2 Statistical postulate of ordinary quantum theory

We treat nonrelativistic quantum mechanics with spin, allowing only pure states described by a state vector \( \psi \) in a Hilbert space \( \mathcal{H} \). (Masani [13] has raised the question of how to put mixed states into Wiener-Siegel theory; we avoid that topic for now.) With \( n \) particles, \( \psi \) may be realized as a wave function \( \psi : \mathbb{R}^n \times U^n \rightarrow C \), where \( U = \{1, -1\} \) is the domain of the spin index of a particle. Reference to the time dependence of \( \psi \) will be suppressed. For each spin state \( \sigma \), \( \psi(x, \sigma) \in \mathcal{L}(\mathbb{R}^n) \). The scalar product in \( \mathcal{H} \) is

\[
(\psi, \phi) = \int \overline{\psi}(x, \sigma) \phi(x, \sigma) \, dx
\]

(1)

Physical observables correspond to self-adjoint operators \( R \). We decompose the spectrum of \( R \) into disjoint sets \( R_i, R_j, \ldots \), and let \( P(R_i) \) be the projection operator onto the subspace of \( \mathcal{H} \) corresponding to \( R_i \). We have

\[
\sum_i P(R_i) = 1 \quad P(R_i)P(R_j) = 0 \quad i \neq j
\]

(2)

The statistical postulate of quantum mechanics is this: If \( r \) is the result of a measurement of the observable corresponding to \( R \), then the probability that \( r \in R_i \) is

\[
p(r \in R_i) = (\psi, P(R_i)\psi)
\]

(3)

The word probability refers to frequency of the observation \( r \in R_i \) among a large number of observations on an ensemble of identically prepared systems. The state vector is the theoretical counterpart of the whole ensemble, not that of a single system.

We have a statistical statement of the usual kind, a prediction of a frequency, for many observations on a physical ensemble that can be prepared in the laboratory. On the other hand, the theory provides no description of this ensemble as a probability space in the mathematical sense. There is no theoretical counterpart of a single system, no candidate for a point in probability space! This striking omission provides the motivation for the Wiener-Siegel theory.

3 Stochastic coordinates of the wave function

The required probability space is derived from a complex Wiener process \( X \) on the configuration space \( \mathbb{R}^n \), one for each spin state \( \sigma \). Note that \( X \) does not depend on the time, \( X : \mathbb{R}^n \times U^n \rightarrow C \). Following Wiener's custom of labeling individual Brownian motions with a parameter \( \alpha \in [0, 1] \), we characterize the process \( X(x, \sigma, \alpha) = X_1 + iX_2 \) in terms of the expectation value \( \mathcal{E} \) (integral over \( \alpha \)) as follows:

\[
X(0, \sigma, \alpha) = 0
\]

(4a)

\[
\mathcal{E}[X_{1}(x, \sigma, \alpha)] = 0
\]

(4b)

\[
\mathcal{E}[X_{2}(x, \sigma, \alpha) - X_{2}(y, \sigma, \alpha)] = \theta_{X_{1}}(x,y)
\]

(4c)

\[
\mathcal{E}[X_{1}(x, \sigma, \alpha) - X_{1}(y, \sigma, \alpha)]\mathcal{E}[X_{1}(u, \tau, \alpha) - X_{1}(v, r, \alpha)] = 0 \quad \text{if} \quad m([x, y] \cap [u, v]) = 0
\]

(4d)
where \( m \) is Lebesgue measure on \( \mathbb{R}^n \) and \([x,y]\) is the \( 3n\)-dimensional interval \( \{x_i \leq t_i \leq y_i \mid i = 1, \ldots, 3n\} \); thus \( m([x,y]) = \prod_{i=1}^{3n} (y_i - x_i) \). Increments of real and imaginary parts are independent, as are increments with different spin states, and general increments for non-overlapping intervals.

Wiener's stochastic integral, generalized to include spin, is written as

\[
\sum_{\sigma} \int \psi(x,\sigma) dX(x,\sigma,\alpha) = \langle \psi(\alpha) \rangle .
\]  

(5)

As we shall see, the Dirac-style notation \( \langle \psi(\alpha) \rangle \) with \( \langle \alpha | \psi \rangle = \psi(\alpha) \) is convenient and heuristically suggestive. Henceforth we also use Dirac notation for the scalar product, \( \langle \psi | \phi \rangle = \psi(\phi) \). Since almost all \( X \) have unbounded variation, the integral is not a Stieltjes integral. It is defined as a limit in the mean through approximation of \( \psi \) by step functions. We divide \( \mathbb{R}^n \) into non-overlapping intervals \([x^i, x^{i+1}]\) by a Cartesian grid, and approximate \( \psi(x) \) by a constant \( \psi_{i\sigma} \) on the \( i\)-th interval, taking \( \psi_{i\sigma} = 0 \) for \( i > N \). For the approximate \( \psi \), the integral is defined as

\[
\sum_{i\sigma} \sum_{i\sigma} \int \psi_{i\sigma} [X(x^{i+1},\sigma,\alpha) - X(x^i,\sigma,\alpha)] dx .
\]

(6)

and by (4b,4c)

\[
E(\langle \psi(\alpha) | \alpha(\phi) \rangle) = \sum_{i\sigma} \psi_{i\sigma} \psi_{j\sigma} E \left[ (X(x^{i+1},\sigma,\alpha) - X(x^i,\sigma,\alpha)) \alpha(x^{i+1},\tau,\alpha) - X(x^i,\tau,\alpha) \right]
\]

(7)

\[
= 2 \sum_{i\sigma} \psi_{i\sigma} |m(x^i, x^{i+1})| = 2 \int |\psi|^2 dx = 2 \langle \psi(\phi) \rangle .
\]

(8)

Invoking the Riesz-Fischer theorem and the fact that the step functions are dense in \( L^2 \), we conclude that the integral exists as a limit in the mean. Similarly,

\[
E(\langle \psi(\alpha) | \alpha(\phi) \rangle) = 2 \langle \psi(\phi) \rangle .
\]

(9)

Now \( \text{Re}(\psi(\phi)) \) and \( \text{Im}(\psi(\phi)) \) are independent Gaussian variables, and each has unit variance if \( \langle \psi | \psi \rangle = 1 \). The following formulas can be used to set up the covariance matrix of a set of such variables:

\[
E(\text{Re}(\psi(\phi)) \text{Re}(\psi(\phi))) = E(\text{Im}(\psi(\phi)) \text{Im}(\psi(\phi))) = \text{Re}(\psi(\phi)) \text{Re}(\psi(\phi)) ;
\]

(10)

\[
E(\text{Im}(\psi(\alpha)) \text{Re}(\psi(\phi))) = \text{Im}(\psi(\phi)) = -\text{Im}(\psi(\phi)) .
\]

(11)

Remarks: (1) The above treatment of spin seems to me an obvious choice, but it differs from that of [2]. As I recall, Siegel had rejected the method of [2] in 1959. His review in [9] avoids spin. (2) The random function \( X(x,\sigma,\alpha) \) is sometimes called a point \( \alpha \) in differential space; its differentials \( dX \) are the significant thing. (3) I propose the name stochastic coordinate of the wave function for \( \psi(\alpha) \), since it may be thought of as the projection of \( \psi \) along the direction in differential space [9] labeled by \( \alpha \). (4) Siegel told me that Wiener resisted strongly the idea of making the Brownian motions independent of time. He had a vision of random motions proceeding in time.

\section{Joint probabilities}

The joint probability for an eigenvalue of \( R \) to lie in \( R_i \) and an eigenvalue of \( S \) to lie in \( S_j \) is the measure of the set of all \( \alpha \) such that both inequalities (12) and (13) hold. As the simplest example we take one electron with spin, with wave function \( \psi(x,\tau) = \psi_1(x) \) and \( \psi(x,-\tau) = \psi_2(x) \). The spin indices \( \pm 1 \) correspond to the eigenvalues \( \pm 1/2 \) of the third component of spin, \( S_z \). The projection operators for \( S_z = \pm 1/2 \) are \( P_{\pm} = 1/2 \pm i/2 \). Then one finds by the usual Pauli spin algebra that

\[
\langle \psi | P_{\pm} | \alpha \rangle = (\psi_\pm | \alpha \rangle , \quad \langle \psi | P_{\mp} | \alpha \rangle = (\psi_\mp | \alpha \rangle ,
\]

(16)

\[
\langle \psi | P_{\pm} | \alpha \rangle = \frac{1}{\sqrt{2}} (\psi_1 | \alpha \rangle + \psi_2 | \alpha \rangle , \quad \langle \psi | P_{\pm} | \alpha \rangle = \frac{1}{\sqrt{2}} (\psi_1 | \alpha \rangle - \psi_2 | \alpha \rangle ,
\]

(17)

where the stochastic integrals on the right entail no spin sum. With \( \langle \psi | \psi \rangle = \langle \psi_1 | \psi_1 \rangle + \langle \psi_2 | \psi_2 \rangle = 1 \), we also have

\[
\langle \psi | P_{\pm} | \psi \rangle = \langle \psi_\pm | \psi \rangle , \quad \langle \psi | P_{\mp} | \psi \rangle = \langle \psi_\mp | \psi \rangle ,
\]

(18)

\[
\langle \psi | P_{\pm} | \psi \rangle = \frac{1}{2} \pm \text{Re}(\psi_\pm | \psi_\mp \rangle .
\]

(19)
We compute the joint probability $p(S_z = \hbar/2 \cap S_x = \hbar/2)$. Define
\[ x_k + iy_k = (\psi_k | \alpha \rangle / a_k, \quad a_k = (\psi_k | \phi_0 \rangle)^{1/2}, \quad b = 2Re(\psi_1 | \psi_2) \, . \] (20)

By Eqs. (12,16,17,18,19) the required probability is
\[ p(S_z = \hbar/2 \cap S_x = \hbar/2) = M(\Omega) = \frac{1}{(2\pi)^4} \int_0^\Omega \exp(-\frac{1}{2} z^2)dz \, , \] (21)
\[ z = (x_1, y_1, x_2, y_2) \, , \] (22)
where $\Omega$ is the set in $\mathbb{R}^4$ on which the following inequalities hold:
\[ x_1^2 + y_1^2 < a_1^2 \] (23)
\[ \frac{(a_1 x_1 + a_2 y_2)^2 + (a_1 y_1 + a_2 x_2)^2}{(1 + b)^2} < \frac{(a_1 x_1 - a_2 y_2)^2 + (a_1 y_1 - a_2 x_2)^2}{(1 - b)^2} \] (24)

To impose the inequalities, we use a Fourier representation of the unit step function,
\[ \theta(z) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i(kz)}}{\lambda - ic} dz \, , \] (25)
choosing $x$ so that $\theta = 1$ when an inequality is satisfied, otherwise $\theta = 0$. Taking limits outside the integral we obtain
\[ M(\Omega) = \lim_{\eta \to 0} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{dz}{\mu - i\eta} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda - ic} \int_{-\infty}^{\infty} \exp(-\frac{1}{2} z^2)dz \, . \] (26)

Here $A$ is a $4 \times 4$ matrix that can be read off from (33,24,25). To justify taking the limits outside the integral, note that the integral in (25) is equal to $\theta(z) \exp(-cz)$ and therefore approximates $\theta(z)$ uniformly on any finite interval for small $\epsilon$. On the other hand, the real part of $A$ is the unit matrix, so that the $z$ integral converges uniformly in $\epsilon$ and $\eta$. For any prescribed accuracy, only a finite region of integration on $x$ is relevant.

The $z$-integral equals $(2\pi)^2 \det^{-1/2} A$, and $\det^{1/2} A$ is a quadratic polynomial in $\lambda$ with roots $\lambda_i(\mu)$. An analysis (not very simple) shows that, for all $\mu$, only one of the roots is in the lower half $\lambda$-plane. Consequently, one can evaluate the $\lambda$-integral as a single residue, and find eventually the formula
\[ M(\Omega) = \frac{a_1^2}{2} + \frac{a_2^2}{2} \int_0^{\infty} \frac{d\mu}{\mu} \text{Im} [\lambda_1(\mu) - \lambda_2(\mu)] \lambda_3(\mu) \, . \] (27)

The integrand is continuous and bounded at all $\mu$, and falls off as $\mu^{-3}$ at infinity. For the rest we resort to numerical integration, taking care to use the correct branches of square roots in the complicated expression for $\lambda_i$.

The joint probability depends on two independent parameters, which we take to be $p(S_z = \hbar/2) = a_1^2$ and $p(S_x = \hbar/2) = (1 + b)/2$. Figure 1 shows the lower limit of $p(S_x = \hbar/2 \cap S_z = \hbar/2)$ as a function of the latter, for a few choices of the former. The upper limit of $p(S_x = \hbar/2)$ is determined by Schwarz's inequality and the normalization $|\psi(\phi)| = a_1^2 + a_2^2 = 1$.

The curve look quite reasonable, showing that the joint probability is sometimes larger and sometimes smaller than the product of the individual probabilities, depending on the spin content of the wave function. A check on the computation comes from the case $p(S_z = \hbar/2) = 1/2$, which can be handled analytically. It yields $p(S_x = \hbar/2, S_z = \hbar/2) = p(S_x = \hbar/2)p(S_z = \hbar/2)$ in agreement with the numerical results.
6 The Einstein-Podolsky-Rosen-Bohm Gedankenexperiment

We consider two spin 1/2 particles in a singlet state (total spin 0). The state vector is a sum of direct products of vectors belonging to the separate particles:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|e(1)\rangle \otimes |e(-1)\rangle - |e(-1)\rangle \otimes |e(1)\rangle)$$

where $|e(\pm 1)\rangle$ is an eigenvector of $S_i \cdot e$ with eigenvalue $\pm \hbar/2$. Here $S_i \cdot e$ is the scalar product of the spin operator for the $i$-th particle with a unit vector $e$. The singlet state is spherically symmetric, which means that the choice of $e$ is arbitrary. The projection operator for $S_i \cdot e$ having eigenvalue $\pm \hbar/2$ is denoted by $P_i(e(\pm 1))$. We have

$$P_i(e(\pm 1)) = |e(1)\rangle \langle e(1)| \pm |e(-1)\rangle \langle e(-1)|$$

where $I_1$ is the unit operator for particle $i$ in spin space.

Usual quantum theory gives the joint probability for $S_1 \cdot e = k\hbar/2$ and $S_2 \cdot f = h\hbar/2$ as

$$p(e(1) \cap P_z(f(1))) = \langle \psi | P_z(e(k)) P_z(f(l)) | \psi \rangle = \frac{k l}{2} \langle e_1(k) | e_1(-l) \rangle \langle e_2(-k) | e_2(l) \rangle$$

where the final equality is obtained by writing $|\psi\rangle$ in terms of $e$ and $|\psi\rangle$ in terms of $f$, then applying Eq.(29).

This hypothesis is unambiguous, thanks to the commutativity of Eq.(30). The Wiener-Siegel theory of course gives the value (31) for the measure of the set of all

$$\mbox{where the final equality is obtained by writing } |\psi\rangle \mbox{ in terms of } e \mbox{ and } |\psi\rangle \mbox{ in terms of } f, \mbox{ then applying Eq.(29).}$

where $\alpha$ is the angle between $e$ and $f$. It is easy to derive the formulas (38) from the usual law of transformation of spinors under rotation. Tractable expressions for the inverse and the determinant of the resulting $8 \times 8$ matrix $A$ can be derived. We can now proceed as in the previous section, enforcing the inequalities (37) by means of the integral representation of the step function (25). As before, the $x$-integral and one of the integrals from the step functions can be evaluated. For the case $\theta = 120^\circ$ the result after those integrations is

$$M_{gg} = M_{z-1} = \frac{1}{2} \cos 2\theta, \quad M_{z-1} = M_{z-1} = \frac{1}{2} 8 \sin^2 \theta.$$

A numerical integration yields $M(\Omega) = 0.26630...$, in clear disagreement with the value 3/8 of ordinary quantum theory.

Several experiments have been made to measure directly the probability we have discussed. There have been several experiments designed to imitate the EPRB Gedankenexperiment as closely as possible, usually using photons from cascade decays of an atom [19]. A difficulty in principle is the ambiguity in the interpretation of missing particles. If particle 1 is observed passing through polarizer 1, particle 2 may be unobserved, for one of two reasons that cannot be distinguished: (i) the detector has very poor efficiency and simply fails to detect the particle or (ii) the particle is blocked by polarizer 2. Certain techniques and assumptions have been introduced to avoid this and other problems, and the general opinion seems to be that the experiments favor quantum theory over hidden-variable theories. A forthcoming experiment by E. Fry at Texas A & M University, which is to observe mercury atoms from dissociation of a mercury molecule, may give a clearer test.

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