of the ring are complicated.¹ When the ring is rotating at a constant angular speed, the equations of motion are still complicated. Consequently, solutions to these equations are difficult even for the most simple boundary conditions. To illustrate this, consider the following.

A segment of a circular ring with a circular cross-section is rotating at a constant angular speed ω_x about the horizontal X axis (e.g., a shaft supported in bearings), refer to Figure 2. The first extensional and torsional natural frequencies are much higher than the first bending natural frequencies. Also, ω_x is not close to the critical angular speed for the rotating circular shaft (refer to Appendix A). A slight amount of damping exists in the ring material; consequently, the starting transients have damped out and all vibration has stopped and only quasi-static motion exists. That is,

¹ When a circular ring is rotating about an axis perpendicular to the plane of the ring, where the axis of rotation passes through the center for the radius of the curvature, the coupled equations of motion are less complicated than the case considered in this report. That is, the coefficients of the dependent variables in the partial differential equations are independent of the spatial coordinate ϕ .

$$\dot{u} = \ddot{u} = \dot{v} = \ddot{v} = \dot{w} = \ddot{w} = \dot{\phi}_x = \ddot{\phi}_x = 0 \quad (173)$$

The acceleration of the ring in the direction of r or y' (refer to Figure 5) is large compared with the local acceleration of gravity g , i.e.,

$$\frac{r\omega_x^2}{g} \gg 1 \quad (174)$$

where

$$\begin{aligned} r &= a(\sin \phi - \sin \alpha) + e \\ e &= 0 \end{aligned} \quad (175)$$

and the local acceleration of gravity is along the negative Y axis; i.e.,

$$mg_Y = mg \cos \gamma_2 = -mg \quad (176)$$

Also,

$$\begin{aligned} \Omega_x &= \omega_x \sin \phi \\ \Omega_y &= -\omega_x \cos \phi \\ \bar{f}(\phi) &= -mg \cos \phi \cos \beta \bar{i} - mg \sin \phi \cos \beta \bar{j} + mg \sin \beta \bar{k} \\ \bar{F}(\phi, t) &= 0 \end{aligned} \quad (177)$$

where ω_x is specified and $\beta = \omega_x t + \beta_o$ where $\beta_o = 0$ at $t = 0$.

Since $e = 0$,

$$\begin{aligned} r_x &= a \sin(\phi - \alpha) \\ r_y &= a[1 - \cos(\phi - \alpha)] \end{aligned} \quad (178)$$

Each end of the ring segment is simply supported (refer to Eqs. (165 a-f)) and an approximate expression for $P(\phi)$ is given by Eq. (179a), refer to Appendix A, Eq. (A.6).

$$P(\phi) = \omega_x^2 \rho A a^2 \left[1 - \left(\frac{\pi}{2} - \phi \right) \sin \alpha \frac{1}{\cos \phi} \right] \quad (179a)$$

and

$$P' = \frac{dP}{d\phi} = -\omega_x^2 \rho A a^2 \sin \alpha \left[-\cos \phi + \left(\frac{\pi}{2} - \phi \right) \sin \phi \right] \frac{1}{\cos^2 \phi} \quad (179b)$$

Substituting Eqs. (173) through (179) into Eqs. (90) through (93) results in Eqs. (180) and (181). Note that since there is no applied torque or out of plane applied forces, twisting and out of plane centroidal deformation does not occur and Eqs. (91) and (93) yield $\phi_x = w = 0$. Eqs. (90) and (92) yield Eqs. (180) and (181).

$$\begin{aligned}
 & m\omega_x^2 \left[\cos^2 \phi + \frac{I_z}{Aa^2} \cos 2\phi \right] u - \frac{EI_z}{a^4} v''' \\
 & + \left[-\frac{EI_z}{a^4} + \frac{\rho I_z}{a^2} \omega_x^2 \cos 2\phi \right] v' + [m\omega_x^2 \sin \phi \cos \phi] v = \\
 & + \omega_x^2 \rho A a \sin \alpha \left[-\cos \phi + \left(\frac{\pi}{2} - \phi \right) \sin \phi \right] \frac{1}{\cos^2 \phi} \\
 & + m\omega_x^2 a \left\{ -[1 - \cos(\phi - \alpha)] \sin \phi \cos \phi - \sin(\phi - \alpha) \cos^2 \phi \right\} - \frac{\rho I_z}{a} \omega_x^2 \sin \phi \cos \phi
 \end{aligned} \tag{180}^2$$

$$\begin{aligned}
 & -\frac{EI_z}{a^4} v''' + \left\{ -\frac{EI_z}{a^4} + \omega_x^2 \rho A \left[1 - \left(\frac{\pi}{2} - \phi \right) \frac{\sin \alpha}{\cos \phi} \right] - \frac{\rho I_z}{a^2} \omega_x^2 \cos 2\phi \right\} v'' \\
 & + \left\{ -\omega_x^2 \rho A \sin \alpha \left[-\cos \phi + \left(\frac{\pi}{2} - \phi \right) \sin \phi \right] \frac{1}{\cos^2 \phi} \right\} v' + [m\omega_x^2 \sin^2 \phi] v \\
 & + \left[\frac{\rho I_z}{a^2} \omega_x^2 \cos 2\phi \right] u' + [m\omega_x^2 \sin \phi \cos \phi] u = \\
 & \omega_x^2 \rho A a \left[1 - \left(\frac{\pi}{2} - \phi \right) \frac{\sin \alpha}{\cos \phi} \right] - m\omega_x^2 a [1 - \cos(\phi - \alpha)] \sin^2 \phi \\
 & - m\omega_x^2 a \sin \phi \cos \phi \sin(\phi - \alpha) + \frac{\rho I_z}{a} \omega_x^2 \cos 2\phi - 2I_z \frac{\rho}{a} \omega_x^2 \cos 2\phi
 \end{aligned} \tag{181}$$

Note that $\left(\frac{d}{d\phi} \right) = \frac{d(\)}{d\phi}$ and that $I_y = I_z$ and that $I_y + I_z = 2I$. The variable coefficients in Eqs. (180) and (181) makes their solution difficult even for this simple case.

When $\omega_x = 0$, Eqs. (180) and (181) result in a trivial solution since the gravity force is neglected for this particular case.

²The last line in Eq. (180) can be rewritten as:

$$+m\omega_x^2 a \left[-1 + \sin \alpha \cos \phi \cos 2\phi - \frac{I_z}{a^2 A} \sin \phi \cos \phi \right].$$

Eqs. (180) and (181) can be further simplified. However, they still remain ordinary differential equations with variable coefficients.

References

1. SAND79-8202, "Derivation of the Coupled Equations of Motion for a Beam Subjected to Three Translational Accelerations and Three Rotational Accelerations," by G. A. Benedetti, February 1979.
2. SAND90-8015, "Dynamic Stability of a Spinning Tube Conveying a Fluid Through a Symmetrical Noncircular Cross-Section," by G. A. Benedetti, November 1990.

Appendix A -- Approximate Critical Angular Speed for Lateral Dynamic Instability of a Rotating Circular Shaft which Supports Several Rotating Circular Ring Segments

Fig. A.1. "Rigid" circular ring segment and rigid link rotating at constant angular speed about X axis. Non-rigid or flexible circular shaft is supported in bearings. Rigid link is attached to shaft and segment through a simply supported connection. The number of ring segments is symmetrical when viewed along the X axis; e.g., three ring segments would be spaced 120° apart. The effect of gravity is neglected. That is, the acceleration of the ring in the y' or r direction is large relative to the local acceleration of gravity or $r\omega_x^2/g \gg 1$ where r is given by Eq. (A.2). Also, refer to Figure 5. Note that "rigid" used here implies no change in ring radius of curvature and that T is a tensile force (not a torque) passing through the centroid of the ring segment cross-section.

Determine the tensile force $T(\alpha)$ acting at the ends of a the rotating "rigid" circular ring segment as a function of α . Equilibrium of forces in the direction of r or y' (refer to Figure 5) results in Eq. (A.1).

$$2T(\alpha)\cos\alpha = \int r\omega_x^2 dm \quad (\text{A.1})$$

where

$$\begin{aligned} r &= a\sin\phi - a\sin\alpha + e \\ dm &= \rho A ds = \rho A a d\phi \end{aligned} \quad (\text{A.2})$$

Substituting Eqs. (A.2) in (A.1) results in Eq. (A.3)

$$2T(\alpha)\cos\alpha = \int_{\phi=\alpha}^{\phi=\pi-\alpha} [a(\sin\phi - \sin\alpha) + e] \omega_x^2 \rho A a d\phi \quad (\text{A.3})$$

Carrying out the integration in Eq. (A.3) results in Eq. (A.4).

$$T(\alpha)\cos\alpha = \omega_x^2 \rho A a^2 \left[\cos\alpha + \left(\frac{\pi}{2} - \alpha \right) \left(\frac{e}{a} - \sin\alpha \right) \right] \quad (\text{A.4})$$

Determine the tensile force $T(\phi)$ in the rotating circular ring segment as a function of ϕ . Equilibrium in the direction of r results in Eq. (A.5).

$$-T(\alpha)\cos\alpha + \int_{\zeta=\alpha}^{\zeta=\phi} (a\sin\zeta - a\sin\alpha + e) \omega_x^2 \rho A a d\zeta + T(\phi)\cos\phi = 0 \quad (\text{A.5})$$

Carrying out the integration in Eq. (A.5) and utilizing Eq. (A.4) results in Eq. (A.6).

$$T(\phi) = \omega_x^2 \rho A a^2 \left[1 + \left(\frac{e}{a} - \sin\alpha \right) \left(\frac{\pi}{2} - \alpha \right) \frac{1}{\cos\phi} \right] \quad (\text{A.6})$$

Determine the compressive axial force P acting on the ends of the flexible (non-rigid) rotating circular shaft. Equilibrium of forces in the X direction results in Eq. (A.7).

$$P = J T(\alpha) \sin\alpha \quad (\text{A.7})$$

or

$$P = J \omega_x^2 \rho A a^2 \left[1 + \left(\frac{e}{a} - \sin\alpha \right) \left(\frac{\pi}{2} - \alpha \right) \frac{1}{\cos\alpha} \right] \sin\alpha \quad (\text{A.8})$$

where $J = 2, 3, 4 \dots k$ is the number of "rigid" symmetrical circular ring segments attached to the rotating flexible shaft.

The N th undamped natural circular frequency for lateral vibration for a simply supported circular beam (shaft) with a compressive axial force applied to its ends and passing through the centroid of the cross-section is given by Eq. (A.9).

$$\omega_n = \left[\frac{EI}{m_b} \left(\frac{\pi}{\ell} \right)^4 \left(n^4 - n^2 \frac{P}{P_e} \right) \right]^{1/2} \quad n = 1, 2, 3, \dots, N \quad (\text{A.9})$$

where

$$P_e = EI \left(\frac{\pi}{\ell} \right)^2 \quad (\text{A.10})$$

For a simply supported beam (shaft) of circular cross-section spinning about its longitudinal axis with a constant angular speed ω_x , the apparent natural frequencies for lateral vibration are given by Eqs. (A.11). (For the case where the cross-section of the beam is not circular, refer to reference 2 for a more thorough treatment for the lateral vibration of a spinning beam).

$$\begin{aligned} {}_1\Omega_n &= \omega_n - \omega_x \\ {}_2\Omega_n &= \omega_n + \omega_x \end{aligned} \quad n = 1, 2, 3, \dots, N \quad (\text{A.11})$$

Lateral dynamic instability results when $\omega_x = \omega_n$. Therefore,

$$\omega_{x_{cr_n}} = \omega_n \quad n = 1, 2, 3, \dots, N \quad (\text{A.12})$$

where

$$\omega_n = \left[\frac{EI}{m_b} \left(\frac{\pi}{\ell} \right)^4 \left(n^4 - n^2 \frac{P}{P_e} \right) \right]^{1/2} \quad n = 1, 2, 3, \dots, N \quad (\text{A.13})$$

$$P_e = EI \left(\frac{\pi}{\ell} \right)^2 \quad (\text{A.14})$$

$$P = J\omega_{x_{cr_n}}^2 \rho A a^2 \left[1 + \left(\frac{e}{a} - \sin \alpha \right) \left(\frac{\pi}{2} - \alpha \right) \frac{1}{\cos \alpha} \right] \sin \alpha \quad (\text{A.15})$$

Note that from Eq. (A.11) the frequency of lateral vibration for dynamic instability is twice the N^{th} critical frequency. That is,

$${}_2\Omega_n = 2\omega_{x_{cr_n}} \quad n = 1, 2, 3, \dots, N \quad (\text{A.16})$$

Also, since damping is zero in this case, the amplitude of lateral vibration increases linearly with respect to time.

Utilizing Eqs. (A.13) through (A.15) in Eq. (A.12) and solving for $\omega_{x_{cr_n}}$ results in Eq. (A.17).

$$\omega_{x_{cr_n}} = \left[\frac{\frac{EI \left(\frac{\pi}{\ell} \right)^4 n^4}{m_b}}{1 + n^2 \left(\frac{\pi}{\ell} \right)^2 J \frac{\rho A a^2}{m_b} \left[1 + \left(\frac{e}{a} - \sin \alpha \right) \left(\frac{\pi}{2} - \alpha \right) \frac{1}{\cos \alpha} \right] \sin \alpha} \right]^{1/2} \quad (\text{A.17})$$

where J is the number of "rigid" symmetrical ring segments attached to the rotating shaft and

$$n = 1, 2, 3, \dots, N$$

Eq. (A.17) yields N (where $N=1$) critical angular speeds of a simply supported flexible rotating beam (circular shaft) which supports k ($J=2, 3, 4, \dots, k$) symmetrical "rigid" rotating circular ring segments each of which is offset, through a rigid link, a distance e from the rotating circular shaft. Refer to Figure A.1.

For Eq. (A.17), it is important to note that the lowest critical angular speed $\omega_{x_{cr_1}}$ for the rotating shaft occurs when $n=1$. Other values of n could be appropriate for use in Eq. (A.17); e.g., if the rotating circular shaft could also be simply supported by a bearing at its midlength, then $n=2$. However, a center support for the rotating shaft is not practical since the rotating ring segments would collide with the center support. Therefore, $N=1$ for Eq. (A.17)

The beam and ring segment parameter definitions are as follows. Recall that the effect of gravity is neglected; that is, $r\omega_x^2/g \gg 1$.

Parameter Definitions for a
Simply Supported Flexible
Beam or Circular Shaft

E ~ modulus of elasticity

I ~ area moment of inertia of
circular cross-section

m_b ~ mass per unit length

ℓ ~ length

n ~ mode number ($n = N = 1$)

$\omega_{x_{cr_n}}$ ~ critical angular speed
(radians/sec)

Parameter Definitions for a
"Rigid" Circular Ring Segment

J ~ Number of symmetrical circular
ring segments. Segment locations
are symmetrical when viewed
along X axis. $J = 2, 3, 4, \dots, k$

ρA ~ mass per unit length; $m = \rho A$

a ~ radius

e ~ offset distance of ring segment

α ~ angle (radians)

The units associated with the above parameters must, of course, be consistent.

For the case where symmetrical circular ring segments rotate about a non-rotating or stationary circular shaft (e.g., the links are attached to the outer-bearing race and the inner-bearing race is attached to the shaft), the lowest critical angular speed $\omega_{x_{cr_1}}$ of the ring segments occurs for $n = 1$ and $P = P_e$. Refer to Eq. (A.9). In this case $\omega_1 = 0$, there is no lateral vibration, and the shaft buckles statically when $n = 1$ and

$$P = P_e \quad (A.18)$$

Utilizing Eq. (A.8) in (A.18) results in Eq. (A.19) for the lowest critical angular speed of symmetrical rotating circular ring segments attached to a non-rotating circular shaft.

$$\omega_{x_{cr_1}} = \left[\frac{EI \left(\frac{\pi}{\ell} \right)^2}{J \rho A a^2 \left[1 + \left(\frac{e}{a} - \sin \alpha \right) \left(\frac{\pi}{2} - \alpha \right) \frac{1}{\cos \alpha} \right] \sin \alpha} \right]^{\frac{1}{2}} \quad (\text{A.19})$$

where $J = 2, 3, 4, \dots k$.

Comparing Eq. (A.17) for $n = 1$ to Eq. (A.19), the rotating shaft has a lower first ($n = 1$) critical speed than the non-rotating shaft, provided corresponding parameters are identical.

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