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Discrete and Continuous Models of Conservation Laws AUG 27 1996

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J. M. Greenberg

This report covers the period 3-1-94 through 12-1-95 and details a series of projects to be worked on during the period 4-1-96 through 9-30-97. These tasks were part of the original proposal of the same title submitted to DOE in the fall of 1993 and originally funded as DOE FG0294ER25198 on March 1, 1994.

1 Conservation Laws with Source Terms.

Two projects have been initiated. The first involves a convex scalar conservation law with delta function source terms and was carried out with Alain LeRoux of the University of Bordeaux. The second involves work on gravity driven flows and is being carried out jointly with my student, Andrew Yershov, who moved with me from Maryland to Carnegie Mellon University in August.

The work with LeRoux dealt with the scalar conservation law

$$u_t + f(u)_x = a_x. \quad (1.1)$$

We restricted our attention to even-convex functions f and our interest was in numerical schemes for (1.1) which preserve stable steady solutions of the form: $f(u(x)) - a(x) \equiv H$, a constant. Such steady solutions make sense when $a(\cdot)$ has simple jump discontinuities.

To build such schemes we were forced to consider solutions of (1.1) satisfying the initial conditions

$$u(x, 0^+) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases} \quad (1.2)$$

when $a(\cdot)$ was a step function

$$a(x) = \begin{cases} a_l, & x < 0 \\ a_r, & x > 0 \end{cases} \quad (1.3)$$

The solution to the Riemann Problem (1.1)-(1.3) must satisfy the unforced equation $u_t + f(u)_x = 0$ away from $x = 0$, the Lax entropy condition that u decreases across any shock propagating with non-zero speed, and finally across the interface $x = 0$

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$$f(u_+) - f(u_-) = a_r - a_l \quad (1.4)$$

where $u_{\pm} = \lim_{x \rightarrow 0^{\pm}} u(x, t)$. Unfortunately these conditions do not uniquely characterize the solution to (1.1)-(1.3). The basic result of [1] was a reformulation of the Riemann Problem (1.1)-(1.3) as an "optimization" problem and the demonstration that this "optimization" problem had a unique minimizer. This minimizer gave us a selection principle for how to choose from amongst the plethora of possible solutions to (1.1)-(1.3). The solutions yielded via this selection principle generate a contraction semi-group in $L_1^{loc}(-\infty, \infty)$ and also provide a generalization of the "Lax-Formula" for solutions of the unforced equation $u_t + f(u)_x = 0$. The solutions to the Riemann Problem also provide the essential ingredient to the sought after numerical method for solving (1.1) with arbitrary $a(\cdot)$, namely the appropriate fluxes $f_{k,\pm} = \lim_{x \rightarrow kh^{\pm}} f(u(x, t))$. A short announcement of these results has already appeared in *Compte Rendu* [2].

The work with Yershov has focused on the shallow water equations

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \quad (1.5)$$

and

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}(hu^2 + \frac{gh^2}{2}) + gh \frac{\partial a}{\partial x} = 0 \quad (1.6)$$

where $a(\cdot)$ describes the bottom topography, h the height of the water above the bottom surface $y = a(x)$, and u is the mean flow velocity. Our goal in this project are numerical schemes which preserve equilibrium solutions

$$(hu)(x) = m \quad (1.7)$$

and

$$(\frac{u^2}{2} + g(h + a))(x) = H \quad (1.8)$$

where $m > 0$ and $H > 0$ are respectively the mass flux and total head.

Our initial efforts on this problem focused on building Riemann solvers for (1.5) and (1.6) when $a(\cdot)$ is a step function

$$a = \begin{cases} a_l, & x < 0 \\ a_r, & x > 0 \end{cases} \quad \text{and} \quad \frac{\partial a}{\partial x} = (a_r - a_l)\delta(x) \quad (1.9)$$

and h and u satisfy

$$(h, u)(x, 0^+) = \begin{cases} (h_l, u_l), & x < 0 \\ (h_r, u_r), & x > 0 \end{cases} \quad (1.10)$$

This problem is poorly posed as stated since it involved taking products of distributions $gh \frac{\partial a}{\partial x}$.

Our first approach to this problem was to regularize $a(\cdot)$ and replace it with

$$a_\epsilon(x) = \begin{cases} a_l, & x \leq 0 \\ a_l + \frac{(a_r - a_l)x}{\epsilon}, & 0 < x \leq \epsilon \\ a_r, & \epsilon < x \end{cases} \quad (1.11)$$

modify the initial data

$$(h_\epsilon, u_\epsilon)(x) = \begin{cases} (h_l, u_l), & x \leq 0 \\ (H_\epsilon, U_\epsilon), & 0 < x \leq \epsilon \\ (h_r, u_r), & \epsilon < x \end{cases} \quad (1.12)$$

and solve (1.5), (1.6), (1.11), and (1.12).

Our strategy was to choose (H_ϵ, U_ϵ) in (1.12) to satisfy the equilibrium relations (1.7) and (1.8) over the interval $0 < x \leq \epsilon$. Once done, this guaranteed that the solution to (1.5), (1.6), (1.11), and (1.12) has the structure

$$(h_\epsilon, u_\epsilon) = \begin{cases} (\tilde{H}, \tilde{U})(x/t), & x \leq 0 \text{ and } t > 0 \\ (H_\epsilon, U_\epsilon)(x), & 0 < x \leq \epsilon \text{ and } t > 0 \\ (\tilde{H}, \tilde{U})(\frac{x-\epsilon}{t}), & \epsilon < x \text{ and } t > 0 \end{cases} \quad (1.13)$$

This program worked for arbitrary states $a_l, a_r, (h_l, u_l)$, and (h_r, u_r) and the resulting solution satisfied the Lax entropy conditions across shocks in the regions $x < 0$ and $x > \epsilon$. Using these solutions we were able to pass to the $\epsilon = 0^+$ limit and these limiting solutions (h_0, u_0) were taken to be the solution to the singular problem (1.5), (1.6), (1.9), and (1.10). For these limiting solutions we were able to compute the flux terms

$$\lim_{x \rightarrow 0^\pm} (h_0 u_0, h_0 u_0^2 + \frac{gh_0^2}{2})(x, t) \quad (1.14)$$

and it is these limits which were incorporated into the desired Riemann solver.

Though the program described above is conceptually sound we have come to the conclusion that its implementation for general data is impractical

except when the solution described by (1.13) has the additional property that the wave speeds

$$c_\epsilon^1 = u_\epsilon - \sqrt{gh_\epsilon} \text{ and } c_\epsilon^2 = u_\epsilon + \sqrt{gh_\epsilon} \quad (1.15)$$

are either both subsonic (< 0) or both supersonic (> 0) for all x and $t > 0$. In this case the algebra involved is manageable and Godunov type schemes using the fluxes defined in (1.14) yield excellent results. When (1.15) cannot be guaranteed we have been forced to limit our class of difference schemes to those of the Lax-Freidrichs type. Here we have to contend with a much simpler Riemann solvers where the initial data for h and u is continuous at the interface; i.e., $(h_l, u_l) = (h_r, u_r)$, and only $a(\cdot)$ jumps at $x \equiv 0$. These schemes are somewhat more diffuse than the Godunov type schemes.

Figures 1-6 show the results of the original scheme for data which is supersonic throughout and for a bottom topography which is a simple ramp

$$a(x) = \begin{cases} 0, & x \leq 0 \\ .4x, & 0 < x \leq 1 \\ .4, & 1 < x \end{cases}$$

A uniform grid with $\Delta x = 1/100$ and $\Delta \tau = 1/400$ was used in all of these computations. The initial data for h and u are

$$(h, u) = \begin{cases} (1, 2), & x < 0 \\ (0, 0), & x > 0 \end{cases}$$

and units are chosen so that $g = 1$.

References

1. (with A. Y. LeRoux), Analysis and Approximation of Conservation Laws with Source Terms, SIAM J. Numerical Analysis (to appear).
2. (with A. Y. LeRoux, R. Baraille, A. Noussair), Analyse et approximation de lois de conservation avec terme source, C. R. Acad. Sci. Paris, 321 Serie I, pp. 1073-1076 (1995).

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

2 General Integration Schemes for Conservation Laws without Source Terms.

This represents new work which seems quite promising. As with much of this subject one can obtain provable theorems in the scalar case yet experimentally the results seem to carry over for systems. Our interest is in integration methods for the system

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \quad (2.1)$$

where $\mathbf{u} \in R^N$ and $\mathbf{f} : R^N$ into R^N . We assume the Jacobian, $\mathbf{f}'$, of $\mathbf{f}$ has N real and distinct eigenvalues

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \dots < \lambda_N(\mathbf{u}) \quad (2.2)$$

for all $\mathbf{u}$ (or at least for all $\mathbf{u}$'s in the range of the solution to (2.1)) and we further assume that $\lambda_1(\mathbf{u}) > 0$. If this is not the case then the simple change of variables

$$y = x + \mu t \text{ and } \tau = t$$

maps (2.1) into

$$\mathbf{u}_\tau + (\mathbf{f}(\mathbf{u}) + \mu \mathbf{u})_y = 0$$

and the eigenvalues of $\mathbf{f}'(\mathbf{u}) + \mu I$, namely the numbers

$$\mu + \lambda_1(\mathbf{u}) < \mu + \lambda_2(\mathbf{u}) < \dots < \mu + \lambda_N(\mathbf{u})$$

satisfy $\mu + \lambda_1(\mathbf{u}) > 0$ for μ large enough. Our interest is in solutions of (2.1) which satisfy

$$\mathbf{u}(x, 0^+) = \mathbf{u}_0(x), \quad -\infty < x < \infty \quad (2.3)$$

and for definitiveness we shall assume that

$$\mathbf{u}_0(x) \equiv \mathbf{u}_-, \quad x < 0 \text{ and } \mathbf{u}_0(x) \equiv \mathbf{u}_+, \quad x > l_0 > 0.$$

Once again we are assuming that $\lambda_1(\mathbf{u}) > 0$ for all $\mathbf{u}$. For L_∞ solutions $\mathbf{u}(x, t)$ to (2.1) and (2.3) we are guaranteed that

$$\mathbf{u}(x, t) \equiv \mathbf{u}_-, \quad x \leq 0 \text{ and } \mathbf{u}(x, t) \equiv \mathbf{u}_+, \quad x \geq l_0 + st \quad (2.4)$$

provided that s is an upper bound for $\lambda_N(\mathbf{u})$. Our problem is to resolve the solution over the cone $0 < x < l_0 + st$ taking note of the fact that

$$\lim_{x \rightarrow 0} \mathbf{u}(x, t) = \mathbf{u}_- \text{ and } \lim_{x \rightarrow (l_0 + st)} \mathbf{u}(x, t) = \mathbf{u}_+. \quad (2.5)$$

One could introduce an expanding grid

$$x_k^n = kh, \quad 0 \leq k \leq N+n \quad \text{and} \quad Nh = l_0$$

with constant time steps $\Delta\tau$ satisfying $\frac{(\Delta\tau)s}{h} < 1$ and obtain approximate solutions at times $t_n = n\Delta\tau$. This will work satisfactorily if all that is desired is the solution over a finite time interval $0 \leq t \leq T$. If one is interested in the large time behavior of the solution to (2.1), (2.3), and (2.4) such an approach seems doomed since the number of degrees of freedom grows linearly with n for any fixed set of approximation parameters h and $\Delta\tau$. What is needed is a scheme that will involve a fixed number of degrees of freedom in x for all t . That such a scheme has a chance of succeeding follows from the fact that for nonlinear f 's the system (2.1) is dispersive and solutions tend to decouple into characteristic wave forms associated with the separated eigenvalues

$$0 < \lambda_1(u) < \lambda_2(u) < \dots < \lambda_N(u).$$

In what follows we let

$$l(t) = l_0 + st, \quad \xi = \frac{x}{l(t)}, \quad \text{and} \quad x = t. \quad (2.6)$$

Under this change of variables (2.1) transforms to

$$u_\tau - \frac{l\xi}{l} u_\xi + \frac{1}{l} f(u)_\xi = 0 \quad (2.7)$$

and the initial data becomes

$$u(\xi, 0^+) = u_0(l_0\xi), \quad 0 < \xi < 1. \quad (2.8)$$

We solve this system (2.7) and (2.8) subject to the boundary conditions

$$u(0, \tau) = u_- \quad \text{and} \quad u(1, \tau) = u_+ \quad (2.9)$$

where u_- and u_+ are the constant states defined in (2.5). We further note that if u satisfies (2.7) and (2.8), then $U(\xi, \tau) \stackrel{\text{def}}{=} \int_0^\xi u(s, \tau) ds$ satisfies

$$U_\tau - \frac{l\xi}{l} U_\xi + \frac{l}{l} U + \frac{1}{l} f(U_\xi) = f(u_-), \quad (2.10)$$

$$U(\xi, 0^+) = \int_0^\xi u_0(l_0 s) ds, \quad 0 \leq \xi \leq 1 \quad (2.11)$$

and

$$U(0, \tau) = 0 \text{ and } f(U_\xi(1, \tau)) = f(u_+). \quad (2.12)$$

We use a fractional step method to integrate (2.7). We assume the approximate solution U is known at time τ_n at the grid points $\xi_k = k\Delta\xi$, $0 \leq k \leq N$ where $N\Delta\xi = 1$; that is $U(\xi_k, \tau_n) = U_k^n$ and we further assume that $U(\cdot, \tau)$ is linear between the grid points. We choose $(\Delta\tau)^{n+1} = \tau_{n+1} - \tau_n$ so that

$$\frac{(\Delta\tau)^{n+1}s}{l_{n+1}\Delta\xi} = 1. \quad (2.13)$$

This implies that $l_{n+1} = l_0 + s\tau_{n+1}$ and l_n are related by

$$l_{n+1} = \frac{l_n}{1 - \Delta\xi} \quad (2.14)$$

and that

$$\tau_{n+1} = \tau_n + \frac{l_{n+1}\Delta\xi}{s}. \quad (2.15)$$

Over the interval $\tau_n \leq \tau \leq \tau_{n+1}$ we first solve

$$U_\tau - \frac{l\xi}{l}U_\xi + \frac{l}{l}U = 0 \quad (2.16)$$

subject to the initial conditions given above. This implies that

$$V_k^{n+1} = U(y_k, \tau_{n+1}) = (1 - \Delta\xi)U_k^n \quad (2.17)$$

where

$$y_k = (1 - \Delta\xi)\xi_k, \quad 0 \leq k \leq N. \quad (2.18)$$

Given the numbers V_k^{n+1} we define the function $U^{n+1/2}(\xi, \tau_{n+1}) = V_k^{n+1} + \frac{(V_{k+1}^{n+1} - V_k^{n+1})}{\Delta\xi(1 - \Delta\xi)}(\xi - y_k)$, $y_k \leq \xi \leq y_{k+1}$, and $k = 0, 1, \dots, N-1$. On the half line $y_N = \xi_{N-1} < \xi$ we let $U^{n+1/2}(\xi, \tau_{n+1}) = V_N^{n+1} + u_+(\xi - \xi_{N-1})$. Finally we let

$$U_k^{n+1/2} = U^{n+1/2}(\xi_k, \tau_{n+1}), \quad 0 \leq k \leq N \quad (2.19)$$

and we use these latter data in the second half step update. Here we solve

$$U_\tau + \frac{1}{l_{n+1}}f(U_\xi) = f(u_-)$$

with the discrete data $U_k^{n+1/2}$ given by (2.19). The fact that all the wave speeds are positive allows us to use a simple donor cell method for this update. Recalling that $\frac{(\Delta\tau)_{n+1}s}{l_{n+1}\Delta\xi} = 1$, the second half-step update takes the form

$$U_k^{n+1} = U_k^{n+1/2} + \frac{\Delta\xi}{s} (f(u_-) - f(\frac{U_k^{n+1/2} - U_{k-1}^{n+1/2}}{\Delta\xi})), \quad 0 \leq k \leq N.$$

The u 's at time τ_{n+1} are then given by

$$u_k^{n+1} = \frac{U_k^{n+1} - U_{k-1}^{n+1}}{\Delta\xi}, \quad 1 \leq k \leq N$$

and these represent the approximate solution on the interval $(k-1)\Delta\xi < \xi < k\Delta\xi$. For small times this scheme does at least as well as standard methods and it also captures the long time behavior of the solution. The algorithm has the other salutary feature that it requires no Riemann solvers. This follows from the fact that all the wave speeds are positive and hence the values of u on the interfaces $\xi = kh$ are the numbers $u_k^{n+1/2} = \frac{U(\xi_k)^{n+1/2} - U(\xi_{k-1})^{n+1/2}}{\Delta\xi}$. The fact that $U(\xi)^{n+1/2} = V_N^{n+1} + u_+(\xi - \xi_{N-1})$ for $\xi > \xi_{N-1}$ guarantees that $u_N^{n+1/2} = u_+$ and this is the fashion in which the boundary condition (2.11)₂ is implemented.

A manuscript on this work is currently in preparation. Figures (7)-(10) show the results of a simulation with these schemes for the unforced shallow water equation (see (1.5) and (1.6)) with initial data

$$(h, u)(x, 0) = \begin{cases} (1, 2), & x < 0 \\ (1, 4), & 0 < x < 1 \\ (1, 2), & 1 < x \end{cases} \quad (2.20)$$

In these calculations $\Delta\xi = 1/1000$, $s = 5$, and $l_0 = 1$.

The top portrait in Figures 7-11 represent $h^{1/2}$ and the bottom portrait the cross-sectional velocity u .

The portraits at time = .1 show slow and fast rarefaction waves emerging from $\xi = 0$ and slow and fast shocks emerging from $\xi = 1$. These waves interact over the time interval $[.1, 1]$ and by time = 1 the flow has stabilized to a slow rarefaction wave overtaking a slow shock and a fast rarefaction wave overtaking a fast shock. This basic wave pattern persists into the future and the portraits at time 1, 10, and 100 demonstrate the decay of these waves.

The quantity labeled length in each of these portraits represents the true length of the computational domain $1 + 5t$.

Further research on these ideas will focus on using nonuniform meshes in ξ or equivalently a nonuniform regridding algorithm instead of the one defined by (2.17). Such adaptivity will exploit the fact that over large intervals $a < \xi < b$, $u(\xi, \tau)$ is essentially constant as $\tau \rightarrow \infty$.

Figure 7

Figure 8

Figure 9

Figure 10

3 Relaxation Methods for Integrating Conservation Laws.

While doing our work on plasticity it struck us that the methodology we were employing - namely a combined relaxation singular perturbation procedure - worked for a much broader class of problems and yielded surprisingly good results. Additionally, *a priori* estimates were obtainable for such methods. One of our goals for future work is to see if these ideas carry over to systems in several space dimensions. Here the problem is coming up with simple semi-linear systems which mimic the given quasilinear one.

Below we shall give a simple overview of the technique within the context of the quasi-linear wave equation

$$u_t - v_x = 0 \text{ and } v_t - \hat{\sigma}(u)_x = 0. \quad (3.1)$$

Here $\hat{\sigma}(u)$ satisfies $\hat{\sigma}'(u) > 0$ and $\hat{\sigma}''(u)$ is not identically zero. We shall restrict our attention to L_∞ solutions of (3.1) and note that for such solutions there is an upper bound c^2 so that $\hat{\sigma}'(u) < c^2$.¹ Our program starts by replacing (3.1) with the semi-linear system

$$\left. \begin{aligned} r_t + cr_x &= C/\epsilon \\ l_t - cl_x &= C/\epsilon \\ d_t &= -2C/\epsilon \end{aligned} \right\} \quad (3.2)$$

where C is as yet unspecified. The number c in (3.2) is the square root of upper bound for $\hat{\sigma}'(u)$. We now introduce the moments

$$\left. \begin{aligned} u &= l + r + 2\delta \\ v &= c(l - r) \\ \sigma &= c^2(l + r) \end{aligned} \right\} \quad (3.3)$$

and note that u , v , and σ satisfy

$$u_t - v_x = 0, \quad v_t - \sigma_x = 0, \quad \text{and} \quad \sigma_t - c^2 v_x = \frac{2c^2 C}{\epsilon}. \quad (3.4)$$

Since our goal is to have solutions of (3.4) mimic those of (3.1) we choose the collision term C to be

$$C = (\hat{\sigma}(u) - \sigma)/2. \quad (3.5)$$

¹In the situation such as gas dynamics where $\hat{\sigma}(\cdot)$ is defined only as $u > 0$ we shall assume that $\hat{\sigma}(u) \rightarrow -\infty$ and $\hat{\sigma}'(u) \rightarrow +\infty$ as $u \rightarrow 0^+$. In this case our interest is in solutions satisfying $0 < u_0 \leq u \leq u_1 < \infty$.

Then, formally the system (3.4) reduces to (3.1) in the limit where $\epsilon \rightarrow 0^+$. The system (3.2) with C given by (3.5) has some advantages over (3.1). The first is that all discontinuities of (3.2) propagate with the speeds $\pm c$ or zero and the second no discontinuities are present if they are not put in initially or are not generated by the interaction of initial and boundary data.

It is easily verified that (3.4) with C given by (3.5) is endowed with an entropy-entropy flux pair

$$\left. \begin{aligned} \eta &= \frac{1}{2}v^2 + \int_1^u (\hat{\sigma}(\eta) - \hat{\sigma}(1))d\eta + \mathcal{P}(\sigma - \hat{\sigma}(u), u) \\ \text{and} \\ q &= \sigma v \end{aligned} \right\} \quad (3.6)$$

where

$$\mathcal{P}(\delta, u) = \int_0^\delta \frac{\eta d\eta}{c^2 - \hat{\sigma}'(\hat{u}(\eta; u, \delta))} = u\delta - \int_0^\delta \hat{u}(\eta; u, \delta)d\eta \quad (3.7)$$

and $\hat{u}(\eta; u, \delta)$ is the unique solution to

$$\frac{d\hat{u}}{d\eta} = \frac{1}{c^2 - \hat{\sigma}'(\hat{u})}, 0 \leq \eta \leq \delta \text{ and } \hat{u}(\delta) = u. \quad (3.8)$$

The pair (η, q) satisfies

$$\eta_t - q_x = -\frac{2c}{\epsilon} \delta \mathcal{P}_\delta(\delta, u) \quad (3.9)$$

where

$$\delta \stackrel{\text{def}}{=} \sigma - \hat{\sigma}(u). \quad (3.10)$$

It is easily verified that so long as $c^2 > \hat{\sigma}'(\hat{u}(\eta; u, \delta))$, $\mathcal{P}$ and $\delta \mathcal{P}_\delta$ satisfy

$$\frac{\delta^2}{(c^2 - \min \hat{\sigma}')} \leq \mathcal{P} \text{ and } \delta \mathcal{P}_\delta \leq \frac{\delta^2}{2(c^2 - \max \hat{\sigma}')}. \quad (3.11)$$

The identity (3.9) and inequality (3.11) provide us with basic L_2 and decay estimates that imply that as $\epsilon \rightarrow 0^+$ $\sigma \rightarrow \hat{\sigma}(u)$. If one imposes convexity requirements on $\hat{\sigma}$ one can find other functionals which are positive definite and nonincreasing on solutions of (3.2) or equivalently (3.4). Some of these functionals distinguish between solutions with shocks ($v_x < 0$) and those with rarefaction waves ($v_x > 0$).²

²This nomenclature is applicable when $\hat{\sigma}''(u) < 0$.

The integration schemes we employ are basically advection-relaxation schemes for the semi-linear system (3.2) (or equivalently (3.4)). We confine ourselves to the situation where initial data is given on $x > 0$; i.e.,

$$(u, v, \sigma)(x, 0^+) = (u_0, v_0, \hat{\sigma}(u_0))(x) \quad (3.12)$$

and

$$(u_0, v_0, \hat{\sigma}(u_0))(x) \equiv (u_+, v_+, \hat{\sigma}(u_+)), \quad x > L_0 \quad (3.13)$$

and we look for a solution of (3.4) in $x > 0$ and $t > 0$ satisfying (3.12) and a boundary condition at $x = 0$ of the type

$$v(0, t) = \bar{v}(t) \text{ or } \sigma(0, t) = \bar{\sigma}(t) \text{ or } u(0, t) = \bar{u}(t). \quad (3.14)$$

We note that in the region

$$x > L(t) \stackrel{\text{def}}{=} L_0 + ct, \quad t > 0 \quad (3.15)$$

the solution will take on the constant values defined in (3.13) and that on the curve: $x = L(t) = L_0 + ct$, l and d satisfy

$$\left. \begin{aligned} l_-(t) &= \lim_{x \rightarrow L^-(t)} l(x, t) = \frac{1}{2} \left(\frac{\hat{\sigma}(u_+)}{c^2} + \frac{v_+}{c} \right) \\ \text{and} \\ d_-(t) &= \lim_{x \rightarrow L^-(t)} d(x, t) = \frac{1}{2} \left(u_+ - \frac{\hat{\sigma}(u_+)}{c^2} \right) \end{aligned} \right\} \quad (3.16)$$

Moreover, if the initial data is continuous and compatible with the boundary conditions (3.14), then the same will be true for all future times and additionally on: $x = L(t) = L_0 + ct$ r will satisfy

$$r_-(t) = \lim_{x \rightarrow L^-(t)} r(v, t) = \frac{1}{2} \left(\frac{\hat{\sigma}(u_+)}{c^2} - \frac{v_+}{c} \right). \quad (3.17)$$

Since all the variation in u, v , and σ will be confined to the cone $0 < x < L(t)$ we again introduce the transformation

$$\xi = \frac{x}{L(t)} \text{ and } \tau = t \quad (3.18)$$

where $L(t)$ is given by (3.15). The system (3.2) then becomes

$$\left. \begin{aligned} r_\tau - \frac{L\xi}{L} r_\xi + \frac{c}{L} r_\xi &= \frac{\hat{\sigma}(u) - \sigma}{2\epsilon} \\ l_\tau - \frac{L\xi}{L} l_\xi - \frac{c}{L} l_\xi &= \frac{\hat{\sigma}(u) - \sigma}{2\epsilon} \\ d_\tau - \frac{L\xi}{L} d_\xi &= \frac{\sigma - \hat{\sigma}(u)}{\epsilon} \end{aligned} \right\} \quad (3.19)$$

where again

$$\left. \begin{aligned} u &= l + r + 2d \\ v &= c(l - r) \\ \sigma &= c^2(l + r) \end{aligned} \right\} \quad (3.20)$$

or equivalently

$$\left. \begin{aligned} l &= \frac{1}{2} \left(\frac{\sigma}{c^2} + \frac{v}{c} \right) \\ r &= \frac{1}{2} \left(\frac{\sigma}{c^2} - \frac{v}{c} \right) \\ d &= \frac{1}{2} \left(u - \frac{\sigma}{c^2} \right) \end{aligned} \right\} \quad (3.21)$$

The relation between the time step $(\Delta\tau)_{n+1} = \tau_{n+1} - \tau_n$ and space step $(\Delta\xi)$ is now taken to be

$$\frac{(\Delta\tau)_{n+1}c}{L_{n+1}(\Delta\xi)} = 1 \quad (3.22)$$

and this yields

$$L_{n+1} = \frac{L_n}{(1 - \Delta\xi)} \text{ and } \tau_{n+1} = \tau_n + \frac{L_{n+1}\Delta\xi}{c}. \quad (3.23)$$

We use a three stage splitting to integrate (3.19). During the regrid stage we solve

$$\tau_\tau - \frac{\dot{L}\xi}{L} \tau_\xi = 0, \quad l_\tau - \frac{\dot{L}\xi}{L} l_\xi = 0, \quad \text{and} \quad d_\tau - \frac{\dot{L}\xi}{L} d_\xi = 0 \quad (3.24)$$

with data (r_k^n, l_k^n, d_k^n) given on the uniform grid $\xi_k = k\Delta\xi$, $k = 0, 1, \dots, N$ with $N\Delta\xi = 1$. These data passively advect to the grid $y_k = (1 - \Delta\xi)\xi_k$ at time τ_{n+1} . These grid values are then linearly interpolated to the original grid $\xi_k = k\Delta\xi$, $k = 0, 1, \dots, N - 1$ and we denote them by $(r_k^{n+1/3}, l_k^{n+1/3}, d_k^{n+1/3})$, $k = 0, 1, \dots, N - 1$. By virtue of (3.16) and (3.17) we assign the values $(r_N^{n+1/3}, l_N^{n+1/3}, d_N^{n+1/3})$ to be

$$\left(\frac{1}{2} \left(\frac{\hat{\sigma}(u_+)}{c^2} - \frac{v_+}{c} \right), \frac{1}{2} \left(\frac{\hat{\sigma}(u_+)}{c^2} + \frac{v_+}{c} \right), \frac{1}{2} (u_+ - \hat{\sigma}(u_+)) \right).$$

In the second step we advect these values passively via the equations

$$r_\tau + \frac{c}{L_{n+1}} r_\xi = 0, \quad l_\tau - \frac{c}{L_{n+1}} l_\xi, \quad \text{and} \quad d_\tau = 0, \quad \tau_n < \tau < \tau_{n+1}.$$

The choice of $(\Delta\tau)_{n+1}$ given by (3.22) implies that

$$d_k^{n+2/3} = d_k^{n+1/3}, \quad 0 \leq k \leq N$$

and

$$l_k^{n+2/3} = \begin{cases} l_{k+1}^{n+1/3}, & 0 \leq k \leq N-1 \\ \frac{1}{2} \left(\frac{\hat{\sigma}(u_+)}{c^2} + \frac{v_+}{c} \right), & k = N \end{cases} \quad (3.25)$$

If we assume, for definiteness, that the boundary condition (3.14) holds then

$$r_k^{n+2/3} = \begin{cases} l_2^{n+1/3} - \bar{v}(\tau_n), & k = 1 \\ r_{k-1}^{n+1/3}, & 2 \leq k \leq N \end{cases} \quad (3.26)$$

In the final stage we solve the relaxation equations

$$u_\tau = 0, \quad v_\tau = 0, \quad \text{and} \quad \sigma_\tau = \frac{c^2(\hat{\sigma}(u) - \sigma)}{\epsilon} \quad (3.27)$$

with data from the previous stage. This guarantees that

$$\text{and} \quad \left. \begin{aligned} u_k^{n+1} &= l_k^{n+2/3} + r_k^{n+2/3} + 2d_k^{n+2/3} \\ v_k^{n+1} &= c(l_k^{n+2/3} - r_k^{n+2/3}) \end{aligned} \right\} \quad 0 \leq k \leq N. \quad (3.28)$$

The beauty of this process is we take $(\Delta\tau)^{n+1} \gg \epsilon$ and therefore set

$$\sigma_k^{n+1} = \hat{\sigma}(u_k^{n+1}), \quad 0 \leq k \leq N. \quad (3.29)$$

These methods seem quite competitive in obtaining the large time behavior of the original system with a fixed degree of freedom algorithm. They also have advantages similar to those of the methods discussed in Section 2 in that they do not require the use of complicated and time consuming Riemann solvers. It remains an open question as to whether these ideas generalize to several space dimensions; specifically whether given a system such as gas dynamics in 2 or 3 dimensions we can come up with a semi-linear system which is easy to integrate and which reduces to the original one as a relaxation parameter tends to zero. This is one of the projects we plan to work on in the coming year.

These relaxation methods also yield interesting results for non-monotone $\hat{\sigma}$'s which may be used to model simple phase changes. In the example shown in Figures 11 through 13 we considered the case where

$$\hat{\sigma}(u) = u(u^2 - 1). \quad (3.30)$$

This corresponds to the double well potential energy

$$\mathcal{E}(u) = \int_1^u (\hat{\sigma}(s) - \hat{\sigma}(1)) ds = \frac{(u^2 - 1)^2}{4}. \quad (3.31)$$

The specific problem under consideration was the system (3.4) and (3.5) with initial data

$$(u, \sigma, v)(x, 0) = (1, 0, -1), \quad x > 0 \quad (3.32)$$

and boundary conditions

$$v(0^+, t) = 0. \quad (3.33)$$

The structure of the wave generated by our computations is described below. We get a precursor rarefaction wave emanating from $(0, 0)$. The leading edge of this wave travels at the ambient sound speed, $\sqrt{2}$, and the trailing edge terminates at the constant state $(u_1, \sigma_1, v_1) = (.6103, -.3830, -.6273)$. The constant state labeled 1 is joined to the state $(u_2, \sigma_2, v_2) = (-1.2213, -.6002, 0)$ by a change of phase wave which propagates with speed $s = .3425$. The speed of this change of phase wave is the same as the speed of the trailing rarefaction wave; that is

$$s = .3425 = \sqrt{\frac{\sigma_2 - \sigma_1}{u_2 - u_1}} = \sqrt{3u_1^2 - 1}. \quad (3.34)$$

Moreover $\sigma_1 = \hat{\sigma}(u_1)$ and $\sigma_2 = \hat{\sigma}(u_2)$ where $\hat{\sigma}(\cdot)$ is defined in (3.30). That such change of phase waves are predicted by the relaxation model is a surprising result. The conventional wisdom for dissipative models is that phase changes take place at constant stress. One of our tasks in the coming year is a detailed analysis of this problem with the goal of explaining the results reported above. We also plan a series of experiments to determine how the speed of the change of phase wave varies with the end states (u_1, σ_1, v_1) and (u_2, σ_2, v_2) and the model parameter c .

The top portrait in Figures 11-13 represent the profiles of u, σ , and v in the self similar variable ξ and the bottom portraits show the difference between $\sigma^{n+2/3} = c^2(l^{n+2/3} + r^{n+2/3})$ and $\hat{\sigma}(u^{n+1})$. This difference represents the discrepancy between $\sigma^{n+2/3}$ and its equilibrium values. The vertical line in each portrait is the location of the maximum value of this discrepancy and is the ξ value of the shock. The speed of the shock is equal to this ξ value, namely .0856, times the speed at which the domain is expanding, namely 4.

Figure 11

Figure 12

Figure 13

4 Materials Processing.

Last summer the PI initiated a project with Yves Demay of the Institute de Nonlineare de Nice (INLN) on extrusion flows for a two fluid system. The two fluids are assumed to be linearly viscous and have the same density but different viscosities. Steady Poiseuille Flows which are independent of the distance along the pipe are known for such systems. For these solutions the less viscous fluid entrains the more viscous and these solutions satisfy a minimum energy principle. Our goals were to develop an approximate "evolutionary equation" for this system where the distance down the pipe plays the role of the "time like" variable and to use these equations to obtain an estimate for the distance down the pipe required to go from an arbitrary inlet configuration where the less viscous fluid is not entraining the more viscous one to the steady entraining Poiseuille flow. This work is ongoing. To date we have implemented this program for a one fluid system in a pipe of circular cross-section. We have shown that if the underlying Poiseuille flow is $(u, v, w) = (0, 0, \frac{G_0(1-r^2)}{4})$, then the decay to this solution goes as $e^{\frac{-64z}{G_0}}$ where z is the distance down the tube from the inlet. Our goal in the two fluid project are decay estimates similar to those in the one fluid case.