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ITEM 1

The multiphase equations in 1-d

Notation:

ϕ_a = volume fraction of phase 'a'
 ρ_a = partial density of phase 'a'
 $\rho_a \phi_a$ = density of phase 'a'
 τ_a = partial pressure of phase 'a'
 p_a = phase pressure of phase 'a'
 ρ = mixture density
 $\sum \rho_a = \rho$

Equations:

$$\phi_g + \phi_l = 1$$

(1)

$[v_a = \text{particle velocity of phase 'a'}$
 $v = \text{mixture velocity} = (\rho_l v_l + \rho_g v_g) / \rho]$

Mass

$$\frac{\partial \rho_a}{\partial t} + v_a \frac{\partial \rho_a}{\partial x} + \rho_a \frac{\partial v_a}{\partial x} = 0 \quad (2)$$

Momentum

$$\rho_a \left[\frac{\partial v_a}{\partial t} + v_a \frac{\partial v_a}{\partial x} \right] = - \frac{\partial \tau_a}{\partial x} + \rho_a g_a - M_a - C_a v_a \quad (3)$$

Energy

$$\rho_a \left[\frac{\partial e_a}{\partial t} + v_a \frac{\partial e_a}{\partial x} \right] = - \tau_a \frac{\partial v_a}{\partial x} - \frac{\partial q_a}{\partial x} + \rho_a \dot{e}_a + \dot{e}_a -$$

$$- m_a v_a - (C_a v_a^2) C_a \quad (4)$$

Additional notation:

b_a = body force / mass

C_a = mass source for phase 'a', mass / vol. - time

M_a = momentum source for phase 'a', mom. / vol. - time

e_a = specific internal energy

q_a = heat flux

r_a = external energy source, energy / mass - time

$\bar{E}_a$ = internal energy source, energy / vol. - time

- Notes:
1. The balance equations follow those for a single fluid.
 2. The summing rules $C_g + C_s = 0$, $M_g + M_s = 0$, $\bar{E}_g + \bar{E}_s = 0$, hold.

The above eqns. are supplemented by the compaction eqn.

$$\frac{\partial \phi_s}{\partial t} + v_s \frac{\partial \phi_s}{\partial x} = F + \frac{C_s}{\gamma_s}$$

Constitutive information:

$$F = \frac{1}{\mu_c} \phi_s \phi_g (p_s - p_g - \beta)$$

$$q_a = -k_a \frac{\partial T_a}{\partial x}$$

$$M_p = p_g \frac{\partial \phi_s}{\partial x} - D + C_s v_s$$

$$D = \left(\delta + \frac{1}{2} C_s \right) (v_p - v_g)$$

$$\begin{aligned} \bar{E}_p = & \left[p_g \frac{\partial \phi_s}{\partial x} - D \right] v_p - (p_s - \beta) F + H \\ & + C_s \left(e_s + \frac{1}{2} v_s^2 \right) \end{aligned}$$

$$H = h (T_g - T_p)$$

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Additional notation still :

δ = drag coefficient

h = heat transfer coefficient

k_a = thermal conductivity

μ_c = compaction viscosity

β_s = solid configuration pressure

Notes: 1. The choices made above for the constitutive eqns. are sufficient (but not necessary) to ensure the satisfaction of the entropy inequality. This is where the model is at its weakest.

2. The configuration pressure β is a measure of the contact forces between the solid grains.

3. When μ_c vanishes, the compaction equation reduces to $p_s = p_g + \beta$.

Let us now suppose that

$$b_a = v_a = 0.$$

(1)

Then, by using the constitutive information, the balance laws can be rewritten.

Solid mass balance:

$$\frac{\partial}{\partial t} (\phi_s \gamma_s) + v_s \frac{\partial}{\partial x} (\phi_s \gamma_s) = C_s - \phi_s \gamma_s \frac{\partial v_s}{\partial x}$$

$$\Rightarrow \phi_s \left\{ \frac{\partial \gamma_s}{\partial t} + v_s \frac{\partial \gamma_s}{\partial x} \right\} + \gamma_s \left\{ \frac{\partial \phi_s}{\partial t} + v_s \frac{\partial \phi_s}{\partial x} \right\} = - \phi_s \gamma_s \frac{\partial v_s}{\partial x}$$

Use of (2.1) $\Rightarrow$

$$\frac{\partial \gamma_s}{\partial t} + v_s \frac{\partial \gamma_s}{\partial x} = - \frac{\gamma_s}{\phi_s} F - \gamma_s \frac{\partial v_s}{\partial x} \quad (2)$$

Gas mass balance:

$$\frac{\partial}{\partial t} (\phi_g \gamma_g) + v_g \frac{\partial}{\partial x} (\phi_g \gamma_g) + \phi_g \gamma_g \frac{\partial v_g}{\partial x} = -C_p$$

$$\Rightarrow \phi_g \left\{ \frac{\partial \gamma_g}{\partial t} + v_g \frac{\partial \gamma_g}{\partial x} \right\} + \gamma_g \left\{ \frac{\partial \phi_g}{\partial t} + v_g \frac{\partial \phi_g}{\partial x} \right\} = -C_p - \phi_g \gamma_g \frac{\partial v_g}{\partial x}$$

On using $\dot{\phi}_g = -\dot{\phi}_p$, $\Rightarrow$

$$\frac{\partial \gamma_g}{\partial t} + v_g \frac{\partial \gamma_g}{\partial x} - \frac{\gamma_g}{\phi_g} \left\{ \frac{\partial \phi_p}{\partial t} + v_p \frac{\partial \phi_p}{\partial x} + (v_g - v_p) \frac{\partial \phi_p}{\partial x} \right\} = -\frac{C_p}{\phi_g} - \gamma_g \frac{\partial v_g}{\partial x}$$

On using (2.1), $\Rightarrow$

$$\frac{\partial \gamma_g}{\partial t} + v_g \frac{\partial \gamma_g}{\partial x} = -\gamma_g \frac{\partial v_g}{\partial x} - \frac{\gamma_g}{\phi_g} (v_p - v_g) \frac{\partial \phi_p}{\partial x} + \frac{\gamma_g}{\phi_g} \left[-\left(1 - \frac{\gamma_g}{\gamma_p}\right) \frac{C_p}{\phi_g} \right]$$

Solid momentum balance:

$$\phi_p \gamma_p \left(\frac{\partial v_p}{\partial t} + v_p \frac{\partial v_p}{\partial x} \right) = -\phi_p \frac{\partial p_p}{\partial x} - p_p \frac{\partial \phi_p}{\partial x} + p_g \frac{\partial \phi_p}{\partial x} - D$$

(on using D) $\Rightarrow$

$$\phi_p \gamma_p \left(\frac{\partial v_p}{\partial t} + v_p \frac{\partial v_p}{\partial x} \right) = -\phi_p \frac{\partial p_p}{\partial x} + (p_g - p_p) \frac{\partial \phi_p}{\partial x} - \left(\delta + \frac{1}{2} C_p \right) \phi_p$$

Gas momentum balance:

$$\phi_g \gamma_g \left(\frac{\partial v_g}{\partial t} + v_g \frac{\partial v_g}{\partial x} \right) = -\phi_g \frac{\partial p_g}{\partial x} - p_g \frac{\partial \phi_g}{\partial x} - m_p + C_p$$

$$= -\phi_g \frac{\partial p_g}{\partial x} + D - C_p v_p + C_p v_g$$

$\Rightarrow$

$$\phi_g \gamma_g \left(\frac{\partial v_g}{\partial t} + v_g \frac{\partial v_g}{\partial x} \right) = - \phi_g \frac{\partial p_g}{\partial x} + \left(\delta - \frac{1}{2} C_p \right) (v_p - v_g). \quad (1)$$

Solid energy balance:

$$\begin{aligned} \phi_p \gamma_p \left(\frac{\partial e_p}{\partial t} + v_p \frac{\partial e_p}{\partial x} \right) = & - \phi_p p_p \frac{\partial v_p}{\partial x} + \frac{\partial}{\partial x} \left(k_p \frac{\partial T_p}{\partial x} \right) \\ & + \dot{E}_p - m_p v_p - \left(e_p - \frac{1}{2} v_p^2 \right) C_p. \end{aligned}$$

Now,

$$\begin{aligned} & \dot{E}_p - m_p v_p - \left(e_p - \frac{1}{2} v_p^2 \right) C_p \\ & = \left[p_g \frac{\partial \phi_p}{\partial x} - \left(\delta + \frac{1}{2} C_p \right) (v_p - v_g) \right] v_p - (p_p - p_g) F \\ & \quad + H + C_p \left(e_p + \frac{1}{2} v_p^2 \right) - \left[p_g \frac{\partial \phi_p}{\partial x} - D \right] v_p - C_p v_p^2 \\ & \quad - \left(e_p - \frac{1}{2} v_p^2 \right) C_p \\ & = h(T_g - T_p) - (p_p - p_g) F \end{aligned}$$

∴ solid energy balance →

$$\begin{aligned} \phi_p \gamma_p \left(\frac{\partial e_p}{\partial t} + v_p \frac{\partial e_p}{\partial x} \right) = & - \phi_p p_p \frac{\partial v_p}{\partial x} + \frac{\partial}{\partial x} \left(k_p \frac{\partial T_p}{\partial x} \right) \\ & - h(T_p - T_g) - (p_p - p_g) F. \end{aligned} \quad (2)$$

Gas energy balance:

$$\begin{aligned} \phi_g \gamma_g \left(\frac{\partial e_g}{\partial t} + v_g \frac{\partial e_g}{\partial x} \right) = & - \phi_g p_g \frac{\partial v_g}{\partial x} + \frac{\partial}{\partial x} \left(k_g \frac{\partial T_g}{\partial x} \right) \\ & - \dot{E}_g + m_p v_g + \left(e_g - \frac{1}{2} v_g^2 \right) C_p \\ & = - \phi_g p_g \frac{\partial v_g}{\partial x} + \frac{\partial}{\partial x} \left(k_g \frac{\partial T_g}{\partial x} \right) + h(T_p - T_g) + (p_p - p_g) F \\ & - (v_p - v_g) p_g \frac{\partial \phi_p}{\partial x} + \delta (v_p - v_g)^2 - (e_p - e_g) C_p. \end{aligned} \quad (3)$$

Equations of State:

So far we have 12 unknowns ($e_a, T_a, v_a, \gamma_a, p_a$, and 8 equations. The deficit is made up by the equation of state. If one uses the canonical eqns. of state, then

$$e_a = e_a(\eta_a, \gamma_a)$$

where η_a is the entropy, whence

$$T_a = \frac{\partial e_a}{\partial \eta_a}, \quad p_a = \gamma_a^2 \frac{\partial e_a}{\partial \gamma_a}.$$

Alternatively one could use a thermal equation of state

$$p_a = p_a(\gamma_a, T_a)$$

and a caloric equation of state,

$$e_a = e_a(\gamma_a, T_a).$$

Let us choose the latter path and set

$$e_a = c_{va} T_a, \quad p_a = \Gamma_a c_{va} \gamma_a T_a \quad (1)$$

Here Γ_a is the Gruneisen coefficient. (Essentially, each phase is treated like an ideal gas.)

We also treat the contact stress β_p to be a prescribed function of ϕ_p :

$$\beta_p = \beta_p(\phi_p). \quad (2)$$

Summary of equations at this stage:

Mass

$$\frac{\partial \gamma_p}{\partial t} + v_p \frac{\partial \gamma_p}{\partial x} = - \frac{\gamma_p}{\phi_p} F - \gamma_p \frac{\partial v_p}{\partial x} \quad \frac{\partial \phi_p}{\partial x} \quad (1)$$

$$\frac{\partial \gamma_g}{\partial t} + v_g \frac{\partial \gamma_g}{\partial x} = + \frac{\gamma_g}{\phi_g} F - \gamma_g \frac{\partial v_g}{\partial x} - \frac{\gamma_g}{\phi_g} (v_p - v_g) - \left(1 - \frac{\gamma_g}{\gamma_p}\right) \frac{C_p}{\phi_g} \quad (2)$$

Momentum

$$\phi_p \gamma_p \left(\frac{\partial v_p}{\partial t} + v_p \frac{\partial v_p}{\partial x} \right) = - \phi_p \frac{\partial p_p}{\partial x} + (p_g - p_p) \frac{\partial \phi_p}{\partial x} - \left(\delta + \frac{1}{2} C_p \right) \times (v_p - v_g) \quad (3)$$

$$\phi_g \gamma_g \left(\frac{\partial v_g}{\partial t} + v_g \frac{\partial v_g}{\partial x} \right) = - \phi_g \frac{\partial p_g}{\partial x} + \left(\delta - \frac{1}{2} C_p \right) (v_p - v_g) \quad (4)$$

Energy

$$\phi_p \gamma_p \left(\frac{\partial e_p}{\partial t} + v_p \frac{\partial e_p}{\partial x} \right) = - \phi_p p_p \frac{\partial v_p}{\partial x} + \frac{\partial}{\partial x} \left(k_p \frac{\partial T_p}{\partial x} \right) - h (T_p - T_g) - (p_p - p_g) F \quad (5)$$

$$\begin{aligned} \phi_g \gamma_g \left(\frac{\partial e_g}{\partial t} + v_g \frac{\partial e_g}{\partial x} \right) = & - \phi_g p_g \frac{\partial v_g}{\partial x} + \frac{\partial}{\partial x} \left(k_g \frac{\partial T_g}{\partial x} \right) \\ & + h (T_p - T_g) + (p_p - p_g) F \\ & - (v_p - v_g) p_g \frac{\partial \phi_p}{\partial x} + \delta (v_p - v_g)^2 \\ & - (e_p - e_g) C_p. \end{aligned} \quad (6)$$

Compaction

$$\frac{\partial \phi_p}{\partial t} + v_p \frac{\partial \phi_p}{\partial x} = F + \frac{C_p}{\gamma_p} \quad (7)$$

$$F = \frac{\phi_p \phi_g}{\mu_c} [\beta_p - \beta_g - \beta_g] \quad (1)$$

$$\phi_g = 1 - \phi_p \quad (2)$$

$$c_a = c_{va} T_a, \quad p_a = T_a c_{va} \gamma_a T_a \quad (3)$$

It remains to specify C_p . The units of C_p are mass/vol.-time. Let us assume that

$$C_p = K \gamma_p \Phi \quad (4)$$

where Φ is a yet unspecified, dimensionless function and K has units of 1/time.

$$\text{sonic speed } a_p^u = (\Gamma_p + 1) \frac{p_p}{\gamma_p} \quad (5)$$

$$a_g^u = (\Gamma_g + 1) \frac{p_g}{\gamma_g} \quad (6)$$

Nondimensionalization

Reference quantities:

$$\bar{p}, \bar{r}, \bar{v} = \sqrt{\bar{p}/\bar{r}}, \bar{t}, \bar{x} = \bar{v}\bar{t}, \bar{\tau}$$

Dimensionless parameters:

$$\tilde{M}_c = \frac{M_c}{\bar{p}\bar{t}},$$

$$\tilde{h} = \bar{t}K$$

$$\tilde{\delta} = \frac{\bar{t}}{\bar{r}}\delta,$$

$$\tilde{C}_{va} = \frac{\bar{r}\bar{\tau}}{\bar{p}} C_{va}$$

$$\tilde{k}_a = \frac{\bar{\tau}}{\bar{p}\bar{v}\bar{x}} k_a$$

$$\tilde{h} = \frac{\bar{\tau}\bar{x}}{\bar{p}\bar{v}} h$$

Dimensionless Equations

$$F = \frac{\phi_p \phi_g}{\tilde{\mu}_c} (p_p - p_g)$$

$$\phi_g = 1 - \phi_p$$

$$p_a = \Gamma_a \tilde{c}_{va} \gamma_a T_a$$

mass

$$\frac{\partial \gamma_p}{\partial t} + v_p \frac{\partial \gamma_p}{\partial x} = - \frac{\gamma_p}{\phi_p} F - \gamma_p \frac{\partial v_p}{\partial x}$$

$$\begin{aligned} \frac{\partial \gamma_g}{\partial t} + v_g \frac{\partial \gamma_g}{\partial x} &= \frac{\gamma_g}{\phi_g} F - \gamma_g \frac{\partial v_g}{\partial x} - (\tilde{c}_{vp} - v_g) \frac{\partial \phi_p}{\partial x} \\ &\quad - \frac{1}{\phi_g} \left(1 - \frac{\gamma_g}{\gamma_p}\right) \tilde{\mu} \gamma_p \Phi \end{aligned}$$

momentum

$$\begin{aligned} \phi_p \gamma_p \left(\frac{\partial v_p}{\partial t} + v_p \frac{\partial v_p}{\partial x} \right) &= - \phi_p \frac{\partial p_p}{\partial x} + \left(\tilde{c}_{vp} - \frac{1}{2} \tilde{\mu} \gamma_p \Phi \right) (v_p - v_g) \\ &\quad - \left(\tilde{\delta} + \frac{1}{2} \tilde{\mu} \gamma_p \Phi \right) (v_p - v_g) \end{aligned}$$

$$\phi_g \gamma_g \left(\frac{\partial v_g}{\partial t} + v_g \frac{\partial v_g}{\partial x} \right) = - \phi_g \frac{\partial p_g}{\partial x} + \left(\tilde{\delta} - \frac{1}{2} \tilde{\mu} \gamma_p \Phi \right) (v_p - v_g)$$

energy

$$\begin{aligned} \phi_p \gamma_p \tilde{c}_{vp} \left(\frac{\partial T_p}{\partial t} + v_p \frac{\partial T_p}{\partial x} \right) &= - \phi_p p_p \frac{\partial v_p}{\partial x} + \frac{\partial}{\partial x} \left(\tilde{k}_p \frac{\partial T_p}{\partial x} \right) \\ &\quad - \tilde{h} (T_p - T_g) - (\tilde{c}_{vp} - \frac{1}{2} \tilde{\mu} \gamma_p \Phi) F \end{aligned}$$

LHS can also be written as

$$\phi_p \gamma_p \left(\frac{\partial e_p}{\partial t} + v_p \frac{\partial e_p}{\partial x} \right)$$

$$\begin{aligned}
\phi_g \gamma_g \tilde{C}_{vg} \left(\frac{\partial T_g}{\partial t} + v_g \frac{\partial T_g}{\partial x} \right) = & - \phi_g p_g \frac{\partial v_g}{\partial x} + \frac{\partial}{\partial x} \left(\tilde{k}_g \frac{\partial T_g}{\partial x} \right) \\
& + \tilde{k} (T_p - T_g) + (p_p - p_g) F + (v_p - v_g) p_g \frac{\partial \phi_p}{\partial x} \\
& + \delta (v_p - v_g)^2 - (\tilde{C}_{vp} h - \tilde{C}_{vg} T_p) \tilde{k} \gamma_p \Phi \quad (1)
\end{aligned}$$

compaction

$$\frac{\partial \phi_p}{\partial t} + v_p \frac{\partial \phi_p}{\partial x} = F + \tilde{k} \Phi \quad (2)$$

We now choose the basic variables to be

$$\underline{u} = (\gamma_p, v_p, p_p, \phi_p, \gamma_g, v_g, p_g) \quad (3)$$

and attempt to write the above system as

$$\frac{\partial \underline{u}}{\partial t} + \underline{A}(\underline{u}) \frac{\partial \underline{u}}{\partial x} = \underline{R}(\underline{u}) + \frac{\partial}{\partial x} \left(\underline{B}(\underline{u}) \frac{\partial \underline{u}}{\partial x} \right). \quad (4)$$

First, note that

$$p_p = \Gamma_a \tilde{C}_{va} \gamma_a T_a \Rightarrow$$

$$dp_p = \Gamma_a \tilde{C}_{va} (\gamma_a dT_a + T_a d\gamma_a)$$

Therefore,

$$\begin{aligned}
\frac{\partial p_p}{\partial t} + v_p \frac{\partial p_p}{\partial x} = & \Gamma_p \tilde{C}_{vp} \gamma_p \left(\frac{\partial T_p}{\partial t} + v_p \frac{\partial T_p}{\partial x} \right) \\
& + \Gamma_p \tilde{C}_{vp} T_p \left(\frac{\partial \gamma_p}{\partial t} + v_p \frac{\partial \gamma_p}{\partial x} \right)
\end{aligned}$$

On using (10.4) and (10.8),

$$\begin{aligned}
 \frac{\partial p_b}{\partial t} + v_b \frac{\partial p_b}{\partial x} &= \frac{\Gamma_b}{\Phi_b} \left[-\Phi_b p_b \frac{\partial v_b}{\partial x} + \frac{\partial}{\partial x} \left(\tilde{k}_b \frac{\partial T_b}{\partial x} \right) \right. \\
 &\quad \left. - \tilde{h} (T_b - T_g) - (p_b - \beta_b) F \right] \\
 &\quad + \Gamma_b \tilde{C}_{v_b} T_b \left[-\frac{\gamma_b}{\Phi_b} F - \gamma_b \frac{\partial v_b}{\partial x} \right] \\
 &= - (1 + \Gamma_b) \frac{\partial v_b}{\partial x} + \frac{\Gamma_b}{\Phi_b} \left[-\tilde{h} (T_b - T_g) - (p_b - \beta_b) F \right] \\
 &\quad - \frac{\Gamma_b}{\Phi_b} F + \frac{\Gamma_b}{\Phi_b} \frac{\partial}{\partial x} \left(\tilde{k}_b \frac{\partial T_b}{\partial x} \right)
 \end{aligned}$$

or,

$$\begin{aligned}
 \frac{\partial p_b}{\partial t} + v_b \frac{\partial p_b}{\partial x} + (1 + \Gamma_b) p_b \frac{\partial v_b}{\partial x} &= - \frac{\Gamma_b}{\Phi_b} \tilde{h} (T_b - T_g) \\
 &\quad - \frac{F}{\Phi_b} \left[(\Gamma_b + 1) p_b - \Gamma_b \beta_b \right] + \frac{\Gamma_b}{\Phi_b} \frac{\partial}{\partial x} \left(\tilde{k}_b \frac{\partial T_b}{\partial x} \right)
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \frac{\partial p_g}{\partial t} + v_g \frac{\partial p_g}{\partial x} &= \frac{\Gamma_g}{\Phi_g} \left[-\Phi_g p_g \frac{\partial v_g}{\partial x} + \frac{\partial}{\partial x} \left(\tilde{k}_g \frac{\partial T_g}{\partial x} \right) \right. \\
 &\quad \left. + \tilde{h} (T_b - T_g) + (p_g - \beta_g) F - (v_b - v_g) p_g \frac{\partial \Phi_g}{\partial x} \right. \\
 &\quad \left. + \tilde{E} (v_b - v_g)^2 - (\tilde{C}_{v_b} T_b - \tilde{C}_{v_g} T_g) \tilde{h} \gamma_b \Phi \right] \\
 &\quad + \Gamma_g \tilde{C}_{v_g} T_g \left[\frac{\gamma_g}{\Phi_g} F - \gamma_g \frac{\partial v_g}{\partial x} - \frac{\gamma_g}{\Phi_g} (v_b - v_g) \frac{\partial \Phi_g}{\partial x} - \left(1 - \frac{\gamma_g}{\gamma_b} \right) \frac{\tilde{h} \gamma_b \Phi}{\Phi_g} \right]
 \end{aligned}$$

$$\begin{aligned}
& \frac{\partial p_g}{\partial t} + v_g \frac{\partial p_g}{\partial x} + (1+\Gamma_g) p_g \frac{\partial v_g}{\partial x} + (1+\Gamma_g) \frac{p_g}{\phi_g} (v_p - v_g) \frac{\partial \phi_g}{\partial x} \\
&= \frac{\Gamma_g}{\phi_g} \left[\tilde{k} (T_p - T_g) + \tilde{\delta} (v_p - v_g)^2 + (p_p - p_g) F \right. \\
&\quad \left. - (\tilde{C}_{vp} T_p - \tilde{C}_{vg} T_g) \tilde{k} \gamma_p \Phi \right] \\
&+ \frac{p_g}{\phi_g} F - \frac{1}{\phi_g} \Gamma_g \tilde{C}_{vg} T_g \left(1 - \frac{\gamma_g}{\gamma_p} \right) \tilde{k} \gamma_p \Phi \\
&+ \frac{\Gamma_g}{\phi_g} \frac{\partial}{\partial x} \left(\tilde{k}_g \frac{\partial T_g}{\partial x} \right). \quad (1)
\end{aligned}$$

We can now write down the matrix A introduced in (11.4):

First, rewrite the governing eqns. in order as

$$\begin{aligned}
& \frac{\partial \gamma_p}{\partial t} + v_p \frac{\partial \gamma_p}{\partial x} + \gamma_p \frac{\partial v_p}{\partial x} = R_1 \\
& \frac{\partial v_p}{\partial t} + v_p \frac{\partial v_p}{\partial x} + \frac{1}{\gamma_p} \frac{\partial p_p}{\partial x} + \frac{p_p - p_g}{\phi_p \gamma_p} \frac{\partial \phi_g}{\partial x} = R_2 \\
& \frac{\partial p_p}{\partial t} + v_p \frac{\partial p_p}{\partial x} + (1+\Gamma_p) p_p \frac{\partial v_p}{\partial x} = R_3 + \frac{\Gamma_p}{\phi_p} \frac{\partial}{\partial x} \left(\tilde{k}_p \frac{\partial T_p}{\partial x} \right) \\
& \frac{\partial \phi_p}{\partial t} + v_p \frac{\partial \phi_p}{\partial x} = R_4 \\
& \frac{\partial \gamma_g}{\partial t} + v_g \frac{\partial \gamma_g}{\partial x} + \gamma_g \frac{\partial v_g}{\partial x} + \frac{\gamma_g}{\phi_g} (v_p - v_g) \frac{\partial \phi_g}{\partial x} = R_5 \\
& \frac{\partial v_g}{\partial t} + v_g \frac{\partial v_g}{\partial x} + \frac{1}{\gamma_g} \frac{\partial p_g}{\partial x} = R_6 \quad \frac{\partial}{\partial x} \left(\tilde{k}_g \frac{\partial T_g}{\partial x} \right) \\
& \frac{\partial p_g}{\partial t} + v_g \frac{\partial p_g}{\partial x} + (1+\Gamma_g) p_g \frac{\partial v_g}{\partial x} + (1+\Gamma_g) \frac{p_g}{\phi_g} (v_p - v_g) \frac{\partial \phi_g}{\partial x} = R_7 + \frac{\Gamma_g}{\phi_g} \left(\frac{\partial}{\partial x} \left(\tilde{k}_g \frac{\partial T_g}{\partial x} \right) \right)
\end{aligned}$$

$$A_{\vec{p}} = \begin{pmatrix} v_p - \lambda & \gamma_p & 0 & 0 & 0 & 0 \\ 0 & v_p - \lambda & 1/\gamma_p & \frac{p_p - p_g}{\phi_p \gamma_p} & 0 & 0 \\ 0 & p_p(1+\beta) & v_p - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & v_p - \lambda & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma_g}{\phi_g}(v_p - v_g) & v_g - \lambda & 0 \\ 0 & 0 & 0 & 0 & v_g - \lambda & 1/\gamma_g \\ 0 & 0 & 0 & \frac{p_g}{\phi_g}(1+\beta)(v_p - v_g) & 0 & (1+\beta)p_g - \lambda + v_g \end{pmatrix}$$

$$\det(A_{\vec{p}} - \lambda I) = (v_p - \lambda)^2 (v_g - \lambda) \left[(v_p - \lambda)^2 - \frac{(1+\beta)p_p}{\gamma_p} \right] \left[(v_g - \lambda)^2 - \frac{(1+\beta)p_g}{\gamma_g} \right] = 0$$

$$\Rightarrow \lambda = v_p, v_p, v_p + a_p, v_p - a_p, v_g, v_g + a_g, v_g - a_g$$

$$(\text{small}) \quad \vec{0}^T = (1+\beta) p_a$$

With the eigenvalues known, let us compute the left eigenvectors, which satisfy

$$(\underline{A}^T - \lambda \underline{I}) \underline{l} = \underline{0}.$$

$$(i) \quad \lambda = \lambda_1 = v_p$$

$$\underline{A}^T - \lambda_1 \underline{I} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_p & 0 & p_p(1+\Gamma_p) & 0 & 0 & 0 & 0 \\ 0 & 1/\gamma_p & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{p_p - p_g}{\phi_p \gamma_p} & 0 & 0 & \frac{\gamma_g}{\phi_g} (v_p - v_g) & 0 & \frac{p_g}{\phi_g} (1+\Gamma_g) (v_p - v_g) \\ 0 & 0 & 0 & 0 & v_g - v_p & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_g & v_g - v_p & (1+\Gamma_g) p_g \\ 0 & 0 & 0 & 0 & 0 & 1/\gamma_g & v_g - v_p \end{pmatrix}$$

$$(\underline{A}^T - \lambda_1 \underline{I}) \underline{l} = 0 \text{ reduces to}$$

$$\gamma_p l_1 + p_p (1+\Gamma_p) l_2 = 0$$

$$l_2 = 0$$

$$\frac{1}{\phi_g} (v_p - v_g) [\gamma_g l_5 + p_g (1+\Gamma_g) l_7] = 0$$

$$(v_g - v_p) l_5 = 0$$

$$\gamma_g l_5 + (v_g - v_p) l_6 + (1+\Gamma_g) p_g l_7 = 0$$

$$\frac{1}{\gamma_g} l_6 + (v_g - v_p) l_7 = 0$$

There are two possibilities

(a) $v_g - v_p \neq 0$. Now, $\lambda = v_p$ has multiplicity 2,

i.e., $\lambda_1 = \lambda_2 = \nu_p$.

We then get $l_2 = l_3 = l_6 = l_7 = 0$, $l_1 = -a_p^2 l_3$

$$\therefore \underline{l}^T = (-a_p^2 l_3, 0, l_3, l_4, 0, 0, 0)$$

$$= l_3 (-a_p^2, 0, 1, 0, 0, 0, 0)$$

and

$$l_4 (0, 0, 0, 1, 0, 0, 0)$$

(b) $\nu_g - \nu_p = 0$. Now, $\lambda = \nu_p$ has multiplicity 3. Then

$$\underline{l}^T = (-a_g^2 l_3, 0, l_3, l_4, -a_g^2 l_7, 0, l_7),$$

yielding the additional eigenvector

$$(0, 0, 0, 0, -a_g^2, 0, 1).$$

(ii) $\lambda = \nu_p \neq a_p$

$$\underline{A}^T - \lambda \underline{I} = \begin{pmatrix} \pm a_p & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_p & \pm a_p & \gamma_p a_p^2 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_p} & \pm a_p & 0 & 0 & 0 & 0 \\ 0 & \frac{p_p - p_g}{\phi_p \gamma_p} & 0 & \pm a_p & \frac{\gamma_g}{\phi_g} (\nu_p - \nu_g) & 0 & \frac{\gamma_g}{\phi_g} a_g^2 \\ 0 & 0 & 0 & 0 & \nu_g - \nu_p \pm a_p & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_g & \nu_g - \nu_p \pm a_p & \gamma_g \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_g} & \nu_g - \nu_p \end{pmatrix}$$

There are two possibilities :

(a) $v_g - v_p \pm a_p \neq 0$. Then,

$$l_1 = 0, \quad l_2 \pm \gamma_p a_p l_3 = 0, \quad l_5 = l_6 = l_7 = 0,$$

$$\frac{p_p - p_g}{\phi_p \gamma_p} l_2 \pm a_p l_4 = 0$$

$$\therefore \underline{l} = (0, \mp \gamma_p a_p l_3, l_3, \frac{p_p - p_g}{\phi_p} l_3, 0, 0, 0)^T$$

Thus, $\lambda = v_p - a_p \Rightarrow$

$$\underline{l} = (0, -\gamma_p a_p, 1, \frac{p_p - p_g}{\phi_p}, 0, 0, 0)^T \quad (1)$$

and $\lambda = v_p + a_p \Rightarrow$

$$(0, \gamma_p a_p, 1, \frac{p_p - p_g}{\phi_p}, 0, 0, 0)^T. \quad (2)$$

(b) If $v_p - v_g = a_p$ or $-a_p$, then, in addition to the eigenvector (1) or (2) found above, there is another one, given by

$$\underline{l} = (0, 0, 0, 0, 0, -a_g^2, 0, 1) \quad (3)$$

The reason for this additional eigenvector is that the eigenvalue $v_p + a_p$ (or $v_p - a_p$) coincides with the eigenvalue v_g .

$$(iii) \lambda = v_g.$$

$$\underline{A} - \lambda \underline{I} = \begin{pmatrix} v_p - v_g & 0 & 0 & 0 & 0 & 0 \\ \gamma_p & v_p - v_g & \gamma_p a_g^2 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_p} & v_p - v_g & 0 & 0 & 0 \\ 0 & \frac{p_p - p_g}{\phi_p \gamma_p} & 0 & v_p - v_g & \frac{\gamma_p}{\phi_g} (v_p - v_g) & 0 & \frac{\gamma_p}{\phi_g} a_g^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_g & 0 & 1 + \gamma_g \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_g} \end{pmatrix}$$

We shall assume $v_p \neq v_g$. (The case $v_p = v_g$ has been dealt with)

$$l_1 = 0 = l_2 = l_3 = l_6$$

$$l_5 + a_g^2 l_7 = 0$$

$$l_4 + \frac{\gamma_g}{\phi_g} l_5 + \frac{\gamma_g}{\phi_g} a_g^2 l_7 = 0 \} \Rightarrow l_4 = 0, \quad l_5 = -a_g^2 l_7$$

$$\therefore \underline{l} = (0, 0, 0, 0, -a_g^2, 0, 1).$$

$$(iv) \lambda = v_g \neq a_g$$

$$\lambda \underline{I} = \begin{pmatrix} v_p - v_g \pm a_g & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_p & v_p - v_g \pm a_g & \gamma_p a_g^2 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_p} & v_p - v_g \pm a_g & 0 & 0 & 0 & 0 \\ 0 & \frac{p_p - p_g}{\phi_p \gamma_p} & 0 & v_p - v_g \pm a_g, \frac{\gamma_g}{\phi_g} (v_p - v_g), 0, \frac{\gamma_g}{\phi_g} a_g^2 (v_p - v_g) \\ 0 & 0 & 0 & 0 & \pm a_g & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_g & \pm a_g & \gamma_g a_g^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_g} & \pm a_g \end{pmatrix}$$

$$(a) \quad v_p - v_g \pm a_g \neq 0$$

$$l_1 = l_2 = l_3 = l_5 = 0$$

$$l_6 \pm \gamma_g a_g l_7 = 0$$

$$(v_p - v_g \pm a_g) l_4 + \frac{\gamma_g}{\phi_g} a_g^2 (v_p - v_g) l_7 = 0$$

$$\therefore \underline{l} = \langle 0, 0, 0, -\frac{\gamma_g a_g^2 (v_p - v_g)}{\phi_g (v_p - v_g \pm a_g)}, 0, \mp \gamma_g a_g, 1 \rangle. \quad (1)$$

$$(b) \quad v_p - v_g \pm a_g = 0. \quad \text{In this case,}$$

$$\underline{l} = \langle 0, 0, 0, 1, 0, 0, 0 \rangle. \quad (2)$$

Summary :

Recall that the eigenvalues are

$$\lambda_1 = \lambda_2 = v_p, \quad \lambda_3 = v_p + a_p, \quad \lambda_4 = v_p - a_p, \quad \lambda_5 = v_g, \\ \lambda_6 = v_g + a_g, \quad \lambda_7 = v_g - a_g.$$

Suppose that there is no further coincidence of eigenvalues. Then the left eigenvectors form a complete set:

$$\underline{l}_1 = \langle -a_p^2, 0, 1, 0, 0, 0, 0 \rangle$$

$$\underline{l}_2 = \langle 0, 0, 0, 1, 0, 0, 0 \rangle$$

$$\underline{l}_3 = \langle 0, \gamma_p a_p, 1, \frac{p_p - p_g}{\phi_p}, 0, 0, 0 \rangle$$

$$\underline{l}_4 = \langle 0, -\gamma_p a_p, 1, \frac{p_p - p_g}{\phi_p}, 0, 0, 0 \rangle$$

$$\underline{l}_5 = \langle 0, 0, 0, 0, -a_g, 0, 1 \rangle$$

$$\underline{l}_6 = \langle 0, 0, 0, -\frac{\gamma_g a_g^2 (v_p - v_g)}{\phi_g (v_p - v_g - a_g)}, 0, \gamma_g a_g, 1 \rangle$$

$$\underline{l}_7 = \langle 0, 0, 0, -\frac{\gamma_g a_g^2 (v_p - v_g)}{\phi_g (v_p - v_g + a_g)}, 0, -\gamma_g a_g, 1 \rangle$$

Now, suppose that $v_g = v_p$, i.e., $\lambda_5 = \lambda_1$. Then, the set remains complete. The same is true if $\lambda_3 = \lambda_2$ or $\lambda_4 = \lambda_5$ or, λ_3 or λ_4 equals λ_6 or λ_7 .

The situation is different if λ_1 equals λ_6 or λ_7 , i.e., if $v_p = v_g \pm a_g$. Then, either $\underline{e}_1$ or $\underline{e}_7$ coincides with $\underline{e}_2$ and completeness is lost.

For the moment, let us assume that the system is not resonant. Then, the equations take the following characteristic form:

$$\underline{e}^T \left[\frac{\partial \underline{u}}{\partial t} + \underline{A} \frac{\partial \underline{u}}{\partial x} \right] = \underline{e}^T \underline{R}(\underline{u}) + \underline{e}^T \frac{\partial}{\partial x} \left[\underline{D}(\underline{u}) \frac{\partial \underline{u}}{\partial x} \right]$$

The expanded form is:

$$\frac{\partial p_s}{\partial t} + v_p \frac{\partial p_s}{\partial x} - a_p^2 \left(\frac{\partial \gamma_p}{\partial t} + v_p \frac{\partial \gamma_p}{\partial x} \right) = \gamma_p R_1 + R_3 + \frac{\Gamma_p}{\Phi_p} \left(\tilde{k}_p \frac{\partial \tau_p}{\partial x} \right) \quad (2)$$

$$\frac{\partial \Phi_p}{\partial t} + v_p \frac{\partial \Phi_p}{\partial x} = R_4 \quad (3)$$

$$\begin{aligned} & \frac{\partial p_p}{\partial t} + (v_p \pm a_p) \frac{\partial p_p}{\partial x} \pm \gamma_p a_p \left[\frac{\partial v_p}{\partial t} + (v_p \pm a_p) \frac{\partial v_p}{\partial x} \right] \\ & + \frac{\Gamma_p - \Gamma_g}{\Phi_p} \left[\frac{\partial \Phi_p}{\partial t} + (v_p \pm a_p) \frac{\partial \Phi_p}{\partial x} \right] \\ & = \pm \gamma_p a_p R_2 + R_3 + \frac{\Gamma_p}{\Phi_p} \frac{\partial}{\partial x} \left(\tilde{k}_p \frac{\partial \tau_p}{\partial x} \right) + \frac{\Gamma_p - \Gamma_g}{\Phi_p} R_4. \end{aligned} \quad (4)$$

$$\begin{aligned} & \frac{\partial p_g}{\partial t} + v_g \frac{\partial p_g}{\partial x} - a_g^2 \left(\frac{\partial \gamma_g}{\partial t} + v_g \frac{\partial \gamma_g}{\partial x} \right) = \gamma_g R_5 + R_7 \\ & + \frac{\Gamma_g}{\Phi_g} \left(\tilde{k}_g \frac{\partial \tau_g}{\partial x} \right) \end{aligned} \quad (5)$$

$$\begin{aligned}
& \frac{\partial p_g}{\partial t} + (v_g \pm a_g) \frac{\partial p_g}{\partial x} \pm \gamma_g a_g \left[\frac{\partial v_g}{\partial t} + (v_g \pm a_g) \frac{\partial v_g}{\partial x} \right] \\
& - \frac{\gamma_g a_g^2 (v_b - v_g)}{\phi_g (v_b - v_g \mp a_g)} \left[\frac{\partial \phi_b}{\partial t} + (v_g \pm a_g) \frac{\partial \phi_b}{\partial x} \right] \\
& = - \frac{\gamma_g a_g^2 (v_b - v_g)}{\phi_g (v_b - v_g \mp a_g)} R_4 \pm \gamma_g a_g R_6 + R_7 + \frac{\beta_g}{\phi_g} \frac{\partial}{\partial x} \left(\tilde{k}_g \frac{\partial T_g}{\partial x} \right)
\end{aligned}$$

Let us take a look at the source terms.

$$\begin{aligned}
S_1 \equiv R_3 - a_b^2 R_1 &= - \frac{\beta_b}{\phi_b} \tilde{h} (T_b - T_g) - \frac{F}{\phi_b} \left[\gamma_b a_b^2 - \beta_b \beta_b \right] \\
&\quad + \frac{\gamma_b a_b^2}{\phi_b} F \\
&= - \frac{\beta_b}{\phi_b} \tilde{h} (T_b - T_g) + \frac{\beta_b \beta_b}{\phi_b} F.
\end{aligned}$$

$$S_2 \equiv R_4 = F + \tilde{h} \Phi$$

$$\begin{aligned}
S_3 \equiv \pm \gamma_b a_b R_2 + R_3 + \frac{\beta_b - \beta_g}{\phi_b} R_4 \\
= \mp \gamma_b a_b \left[\left(\tilde{\delta} + \frac{1}{2} \tilde{h} \gamma_b \Phi \right) (v_b - v_g) \right] - \frac{\beta_b}{\phi_b} \tilde{h} (T_b - T_g) \\
- \frac{F}{\phi_b} (\gamma_b a_b^2 - \beta_b \beta_b) + \frac{\beta_b - \beta_g}{\phi_b} (F + \tilde{h} \Phi).
\end{aligned}$$

$$\begin{aligned}
S_5 \equiv - a_g^2 R_5 + R_7 &= - a_g^2 \left[\frac{\gamma_g}{\phi_g} F - \left(1 - \frac{\gamma_g}{\gamma_b} \right) \tilde{h} \gamma_b \Phi \right] \\
&\quad + \frac{\beta_g}{\phi_g} \left[\tilde{h} (T_b - T_g) + \tilde{\delta} (v_b - v_g)^2 + (\beta_b - \beta_g) F - (\tilde{c}_{vb} T_b - \tilde{c}_{vg} T_g) \tilde{h} \right]
\end{aligned}$$

$$+ \frac{\beta_g}{\phi_g} F - \frac{1}{\phi_g} \Gamma_g \tilde{C}_{vg} T_g \left(1 - \frac{\gamma_g}{\gamma_p}\right) \tilde{h} \gamma_p \Phi. \quad (1)$$

$$S_{67} \equiv - \frac{\gamma_g a_g^2 (\nu_p - \nu_g)}{\phi_g (\nu_p - \nu_g \mp a_g)} R_4 \pm \gamma_g a_g R_6 + R_7$$

$$= - \frac{\gamma_g a_g^2 (\nu_p - \nu_g)}{\phi_g (\nu_p - \nu_g \mp a_g)} (F + \tilde{h} \Phi) \pm \frac{a_g}{\phi_g} \left(\tilde{\delta} - \frac{1}{2} \tilde{h} \gamma_p \Phi \right) (\nu_p - \nu_g)$$

$$+ \frac{\Gamma_g}{\phi_g} \left[\tilde{L} (T_p - T_g) + \tilde{\delta} (\nu_p - \nu_g)^2 + (\beta_p - \beta_g) F - \right.$$

$$\left. - (\tilde{C}_{vp} T_p - \tilde{C}_{vg} T_g) \tilde{h} \gamma_p \Phi \right] + \frac{\beta_g}{\phi_g} F - \frac{1}{\phi_g} \Gamma_g \tilde{C}_{vg} T_g \left(1 - \frac{\gamma_g}{\gamma_p}\right) \tilde{h} \gamma_p \Phi \quad (2)$$

Equations in near-conservation form.

Start with the dimensionless eqns. on p.10.
Note the following:

Mass

$$\begin{aligned} & \frac{\partial}{\partial t} (\phi_p \gamma_p) + \frac{\partial}{\partial x} (\phi_p \gamma_p v_p) \\ &= \phi_p \left(\frac{\partial \gamma_p}{\partial t} + \frac{\partial}{\partial x} (\gamma_p v_p) \right) + \gamma_p \left(\frac{\partial \phi_p}{\partial t} + v_p \frac{\partial \phi_p}{\partial x} \right) \\ &= -\cancel{\gamma_p F} + \cancel{\gamma_p / F} + \gamma_p \tilde{h} \tilde{\Phi} \end{aligned}$$

Similarly,

$$\begin{aligned} & \frac{\partial}{\partial t} (\phi_g \gamma_g) + \frac{\partial}{\partial x} (\phi_g \gamma_g v_g) \\ &= \phi_g \left(\frac{\partial \gamma_g}{\partial t} + \frac{\partial}{\partial x} (\gamma_g v_g) \right) + \gamma_g \left(\frac{\partial \phi_g}{\partial t} + v_g \frac{\partial \phi_g}{\partial x} \right) \\ &= \cancel{\gamma_g / F} - \cancel{\gamma_g (v_p - v_g)} \frac{\partial \phi_p}{\partial x} - \left(1 - \frac{\cancel{\gamma_g}}{\cancel{\gamma_p}} \right) \tilde{h} \cancel{\gamma_p} \tilde{\Phi} \\ & \quad - \gamma_g \left[\cancel{(v_g - v_p)} \frac{\partial \phi_p}{\partial x} + \cancel{F} + \cancel{\tilde{h} \tilde{\Phi}} \right] \end{aligned}$$

Momentum

$$\begin{aligned} & \frac{\partial}{\partial t} (\phi_p \gamma_p v_p) + \frac{\partial}{\partial x} [\phi_p \gamma_p v_p^2 + \phi_p p_p] \\ &= v_p \left[\frac{\partial}{\partial t} (\phi_p \gamma_p) + \frac{\partial}{\partial x} (\phi_p \gamma_p v_p) \right] \\ & \quad + \phi_p \gamma_p \left[\frac{\partial v_p}{\partial t} + v_p \frac{\partial v_p}{\partial x} \right] + \phi_p \frac{\partial p_p}{\partial x} + p_p \frac{\partial \phi_p}{\partial x} \end{aligned}$$

$$\begin{aligned}
&= v_p \gamma_p \tilde{h} \Phi - \cancel{\phi_p \frac{\partial p_p}{\partial x}} + (p_g - p_p) \frac{\partial \phi_p}{\partial x} \\
&\quad - \left(\tilde{\delta} + \frac{1}{2} \tilde{h} \gamma_p \Phi \right) (v_p - v_g) + \cancel{\phi_p \frac{\partial p_p}{\partial x}} + \cancel{p_p \frac{\partial \phi_p}{\partial x}} \\
&= p_g \frac{\partial \phi_p}{\partial x} + v_p \tilde{h} \gamma_p \Phi - \left(\tilde{\delta} + \frac{1}{2} \tilde{h} \gamma_p \Phi \right) (v_p - v_g).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{\partial}{\partial t} (\phi_g \gamma_g v_g) + \frac{\partial}{\partial x} [\phi_g \gamma_g v_g^2 + \phi_g p_g] \\
&= v_g \left[\frac{\partial}{\partial t} (\phi_g \gamma_g) + \frac{\partial}{\partial x} (\phi_g \gamma_g v_g) \right] \\
&\quad + \phi_g \gamma_g \left[\frac{\partial v_g}{\partial t} + v_g \frac{\partial v_g}{\partial x} \right] + \phi_g \frac{\partial p_g}{\partial x} + p_g \frac{\partial \phi_g}{\partial x} \\
&= - v_g \tilde{h} \gamma_p \Phi - \cancel{\phi_g \frac{\partial p_g}{\partial x}} + \left(\tilde{\delta} - \frac{1}{2} \tilde{h} \gamma_p \Phi \right) (v_p - v_g) \\
&\quad + \cancel{\phi_g \frac{\partial p_g}{\partial x}} + \cancel{p_g \frac{\partial \phi_g}{\partial x}} \\
&= - v_p \tilde{h} \gamma_p \Phi + \left(\tilde{\delta} + \frac{1}{2} \tilde{h} \gamma_p \Phi \right) (v_p - v_g) + p_g \frac{\partial \phi_g}{\partial x}.
\end{aligned}$$

Energy

$$E_a = e_a + \frac{1}{2} v_a^2 \quad (\text{dimensional})$$

In dimensionless form, $E_a = \tilde{c}_{va} T_a + \frac{1}{2} v_a^2$.

$$\text{Now, } \frac{\partial}{\partial t} (\phi_p \gamma_p E_p) + \frac{\partial}{\partial x} [\phi_p \gamma_p E_p v_p + \phi_p p_p v_p]$$

$$\begin{aligned}
&= \frac{\partial}{\partial t} (\phi_p \gamma_p e_p + \phi_p \gamma_p \frac{1}{2} v_p^2) \\
&\quad + \frac{\partial}{\partial x} (\phi_p \gamma_p e_p v_p + \phi_p \gamma_p \frac{1}{2} v_p^2 v_p + \phi_p p_p v_p) \\
&= (e_p + \frac{1}{2} v_p^2) \left[\frac{\partial}{\partial t} (\phi_p \gamma_p) + \frac{\partial}{\partial x} (\phi_p \gamma_p v_p) \right] \\
&\quad + \phi_p \gamma_p \left[\frac{\partial e_p}{\partial t} + v_p \frac{\partial e_p}{\partial x} \right] + \phi_p \gamma_p v_p \left[\frac{\partial v_p}{\partial t} + v_p \frac{\partial v_p}{\partial x} \right] \\
&\quad + \frac{\partial}{\partial x} (\phi_p p_p v_p)
\end{aligned}$$

On using the appropriate expressions from pages 10 and

$$\begin{aligned}
&= (e_p + \frac{1}{2} v_p^2) \gamma_p \tilde{\kappa} \tilde{\Phi} - \phi_p p_p \frac{\partial v_p}{\partial x} + \frac{\partial}{\partial x} (\tilde{k}_p \frac{\partial T_p}{\partial x}) \\
&\quad - \tilde{\kappa} (T_p - T_g) - (p_p - p_g) F + v_p \left[-\phi_p \frac{\partial p_p}{\partial x} + (p_g - p_p) \frac{\partial \phi_p}{\partial x} \right. \\
&\quad \left. - (\tilde{\delta} + \frac{1}{2} \tilde{\kappa} \gamma_p \tilde{\Phi}) (v_p - v_g) \right] + \frac{\partial}{\partial x} (\phi_p p_p v_p)
\end{aligned}$$

Finally, as above,

$$\begin{aligned}
&\frac{\partial}{\partial t} (\phi_g \gamma_g e_g) + \frac{\partial}{\partial x} [\phi_g \gamma_g e_g v_g + \phi_g p_g v_g] \\
&= \frac{\partial}{\partial t} (\phi_g \gamma_g e_g + \phi_g \gamma_g \frac{1}{2} v_g^2) + \frac{\partial}{\partial x} [\phi_g \gamma_g e_g v_g + \phi_g \gamma_g \frac{1}{2} v_g^2 v_g \\
&\quad + \phi_g p_g v_g] \\
&= \cancel{(e_g + \frac{1}{2} v_g^2)} \left[-\gamma_p \tilde{\kappa} \tilde{\Phi} \right] - \phi_g p_g \frac{\partial v_g}{\partial x} + \frac{\partial}{\partial x} (\tilde{k}_g \frac{\partial T_g}{\partial x})
\end{aligned}$$

Summary : near-conservation form of equations

$$\frac{\partial}{\partial t} (\phi_p \gamma_p) + \frac{\partial}{\partial x} (\phi_p \gamma_p v_p) = C_p^* \quad (1)$$

$$\frac{\partial}{\partial t} (\phi_g \gamma_g) + \frac{\partial}{\partial x} (\phi_g \gamma_g v_g) = -C_g^* \quad (2)$$

$$\frac{\partial}{\partial t} (\phi_p \gamma_p v_p) + \frac{\partial}{\partial x} [\phi_p \gamma_p v_p^2 + \phi_p p_p] - p_p \frac{\partial \phi_p}{\partial x} = m_p^* \quad (3)$$

$$\frac{\partial}{\partial t} (\phi_g \gamma_g v_g) + \frac{\partial}{\partial x} [\phi_g \gamma_g v_g^2 + \phi_g p_g] - p_g \frac{\partial \phi_g}{\partial x} = -m_g^* \quad (4)$$

$$\frac{\partial}{\partial t} (\phi_p \gamma_p E_p) + \frac{\partial}{\partial x} [\phi_p \gamma_p E_p v_p + \phi_p p_p v_p] - p_p \frac{\partial \phi_p}{\partial x} v_p = \frac{\partial}{\partial x} (\tilde{k}_p \frac{\partial T_p}{\partial x}) + \tilde{E}_p^* \quad (5)$$

$$\frac{\partial}{\partial t} (\phi_g \gamma_g E_g) + \frac{\partial}{\partial x} [\phi_g \gamma_g E_g v_g + \phi_g p_g v_g] - p_g \frac{\partial \phi_g}{\partial x} v_g = \frac{\partial}{\partial x} (\tilde{k}_g \frac{\partial T_g}{\partial x}) - \tilde{E}_g^* \quad (6)$$

Here,

$$C_p^* = \tilde{h} \gamma_p \Phi \quad (7)$$

$$m_p^* = v_p C_p^* - (\tilde{\delta} + \frac{1}{2} C_p^*) (v_p - v_g) \quad (8)$$

$$\tilde{E}_p^* = E_p C_p^* - \tilde{h} (T_p - T_g) - (p_p - p_g) F - (\tilde{\delta} + \frac{1}{2} C_p^*) (v_p - v_g) \quad (9)$$

$$E_a = \tilde{c}_v a T_a + \frac{1}{2} v_a^2 ; \quad e_a = \tilde{c}_v a T_a. \quad (10)$$

Also,

$$\frac{\partial \gamma_p}{\partial t} + \frac{\partial}{\partial x} (\gamma_p v_p) = -\frac{\gamma_p}{\phi_p} F. \quad (11)$$

$$\begin{aligned}
& + \tilde{h} (T_p - T_g) + (\cancel{p_p} - \beta_p) F - (\cancel{v_p} - \cancel{v_g}) \cancel{p_g} \frac{\partial \phi_p}{\partial x} \\
& + \tilde{\delta} (\cancel{v_p} - \cancel{v_g})^2 - (\cancel{e_p} - \cancel{e_g}) \tilde{h} \gamma_p \Phi + \cancel{v_g} \left[-\cancel{\phi_g} \frac{\partial \cancel{p_g}}{\partial x} \right. \\
& \left. + \left(\tilde{\delta} - \frac{1}{2} \tilde{h} \gamma_p \Phi \right) (\cancel{v_p} - \cancel{v_g}) \right] + \frac{\partial}{\partial x} (\cancel{\phi_g} \cancel{p_g} \cancel{v_g}) \\
& = \frac{\partial}{\partial x} \left(\tilde{h}_g \frac{\partial T_g}{\partial x} \right) + \tilde{h} (T_p - T_g) + (\cancel{p_p} - \beta_p) F + \cancel{v_p} \cancel{p_g} \frac{\partial}{\partial x} \\
& + \left\{ \tilde{\delta} (\cancel{v_p} - \cancel{v_g})^2 + \cancel{v_g} \left(\tilde{\delta} - \frac{1}{2} \tilde{h} \gamma_p \Phi \right) (\cancel{v_p} - \cancel{v_g}) \right. \\
& \left. - \frac{1}{2} \cancel{v_g}^2 \gamma_p \tilde{h} \Phi \right\} - \cancel{e_p} \tilde{h} \gamma_p \Phi
\end{aligned}$$

in $\{ \}$

The three terms can be combined to yield

$$-\frac{1}{2} \cancel{v_p}^2 \gamma_p \tilde{h} \Phi + \cancel{v_p} \left(\tilde{\delta} + \frac{1}{2} \tilde{h} \gamma_p \Phi \right) (\cancel{v_p} - \cancel{v_g}).$$

ITEM 2

On the Solid Equation of State.

A-N (1986, p. 871, eq. (61)) start with the Helmholtz free-energy function ψ , in the following dimensional form:

$$\psi_p(\gamma_p, T_p) = C_v^p (T_p - T_0) + C_v^p (T_p - T_0) \Gamma_p \left(\frac{\gamma_p}{\gamma_{p0}} - 1 \right) - C_v^p T_p \ln \frac{T_p}{T_{p0}} + \frac{1}{\gamma_{p0}^2} \frac{K_T}{N(N-1)} \left[\left(\frac{\gamma_p}{\gamma_{p0}} \right)^{N-1} - (N-1) \left(1 - \frac{\gamma_{p0}}{\gamma_p} \right) - 1 \right]. \quad (1)$$

$$\frac{\partial \psi_p}{\partial T_p} = \cancel{C_v^p} + C_v^p \Gamma_p \left(\frac{\gamma_p}{\gamma_{p0}} - 1 \right) - C_v^p \ln \frac{T_p}{T_{p0}} - \cancel{C_v^p} \quad (2)$$

$$\frac{\partial \psi_p}{\partial \gamma_p} = C_v^p (T_p - T_0) \frac{\Gamma_p}{\gamma_{p0}^2} + \frac{K_T}{N \gamma_{p0}^2} \left[\left(\frac{\gamma_p}{\gamma_{p0}} \right)^N - 1 \right]. \quad (3)$$

Now,

$$p = \gamma_p^2 \frac{\partial \psi_p}{\partial \gamma_p} = \gamma_p^2 C_v^p (T_p - T_0) \frac{\Gamma_p}{\gamma_{p0}^2} + \frac{K_T}{N} \left[\left(\frac{\gamma_p}{\gamma_{p0}} \right)^N - 1 \right]. \quad (4)$$

$$e_p = \psi_p - T_p \frac{\partial \psi_p}{\partial T_p} = C_v^p (T_p - T_0) - C_v^p T_0 \Gamma_p \left(\frac{\gamma_p}{\gamma_{p0}} - 1 \right) + \frac{1}{\gamma_{p0}^2} \frac{K_T}{N(N-1)} \left[\left(\frac{\gamma_p}{\gamma_{p0}} \right)^{N-1} - (N-1) \left(1 - \frac{\gamma_{p0}}{\gamma_p} \right) - 1 \right]. \quad (5)$$

Eliminating T_p from (4) and (5),

$$p = \frac{\Gamma_p \gamma_p^2}{\gamma_{p0}^2} e_p + C_v^p T_0 \Gamma_p^2 \frac{\gamma_p^2}{\gamma_{p0}^2} \left(\frac{\gamma_p}{\gamma_{p0}} - 1 \right) - \frac{\Gamma_p \gamma_p^2}{\gamma_{p0}^2} \frac{K_T}{N(N-1)} \left[\left(\frac{\gamma_p}{\gamma_{p0}} \right)^{N-1} - (N-1) \left(1 - \frac{\gamma_{p0}}{\gamma_p} \right) - 1 \right] + \frac{K_T}{N} \left[\left(\frac{\gamma_p}{\gamma_{p0}} \right)^N - 1 \right]. \quad (6)$$

The last result can be rewritten as

$$p_b = \Gamma_b \gamma_b \left(\frac{\gamma_b}{\gamma_b^0} \right) \left[e_b + \Gamma_b C_v^b T_b^0 \left(\frac{\gamma_b}{\gamma_b^0} - 1 \right) - \frac{1}{\gamma_b^0} \frac{K_T}{N(N-1)} \left\{ \left(\frac{\gamma_b}{\gamma_b^0} \right)^{N-1} - (N-1) \left(1 - \frac{\gamma_b^0}{\gamma_b} \right) - 1 \right\} \right] + \frac{K_T}{N} \left[\left(\frac{\gamma_b}{\gamma_b^0} \right)^N - 1 \right].$$

This expression is precisely the same as in Baer (1988 p. 38, eq. (7)), if the factor $\left(\frac{\gamma_b}{\gamma_b^0} \right)$ outside the square brackets is replaced by 1.

In fact, unless the above modification is made, the above equation fails to yield

$$\Gamma_b = \frac{1}{\gamma_b} \left(\frac{\partial p_b}{\partial e_b} \right)_{\gamma_b}.$$

With the modification, consistency requires that (4) be changed

$$p_b = \Gamma_b \gamma_b C_v^b (T_b - T_b^0) + \frac{K_T}{N} \left[\left(\frac{\gamma_b}{\gamma_b^0} \right)^N - 1 \right]. \quad (8)$$

Eq. (5) remains unaltered, while (7) $\Rightarrow$

$$p_b = \Gamma_b \gamma_b \left[e_b + \Gamma_b C_v^b T_b^0 \left(\frac{\gamma_b}{\gamma_b^0} - 1 \right) - \frac{1}{\gamma_b^0} \frac{K_T}{N(N-1)} \left\{ \left(\frac{\gamma_b}{\gamma_b^0} \right)^{N-1} - (N-1) \left(1 - \frac{\gamma_b^0}{\gamma_b} \right) - 1 \right\} \right] + \frac{K_T}{N} \left[\left(\frac{\gamma_b}{\gamma_b^0} \right)^N - 1 \right]. \quad (9)$$

let us define

$$\frac{K_T}{N(N-1)} \left\{ \sigma^{N-1} - (N-1) \left(1 - \frac{1}{\sigma}\right) - 1 \right\} \equiv g(\sigma) \quad (10)$$

$$\frac{K_T}{N} \left[\sigma^{N-1} \right] = h(\sigma). \quad (10')$$

Note that $g'(\sigma) = \frac{h(\sigma)}{\sigma^2}$, $h'(\sigma) = K_T \sigma^{N-1}$. (11)

Then,

$$p_p = \Gamma_p \gamma_p C_v^p (T_p - T_{p0}^0) + h(\gamma_p / \gamma_{p0}) \quad (12)$$

and

$$p_p = \Gamma_p \gamma_p \left[e_p + \Gamma_p C_v^p T_{p0}^0 \left(\frac{\gamma_p}{\gamma_{p0}} - 1 \right) - \frac{1}{\gamma_{p0}^2} g(\gamma_p / \gamma_{p0}) \right] + h(\gamma_p / \gamma_{p0}). \quad (13)$$

From (13),

$$\begin{aligned} \left(\frac{\partial p_p}{\partial \gamma_p} \right)_{e_p} &= \Gamma_p \left[\frac{p_p - h}{\Gamma_p \gamma_p} \right] + \frac{1}{\gamma_{p0}} h' \left(\frac{\gamma_p}{\gamma_{p0}} \right) \\ &\quad + \Gamma_p \gamma_p \left[\Gamma_p C_v^p T_{p0}^0 \frac{1}{\gamma_{p0}^2} - \frac{1}{\gamma_{p0}^2} g' \left(\frac{\gamma_p}{\gamma_{p0}} \right) \right] \\ &= \frac{p_p}{\gamma_p} - \frac{h}{\gamma_p} + \Gamma_p^2 C_v^p T_{p0}^0 \frac{\gamma_p}{\gamma_{p0}^2} \\ &\quad + \frac{1}{\gamma_{p0}} h' \left(\frac{\gamma_p}{\gamma_{p0}} \right) - \Gamma_p \frac{\gamma_p}{\gamma_{p0}^2} g' \left(\frac{\gamma_p}{\gamma_{p0}} \right) \quad (14) \end{aligned}$$

On using (14),

$$a_p^2 = \frac{(\Gamma_p + 1) p_p}{\gamma_p} + \Gamma_p^2 C_v^p T_p^0 \frac{\gamma_p}{\gamma_p^0} \\ + \frac{1}{\gamma_{p0}} \left[h' \left(\frac{\gamma_p}{\gamma_{p0}} \right) - \Gamma_p \frac{\gamma_p}{\gamma_{p0}} g' \left(\frac{\gamma_p}{\gamma_{p0}} \right) \right]$$

or

$$a_p^2 = \frac{(\Gamma_p + 1) p_p}{\gamma_p} + \Gamma_p^2 C_v^p T_p^0 \frac{\gamma_p}{\gamma_p^0} \\ + \frac{K_T}{\gamma_{p0}} \left[\left(\frac{\gamma_p}{\gamma_{p0}} \right)^{N-1} - \Gamma_p \frac{\gamma_{p0}}{\gamma_p} \frac{1}{N} \left\{ \left(\frac{\gamma_p}{\gamma_{p0}} \right)^N - 1 \right\} \right]$$

or

$$a_p^2 = \frac{(\Gamma_p + 1) p_p}{\gamma_p} + \Gamma_p^2 C_v^p T_p^0 \frac{\gamma_p}{\gamma_p^0} \\ + \frac{K_T}{\gamma_p} \left[\left(\frac{\gamma_p}{\gamma_{p0}} \right)^N \left(1 - \frac{\Gamma_p}{N} \right) + \frac{\Gamma_p}{N} \right].$$

Let us define

$$\frac{K_T}{N(N-1)} \left\{ \sigma^{N-1} - (N-1) \left(1 - \frac{1}{\sigma}\right) - 1 \right\} \equiv g(\sigma) \quad (10)$$

$$\frac{K_T}{N} \left[\sigma^{N-1} \right] = h(\sigma). \quad (10')$$

Note that $g'(\sigma) = \frac{h(\sigma)}{\sigma^2}$, $h'(\sigma) = K_T \sigma^{N-1}$. (11)

Then,

$$p_p = \Gamma_p \gamma_p C_v^p (T_p - T_p^0) + h(\gamma_p / \gamma_{p0}) \quad (12)$$

and

$$p_p = \Gamma_p \gamma_p \left[e_p + \Gamma_p C_v^p T_p^0 \left(\frac{\gamma_p}{\gamma_{p0}} - 1 \right) - \frac{1}{\gamma_{p0}} g(\gamma_p / \gamma_{p0}) \right] + h(\gamma_p / \gamma_{p0}). \quad (13)$$

From (13),

$$\begin{aligned} \left(\frac{\partial p_p}{\partial \gamma_p} \right)_{e_p} &= \Gamma_p \left[\frac{p_p - h}{\Gamma_p \gamma_p} \right] + \frac{1}{\gamma_{p0}} h' \left(\frac{\gamma_p}{\gamma_{p0}} \right) \\ &\quad + \Gamma_p \gamma_p \left[\Gamma_p C_v^p T_p^0 \frac{1}{\gamma_{p0}} - \frac{1}{\gamma_{p0}^2} g' \left(\frac{\gamma_p}{\gamma_{p0}} \right) \right] \\ &= \frac{p_p}{\gamma_p} - \frac{h}{\gamma_p} + \Gamma_p^2 C_v^p T_p^0 \frac{\gamma_p}{\gamma_{p0}} \\ &\quad + \frac{1}{\gamma_{p0}} h' \left(\frac{\gamma_p}{\gamma_{p0}} \right) - \Gamma_p \frac{\gamma_p}{\gamma_{p0}} \frac{1}{\gamma_{p0}} g' \left(\frac{\gamma_p}{\gamma_{p0}} \right) \quad (14) \end{aligned}$$

Now, sound speed is given by

$$a_p^2 = \left(\frac{\partial p_p}{\partial \gamma_p} \right)_{e_p} + \frac{\Gamma_p p_p}{\gamma_p}. \quad (15)$$

$$a_p^2 = \frac{y_p}{(1+y_p)} + \frac{y_p}{k} \left[\left(\frac{y_0}{y_p} \right)^N - 1 \right] + \frac{N}{y_p}$$

$$a_p^2 = \frac{y_p}{(1+y_p)} + \frac{y_p}{k} \left[\left(\frac{y_0}{y_p} \right)^{N-1} - \left(\frac{y_0}{y_p} \right)^N \right] + \frac{N}{y_p} \left\{ 1 - \left(\frac{y_0}{y_p} \right)^N \right\}$$

$$a_p^2 = \frac{y_p}{(1+y_p)} + \frac{y_p}{k} \left[\left(\frac{y_0}{y_p} \right)^N - \left(\frac{y_0}{y_p} \right)^{N+1} \right] + \frac{N}{y_p} \left(\frac{y_0}{y_p} \right)^N$$

On using (14),

A simplified nonideal equation of state.

Let us assume that

$$p_p = \Gamma_p \gamma_p C_v^p T_p + h\left(\frac{\gamma_p}{\gamma_{p0}}\right) \quad (17)$$

$$e_p = C_v^p T_p + \frac{1}{\gamma_{p0}} g\left(\frac{\gamma_p}{\gamma_{p0}}\right) \quad (18)$$

Let us see if these equations are consistent with a Helmholtz free-energy function.

Define

$$\psi_p = C_v^p T_p [\Gamma_p \ln \gamma_p - \ln T_p] + \frac{1}{\gamma_{p0}} g\left(\frac{\gamma_p}{\gamma_{p0}}\right) \quad (19)$$

Then,

$$\frac{\partial \psi_p}{\partial \gamma_p} = C_v^p T_p \Gamma_p \frac{1}{\gamma_p} + \frac{1}{\gamma_{p0}^2} g'\left(\frac{\gamma_p}{\gamma_{p0}}\right)$$

and

$$\begin{aligned} p_p &= \gamma_p^2 \frac{\partial \psi_p}{\partial \gamma_p} = C_v^p T_p \Gamma_p \gamma_p + \frac{\gamma_p^2}{\gamma_{p0}^2} g'\left(\frac{\gamma_p}{\gamma_{p0}}\right) \\ &= C_v^p T_p \Gamma_p \gamma_p + h\left(\frac{\gamma_p}{\gamma_{p0}}\right) \quad \text{ok} \end{aligned}$$

assuming $h(\sigma) = \sigma^2 g'(\sigma)$.

$$\text{Also, } e_p = \psi_p - T_p \frac{\partial \psi_p}{\partial T_p} = C_v^p T_p - \frac{1}{\gamma_{p0}} g\left(\frac{\gamma_p}{\gamma_{p0}}\right)$$

$$- T_p \left[C_v^p \right]$$

$$\begin{aligned} &= C_v^p T_p \Gamma_p \ln \gamma_p - C_v^p T_p \ln T_p + \frac{1}{\gamma_{p0}} g\left(\frac{\gamma_p}{\gamma_{p0}}\right) \\ &- T_p [C_v^p \Gamma_p \ln \gamma_p - C_v^p \ln T_p - C_v^p] = C_v^p T_p + \frac{1}{\gamma_{p0}} g\left(\frac{\gamma_p}{\gamma_{p0}}\right). \quad \text{ok} \end{aligned}$$

Characteristic analysis revisited in view of more general equations of state.

Consider the dimensional eqns. of state

$$p_a = \Gamma_a \gamma_a C_{va} T_a + h_a \left(\frac{\gamma_a}{\gamma_a^0} \right) \quad (1)$$

$$e_a = C_{va} T_a + \frac{1}{\gamma_a^0} g_a \left(\frac{\gamma_a}{\gamma_a^0} \right) \quad (2)$$

Nondimensionalize:

Units are $\bar{p}$ for p , $\bar{T}$ for T , $\bar{\gamma}$ for γ ,
 $\bar{p}/\bar{\gamma}$ for e , $\bar{p}/\bar{\gamma}\bar{T}$ for C_{va} . Then, (3)

$$p_a = \Gamma_a \gamma_a \tilde{C}_{va} T_a + H_a (\gamma_a / \gamma_a^0) \quad (4)$$

$$e_a = \tilde{C}_{va} T_a + \frac{1}{\gamma_a^0} G_a (\gamma_a / \gamma_a^0) \quad (5)$$

where $H_a = h / \bar{p}$, $G_a = g / \bar{p}$. (6) [Also, $\tilde{C}_p = \frac{\Gamma_p + 1}{\gamma_p} (p_p - H_p) + \frac{H'_p}{\gamma_p^0}$]

We also have

$$H_a(\sigma) = \sigma^2 G'_a(\sigma). \quad (8)$$

Then,

$$\begin{aligned} \dot{p}_a &= \Gamma_a \gamma_a \tilde{C}_{va} \dot{T}_a + \Gamma_a \tilde{C}_{va} T_a \dot{\gamma}_a + \frac{1}{\gamma_a^0} H'_a \dot{\gamma}_a \\ &= \Gamma_a \gamma_a \tilde{C}_{va} \dot{T}_a + \left[\Gamma_a \tilde{C}_{va} T_a + \frac{1}{\gamma_a^0} H'_a \right] \dot{\gamma}_a \end{aligned} \quad (9)$$

Also, one may write

$$p_a = \Gamma_a \gamma_a \left[e_a - \frac{1}{\gamma_a^0} G_a \right] + H_a \quad (10)$$

Then we have used the defn. of a_2^- on the previous page. Also, eqn. (iii) has been used to eliminate e_2 from the total energy eqn. Note that the 3 po

$$\begin{aligned} \frac{\partial p_2}{\partial t} + v_2 \frac{\partial p_2}{\partial x} + \gamma_2 a_2^- \frac{\partial v_2}{\partial x} &= R_3 \\ \frac{\partial v_2}{\partial t} + v_2 \frac{\partial v_2}{\partial x} + \frac{1}{\gamma_2} \frac{\partial p_2}{\partial x} + \frac{p_2 - p_0}{\phi_2 \gamma_2} \frac{\partial \phi_2}{\partial x} &= R_2 \\ \frac{\partial \gamma_2}{\partial t} + v_2 \frac{\partial \gamma_2}{\partial x} + \gamma_2 \frac{\partial v_2}{\partial x} &= R_1 \end{aligned}$$

Let us now consider the basic eqns. on p. 10, ignoring the diffusion terms and the nondifferentiated source terms, to get

$$\begin{aligned} \dot{p}_2 &= \dot{a}_2 \gamma_2 e_a + \frac{\dot{\gamma}_2}{\gamma_2^0} \left[\frac{\gamma_2^0}{\gamma_2} p_a + \gamma_2' - (\dot{\gamma}_2 + 1) \frac{\gamma_2^0}{\gamma_2} H \right] \\ &= \dot{a}_2 \gamma_2 e_a + \frac{\dot{\gamma}_2}{\gamma_2^0} \left[\frac{\gamma_2^0}{\gamma_2} p_a - \frac{\gamma_2^0}{\gamma_2} H - \dot{a}_2 \frac{\gamma_2^0}{\gamma_2} G \right] \\ &\quad + \frac{\dot{\gamma}_2}{\gamma_2^0} \gamma_2' H' \\ &= \frac{\dot{\gamma}_2}{\gamma_2^0} (\dot{p}_a - H) + \dot{a}_2 \gamma_2 e_a - \dot{a}_2 \gamma_2 \frac{\gamma_2^0}{\gamma_2^2} G' \\ &\quad + \frac{\dot{\gamma}_2}{\gamma_2^0} \gamma_2' H' \\ &= \dot{a}_2 \gamma_2 \left[e_a - \frac{G}{\gamma_2^0} \right] + \dot{a}_2 \gamma_2 \left[e_a - \frac{\gamma_2^0}{\gamma_2} G' \right] \end{aligned}$$

eqns. above correspond precisely to their counterparts on p. 13, provided the generalized defn. of α_i is used. Thus, the characteristic speeds are the same as before. The definitions of the source terms R_i , will, of course, reflect the changes in the eqn. of state.

ITEM 3

Asymptotic Reduction of Two-Phase DDT Models

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The Baer-Nunziato Model

Mass

$$\begin{aligned}\frac{\partial}{\partial t}(\phi_s \rho_s) + \frac{\partial}{\partial x}(\phi_s \rho_s v_s) &= \mathcal{C} \\ \frac{\partial}{\partial t}(\phi_g \rho_g) + \frac{\partial}{\partial x}(\phi_g \rho_g v_g) &= -\mathcal{C}\end{aligned}$$

Momentum

$$\begin{aligned}\frac{\partial}{\partial t}(\phi_s \rho_s v_s) + \frac{\partial}{\partial x}[\phi_s(p_s + \rho_s v_s^2)] &= p_s \frac{\partial \phi_s}{\partial x} + \mathcal{M} \\ \frac{\partial}{\partial t}(\phi_g \rho_g v_g) + \frac{\partial}{\partial x}[\phi_g(p_g + \rho_g v_g^2)] &= -p_s \frac{\partial \phi_s}{\partial x} - \mathcal{M}\end{aligned}$$

Energy

$$\begin{aligned}\frac{\partial}{\partial t}[\phi_s \rho_s \{e_s + (1/2)v_s^2\}] + \frac{\partial}{\partial x}[\phi_s \rho_s v_s \{e_s + (p_s/\rho_s) + (1/2)v_s^2\}] \\ = \frac{\partial}{\partial x} \left(\kappa_s \frac{\partial T_s}{\partial x} \right) - (p_s - \beta_s) \mathcal{F} + p_g v_s \frac{\partial \phi_s}{\partial x} + \mathcal{E} \\ \frac{\partial}{\partial t}[\phi_g \rho_g \{e_g + (1/2)v_g^2\}] + \frac{\partial}{\partial x}[\phi_g \rho_g v_g \{e_g + (p_g/\rho_g) + (1/2)v_g^2\}] \\ = \frac{\partial}{\partial x} \left(\kappa_g \frac{\partial T_g}{\partial x} \right) + (p_s - \beta_s) \mathcal{F} - p_g v_s \frac{\partial \phi_s}{\partial x} - \mathcal{E}\end{aligned}$$

Compaction

$$\frac{\partial \phi_s}{\partial t} + v_s \frac{\partial \phi_s}{\partial x} = \mathcal{F} + \frac{\mathcal{C}}{\rho_s}$$

Saturation

$$\phi_s + \phi_g = 1$$

Interaction terms

$$C = \rho_s \mathcal{R}$$

$$\mathcal{M} = C \frac{v_s + v_g}{2} + \delta(v_g - v_s)$$

$$\mathcal{E} = \left(e_s + \frac{1}{2} v_s^2 \right) C - H(T_s - T_g) - \left(\delta + \frac{1}{2} C \right) (v_s - v_g) v_s$$

Compaction term

$$\mathcal{F} = \frac{\phi_s \phi_g}{\mu_c} (p_s - \beta_s - p_g)$$

Configuration stress

$$\beta_s = \beta(\phi_s), \beta'_s > 0$$

Equations of state

Gas

$$e_g = \frac{p_g}{\Gamma_g \rho_g}, \quad p_g = \Gamma_g \rho_g C_{vg} T_g$$

Solid (general)

$$e_s = C_{vs} T_s + \frac{\Lambda}{\rho_{s0}} g(\rho_s / \rho_{s0}), \quad p_s = \Gamma_s \rho_s C_{vs} T_s - \Lambda h(\rho_s / \rho_{s0})$$

$$e_s = \frac{p_s + \Lambda G(\rho_s / \rho_{s0})}{\Gamma_s \rho_s}$$

provided

$$h(\sigma) = -\sigma^2 g'(\sigma), \quad G(\sigma) = \Gamma_s \sigma g(\sigma) - \sigma^2 g'(\sigma)$$

Solid for illustration

$$g(\sigma) = \frac{1}{\sigma}$$

whence

$$e_s = \frac{p_s + \Lambda(\Gamma_s + 1)}{\Gamma_s \rho_s}, \quad p_s = \Gamma_s \rho_s C_{vs} T_s - \Lambda$$

Λ measures the degree of deviation from nonideal behavior.

Remarks

- Model founded on mixture-theory arguments
- Seventh-order system

$$\rho_s, v_s, p_s, \phi_s, \rho_g, v_g, p_g$$

- Input needed for $\mathcal{R}$, δ , H and β_s
- nonconservative (R-H conditions not available; even inert, steady compaction wave is a third-order system)

The Stewart-Prasad-Asay Model

Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

Momentum

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) = \mu \frac{\partial^2 v}{\partial x^2}$$

Energy

$$\frac{\partial}{\partial t}[\rho\{e + (1/2)v^2\}] + \frac{\partial}{\partial x}[\rho v\{e + (1/2)v^2\} + pv] = \mu \frac{\partial^2}{\partial x^2}[(1/2)v^2]$$

Compaction

$$\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x} = r_\phi$$

Reaction

$$\frac{\partial \lambda}{\partial t} + v \frac{\partial \lambda}{\partial x} = r_\lambda(\lambda, p, \rho)$$

Compaction rate

$$r_\phi = \begin{cases} k_\phi(Ap - \phi) & \text{for } Ap - \phi > 0, \text{ and } \phi < 1 \\ 0 & \text{for } Ap - \phi < 0, \text{ and } \phi < 1 \\ 0 & \text{for } \phi = 1 \end{cases}$$

Equation of state

$$e = \frac{1}{\gamma - 1} \frac{p}{\rho} - Q_\phi(1 - \phi) - Q_\lambda \lambda$$

Remarks

- A postulated model
- Fifth-order system

$$\rho, v, p, \phi, \lambda$$

- Input needed for r_λ , Q_λ , Q_ϕ , k_ϕ , A
- Nonconservative (Inert, steady compaction first-order)

Questions

- Connections between the two models?
- Any parametric limits under which B-N reduces to an effectively single-phase description? Are these limits reasonable?
- Properties of the reduced system.

Figure 6. Temporal and spatial variations of (a) gas and (b) solid velocity during combustion of 70% dense HMX.

Asymptotic Reduction of B-N

Numerical calculations with B-N suggest that to an excellent approximation, the two phases move together. In the model, this behavior is ensured by the limit of large drag coefficient,

$$\delta \gg 1.$$

Then,

$$v_g = v + O(1/\delta), \quad v_s = v + O(1/\delta).$$

Useful mixture properties

$$\rho = \phi_s \rho_s + \phi_g \rho_g$$

$$p = \phi_s p_s + \phi_g p_g$$

$$\rho e = \phi_s \rho_s e_s + \phi_g \rho_g e_g$$

Reformulation of B-N

Mixture Equations

Mass

$$\frac{\partial}{\partial t}(\phi_s \rho_s + \phi_g \rho_g) + \frac{\partial}{\partial x}[(\phi_s \rho_s + \phi_g \rho_g)v] = 0$$

Momentum

$$\frac{\partial}{\partial t}[(\phi_s \rho_s + \phi_g \rho_g)v] + \frac{\partial}{\partial x}[\{\phi_s p_s + \phi_g p_g\} + \{\phi_s \rho_s + \phi_g \rho_g\}v^2] = 0$$

Energy

$$\begin{aligned} \frac{\partial}{\partial t}[(\phi_s \rho_s e_s + \phi_g \rho_g e_g) + (1/2)(\phi_s \rho_s + \phi_g \rho_g)v^2] + \frac{\partial}{\partial x}[\{(\phi_s \rho_s e_s + \phi_g \rho_g e_g) \\ + (1/2)(\phi_s \rho_s + \phi_g \rho_g)v^2 + (\phi_s p_s + \phi_g p_g)\}v] = \frac{\partial}{\partial x} \left(\kappa_s \frac{\partial T_s}{\partial x} + \kappa_g \frac{\partial T_g}{\partial x} \right) \end{aligned}$$

In terms of the mixture variables,

Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

Momentum

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) = 0$$

Energy

$$\frac{\partial}{\partial t}[\rho\{e + (1/2)v^2\}] + \frac{\partial}{\partial x}[\rho\{e + (1/2)v^2\} + pv] = \frac{\partial}{\partial x} \left(\kappa_s \frac{\partial T_s}{\partial x} + \kappa_g \frac{\partial T_g}{\partial x} \right)$$

Substantial simplification under the additional limit

$$H \gg 1,$$

leading to

$$T_s = T_g = T$$

except, possibly, in regions of rapid variation.

Summary of reduced model

Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

Momentum

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) = 0$$

Energy

$$\frac{\partial}{\partial t}[\rho\{e + (1/2)v^2\}] + \frac{\partial}{\partial x}[(\rho v\{e + (1/2)v^2\} + pv)] = \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right)$$

Solid mass

$$\frac{\partial}{\partial t}(\phi_s \rho_s) + \frac{\partial}{\partial x}(\phi_s \rho_s v) = \mathcal{C}$$

Compaction

$$\frac{\partial \phi_s}{\partial t} + v \frac{\partial \phi_s}{\partial x} = \mathcal{F} + \frac{\mathcal{C}}{\rho_s}$$

Equality of temperatures

$$T_s = T_g = T$$

implies

$$p_s = p_s(p, \rho, \phi_s, \rho_s)$$

For the demonstration solid,

$$\frac{p_s + \Lambda}{\Gamma_s C_{vs} \rho_s} = \frac{p - \phi_s p_s}{\Gamma_g C_{vg} (\rho - \phi_s \rho_s)}$$

Equation of state

$$\rho e = \frac{p}{\Gamma_g} + \left(\frac{1}{\Gamma_s} - \frac{1}{\Gamma_g}\right) p_s \phi_s + \frac{\Lambda(\Gamma_s + 1)}{\Gamma_s} \phi_s$$

Remarks

- Asymptotic limits $\delta \gg 1$, $H \gg 1$
- Fifth-order system

$$p, \rho, v, \phi_s, \rho_s$$

- Formally, SPA-like; p_s playing the role of λ
- Frozen sound speed is

$$c^2 = \left(\frac{1}{\rho \partial e / \partial p} \right) \left(\frac{p}{\rho} - \rho \frac{\partial e}{\partial \rho} - p \frac{\partial e}{\partial p} \right)$$

- Hyperbolic system, characteristic speeds

$$v, v, v, v \pm c$$

Inert, Steady Compaction

Upstream data

ϕ_{s0} , T_0 , and p_{s0} are prescribed.

Solid equation of state determines ρ_{s0} .

Compaction equilibrium determines p_{g0} .

Gas equation of state determines ρ_{g0} .

Definitions determine p_0 and ρ_0 .

Wave speed V is given. Piston speed v is to be found.

Using a wave-fixed frame, $u = v - V$.

Jump conditions

Mass

$$\rho u = -\rho_0 V$$

Momentum

$$p - \rho_0 V u = p_0 + \rho_0 V^2$$

Solid mass

$$\phi_s \rho_s u = -\phi_{s0} \rho_{s0} V$$

Compaction equilibrium

$$p = p_s - (1 - \phi_s) \beta_s$$

Energy

$$e + \frac{1}{2} u^2 + \frac{p}{\rho} = e_0 + \frac{1}{2} V^2 + \frac{p_0}{\rho_0}$$

Equal temperatures

$$p = \phi_s p_s + \epsilon(p_s + \Lambda)\phi_s$$

where

$$\epsilon = \frac{\Gamma_g C_{vg}(\rho_0 - \phi_{s0}\rho_{s0})}{\Gamma_s C_{vs}\phi_{s0}\rho_{s0}}$$

Equation of state

$$e = \alpha \frac{p}{\rho} + \Lambda(1 + \alpha) \frac{\phi_s}{\rho}$$

where

$$\alpha = \frac{1}{1 + \epsilon} \left(\frac{1}{\Gamma_s} + \frac{\epsilon}{\Gamma_g} \right)$$

Energy equation

$$\begin{aligned} \frac{u^2}{V^2} + \frac{\alpha + 1}{\alpha + (1/2)} \frac{u}{V} \left(1 + \frac{p_0}{\rho_0 V^2} + \frac{\Lambda \phi_s}{\rho_0 V^2} \right) \\ + \frac{\alpha + 1}{\alpha + (1/2)} \left(\frac{1}{2} + \frac{p_0}{\rho_0 V^2} + \frac{\Lambda \phi_{s0}}{\rho_0 V^2} \right) = 0 \end{aligned}$$

Sound speed

$$c^2 = \frac{\alpha + 1}{\alpha} \frac{p + \Lambda \phi_s}{\rho}$$

ITEM 4

①

1.

Characteristic analysis of the equal-temp. model

In nonconservative form the equations are

$$p_t + v p_x + p v_x = 0 \quad (1)$$

$$v_t + v v_x + \frac{1}{\rho} p_x = 0 \quad (2)$$

$$e_t + v e_x + \frac{p}{\rho} v_x = 0 \quad (3)$$

$$(\phi_p)_t + v (\phi_p)_x = \mathcal{F} + \frac{c}{\rho_p} \quad (4)$$

$$(p_p)_t + v (p_p)_x + p_p v_x = - \frac{p_p}{\phi_p} \mathcal{F} \quad (5)$$

$$\text{Also, } e = e(p, \rho, p_p, \phi_p) \quad (6)$$

and the equal-temp. condition gives

$$p_p = p_p(p, \rho, p_p, \phi_p) \quad (7)$$

$$\text{Now, } de = e_p dp + e_\rho d\rho + e_{p_p} dp_p + e_{\phi_p} d\phi_p$$

$$\rightarrow \frac{de}{dt} = e_p \frac{dp}{dt} + e_\rho \frac{d\rho}{dt} + e_{p_p} \frac{dp_p}{dt} + e_{\phi_p} \frac{d\phi_p}{dt}$$

$$\text{If } \frac{de}{dt} = \frac{\partial e}{\partial t} + v \frac{\partial e}{\partial x}, \text{ then}$$

$$\begin{aligned} \frac{de}{dt} &= e_p \frac{dp}{dt} - e_p \rho v_x - e_{p_p} \left[p_p v_x + \frac{p_p}{\phi_p} \mathcal{F} \right] \\ &\quad + e_{\phi_p} \left[\mathcal{F} + \frac{c}{\rho_p} \right] \end{aligned}$$

One may now replace (3) by

$$\begin{aligned} p_t + v p_x &= \frac{1}{e_p} \left[-\frac{p}{\rho} + \rho e_p + p_p e_{p_p} \right] v_x \\ &\quad + \frac{1}{e_p} \left[e_{p_p} \frac{p_p}{\phi_p} \mathcal{F} - e_{\phi_p} \left(\mathcal{F} + \frac{c}{\rho_p} \right) \right] \end{aligned}$$

$$\text{Define } \rho C^2 \equiv -\frac{1}{e_p} \left[-\frac{p}{\rho} + \rho e_p + p_p e_{p_p} \right] \quad (8)$$

(2)

$$\text{Then, } p_t + v p_x + \rho c^2 v_x = \frac{1}{e_p} \left[\frac{p_p}{\phi_p} \left(\frac{p}{\phi} \right) - e_{\phi} \left\{ F + \frac{c}{p} \right\} \right]$$

Now the eqns. may be written as

$$\underline{U}_t + \underline{A} \underline{U}_x = \underline{S}$$

where

$$\underline{U} = \langle p, v, p_p, p, \phi_p \rangle$$

$$\underline{A} = \begin{pmatrix} v & p & 0 & 0 & 0 \\ 0 & v & 0 & \frac{1}{\rho} & 0 \\ 0 & p_p & v & 0 & 0 \\ 0 & \rho c^2 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & v \end{pmatrix}$$

For eigenvalues, $\det (\underline{A} - \lambda \underline{I}) = 0$

$$\Rightarrow (v - \lambda)^3 \left[(v - \lambda)^2 - c^2 \right] = 0$$

$$\Rightarrow \lambda = v, v, v, v \pm c.$$

Characteristic Analysis of the 6 eqn. model ($v_p = v_g = v$)

Ignoring the source terms, the equations are

$$\rho_t + v \rho_x + \rho v_x = 0$$

$$v_t + v v_x + \frac{1}{\rho} p_x = 0$$

$$e_t + v e_x + \frac{p}{\rho} v_x = 0$$

$$(\Phi_p)_t + v (\Phi_p)_x = 0$$

$$(\rho_p)_t + v (\rho_p)_x + \rho_p v_x = 0$$

$$(e_p)_t + v (e_p)_x + \frac{p_p}{\rho_p} v_x = 0$$

$$\text{Let } (\dot{}) = \frac{\partial()}{\partial t} + v \frac{\partial()}{\partial x}$$

$$\text{Now, } e_p = e_p(\rho_p, p_p)$$

$$\therefore \dot{e}_p = \frac{\partial e_p}{\partial \rho_p} \dot{\rho}_p + \frac{\partial e_p}{\partial p_p} \dot{p}_p$$

$$= \frac{\partial e_p}{\partial \rho_p} \dot{\rho}_p - \frac{\partial e_p}{\partial p_p} \rho_p v_x$$

$$\therefore \dot{\rho}_p = \frac{1}{\partial e_p / \partial \rho_p} \left[\dot{e}_p + \frac{\partial e_p}{\partial p_p} \rho_p v_x \right]$$

$$= \frac{1}{\partial e_p / \partial \rho_p} \left[-\frac{p_p}{\rho_p} + \frac{\partial e_p}{\partial p_p} \rho_p \right] v_x$$

Define

$$\rho_p C_s^2 = \frac{1}{\partial e_p / \partial \rho_p} \left[\frac{p_p}{\rho_p} - \frac{\partial e_p}{\partial p_p} \rho_p \right]$$

④

Then, solid energy eqn. may be replaced by

$$(\dot{p}_s)_t + v(\dot{p}_s)_x + \rho_s C_p^2 \dot{v}_x = 0. \quad (1)$$

Similarly,

$$e = e(p, \rho, p_s, \rho_s, \phi_s) \quad (2)$$

$$\dot{e} = \frac{\partial e}{\partial p} \dot{p} + \frac{\partial e}{\partial \rho} \dot{\rho} + \frac{\partial e}{\partial p_s} \dot{p}_s + \frac{\partial e}{\partial \rho_s} \dot{\rho}_s$$

(In the absence of source terms, $\dot{\phi}_s = 0$)

$$= \frac{\partial e}{\partial p} \dot{p} + \dot{v}_x \left[-\frac{\partial e}{\partial \rho} \rho - \frac{\partial e}{\partial p_s} \rho_s C_p^2 - \frac{\partial e}{\partial \rho_s} \rho_s \right]$$

$$\therefore \dot{p} = \frac{-1}{e_p} \left[+\frac{p}{\rho} - \rho \frac{\partial e}{\partial \rho} - \rho_s C_p^2 \frac{\partial e}{\partial p_s} - \rho_s \frac{\partial e}{\partial \rho_s} \right]$$

Define

$$\rho C^2 = \frac{1}{e_p} \left[\frac{p}{\rho} - \rho \frac{\partial e}{\partial \rho} - \rho_s \frac{\partial e}{\partial p_s} - \rho_s C_p^2 \frac{\partial e}{\partial \rho_s} \right]$$

Then the bulk energy eqn. may be replaced by

$$\dot{p}_t + v \dot{p}_x + \rho C^2 \dot{v}_x = 0. \quad (4)$$

⑤

5.

Now define $\underline{U} = \langle \rho, v, p, \phi, p_s, p_p \rangle$.

Then the eqns. become

$$\underline{U}_t + \underline{A} \underline{U}_x = 0$$

where now,

$$\underline{A} = \begin{pmatrix} v & \rho & 0 & 0 & 0 & 0 \\ 0 & v & \frac{1}{\rho} & 0 & 0 & 0 \\ 0 & \rho c^2 & v & 0 & 0 & 0 \\ 0 & 0 & 0 & v & 0 & 0 \\ 0 & p_s & 0 & 0 & v & 0 \\ 0 & p_s c_s^2 & 0 & 0 & 0 & v \end{pmatrix}$$

$$\det \underline{A} - \lambda \underline{I} = 0 \Rightarrow$$

$$(v - \lambda)^4 [(v - \lambda)^2 - c^2] = 0$$

$$\Rightarrow \lambda = v, v, v, v, v \pm c.$$

Note that the definition (4.3) for C , valid for the 6-eqn. model, reduces to the defn. (1.8) for the 5-eqn. model because for the latter case, c does not depend explicitly on p_s .

Computation of sound speeds
I The solid sound speed, c_s

Recall that

$$(1) \quad \rho_s c_s^2 = \frac{1}{\rho_s} \frac{\partial e_s}{\partial p_s} \left[\frac{p_s}{\rho_s} - \rho_s \frac{\partial e_s}{\partial \rho_s} \right]$$

For the solid, Radzil gives the following eqn:

$$p_s - p_{int} = G_{ov} \frac{\rho_s}{\rho_{s0}} \left[e_s + T_3 \left(1 - \frac{\rho_s}{\rho_{s0}} \right) - \right.$$

$$\left. - T_4 \left\{ \left(\frac{\rho_s}{\rho_{s0}} \right)^{n-1} - \left(1 - \frac{\rho_s}{\rho_{s0}} \right) - 1 \right\} \right]$$

$$(2) \quad + A k_{to} \frac{1}{n} \left\{ \left(\frac{\rho_s}{\rho_{s0}} \right)^n - 1 \right\}$$

Solving for e_s ,

$$e_s = -T_3 \left(1 - \frac{\rho_s}{\rho_{s0}} \right) + T_4 \left\{ \left(\frac{\rho_s}{\rho_{s0}} \right)^{n-1} - \left(1 - \frac{\rho_s}{\rho_{s0}} \right) - 1 \right\}$$

$$(3) \quad + \frac{1}{G_{ov}} \frac{\rho_{s0}}{\rho_s} \left[p_s - p_{int} - A k_{to} \frac{1}{n} \left\{ \left(\frac{\rho_s}{\rho_{s0}} \right)^n - 1 \right\} \right]$$

$$(4) \quad \therefore \frac{\partial e_s}{\partial p_s} = \frac{1}{G_{ov}} \frac{\rho_{s0}}{\rho_s}$$

⑦

$$\begin{aligned} \frac{\partial e_s}{\partial p_s} = & \frac{1}{p_{s0}} \left(\frac{p_{s0}}{p_s} \right)^2 \left[-T_3 + (n-1) T_4 \left\{ \left(\frac{p_s}{p_{s0}} \right)^{n-1} \right\} - \right. \\ & - \frac{1}{G_{ov}} \left\{ p_s - p_{int} - A_{kto} \frac{1}{n} \left[\left(\frac{p_s}{p_{s0}} \right)^n - 1 \right] \right\} \\ & \left. - \frac{A_{kto}}{G_{ov}} \left(\frac{p_s}{p_{s0}} \right)^n \right] \end{aligned}$$

Thus, we can compute $p_s c_s^2$ by substituting for p_s from (1.2), $\partial e_s / \partial p_s$ from (1.4), and $\partial \tilde{e}_s / \partial p_s$ (2.1), into the definition (1.1).

II The mixture sound speed, c

Recall the definition

$$p c^2 = \frac{1}{\partial e / \partial p} \left[\frac{p}{p} - p \frac{\partial e}{\partial p} - p_s \frac{\partial e}{\partial p_s} - p_s c_s^2 \frac{\partial e}{\partial p_s} \right]$$

Here, it is assumed that

$$e = e(p, p, p_s, p_s, \phi)$$

and

$$c_p = c_p(p_p, p_s).$$

Also, the mixture variables are defined as

$$(1) \quad p = p_g \phi_g + p_l \phi_l, \quad \rho = \rho_g \phi_g + \rho_l \phi_l,$$

$$(2) \quad p = p_g \phi_g + p_l \phi_l,$$

$$(3) \quad p = p_g \phi_g + p_l \phi_l$$

Here, $\phi_g = 1 - \phi_l$ and it is assumed that

$$(4) \quad \epsilon_g = \epsilon_g(p_g, p_l).$$

Adair gives the following eqn. for ϵ_g :

$$\epsilon_g = -h_{ul} - c_{vg} T_{g0} + \frac{\phi_{g0} p_{g0}}{v_g} \left[\frac{v_g}{\omega} \left\{ \frac{p_{g0}}{p_{l0}} - \frac{p_{g0}}{p_{l0}} \right\} + \frac{R_1}{A} e^{-R_1 v_g} + \frac{R_2}{B} e^{-R_2 v_g} \right]$$

with

$$(6) \quad v_g = \frac{\phi_{g0} p_{g0}}{p_g}.$$

Also, from (2) and (3) above,

$$(7) \quad p_g = \frac{p - p_l \phi_l}{1 - \phi_l}, \quad p_l = \frac{p - p_g \phi_g}{1 - \phi_g}.$$

⑨

The last equations give

$$\frac{\partial p_g}{\partial p} = \frac{1}{1-\phi_s}, \quad \frac{\partial p_g}{\partial p_s} = \frac{-\phi_s}{1-\phi_s},$$

$$\frac{\partial p_g}{\partial p} = \frac{1}{1-\phi_s}, \quad \frac{\partial p_g}{\partial p_s} = \frac{-\phi_s}{1-\phi_s}.$$

Now, the formula (2.2) for C requires the derivatives

$$\frac{\partial e}{\partial p}, \quad \frac{\partial e}{\partial p_s}, \quad \frac{\partial e}{\partial p_s}, \quad \frac{\partial e}{\partial p_s},$$

where, from (3.1),

$$e = \frac{p_s \phi_s}{p} e_s(p_s, p_s) + \frac{1}{p} (p - p_s \phi_s) e_g(p_s, p_s)$$

Now,

$$\frac{\partial e}{\partial p} = \frac{1}{p} (p - p_s \phi_s) \frac{\partial e_g}{\partial p_g} \frac{\partial p_g}{\partial p}$$

On using the defn of $\partial p_g / \partial p$ from (2) above,

$$\frac{\partial e}{\partial p} = \frac{p - p_s \phi_s}{p} \frac{\partial e_g}{\partial p_g} \frac{1}{1-\phi_s}.$$

Similarly,

$$\frac{\partial e}{\partial p_s} = - \frac{p_s \phi_s}{p^2} e_s + \frac{p_s \phi_s}{p^2} e_g + \left(1 - \frac{p_s \phi_s}{p}\right) \frac{\partial e_g}{\partial p_g} \frac{1}{1-\phi_s}$$

$$\frac{\partial e}{\partial p_s} = \frac{\phi_s}{p} e_s + \frac{p_s \phi_s}{p} \frac{\partial e_s}{\partial p_s} - \frac{\phi_s}{p} e_g + \left(1 - \frac{p_s \phi_s}{p}\right) \frac{\partial e_g}{\partial p_g} \left(\frac{-\phi_s}{1-\phi_s}\right) \quad (1)$$

$$\frac{\partial e}{\partial p_s} = \frac{p_s \phi_s}{p} \frac{\partial e_s}{\partial p_s} + \left(1 - \frac{p_s \phi_s}{p}\right) \frac{\partial e_g}{\partial p_g} \left(\frac{-\phi_s}{1-\phi_s}\right) \quad (2)$$

The derivatives $\partial e_s / \partial p_s$ and $\partial e_s / \partial p_g$ have already been computed in (1.4) and (2.1). It remains to compute $\partial e_g / \partial p_g$ and $\partial e_g / \partial p_s$ from (3.5). We get

$$\frac{\partial e_g}{\partial p_g} = \frac{1}{\phi_{s0} p_{s0}} \frac{v_g}{\omega} = \frac{1}{\omega p_g} \quad (3)$$

$$\begin{aligned} \frac{\partial e_g}{\partial p_g} = & - \frac{1}{p_g^2} \left[\frac{1}{\omega} \left\{ p_g - A e^{-R_1 v_g} - B e^{-R_2 v_g} \right\} + \right. \\ & + \frac{v_g}{\omega} \left(A R_1 e^{-R_1 v_g} + B R_2 e^{-R_2 v_g} \right) \\ & \left. - A e^{-R_1 v_g} - B e^{-R_2 v_g} \right] \quad (4) \end{aligned}$$

We now have all the info. needed to compute c^2 .

①
Characteristic analysis of the 6-eqn. model
in the solid/gas form rather than the
solid/mixture form

The equations are:

mass

$$\frac{\partial}{\partial t} (\phi_s \rho_s) + \frac{\partial}{\partial x} (\phi_s \rho_s v) = C$$

$$\frac{\partial}{\partial t} (\phi_g \rho_g) + \frac{\partial}{\partial x} (\phi_g \rho_g v) = -C$$

momentum

$$\frac{\partial}{\partial t} [(\phi_s \rho_s + \phi_g \rho_g) v] + \frac{\partial}{\partial x} [(\phi_s \rho_s + \phi_g \rho_g) v^2 + (\phi_s p_s + \phi_g p_g)] = 0$$

energy

$$\frac{\partial}{\partial t} (\phi_s \rho_s e_s) + \frac{\partial}{\partial x} (\phi_s \rho_s e_s v) + \phi_s p_s \frac{\partial v}{\partial x} = C e_s - H(T_s - T_g) - (p_s - p_g) F$$

$$\frac{\partial}{\partial t} (\phi_g \rho_g e_g) + \frac{\partial}{\partial x} (\phi_g \rho_g e_g v) + \phi_g p_g \frac{\partial v}{\partial x} = -C e_s + H(T_s - T_g) + (p_g - p_s) F$$

compaction

$$\frac{\partial \phi_s}{\partial t} + v \frac{\partial \phi_s}{\partial x} = \frac{\partial \phi_s}{\partial t} - \frac{C}{\rho_s}$$

An alternate form is

mass

$$\frac{\partial}{\partial t} (\rho_s) + v \frac{\partial}{\partial x} (\rho_s) + \rho_s \frac{\partial v}{\partial x} = - \frac{\rho_s}{\phi_s} \mathcal{F}$$

$$\frac{\partial}{\partial t} (\rho_g) + v \frac{\partial}{\partial x} (\rho_g) + \rho_g \frac{\partial v}{\partial x} = \frac{\rho_g \mathcal{F}}{\phi_g} - (1 - \frac{\rho_s}{\rho_g}) \mathcal{F}$$

momentum

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0$$

where

$$\rho \equiv \phi_s \rho_s + \phi_g \rho_g ; \quad p \equiv \phi_s p_s + \phi_g p_g$$

energy

$$\frac{\partial e_s}{\partial t} + v \frac{\partial e_s}{\partial x} = - \frac{1}{\phi_s \rho_s} \left[H(T_s - T_g) + (p_s - p_g) \mathcal{F} - (p_s / \rho_s) v_x \right]$$

$$\frac{\partial e_g}{\partial t} + v \frac{\partial e_g}{\partial x} = \frac{1}{\phi_g \rho_g} \left[(e_g - e_s) c + H(T_s - T_g) + (p_s - p_g) \mathcal{F} - (p_g / \rho_g) v_x \right]$$

compaction

$$\frac{\partial \phi_s}{\partial t} + v \frac{\partial \phi_s}{\partial x} = \mathcal{F} + \frac{c}{\rho_s}$$

Also, eqn. of state:

$$e_g = e_g(p_g, \rho_g), \quad e_s = e_s(p_s, \rho_s)$$

With a dot denoting convective derivative, i.e., with

$$(\dot{}) \equiv \frac{\partial}{\partial t}() + v \frac{\partial}{\partial x}(),$$

$$\dot{e}_s = \frac{\partial e_s}{\partial p_s} \dot{p}_s + \frac{\partial e_s}{\partial f_s} \dot{f}_s$$

$$\Rightarrow \dot{p}_s = \frac{1}{\partial e_s / \partial p_s} \left[\dot{e}_s - \frac{\partial e_s}{\partial f_s} \dot{f}_s \right]$$

$$= \frac{1}{\partial e_s / \partial p_s} \left[-\frac{p_s}{f_s} \frac{\partial v}{\partial x} + \frac{\partial e_s}{\partial f_s} v_x \right] + \text{source terms}$$

or, with

$$p_s C_s^2 \equiv \frac{1}{\partial e_s / \partial p_s} \left[\frac{p_s}{f_s} - \frac{\partial e_s}{\partial f_s} \right],$$

$$\frac{\partial p_s}{\partial t} + v \frac{\partial p_s}{\partial x} + p_s C_s^2 \frac{\partial v}{\partial x} = \text{source terms.}$$

Similarly, for the gas,

$$\frac{\partial p_g}{\partial t} + v \frac{\partial p_g}{\partial x} + p_g C_g^2 \frac{\partial v}{\partial x} = \text{source terms,}$$

with C_g defined by an expression analogous to C_s .

Dropping the source terms altogether, the system takes the form

$$\frac{\partial}{\partial t} \underline{U} + \underline{A} \frac{\partial}{\partial x} \underline{U} = 0$$

where

$$(\underline{U})^T = (p_s, p_g, v, p_s, p_g, \phi_s)$$

and

$$\underline{A} = \begin{bmatrix} v & 0 & p_s & 0 & 0 & 0 \\ 0 & v & p_g & 0 & 0 & 0 \\ 0 & 0 & v & \frac{\phi_s}{\rho} & \frac{\phi_g}{\rho} & \frac{1}{\rho}(p_s - p_g) \\ 0 & 0 & p_s c_s^2 & v & 0 & 0 \\ 0 & 0 & p_g c_g^2 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & 0 & v \end{bmatrix}$$

The eigenvalues are given by

$$(v - \lambda)^4 \left[(v - \lambda)^2 - \frac{\phi_s p_s c_s^2 + \phi_g p_g c_g^2}{\rho} \right] = 0$$

$$\Rightarrow \lambda = v, v, v, v, v \pm c$$

where $\rho c^2 = \phi_s p_s c_s^2 + \phi_g p_g c_g^2$.

In the mixture formulation, the mixture ^{sound} speed is given by

$$\rho c^2 = \frac{1}{\partial e / \partial p} \left[\frac{p}{\rho} - \rho \frac{\partial e}{\partial \rho} - p_s \frac{\partial e}{\partial p_s} - p_s c_s^2 \frac{\partial e}{\partial p_s} \right] \quad (1)$$

Here,

$$p_s c_s^2 = \frac{1}{\partial e_s / \partial p_s} \left[\frac{p_s}{\rho_s} - \rho_s \frac{\partial e_s}{\partial \rho_s} \right] \quad (2)$$

$$\text{Also, } e = e(p, \rho, p_s, \rho_s, \phi_s) \quad (3)$$

$$e_s = e_s(p_s, \rho_s). \quad (4)$$

Let us attempt to change the above relation for c into one involving a gas/solid rather than a mixture/solid formulation. Recall that

$$\rho e = \rho_s \phi_s e_s + \rho_g \phi_g e_g \quad (5)$$

$$\rho = \rho_s \phi_s + \rho_g \phi_g \quad (6)$$

$$p = p_s \phi_s + p_g \phi_g \quad (7)$$

$$e_g = e_g(p_g, \rho_g) \quad (8)$$

We now compute the following derivatives of e by using eqn. (5) above. Note the arguments of e in eqn. (3).

$$\rho \frac{\partial e}{\partial p} = \rho_g \phi_g \frac{\partial e_g}{\partial p_g} \frac{\partial p_g}{\partial p} = \rho_g \phi_g \frac{\partial e_g}{\partial p_g} \frac{1}{\phi_g}$$

$$\text{or } \frac{\partial e}{\partial p} = \frac{\rho_g}{\rho} \frac{\partial e_g}{\partial p_g}$$

(16)

In computing $\partial e / \partial p_s$ at fixed p, p_s, p_s and ϕ_s , note that in eqn. (1.5), p_g varies. Thus,

$$p \frac{\partial e}{\partial p_s} = \phi_s e_s + p_s \phi_s \frac{\partial e_s}{\partial p_s} + \phi_g \left[e_g + p_g \frac{\partial e_g}{\partial p_g} \right] \frac{\partial p_g}{\partial p_s}$$

where, from (1.6),

$$0 = \phi_s + \phi_g \frac{\partial p_g}{\partial p_s} \Rightarrow \frac{\partial p_g}{\partial p_s} = - \frac{\phi_s}{\phi_g}$$

$$\therefore p \frac{\partial e}{\partial p_s} = \phi_s e_s + p_s \phi_s \frac{\partial e_s}{\partial p_s} - \phi_s e_g - \phi_s p_g \frac{\partial e_g}{\partial p_g}$$

Now we compute $\frac{\partial e}{\partial p_s}$ at fixed p, p_s, ϕ_s and p , but with p_g varying, such that

$$\frac{\partial p_g}{\partial p_s} = - \frac{\phi_s}{\phi_g} \quad (17)$$

Therefore,

$$p \frac{\partial e}{\partial p_s} = p_s \phi_s \frac{\partial e_s}{\partial p_s} + p_g \phi_g \frac{\partial e_g}{\partial p_g} \left(- \frac{\phi_s}{\phi_g} \right)$$

$$\text{or, } p \frac{\partial e}{\partial p_s} = \phi_s p_s \frac{\partial e_s}{\partial p_s} - \phi_s p_g \frac{\partial e_g}{\partial p_g}$$

Finally, we compute $\frac{\partial e}{\partial p}$ at fixed p, p_s, p_s, ϕ_s , but with varying p_g , with $\partial p_g / \partial p = 1 / \phi_g$. Therefore,

$$e + p \frac{\partial e}{\partial p} = \left[\phi_g e_g + p_g \phi_g \frac{\partial e_g}{\partial p_g} \right] \frac{1}{\phi_g}$$

or,

$$p \frac{\partial e}{\partial p} = -e + e_g + p_g \frac{\partial e_g}{\partial p_g} \quad (18)$$

We shall now substitute these derivatives in (1.1) to see what the simplified expression for c is.

Now, (1.1) $\Rightarrow$

$$\begin{aligned} \rho c^2 \frac{\partial e}{\partial p} &= \frac{p}{\rho} - \rho \frac{\partial e}{\partial \rho} - \rho_s \frac{\partial e}{\partial \rho_s} - \rho_s c_s^2 \frac{\partial e}{\partial \rho_s} \\ &= \frac{p}{\rho} + \left[e \cdot e_g - \rho_g \frac{\partial e_g}{\partial \rho_g} \right] \\ &\quad - \frac{\rho_s}{\rho} \left[\phi_s e_s + \rho_s \phi_s \frac{\partial e_s}{\partial \rho_s} - \phi_s e_g - \phi_s \rho_g \frac{\partial e_g}{\partial \rho_g} \right] \\ &\quad - \rho_s c_s^2 \frac{\partial e}{\partial \rho_s} \end{aligned}$$

Now,

$$\begin{aligned} e - e_g - \frac{\rho_s}{\rho} \phi_s [e_s - e_g] &= \frac{1}{\rho} [\rho e - \rho_s \phi_s e_s - \{\rho - \rho_s \phi_s\} e_g] \\ &= \frac{1}{\rho} [\rho e - \rho_s \phi_s e_s - \rho_g \phi_g e_g] = 0. \end{aligned}$$

$$\begin{aligned} \rho c^2 \frac{\partial e}{\partial p} &= \frac{p}{\rho} - \rho_g \frac{\partial e_g}{\partial \rho_g} \left\{ 1 - \frac{\rho_s \phi_s}{\rho} \right\} - \frac{\rho_s^2}{\rho} \phi_s \frac{\partial e_s}{\partial \rho_s} \\ &\quad - \rho_s c_s^2 \frac{1}{\rho} \left[\phi_s \rho_s \frac{\partial e_s}{\partial \rho_s} - \phi_s \rho_g \frac{\partial e_g}{\partial \rho_g} \right] \\ &= \frac{\rho_s \phi_s}{\rho} - \frac{\rho_s^2 \phi_s}{\rho} \frac{\partial e_s}{\partial \rho_s} + \frac{\rho_g \phi_g}{\rho} - \frac{\rho_s^2 \phi_g}{\rho} \frac{\partial e_g}{\partial \rho_g} \\ &\quad - \frac{\rho_s \phi_s c_s^2}{\rho} \left[\rho_s \frac{\partial e_s}{\partial \rho_s} - \rho_g \frac{\partial e_g}{\partial \rho_g} \right] \\ &= \frac{\rho_s \phi_s}{\rho} \left[\frac{\rho_s}{\rho_s} - \rho_s \frac{\partial e_s}{\partial \rho_s} \right] + \frac{\rho_g \phi_g}{\rho} \left[\frac{\rho_g}{\rho_g} - \rho_g \frac{\partial e_g}{\partial \rho_g} \right] \end{aligned}$$

$$- \frac{\rho_s \phi_s c_s^2}{\rho} \left[\rho_s \frac{\partial e_s}{\partial p_s} - \rho_g \frac{\partial e_g}{\partial p_g} \right]$$

Now use the definition

$$\frac{\partial e_s}{\partial p_s} \rho_s c_s^2 = \frac{p_s}{\rho_s} - \rho_s \frac{\partial e_s}{\partial \rho_s}$$

and a similar one for ρ_g . Then,

$$\begin{aligned} \rho c^2 \frac{\partial e}{\partial p} &= \rho_g c^2 \frac{\partial e_g}{\partial p_g} = \frac{\rho_s^2 \phi_s c_s^2}{\rho} \frac{\partial e_s}{\partial p_s} \\ &\quad + \frac{\rho_g^2 \phi_g c_g^2}{\rho} \frac{\partial e_g}{\partial p_g} \\ &\quad - \frac{\rho_s \phi_s c_s^2}{\rho} \left[\rho_s \frac{\partial e_s}{\partial p_s} - \rho_g \frac{\partial e_g}{\partial p_g} \right] \end{aligned}$$

$$\text{or, } \rho_g c^2 \frac{\partial e_g}{\partial p_g} = \frac{\rho_g}{\rho} \left[\rho_g \phi_g c_g^2 \frac{\partial e_g}{\partial p_g} + \rho_s \phi_s c_s^2 \frac{\partial e_g}{\partial p_g} \right]$$

$$\text{or, } \frac{1}{\rho} \rho_g \frac{\partial e_g}{\partial p_g} \left[\rho c^2 - \rho_s \phi_s c_s^2 - \rho_g \phi_g c_g^2 \right] = 0$$

$\Rightarrow$

$$\rho c^2 = \rho_s \phi_s c_s^2 + \rho_g \phi_g c_g^2$$

as obtained earlier.

The Six-Equation Model in Characteristic Form

We start with the familiar, near-conservative form of the equations:

$$\frac{\partial}{\partial t}(\phi_s \rho_s) + \frac{\partial}{\partial x}(\phi_s \rho_s v) = C_s, \quad (1)$$

$$\frac{\partial}{\partial t}(\phi_g \rho_g) + \frac{\partial}{\partial x}(\phi_g \rho_g v) = -C_s, \quad (2)$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) = 0, \quad (3)$$

$$\frac{\partial}{\partial t}(\phi_s \rho_s e_s) + \frac{\partial}{\partial x}(\phi_s \rho_s e_s v) + \phi_s p_s \frac{\partial v}{\partial x} = C_s e_s - H(T_s - T_g) - (p_s - \beta_s) \mathcal{F}, \quad (4)$$

$$\frac{\partial}{\partial t}(\phi_g \rho_g e_g) + \frac{\partial}{\partial x}(\phi_g \rho_g e_g v) + \phi_g p_g \frac{\partial v}{\partial x} = -C_s e_s + H(T_s - T_g) + (p_s - \beta_s) \mathcal{F}, \quad (5)$$

$$\frac{\partial}{\partial t}(\phi_s) + v \frac{\partial}{\partial x}(\phi_s) = \mathcal{F} + \frac{C_s}{\rho_s}. \quad (6)$$

The mixture quantities appearing above are defined as

$$\rho = \phi_s \rho_s + \phi_g \rho_g, \quad p = \phi_s p_s + \phi_g p_g \quad (7)$$

The compaction function is

$$\mathcal{F} = \frac{\phi_s \phi_g}{\mu_c} (p_s - p_g - \beta_s) \quad (8)$$

For characteristic calculations, it is more convenient to rewrite the equations in the following alternate form:

Mass conservation

$$\frac{\partial}{\partial t}(\rho_s) + v \frac{\partial}{\partial x}(\rho_s) + \rho_s \frac{\partial v}{\partial x} = -\frac{\rho_s}{\phi_s} \mathcal{F} \quad (9)$$

$$\frac{\partial}{\partial t}(\rho_g) + v \frac{\partial}{\partial x}(\rho_g) + \rho_g \frac{\partial v}{\partial x} = \frac{\rho_g}{\phi_g} \mathcal{F} - \left(1 - \frac{\rho_g}{\rho_s}\right) \frac{C_s}{\phi_g} \quad (10)$$

Momentum balance

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \left\{ \phi_s \frac{\partial}{\partial x}(p_s) + \phi_g \frac{\partial}{\partial x}(p_g) + (p_s - p_g) \frac{\partial}{\partial x}(\phi_s) \right\} = 0 \quad (11)$$

Here, the mixture pressure p has been replaced by its definition (7b).

Energy balance

$$\frac{\partial}{\partial t}(e_s) + v \frac{\partial}{\partial x}(e_s) + \frac{p_s}{\rho_s} \frac{\partial v}{\partial x} = -\frac{1}{\phi_s \rho_s} \{H(T_s - T_g) + (p_s - \beta_s) \mathcal{F}\} \quad (12)$$

$$\frac{\partial}{\partial t}(e_g) + v \frac{\partial}{\partial x}(e_g) + \frac{p_g}{\rho_g} \frac{\partial v}{\partial x} = \frac{1}{\phi_g \rho_g} \{H(T_s - T_g) + (p_s - \beta_s) \mathcal{F} + (e_g - e_s) C_s\} \quad (13)$$

The equations of state $e_s = e_s(p_s, \rho_s)$ and $e_g = e_g(p_g, \rho_g)$ can be used to recast the energy equations further. Note that

$$\dot{e}_s = \frac{\partial e_s}{\partial p_s} \dot{p}_s + \frac{\partial e_s}{\partial \rho_s} \dot{\rho}_s$$

whence

$$\dot{p}_s = \frac{1}{\partial e_s / \partial p_s} \left\{ \dot{e}_s - \frac{\partial e_s}{\partial \rho_s} \dot{\rho}_s \right\} \quad (14)$$

The solid energy equation now yields

$$\frac{\partial}{\partial t}(p_s) + v \frac{\partial}{\partial x}(p_s) = \frac{1}{\partial e_s / \partial p_s} \left[-\frac{p_s}{\rho_s} \frac{\partial v}{\partial x} - \frac{1}{\phi_s \rho_s} \{H(T_s - T_g) + (p_s - \beta_s) \mathcal{F}\} + \frac{\partial e_s}{\partial \rho_s} \left\{ \frac{\partial v}{\partial x} + \frac{\mathcal{F}}{\phi_s} \right\} \right]$$

which, in view of the definition

$$\rho_s c_s^2 = \frac{1}{\partial e_s / \partial p_s} \left\{ \frac{p_s}{\rho_s} - \rho_s \frac{\partial e_s}{\partial \rho_s} \right\}, \quad (15)$$

becomes

$$\frac{\partial}{\partial t}(p_s) + v \frac{\partial}{\partial x}(p_s) + \rho_s c_s^2 \frac{\partial v}{\partial x} = E_s \quad (16)$$

where

$$E_s = \frac{1}{\partial e_s / \partial p_s} \left[-\frac{1}{\phi_s \rho_s} \{H(T_s - T_g) + (p_s - \beta_s) \mathcal{F}\} + \frac{\partial e_s}{\partial \rho_s} \frac{\rho_s}{\phi_s} \left\{ \frac{\partial v}{\partial x} + \frac{\mathcal{F}}{\phi_s} \right\} \right] \quad (17)$$

Equation (16) will now be treated as an alternate form of the solid energy equation. Analogously, the gas energy equation turns out to be

$$\frac{\partial}{\partial t}(p_g) + v \frac{\partial}{\partial x}(p_g) + \rho_g c_g^2 \frac{\partial v}{\partial x} = E_g \quad (18)$$

with

$$E_g = \frac{1}{\partial e_g / \partial p_g} \left[\frac{1}{\phi_g \rho_g} \{H(T_s - T_g) + (p_s - \beta_s) \mathcal{F} + (e_g - e_s) \mathcal{C}_s\} + \frac{\partial e_g}{\partial \rho_g} \left\{ -\frac{\rho_g}{\phi_g} \frac{\partial v}{\partial x} + \left(1 - \frac{\rho_g}{\rho_s} \right) \frac{\mathcal{C}_s}{\phi_g} \right\} \right] \quad (19)$$

and a definition similar to (15) for c_g .

To summarize, the 6-equation model now consists of

$$\frac{\partial}{\partial t}(\rho_s) + v \frac{\partial}{\partial x}(\rho_s) + \rho_s \frac{\partial v}{\partial x} = -\frac{\rho_s}{\phi_s} \frac{\partial \phi_s}{\partial x} \quad (20)$$

$$\frac{\partial}{\partial t}(\rho_g) + v \frac{\partial}{\partial x}(\rho_g) + \rho_g \frac{\partial v}{\partial x} = \frac{\rho_g}{\phi_g} \mathcal{F} - \left(1 - \frac{\rho_g}{\rho_s} \right) \frac{\partial \phi_s}{\partial x} \quad (21)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \left\{ \phi_s \frac{\partial}{\partial x}(p_s) + \phi_g \frac{\partial}{\partial x}(p_g) + (p_s - p_g) \frac{\partial}{\partial x}(\phi_s) \right\} = 0 \quad (22)$$

$$\frac{\partial}{\partial t}(p_s) + v \frac{\partial}{\partial x}(p_s) + \rho_s c_s^2 \frac{\partial v}{\partial x} = E_s \quad (23)$$

$$\frac{\partial}{\partial t}(p_g) + v \frac{\partial}{\partial x}(p_g) + \rho_g c_g^2 \frac{\partial v}{\partial x} = E_g \quad (24)$$

$$\frac{\partial}{\partial t}(\phi_s) + v \frac{\partial}{\partial x}(\phi_s) = \mathcal{F} + \frac{\mathcal{C}_s}{\rho_s} \quad (25)$$

This hyperbolic system for the vector

$$\mathbf{U} = \langle \rho_s, \rho_g, v, p_s, p_g, \phi_s \rangle$$

has eigenvalues v (of multiplicity 4) and $v \pm c$. The corresponding left eigenvectors are

$$L_1 = \langle 0, 0, 0, 0, 0, 1 \rangle, \quad (26)$$

$$L_2 = \langle -\rho_g, \rho_s, 0, 0, 0, 0 \rangle, \quad (27)$$

$$L_3 = \langle -c_s^2, 0, 0, 1, 0, 0 \rangle, \quad (28)$$

$$L_4 = \langle 0, -c_g^2, 0, 0, 1, 0 \rangle \quad (29)$$

for the eigenvalue v , and

$$L_5 = \langle 0, 0, \rho c, \phi_s, \phi_g, p_s - p_g \rangle, \quad (30)$$

$$L_6 = \langle 0, 0, -\rho c, \phi_s, \phi_g, p_s - p_g \rangle, \quad (31)$$

for the acoustic eigenvalues $v \pm c$. The equivalent characteristic system can now be obtained by pre-multiplying (20 - 25) with the left eigenvectors. After some manipulation, the following set emerges:

$$\frac{d\phi_s}{dt} = \mathcal{F} + \frac{C_s}{\rho_s} \quad (32)$$

$$\rho_s \frac{d\rho_g}{dt} - \rho_g \frac{d\rho_s}{dt} = \frac{\rho_s \rho_g}{\phi_s \phi_g} \mathcal{F} - \frac{\rho_s - \rho_g}{\phi_g} C_s \quad (33)$$

$$\frac{dp_s}{dt} - c_s^2 \frac{d\rho_s}{dt} = E_s + \frac{\rho_s c_s^2}{\phi_s} \mathcal{F} \quad (34)$$

$$\frac{dp_g}{dt} - c_g^2 \frac{d\rho_g}{dt} = E_g - \frac{\rho_g c_g^2}{\phi_g} \mathcal{F} + \left(1 - \frac{\rho_g}{\rho_s}\right) \frac{c_g^2}{\phi_g} C_s \quad (35)$$

$$\frac{d^+ p}{dt} + \rho c \frac{d^+ v}{dt} = \phi_s E_s + \phi_g E_g + (p_s - p_g) \left(\mathcal{F} + \frac{C_s}{\rho_s} \right) \quad (36)$$

$$\frac{d^- p}{dt} - \rho c \frac{d^- v}{dt} = \phi_s E_s + \phi_g E_g + (p_s - p_g) \left(\mathcal{F} + \frac{C_s}{\rho_s} \right) \quad (37)$$

where d/dt denotes rate of change along the particle path and $d^\pm/dt$ the rate of change along the sonic lines.

characteristics of BKS, including compaction

For $\delta \rightarrow \infty$ and $\mu_c \rightarrow 0$, BKS has the following form:

$$\frac{\partial}{\partial t} (\phi_s \rho_s) + \frac{\partial}{\partial x} (\phi_s \rho_s v) = C \quad (1)$$

$$\frac{\partial}{\partial t} (\phi_g \rho_g) + \frac{\partial}{\partial x} (\phi_g \rho_g v) = -C \quad (2)$$

$$\frac{\partial}{\partial t} (\rho v) + \frac{\partial}{\partial x} (\rho + \rho v^2) = 0 \quad (3)$$

where $\rho = \phi_s \rho_s + \phi_g \rho_g$; $\rho = \phi_s \rho_s + \phi_g \rho_g$

$$\begin{aligned} \frac{\partial}{\partial t} (\phi_s \rho_s e_s) + \frac{\partial}{\partial x} (\phi_s \rho_s e_s v) + \phi_s \rho_s \frac{\partial v}{\partial x} \\ + \rho_g \left(\frac{\partial \phi_s}{\partial t} + v \frac{\partial \phi_s}{\partial x} \right) = C \left(e_s + \frac{\rho_g}{\rho_s} \right) - H(T_s - T_g) \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{\partial}{\partial t} (\phi_g \rho_g e_g) + \frac{\partial}{\partial x} (\phi_g \rho_g e_g v) + \phi_g \rho_g \frac{\partial v}{\partial x} \\ - \rho_g \left(\frac{\partial \phi_s}{\partial t} + v \frac{\partial \phi_s}{\partial x} \right) = C \left(-e_s - \frac{\rho_g}{\rho_s} \right) + H(T_s - T_g) \end{aligned} \quad (5)$$

$$\rho_s - \rho_g - \rho_s (\phi_s) = 0. \quad (6)$$

Before computing the characteristics, these equations need to be recast in the form

$$U_t + A U_x = R \quad (7)$$

We shall take to be the 5-vector

$$U = \langle \phi_s \rho_s, \phi_g \rho_g, v, \rho_s, \phi_s \rangle.$$

Note that since (6) can be thought of as the definition of p_g , the system is effectively 5th order.

Write (1.1) and (1.2) as

$$\frac{\partial}{\partial t} (\phi_s p_s) + v \frac{\partial}{\partial x} (\phi_s p_s) + \phi_s p_s \frac{\partial v}{\partial x} = C \quad (1)$$

$$\frac{\partial}{\partial t} (\phi_g p_g) + v \frac{\partial}{\partial x} (\phi_g p_g) + \phi_g p_g \frac{\partial v}{\partial x} = -C \quad (2)$$

Mixture mass equation, $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0$, reduces (1.3) to

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0.$$

With $p = \phi_s p_s + (1 - \phi_s) (p_s - \beta_s) = p_s - (1 - \phi_s) \beta_s$, the above reduces further to

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p_s}{\partial x} + \left\{ \frac{\beta_s}{\rho} - \phi_g \frac{\beta'_s(\phi_s)}{\rho} \right\} \frac{\partial \phi_s}{\partial x} = 0 \quad (3)$$

Consider now the solid energy eqn. (1.4). With

$$\dot{(\quad)} = \frac{\partial}{\partial t} (\quad) + v \frac{\partial}{\partial x} (\quad),$$

it can be rearranged as (upon using (1.1))

$$\dot{e}_s + \frac{p_s}{\rho_s} v_x + \frac{p_g}{\phi_s \rho_s} \dot{\phi}_s = \frac{1}{\phi_s \rho_s} \left[\frac{p_g}{\rho_s} C - H(T_s - T_g) \right]$$

Now, with $e_s = e_s(p_s, \rho_s)$,

$$\begin{aligned} \dot{p}_s &= \frac{1}{\partial e_s / \partial p_s} \left[\dot{e}_s - \frac{\partial e_s}{\partial \rho_s} \dot{\rho}_s \right] \\ &= \frac{1}{\partial e_s / \partial p_s} \left[-\frac{p_s}{\rho_s} v_x - \frac{p_g}{\phi_s \rho_s} \dot{\phi}_s - \frac{\partial e_s}{\partial \rho_s} \frac{1}{\phi_s} \left\{ -\rho_s \dot{\phi}_s - \phi_s \rho_s v_x + C \right\} \right. \\ &\quad \left. + \frac{1}{\phi_s \rho_s} \left\{ \frac{p_g}{\rho_s} C - H(T_s - T_g) \right\} \right] \end{aligned}$$

Recall that

$$p_s C_s^2 = \frac{1}{\partial e_s / \partial p_s} \left[\frac{p_s}{p_s} - p_s \frac{\partial e_s}{\partial p_s} \right]$$

Thus,

$$\begin{aligned} \dot{p}_s &= -p_s C_s^2 \frac{\partial v}{\partial x} + \frac{1}{\phi_s} \frac{\dot{\phi}_s}{\partial e_s / \partial p_s} \left[p_s \frac{\partial e_s}{\partial p_s} - \frac{p_s}{p_s} + \frac{p_s - p_g}{p_s} \right] \\ &= \frac{\partial e_s / \partial p_s}{\partial e_s / \partial p_s} \frac{C}{\phi_s} + \frac{1}{\phi_s p_s} \left[\frac{p_g}{p_s} C - H(T_s - T_g) \right] \frac{1}{\partial e_s / \partial p_s} \end{aligned}$$

or,

$$\begin{aligned} \dot{p}_s + p_s C_s^2 \frac{\partial v}{\partial x} + \frac{\dot{\phi}_s}{\phi_s} \left[p_s C_s^2 - \frac{1}{\partial e_s / \partial p_s} \frac{p_s - p_g}{p_s} \right] \\ = - \frac{\partial e_s / \partial p_s}{\partial e_s / \partial p_s} \frac{C}{\phi_s} + \frac{1}{\phi_s p_s} \left[\frac{p_g}{p_s} C - H(T_s - T_g) \right] \frac{1}{\partial e_s / \partial p_s} \quad (1) \end{aligned}$$

We now turn to the gas energy eqn. (1.5). Proceeding as with the solid energy,

$$\begin{aligned} \dot{e}_g + \frac{p_g}{p_g} v_x - \frac{p_g}{\phi_g p_g} \dot{\phi}_s &= \frac{1}{\phi_g p_g} \left[-\frac{p_g}{p_s} C + H(T_s - T_g) \right] \quad (2) \\ \dot{p}_g &= \frac{1}{\partial e_g / \partial p_g} \left[\dot{e}_g - \frac{\partial e_g}{\partial p_g} \dot{p}_g \right] \\ &= \frac{1}{\partial e_g / \partial p_g} \left[-\frac{p_g}{p_s} v_x + \frac{p_g}{\phi_g p_g} \dot{\phi}_s - \frac{\partial e_g}{\partial p_g} \frac{1}{\phi_g} \left\{ + p_g \dot{\phi}_s \right. \right. \\ &\quad \left. \left. - \phi_g p_g v_x - C \right\} + \frac{1}{\phi_g p_g} \left\{ -\frac{p_g}{p_s} C + H(T_s - T_g) \right\} \right] \\ &= -p_g C_g^2 v_x + \frac{\dot{\phi}_s}{\phi_g} \frac{1}{\partial e_g / \partial p_g} \left\{ \frac{p_g}{p_s} - p_g \frac{\partial e_g}{\partial p_g} \right\} \\ &\quad + \frac{1}{\partial e_g / \partial p_g} \left[\left(\frac{\partial e_g}{\partial p_g} \right) \frac{C}{\phi_g} + \frac{1}{\phi_g p_g} \left\{ -\frac{p_g}{p_s} C + H(T_s - T_g) \right\} \right] \end{aligned}$$

or,

$$\dot{p}_g + p_g \dot{g}^2 v_x + \frac{\dot{\phi}_g}{\phi_g} \left\{ -p_g \dot{g}^2 \right\} \\ = \frac{1}{\partial e_g / \partial p_g} \left[\frac{C}{\phi_g} \frac{\partial e_g}{\partial p_g} + \frac{1}{\phi_g p_g} \left\{ -\frac{p_g}{p_s} C + H(T_s - T_g) \right\} \right] \quad (1)$$

Adding (3.1) and -(4.1),

$$\dot{p}_s - \dot{p}_g + (p_s \dot{c}_s^2 - p_g \dot{g}^2) v_x + \dot{\phi}_s \left[\frac{p_s \dot{c}_s^2}{\phi_s} + \frac{p_g \dot{g}^2}{\phi_g} \right. \\ \left. - \frac{1}{\partial e_s / \partial p_s} \frac{p_s - p_g}{\phi_s p_s} \right] = \frac{1}{\phi_s \partial e_s / \partial p_s} \left[-\frac{\partial e_s}{\partial p_s} C \right. \\ \left. + \frac{1}{p_s} \left\{ \frac{p_g}{p_s} C - H(T_s - T_g) \right\} \right] - \frac{1}{\phi_g \partial e_g / \partial p_g} \left[\frac{\partial e_g}{\partial p_g} C \right. \\ \left. + \frac{1}{p_g} \left\{ -\frac{p_g}{p_s} C + H(T_s - T_g) \right\} \right] = R_{sg}, \text{ say.} \quad (2)$$

With $p_s - p_g = \beta_s(\phi_s)$, the above becomes

$$\left[\beta'_s + \frac{p_s \dot{c}_s^2}{\phi_s} + \frac{p_g \dot{g}^2}{\phi_g} - \frac{1}{\partial e_s / \partial p_s} \frac{\beta_s}{\phi_s p_s} \right] \dot{\phi}_s + (p_s \dot{c}_s^2 - p_g \dot{g}^2) v_x \\ = R_{sg} \quad (3)$$

Eqs. (2.1), (2.2), (2.3), (3.1) and (4.3) now constitute the recast 5th-order system.

With

$$A \equiv \frac{p_s \dot{c}_s^2}{\phi_s} + \frac{p_g \dot{g}^2}{\phi_g} + \beta'_s - \frac{1}{\partial e_s / \partial p_s} \frac{\beta_s}{\phi_s p_s}, \quad (4)$$

$$\text{and } B \equiv \frac{1}{\phi_s} \left[p_s \dot{c}_s^2 - \frac{1}{\partial e_s / \partial p_s} \frac{\beta_s}{p_s} \right], \quad (5)$$

the system becomes

$$\frac{\partial}{\partial t} (\phi_s p_s) + v \frac{\partial}{\partial x} (\phi_s p_s) + \phi_s p_s v_x = C \quad (1)$$

$$\frac{\partial}{\partial t} (\phi_g p_g) + v \frac{\partial}{\partial x} (\phi_g p_g) + \phi_g p_g v_x = -C \quad (2)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p_s}{\partial x} + \frac{\beta_s - \phi_g \beta_g'}{\rho} \frac{\partial \phi_g}{\partial x} = 0, \quad (3)$$

$$\frac{\partial p_s}{\partial t} + v \frac{\partial p_s}{\partial x} + p_s c_s^2 v_x + \beta \left(\frac{\partial \phi_g}{\partial t} + v \frac{\partial \phi_g}{\partial x} \right) = \dots \quad (4)$$

$$\frac{\partial \phi_s}{\partial t} + v \frac{\partial \phi_s}{\partial x} + \frac{p_s c_s^2 - \beta_g c_g^2}{A} v_x = \dots \quad (5)$$

The 4th eqn. above may be rewritten as

$$\frac{\partial p_s}{\partial t} + v \frac{\partial p_s}{\partial x} + v_x \left[p_s c_s^2 - \frac{\beta}{A} (p_s c_s^2 - \beta_g c_g^2) \right] = \dots \quad (4')$$

Note that

$$A = \beta + \frac{\beta_g c_g^2}{\phi_g} + \beta_s'$$

$$\therefore p_s c_s^2 - \frac{\beta}{A} (p_s c_s^2 - \beta_g c_g^2)$$

$$= \frac{1}{A} \left[\cancel{p_s c_s^2 \beta} + \frac{p_s c_s^2 \beta_g c_g^2}{\phi_g} + p_s c_s^2 \beta_s' - \cancel{p_s c_s^2 \beta} - \beta_g c_g^2 \beta \right]$$

$$= \frac{1}{A} \left[\frac{p_s c_s^2 \beta_g c_g^2}{\phi_g} + p_s c_s^2 \beta_s' + \frac{\beta_g c_g^2}{\phi_g} - \frac{\beta_g c_g^2}{\phi_g} \frac{\beta}{p_s \phi_s} \frac{1}{\partial \phi_s / \partial p_s} \right]$$

$$= \frac{1}{A} \left[\frac{p_s c_s^2 \beta_g c_g^2}{\phi_g \phi_g} + p_s c_s^2 \beta_s' - \frac{\beta_g c_g^2}{p_s \phi_s} \beta_s \frac{1}{\partial \phi_s / \partial p_s} \right]$$

$$= C/A, \text{ say.} \quad (6)$$

The system now takes the form $\underline{U}_t + \underline{A} \underline{U}_1 = \dots$
where

$$\underline{A} = \begin{bmatrix} v & 0 & \phi_s \rho_s & 0 & 0 \\ 0 & v & \phi_g \rho_g & 0 & 0 \\ 0 & 0 & v & \frac{1}{\rho} & \frac{\beta_s - \phi_g \beta'_s}{\rho} \\ 0 & 0 & \frac{C}{A} & v & 0 \\ 0 & 0 & \frac{\rho_s c_s^2 - \rho_g c_g^2}{A} & 0 & v \end{bmatrix}$$

The eigenvalues are given by

$$(v-\lambda)^2 \left[-\frac{\rho_s c_s^2 - \rho_g c_g^2}{A} (v-\lambda) \frac{\beta_s - \phi_g \beta'_s}{\rho} + (v-\lambda) \left\{ (v-\lambda)^2 - \frac{C}{A\rho} \right\} \right] = 0$$

$$(v-\lambda)^3 \left[(v-\lambda)^2 - \frac{C}{A\rho} - \frac{(\rho_s c_s^2 - \rho_g c_g^2)(\beta_s - \phi_g \beta'_s)}{A\rho} \right] = 0$$

consider $D^2 \equiv$

$$\left[C + (\rho_s c_s^2 - \rho_g c_g^2)(\beta_s - \phi_g \beta'_s) \right]$$

$$= \left[\frac{\rho_s c_s^2 \cdot \rho_g c_g^2}{\phi_s \phi_g} + \rho_s c_s^2 \beta'_s - \frac{\rho_g c_g^2}{\rho_s \phi_s} \rho_s \frac{1}{\partial \epsilon_s / \partial \rho_s} - \phi_g \beta'_s (\rho_s c_s^2 - \rho_g c_g^2) + \beta_s (\rho_s c_s^2 - \rho_g c_g^2) \right] \frac{1}{A\rho}$$

$$\frac{1}{A\rho} \left\{ \frac{\rho_s c_s^2 \cdot \rho_g c_g^2}{\phi_s \phi_g} + \beta'_s (\rho_s \phi_s c_s^2 + \rho_g \phi_g c_g^2) + \beta_s \left[\rho_s c_s^2 - \rho_g c_g^2 - \frac{\rho_g c_g^2}{\rho_s \phi_s \partial \epsilon_s / \partial \rho_s} \right] \right\}$$

$$= \frac{1}{\rho} \frac{\frac{\rho_s c_s^2 \cdot \rho_g G^2}{\phi_s \phi_g} + \beta'_s (\rho_s \phi_s c_s^2 + \rho_g \phi_g G^2) + \beta_s \left\{ \rho_s c_s^2 - \rho_g G^2 - \frac{\rho_g G^2}{\rho_s \phi_s \partial c_s / \partial p_s} \right\}}{\frac{\rho_s c_s^2}{\phi_s} + \frac{\rho_g G^2}{\phi_g} + \beta'_s - \frac{\beta_s}{\phi_s \rho_s \partial c_s / \partial p_s}}$$

$$\frac{1}{\rho^2} = \frac{\frac{\rho_s c_s^2 \phi_g + \rho_g G^2 \phi_s}{\phi_s \phi_g} + \beta'_s - \frac{\beta_s}{\phi_s \rho_s \partial c_s / \partial p_s}}{\frac{\rho_s c_s^2}{\phi_s} + \frac{\rho_g G^2}{\phi_g} + \beta'_s - \frac{\beta_s}{\phi_s \rho_s \partial c_s / \partial p_s}}$$

$$\frac{1}{\rho^2} = \frac{\frac{\rho_s c_s^2 \rho_g G^2}{\phi_s \phi_g} + \beta'_s (\rho_s \phi_s c_s^2 + \rho_g \phi_g G^2) + \beta_s \left\{ \rho_s c_s^2 - \rho_g G^2 - \frac{\rho_g G^2}{\phi_s \rho_s \partial c_s / \partial p_s} \right\}}{\frac{\rho_s c_s^2}{\phi_s} + \frac{\rho_g G^2}{\phi_g} + \beta'_s - \frac{\beta_s}{\phi_s \rho_s \partial c_s / \partial p_s}}$$

$$\frac{1}{\rho^2} = \left(\frac{\phi_g}{\rho_g G^2} + \frac{\phi_s}{\rho_s c_s^2} \right) + \frac{\phi_s \phi_g}{\rho_s c_s^2 \rho_g G^2} \left\{ \beta'_s - \frac{\beta_s}{\phi_s \rho_s \partial c_s / \partial p_s} \right\}$$

$$\frac{1}{\rho D^2} = \frac{1 + \phi_s \phi_g \beta'_s \left\{ \frac{\phi_s}{\rho_g G^2} + \frac{\phi_g}{\rho_s c_s^2} \right\} + \beta_s \phi_s \phi_g \frac{\rho_s c_s^2 - \rho_g G^2 - \frac{\rho_g G^2}{\phi_s \rho_s \partial c_s / \partial p_s}}{\rho_s c_s^2 \rho_g G^2}}{1 + \phi_s \phi_g \beta'_s \left\{ \frac{\phi_s}{\rho_g G^2} + \frac{\phi_g}{\rho_s c_s^2} \right\} + \beta_s \phi_s \phi_g \frac{\rho_s c_s^2 - \rho_g G^2 - \frac{\rho_g G^2}{\phi_s \rho_s \partial c_s / \partial p_s}}{\rho_s c_s^2 \rho_g G^2}}$$

For $\beta_s = \beta'_s = 0$, get

$$\frac{1}{\rho D^2} = \frac{\phi_g}{\rho_g G^2} + \frac{\phi_s}{\rho_s c_s^2}, \text{ the equilibrium sound speed.}$$

The eigenvalues are, now,

$$\lambda = v, v, v, \quad v \pm \mathcal{D}.$$

ITEM 5

Eqn. of state for solid:

$$e_s = -T_3 \left(1 - \frac{p_{s0}}{p_s}\right) + T_4 \left\{ \left(\frac{p_s}{p_{s0}}\right)^{n-1} \left(1 - \frac{p_{s0}}{p_s}\right) - 1 \right\} \\ + \frac{1}{G} \frac{p_{s0}}{p_s} \left[p_s - p_{s0} - \frac{A_s}{n} \left\{ \left(\frac{p_s}{p_{s0}}\right)^n - 1 \right\} \right]$$

Units for e are ergs/gm.

$$n = 9.8$$

$$T_3 \text{ (ergs/gm)} = \frac{C_{vs} T_{s0} G}{p_{s0}}$$

$$T_4 \text{ (ergs/gm)} = \frac{A_s}{p_{s0} n (n-1)}$$

$$G \text{ (gm/cm}^3\text{)} = 2.1$$

$$A_s \text{ (dynes/cm}^2\text{)} = 1.35 \times 10^{11}$$

$$T_{s0} = 300^\circ\text{K}, \quad p_{s0} = 10^6 \text{ dynes/cm}^2$$

$$C_{vs} \text{ (ergs/gm } ^\circ\text{K)} = 15 \times 10^6$$

$$C_{vs} (T_s - T_{s0}) = e_s + T_3 \left(\frac{p_s}{p_{s0}} - 1 \right) \frac{p_{s0}}{p_s} \\ - T_4 \left\{ \left(\frac{p_s}{p_{s0}} \right)^{n-1} \left(1 - \frac{p_{s0}}{p_s} \right) - 1 \right\}$$

Eqn. of state for gas:

$$e_g = -h_{rel} - C_{vg} T_{g0} + \frac{1}{\phi_{s0} \rho_{s0}} \left[\frac{v_g}{\omega} \left\{ p_g - A e^{-R_1 v_g} - B e^{-R_2 v_g} \right\} + \frac{A}{R_1} e^{-R_1 v_g} + \frac{B}{R_2} e^{-R_2 v_g} \right]$$

Here,

$$v_g = \frac{\phi_{s0} \rho_{s0}}{\rho_g}$$

$$C_{vg} = [2.4 - 0.28 (\phi_{s0} \rho_{s0} - 1.3)] \times 10^7 \text{ ergs/gm}^\circ$$

$$T_{g0} = 300^\circ K$$

$$R_1 = 4.2, R_2 = 1.0, \omega = 0.25$$

$$A \text{ (dynes/cm}^2\text{)} = [-9.005 + 21.39 \phi_{s0} \rho_{s0} - 16.23 (\phi_{s0} \rho_{s0})^2 + 4.595 (\phi_{s0} \rho_{s0})^3] \times 10^{12}$$

$$B \text{ (dynes/cm}^2\text{)} = [-0.014 + 0.0349 \phi_{s0} \rho_{s0} + 0.0156 (\phi_{s0} \rho_{s0})^2 - 0.026 (\phi_{s0} \rho_{s0})^3]$$

h_{rel} = const, adjusted so that the undisturbed material has $e_g = 0$.

$$C_{vg} (T_g - T_{g0}) = C_g + h_{rel} - \frac{1}{\phi_{g0} \rho_{g0}} \left\{ \frac{A}{R_1} e^{-R_1 V_g} + \frac{B}{R_2} e^{-R_2 V_g} \right\}.$$

Gas / Solid Velocity

Mixture Density

Mixture Pressure

Phi

Mixture Density

Gas/Solid Velocity

ITEM 6

On the 6-eqn. equivelocity model

J. Bdziel has recently written down a 6-eqn. model by starting with the 7-eqn. B-N model and setting $v_s = v_g$. Essentially the two individual momentum eqns. are replaced by a single overall momentum eqn; hence the reduction to a set of 6 eqns.

Such a point of view seems inconsistent to me. In general, the equal-velocity requirement will not agree with one of the momentum equations that has been discarded. A rational scheme for deriving the equivelocity model would be to employ the limit $\delta \rightarrow \infty$. Then, one is not discarding any equation; simply replacing it by a reduced form. Further, one is recognizing the fact that mechanical equilibrium ($v_g = v_s$) does not necessarily imply absence of energy exchange between phases due to drag.

As the following calculations show, the reduced model so obtained is nonconservative.

The 6-eqn. equivelocity model, as derived from
Baer-Nunziato

Start with BN: Mass

$$\frac{\partial}{\partial t} (\phi_s \rho_s) + \frac{\partial}{\partial x} (\phi_s \rho_s v_s) = C$$

$$\frac{\partial}{\partial t} (\phi_g \rho_g) + \frac{\partial}{\partial x} (\phi_g \rho_g v_g) = -C$$

Momentum

$$\begin{aligned} \frac{\partial}{\partial t} (\phi_s \rho_s v_s) + \frac{\partial}{\partial x} [\phi_s \rho_s + \phi_s \rho_s v_s^2] - p_s \frac{\partial \phi_s}{\partial x} \\ \frac{\partial}{\partial t} (\phi_g \rho_g v_g) + \frac{\partial}{\partial x} [\phi_g \rho_g + \phi_g \rho_g v_g^2] + p_g \frac{\partial \phi_g}{\partial x} \end{aligned}$$

Also write down the overall equation

$$\begin{aligned} \frac{\partial}{\partial t} [\phi_s \rho_s v_s + \phi_g \rho_g v_g] + \frac{\partial}{\partial x} [(\phi_s \rho_s + \phi_g \rho_g) \\ + (\phi_s \rho_s v_s^2 + \phi_g \rho_g v_g^2)] = 0. \end{aligned}$$

Energy

$$\begin{aligned} \frac{\partial}{\partial t} [\phi_s \rho_s \{ e_s + \frac{1}{2} v_s^2 \}] + \frac{\partial}{\partial x} [\phi_s \rho_s v_s \{ e_s + \\ + \frac{1}{2} v_s^2 + \frac{p_s}{\rho_s} \}] = - (p_s - p_g) F + p_g v_g \frac{\partial \phi_g}{\partial x} + E \end{aligned}$$

$$\frac{\partial}{\partial t} \left[\phi_g p_g \left\{ e_g + \frac{1}{2} v_g^2 \right\} \right] + \frac{\partial}{\partial x} \left[\phi_g p_g v_g \left\{ e_g + \frac{1}{2} v_g^2 + \frac{p_g}{\rho_g} \right\} \right] = (p_s - p_g) \mathcal{F} - p_g v_s \frac{\partial \phi_s}{\partial x} - \mathcal{E} \quad (7)$$

Compaction

$$\frac{\partial \phi_s}{\partial t} + v_s \frac{\partial \phi_s}{\partial x} = \mathcal{F} + \frac{C}{\rho_s} \quad (8)$$

Interaction terms

$$\mathcal{M} = \delta (v_g - v_s) + \frac{1}{2} C (v_g + v_s) \quad (9)$$

$$\mathcal{E} = (e_s + \frac{1}{2} v_s^2) C - \mathcal{H} (T_s - T_g) + (\delta + \frac{1}{2} C) v_s (v_g - v_s) \quad (10)$$

$$\mathcal{F} = \frac{\phi_s \phi_g}{\mu_c} (p_s - p_g - \beta_s) \quad (11)$$

Overall energy eqn.

$$\begin{aligned} & \frac{\partial}{\partial t} \left[(\phi_s p_s e_s + \phi_g p_g e_g) + \frac{1}{2} (\phi_s p_s v_s^2 + \phi_g p_g v_g^2) \right] \\ & + \frac{\partial}{\partial x} \left[(\phi_s p_s v_s e_s + \phi_g p_g v_g e_g) + \frac{1}{2} (\phi_s p_s v_s^3 + \phi_g p_g v_g^3) \right. \\ & \left. + (\phi_s v_s p_s + \phi_g v_g p_g) \right] = 0 \quad (12) \end{aligned}$$

Solid energy eqn. - $v_s \times$ solid momentum eqn. $\Rightarrow$

$$\frac{\partial}{\partial t} (\phi_s \rho_s e_s) + \frac{\partial}{\partial x} (\phi_s \rho_s v_s e_s) + \phi_s \rho_s (v_s) = C e_s - H (T_s - T_g) - (p_s - p_g) F$$

Further simplification $\Rightarrow$

$$\phi_s \rho_s \left[\frac{\partial e_s}{\partial t} + v_s \frac{\partial e_s}{\partial x} \right] = -H (T_s - T_g) - (p_s - p_g) F - \phi_s \rho_s \frac{\partial v_s}{\partial x}$$

Note the absence of drag terms.

Gas energy eqn. - $v_g \times$ gas momentum eqn. $\Rightarrow$

$$\begin{aligned} \frac{\partial}{\partial t} (\phi_g \rho_g e_g) + \frac{\partial}{\partial x} (\phi_g \rho_g v_g e_g) + \phi_g \rho_g (v_g) &= H (T_s - T_g) + (p_s - p_g) F - C e_s \\ &+ p_g (v_g - v_s) \frac{\partial \phi_s}{\partial x} + \delta (v_g - v_s)^2 \end{aligned}$$

Note the asymmetry of these equations. Further simplification of the gas eqn. $\Rightarrow$

$$\begin{aligned} \phi_g \rho_g \left[\frac{\partial e_g}{\partial t} + v_g \frac{\partial e_g}{\partial x} \right] + \phi_g \rho_g \frac{\partial v_g}{\partial x} &= C (e_g - e_s) \\ &- p_g (v_g - v_s) \frac{\partial \phi_s}{\partial x} + \delta (v_g - v_s)^2 + H (T_s - T_g) + (p_s - p_g) F \end{aligned}$$

Now consider the limit $\delta \rightarrow \infty$. Each of the two individual momentum equations, (1.3) and (1.4), reduces to

$$v_s - v_g = 0 \Rightarrow v_s = v_g = v \quad (1)$$

to leading order. In order to get a second independent equation, one turns to the overall momentum eqn. (1.5), which is independent of δ .

The two mass eqns. remain the same, with the exception of $v_s = v_g = v$. The energy equations (1.6) and (2.7) do not simplify immediately because each contains the $O(1)$ term $\delta(v_g - v_s)$. The overall energy eqn. (2.12) does, however, simplify.

The reduced solid energy eqn. (3.1) or (3.2) simplifies since δ does not appear there. The reduced gas energy eqn. (3.3) or (3.4) simplifies as well, because δ is multiplied by $(v_g - v_s)^2$.

One may now write down the following reduced set:

mass

$$\frac{\partial}{\partial t} (\phi_s \rho_s) + \frac{\partial}{\partial x} (\phi_s \rho_s v) = C \quad (1)$$

$$\frac{\partial}{\partial t} (\phi_g \rho_g) + \frac{\partial}{\partial x} (\phi_g \rho_g v) = -C \quad (2)$$

momentum

$$\begin{aligned} \frac{\partial}{\partial t} [(\phi_s \rho_s + \phi_g \rho_g) v] + \frac{\partial}{\partial x} [(\phi_s \rho_s + \phi_g \rho_g) v^2 \\ + (\phi_s p_s + \phi_g p_g)] = 0. \end{aligned} \quad (3)$$

energy

$$\frac{\partial}{\partial t} (\phi_s \rho_s e_s) + \frac{\partial}{\partial x} (\phi_s \rho_s e_s v) + \phi_s \rho_s v_x = C e_s - H(T_s - T_g) - (p_s - p_g) F \quad (11)$$

$$\frac{\partial}{\partial t} (\phi_g \rho_g e_g) + \frac{\partial}{\partial x} (\phi_g \rho_g e_g v) + \phi_g \rho_g v_x = -C e_s + H(T_s - T_g) + (p_s - p_g) F$$

Compaction

$$\frac{\partial \phi_s}{\partial t} + v \frac{\partial \phi_s}{\partial x} = F + \frac{C}{\rho_s} \quad (13)$$

$$\text{or } \frac{\partial}{\partial t} (\rho \phi_s) + \frac{\partial}{\partial x} (\rho \phi_s v) = \rho F + \frac{\rho}{\rho_s} C$$

Either of the two mass eqns. may be replaced by an overall equation; similarly for energy.

The overall energy eqn. can be written in a conservative form by the individual energy eqns. can not, since the individual momentum eqns. are no longer available.

ITEM 7

The Six-Equation, Equal-Velocity Model: Scaling

1 Introduction

In dimensional form, the basic equations are:

$$\frac{\partial}{\partial t}(\phi_s \gamma_s) + \frac{\partial}{\partial x}(\phi_s \gamma_s u) = C_s, \quad (1)$$

$$\frac{\partial}{\partial t}(\phi_g \gamma_g) + \frac{\partial}{\partial x}(\phi_g \gamma_g u) = -C_s, \quad (2)$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(p + \rho u^2) = 0, \quad (3)$$

$$\frac{\partial}{\partial t}(\phi_s \gamma_s e_s) + \frac{\partial}{\partial x}(\phi_s \gamma_s e_s u) = -\phi_s p_s \frac{\partial u}{\partial x} + C_s e_s - H(T_s - T_g) - (p_s - \beta_s) \mathcal{F}, \quad (4)$$

$$\frac{\partial}{\partial t}(\phi_g \gamma_g e_g) + \frac{\partial}{\partial x}(\phi_g \gamma_g e_g u) = -\phi_g p_g \frac{\partial u}{\partial x} - C_s e_s + H(T_s - T_g) + (p_s - \beta_s) \mathcal{F}, \quad (5)$$

$$\frac{\partial}{\partial t}(\rho \phi_s) + \frac{\partial}{\partial x}(\rho \phi_s u) = \rho \mathcal{F} + \frac{\rho}{\gamma_s} C_s. \quad (6)$$

The mixture quantities appearing above are defined as

$$\rho = \phi_s \gamma_s + \phi_g \gamma_g, \quad p = \phi_s p_s + \phi_g p_g \quad (7)$$

The compaction function is

$$\mathcal{F} = \frac{\phi_s \phi_g}{\mu_c} (p_s - p_g - \beta_s) \quad (8)$$

Solid EOS

The equation of state for the solid is

$$e_s = -T_3 \left(1 - \frac{1}{r}\right) + T_4 \left\{ r^{n-1} - (n-1) \left(1 - \frac{1}{r}\right) - 1 \right\} + \frac{1}{G_{ov} r} \left\{ p_s - p_{int} - \frac{A_{kt0}}{n} (r^n - 1) \right\} \quad (9)$$

where r is the density ratio

$$r = \frac{\gamma_s}{\gamma_{s0}} \quad (10)$$

and the coefficients T_3, T_4 are given by

$$T_3 = \frac{c_{vs} T_{s0} G_{ov}}{\gamma_{s0}}, \quad T_4 = \frac{A_{kt0}}{n(n-1)\gamma_{s0}} \quad (11)$$

The eos yields the expression

$$p_s - p_{int} = G_{ov} r \left[e_s + T_3 \left(1 - \frac{1}{r}\right) - T_4 \left\{ r^{n-1} - (n-1) \left(1 - \frac{1}{r}\right) - 1 \right\} \right] + \frac{A_{kt0}}{n} (r^n - 1) \quad (12)$$

for the pressure p_s and

$$c_s^2 = \frac{1}{\partial e_s / \partial p_s} \left(\frac{p_s}{\gamma_s^2} - \frac{\partial e_s}{\partial \gamma_s} \right) \quad (13)$$

for the solid sound speed, c_s . The temperature T_s is given by

$$T_s - T_{s0} = \frac{1}{c_{vs}} \left[e_s + T_3 \left(1 - \frac{1}{r} \right) - T_4 \left\{ r^{n-1} - (n-1) \left(1 - \frac{1}{r} \right) - 1 \right\} \right] \quad (14)$$

In the MKS units, the coefficients in the eos have the values

$$c_{vs} = 0.15 \text{ e4 Nm/Kg-degree K}, \quad n = 9.8, \quad G_{ov} = 2.1 \text{ e3 Kg/m}^3, \quad A_{s0} = 1.35 \text{ e10 N/m}^2 \quad (15)$$

The initial values appearing above are taken to be

$$T_{s0} = 300 \text{ degrees K}, \quad p_{int} = 1.0 \text{ e5 N/m}^2, \quad \gamma_{s0} = 1.9 \text{ e3 Kg/m}^3 \quad (16)$$

The coefficients T_3, T_4 have units of energy per unit mass (i.e., velocity squared).

Gas EOS

The eos for the gas is

$$e_g = -h_{rel} - c_{vg} T_{g0} + \frac{1}{\phi_{s0} \gamma_{s0}} \left[\frac{\nu_g p_g}{\omega} + \left\{ \frac{1}{R_1} - \frac{\nu_g}{\omega} \right\} A e^{-R_1 \nu_g} + \left\{ \frac{1}{R_2} - \frac{\nu_g}{\omega} \right\} B e^{-R_2 \nu_g} \right] \quad (17)$$

where

$$\nu_g = \frac{\phi_{s0} \gamma_{s0}}{\gamma_g} \quad (18)$$

Temperature T_g is given by the expression

$$T_g - T_{g0} = \frac{1}{c_{vg}} \left[e_g + h_{rel} - \frac{1}{\phi_{s0} \gamma_{s0}} \left\{ \frac{A}{R_1} e^{-R_1 \nu_g} + \frac{B}{R_2} e^{-R_2 \nu_g} \right\} \right] \quad (19)$$

and pressure p_g by

$$p_g = \left\{ 1 - \frac{\omega}{R_1 \nu_g} \right\} A e^{-R_1 \nu_g} + \left\{ 1 - \frac{\omega}{R_2 \nu_g} \right\} B e^{-R_2 \nu_g} + \omega \gamma_g (e_g + h_{rel} + c_{vg} T_{g0}) \quad (20)$$

or equivalently, by

$$p_g = A e^{-R_1 \nu_g} + B e^{-R_2 \nu_g} + \omega \gamma_g c_{vg} T_g \quad (21)$$

while the sound speed c_g is computed from

$$c_g^2 = \frac{1}{\partial e_g / \partial p_g} \left(\frac{p_g}{\gamma_g^2} - \frac{\partial e_g}{\partial \gamma_g} \right) \quad (22)$$

The dimensional coefficients appearing above have the following expressions in MKS units:

$$c_{vg} = [2.4 - 0.28(10^{-3} \phi_{s0} \gamma_{s0} - 1.3)] 10^3 \text{ Nm/Kg-degree K} \quad (23)$$

$$h_{rel} = [7.91 - 4.33(10^{-3} \phi_{s0} \gamma_{s0} - 1.3)^2 - 0.934(10^{-3} \phi_{s0} \gamma_{s0} - 1.3)^3] 10^6 \text{ Nm/Kg} \quad (24)$$

$$A = [-8.005 + 21.39(10^{-3} \phi_{s0} \gamma_{s0}) - 16.23(10^{-3} \phi_{s0} \gamma_{s0})^2 + 4.595(10^{-3} \phi_{s0} \gamma_{s0})^3] 10^{11} \text{ N/m}^2 \quad (25)$$

$$B = [-0.014 + 0.0349(10^{-3} \phi_{s0} \gamma_{s0}) + 0.0156(10^{-3} \phi_{s0} \gamma_{s0})^2 - 0.026(10^{-3} \phi_{s0} \gamma_{s0})^3] 10^{11} \text{ N/m}^2 \quad (26)$$

The coefficients

$$R_1 = 4.2, \quad R_2 = 1.0, \quad \omega = 0.25 \quad (27)$$

are dimensionless. Initial values are

$$T_{g0} = 300 \text{ degrees K}, \quad p_{g0} = 1.0 \text{ e5 N/m}^2 \quad (28)$$

Compaction Law

The compaction law is

$$\beta_s = \begin{cases} -\tau(\phi_s - \phi_{s0}) \frac{\ln \phi_g}{\phi_g} & \text{if } \phi_s \geq \phi_{s0} \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

with the pressure coefficient

$$\tau = 1.2 \text{ e7 N/m}^2 \quad (30)$$

and the compaction viscosity

$$\mu_c = 1.0 \text{ e2 Ns/m}^2. \quad (31)$$

Chemical Rate

The chemical source term is specified as

$$C_s = -3\gamma_s \phi_s A_{coef} (p_{res}/A_{res})^{b_n} e_{ti} (1 - f_{dv})^{2/3}/a \quad (32)$$

where

$$p_{res} = \text{amax1}(p_g, p_{int}), \quad f_{dv} = \text{amax1}\left(\frac{\phi_s - \phi_{s0}}{1 - \phi_{s0}}, 0\right), \quad (33)$$

$$a = (7.5 \text{ e} - 5) \text{ amax1}(1, \phi_{s0}/\phi_s)^{1/3} \text{ m}. \quad (34)$$

The coefficients A_{res} , A_{coef} and b_n are defined below:

$$A_{res} = 1.0 \text{ e8 N/m}^2, \quad (35)$$

$$A_{coef} = 0.1227 \text{ m/s}, \quad b_n = 1 \text{ for } p_{res} \leq 6.9 \text{ e7 N/m}^2, \quad (36)$$

$$A_{coef} = 0.1139 \text{ m/s}, \quad b_n = 0.8 \text{ for } p_{res} > 6.9 \text{ e7 N/m}^2. \quad (37)$$

Finally,

$$e_{ti} = \begin{cases} 0 & \text{for } p_{mix} < p_{ign} \\ 1 & \text{for } p_{mix} \geq p_{ign} \end{cases} \quad (38)$$

where

$$p_{mix} = p - p_{int}, \quad p_{ign} = 1.0 \text{ e7 N/m}^2 \quad (39)$$

2 Scaling

I prefer to use a scaled version of the equations for purposes of computation. Let the basic scales be x_* for length, t_* for time, γ_* for density and c_* for specific heat. Scales for the remaining variables are taken to be $u_* = x_*/t_*$ for speed, $p_* = \gamma_* u_*^2$ for pressure, $e_* = u_*^2$ for internal energy and $T_* = u_*^2/c_*$ for temperature. The chemical source term has the scale $C_{s*} = \gamma_*/t_*$, and the compaction term scales as $\mathcal{F}_* = 1/t_*$.

The scales for the compaction viscosity and the convection heat transfer coefficient are $\mu_{c*} = \gamma_* u_*^2 t_*$ and $H_* = \gamma_* c_*/t_*$.

With these choices, the governing equations (1) - (8) remain as before, with the symbols now denoting scaled variables. The solid EOS is again represented by (9), with G_{ov} scaled by γ_* , T_3 and T_4 by u_*^2 and A_{kt0} by p_* . The sound speed formula (13) and the temperature equation (14) survive unscathed.

The gas EOS (17), the temperature equation (19) and the pressure equations (20), (21) also remain unchanged, provided A and B are scaled by p_* . The sound-speed equation (22) holds as well.

In (29), τ is scaled by p_* .

In the chemical rate term, given by (32), A_{coef} is scaled by u_* and a by x_* . The pressure p_{res} and the coefficient A_{res} are, of course, scaled by p_* .

Sound-speed computation (revised)

I. SOLID

The solid eos may be written as follows:

$$\text{Let } \kappa = \gamma_s / \gamma_{s0}. \quad (1)$$

Then,

$$\begin{aligned} e_s = & -T_3 \left(1 - \frac{1}{\kappa}\right) + T_4 \left\{ \kappa^{n-1} - (n-1) \left(1 - \frac{1}{\kappa}\right) - 1 \right\} \\ & + \frac{1}{G_{0V}} \frac{1}{\kappa} \left[p_s - p_{int} - A_{kto} \frac{1}{n} (\kappa^n - 1) \right] \quad (2) \end{aligned}$$

Now,

$$\gamma_s c_s^2 = \frac{1}{\partial e_s / \partial p_s} \left[\frac{p_s}{\gamma_s} - \gamma_s \frac{\partial e_s}{\partial \gamma_s} \right] \quad (3)$$

Now,

$$\frac{\partial e_s}{\partial p_s} = \frac{1}{G_{0V}} \frac{1}{\kappa} \quad (4)$$

$$\begin{aligned} \frac{\partial e_s}{\partial \gamma_s} &= \frac{1}{\gamma_{s0}} \frac{\partial e_s}{\partial \kappa} \\ &= \frac{1}{\gamma_{s0}} \left[-\frac{T_3}{\kappa^2} + T_4 \left\{ (n-1) \kappa^{n-2} - \frac{n-1}{\kappa^2} \right\} \right. \\ &\quad \left. - \frac{1}{G_{0V}} \frac{1}{\kappa^2} \left\{ p_s - p_{int} - \frac{A_{kto}}{n} (\kappa^n - 1) \right\} \right. \\ &\quad \left. + \frac{1}{G_{0V}} \frac{1}{\kappa} (-A_{kto}) \kappa^{n-1} \right] \\ &= \frac{1}{\gamma_{s0} \kappa^2} \left[-T_3 + (n-1) T_4 (\kappa^n - 1) - \right. \\ &\quad \left. - \frac{1}{G_{0V}} \left\{ p_s - p_{int} - \frac{A_{kto}}{n} (\kappa^n - 1) \right\} - \frac{A_{kto}}{G_{0V}} \kappa^n \right] \quad (5) \end{aligned}$$

Solving for p_s from (1.2),

$$p_s = p_{int} + \frac{A \cdot k}{n} (n^{n-1} - 1) + n \cdot Gov [e_s + T_s (1 - \frac{1}{n})] - T_4 \{ n^{n-1} - (n-1) (1 - \frac{1}{n}) - 1 \} \quad (11)$$

On substituting for the relevant quantities in (1.3),

$$c_s^2 = \frac{1}{p_s} \left(\frac{p_s}{r_s} - r_s \frac{\partial c_s}{\partial r_s} \right) \quad (12)$$

II. GAS

$$e_g = -h_{int} - c_g T_{g0} + \frac{1}{p_g} \left[\frac{p_g}{r_g} \{ p_g - A e^{-R_1 V_g} + \frac{R_1}{B} e^{-R_2 V_g} \} - B e^{-R_2 V_g} \right] \quad (13)$$

with
$$r_g = \frac{\phi_{50} r_{50}}{p_g} \quad (14)$$

Now,
$$\frac{\partial e_g}{\partial p_g} = \frac{1}{p_g} \frac{\partial \phi_{50}}{\partial p_g} = \frac{1}{\omega r_g} \quad (15)$$

$$\frac{\partial e_g}{\partial r_g} = - \frac{p_g}{r_g} \frac{\partial c_g}{\partial r_g} = - \frac{p_g}{r_g} \frac{\partial e_g}{\partial r_g}$$

$$= - \frac{p_g}{r_g} \left[\frac{1}{\phi_{50}} \left\{ \frac{p_g}{r_g} - A e^{-R_1 V_g} - B e^{-R_2 V_g} \right\} \right]$$

$$+ \frac{\omega}{p_g} \left\{ A R_1 e^{-R_1 V_g} + B R_2 e^{-R_2 V_g} - A e^{-R_1 V_g} - B e^{-R_2 V_g} \right\}$$

$$= - \frac{1}{r_g} \left[\frac{p_g}{r_g} - A e^{-R_1 V_g} \left\{ \frac{1}{\omega} - \frac{R_1 V_g}{\omega} + 1 \right\} - B e^{-R_2 V_g} \left\{ \frac{1}{\omega} - \frac{R_2 V_g}{\omega} \right\} \right] \quad (16)$$

The gas sound speed may now be computed from

$$c_g^2 = \frac{1}{\gamma_g \frac{\partial e_g}{\partial p_g}} \left\{ \frac{p_g}{\gamma_g} - \gamma_g \frac{\partial e_g}{\partial \gamma_g} \right\} \quad (1)$$

Finally, the mixture speed is given by

$$c_m^2 = \frac{\phi_g \gamma_g c_g^2 + \phi_s \gamma_s c_s^2}{\phi_g \gamma_g + \phi_s \gamma_s} \quad (2)$$

Alternate Scaling (on t rather than x)

velocity scale u_* , length scale $l_* = 1$
time scale $t_* = l_*/u_* = 1/u_*$
pressure $p_* = u_*^2$, also $e_* = T_* = u_*^2$

Mass equations

$$\frac{\partial}{\partial t} (\phi_s \gamma_s) + \frac{\partial}{\partial x} (\phi_s \gamma_s \bar{u}) = \bar{C}_s$$

$$\frac{\partial}{\partial t} (\phi_g \gamma_g) + \frac{\partial}{\partial x} (\phi_g \gamma_g \bar{u}) = -\bar{C}_s$$

Momentum

$$\frac{\partial}{\partial t} (\rho \bar{u}) + \frac{\partial}{\partial x} (\rho \bar{u}^2 + \bar{p}) = 0$$

Compaction

$$\frac{\partial}{\partial t} (\rho \phi_s) + \frac{\partial}{\partial x} (\rho \phi_s \bar{u}) = \rho \bar{F} + \frac{\rho}{\gamma_s} \bar{C}_s$$

Energy

$$\begin{aligned} \frac{\partial}{\partial t} (\phi_s \gamma_s \bar{e}_s) + \frac{\partial}{\partial x} (\phi_s \gamma_s \bar{e}_s \bar{u}) + \phi_s \bar{p}_s \frac{\partial \bar{u}}{\partial x} \\ = \bar{C}_s \bar{e}_s - \bar{H} (\bar{T}_s - \bar{T}_g) - (\bar{p}_s - \bar{\beta}_s) \bar{F} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} (\phi_g \gamma_g \bar{e}_g) + \frac{\partial}{\partial x} (\phi_g \gamma_g \bar{e}_g \bar{u}) + \phi_g \bar{p}_g \frac{\partial \bar{u}}{\partial x} \\ = -\bar{C}_s \bar{e}_s + \bar{H} (\bar{T}_s - \bar{T}_g) - (\bar{p}_g - \bar{\beta}_g) \bar{F} \end{aligned}$$

$$\bar{H} = H/u_*, \quad \bar{F} = \frac{\phi_s \phi_g u_*}{\mu_c} (\bar{p}_s - \bar{p}_g - \bar{\beta}_s)$$

$$\bar{C}_s = C_s/u_*, \quad \bar{t} = t u_*$$

The equal-velocity model (Kapila notation)

$$\frac{\partial}{\partial t} (\phi_s \gamma_s) + \frac{\partial}{\partial x} (\phi_s \gamma_s v) = C_s$$

$$\frac{\partial}{\partial t} (\phi_g \gamma_g) + \frac{\partial}{\partial x} (\phi_g \gamma_g v) = -C_s$$

$$\frac{\partial}{\partial t} (p v) + \frac{\partial}{\partial x} (p v^2 + p) = 0$$

$$\begin{aligned} \frac{\partial}{\partial t} (\phi_s \gamma_s e_s) + \frac{\partial}{\partial x} (\phi_s \gamma_s e_s v) + \phi_s p_s \frac{\partial v}{\partial x} \\ = C_s e_s - H(T_s - T_g) - (p_s - p_g) \mathcal{F} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} (\phi_g \gamma_g e_g) + \frac{\partial}{\partial x} (\phi_g \gamma_g e_g v) + \phi_g p_g \frac{\partial v}{\partial x} \\ = -C_s e_s + H(T_s - T_g) + (p_s - p_g) \mathcal{F} \end{aligned}$$

$$\frac{\partial}{\partial t} (p \phi_s) + \frac{\partial}{\partial x} (p \phi_s v) = p \mathcal{F} + \frac{p}{\gamma_s} C_s$$

$$p = \phi_s \gamma_s + \phi_g \gamma_g, \quad p = \phi_s p_s + \phi_g p_g$$

$$\mathcal{F} = \frac{\phi_s \phi_g}{\mu_c} (p_s - p_g - \beta_s)$$

Scaled Equations

Take time scale $t_* = 1$

velocity scale $= u_*$

length scale $= u_* t_* = u_*$

pressure scale $p_* = u_*^2$

energy scale $e_* = u_*^2 = \text{temp. scale } T_*$

Let $u = u_* \bar{u}$, $p = u_*^2 \bar{p}$, etc

Mass eqns. are unchanged.

$$\frac{\partial}{\partial t} (\phi_s \gamma_s) + \frac{\partial}{\partial \bar{x}} (\phi_s \gamma_s \bar{u}) = C_s$$

$$\frac{\partial}{\partial t} (\phi_g \gamma_g) + \frac{\partial}{\partial \bar{x}} (\phi_g \gamma_g \bar{u}) = -C_s$$

Momentum eqn. is unchanged.

$$\frac{\partial}{\partial t} (p \bar{u}) + \frac{\partial}{\partial \bar{x}} (p \bar{u}^2 + \bar{p}) = 0.$$

Energy eqns:

$$\begin{aligned} \frac{\partial}{\partial t} (\phi_s \gamma_s \bar{e}_s) + \frac{\partial}{\partial \bar{x}} [\phi_s \gamma_s \bar{e}_s \bar{u}] + \phi_s \bar{p} \frac{\partial \bar{u}}{\partial \bar{x}} \\ = C_s \bar{e}_s - H(\bar{T}_s - \bar{T}_g) - (\bar{p}_s - \bar{p}_g) \bar{u} \end{aligned}$$

$$\frac{\partial}{\partial t} (\phi_g \gamma_g \bar{e}_g) + \frac{\partial}{\partial x} [\phi_g \gamma_g \bar{e}_g \bar{u}] + \phi_g \bar{p}_g \frac{\partial \bar{u}}{\partial x} \\ = -C_s \bar{e}_g + H (\bar{T}_s - \bar{T}_g) + (\bar{p}_s - \bar{p}_g) \bar{T}$$

Compaction

$$\frac{\partial}{\partial t} (P \phi_s) + \frac{\partial}{\partial x} (P \phi_s \bar{u}) = P \bar{T} + \frac{P}{\gamma_s} C_s$$

$$P = \phi_s \gamma_s + \phi_g \gamma_g$$

$$\bar{p} = \phi_s \bar{p}_s + \phi_g \bar{p}_g$$

$$\bar{T} = \frac{\phi_s \phi_g \mu_*^2}{\mu_c} [\bar{p}_s - \bar{p}_g - \bar{p}_s]$$

Machine Notation

$$\text{rho}(1, n) = \gamma_s \phi_s, \quad \text{rho}(2, n) = \gamma_g \phi_g$$

$$\text{rho}u(n) = (\gamma_s \phi_s + \gamma_g \phi_g) u$$

$$\text{rhophi}(1, n) = (\gamma_s \phi_s + \gamma_g \phi_g) \phi_s$$

$$\text{rho}e(1, n) = (\gamma_s \phi_s + \gamma_g \phi_g) e_s$$

$$\text{rho}e(2, n) = (\gamma_s \phi_s + \gamma_g \phi_g) e_g$$

$$p(1, n) = p_s, \quad p(2, n) = p_g$$

$$\text{temp}(1, n) = T_s, \quad \text{temp}(2, n) = T_g$$

$$\begin{array}{cccc} \text{gamma}_{s0} & \text{gamma}_{g0} & \text{phi}_{s0} & \text{phi}_{g0} \end{array}$$

$$\begin{array}{cccc} t_{s0} & t_{g0} & \text{rho}_{s0} & \text{rho}_{g0} \\ T_{s0} & T_{g0} & \gamma_{s0} \phi_{s0} & \gamma_{g0} \phi_{g0} \end{array}$$

$$\text{den} = \gamma_s \phi_s + \gamma_g \phi_g, \quad u = \text{rho}u / \text{den}, \quad \phi_s = \text{rhophi},$$

The 6 basic variables are

$$\text{rho}(1, n), \text{rho}(2, n), \text{rho}u(n), \text{rho}e(1, n), \text{rho}e(2, n), \text{rhophi}(1, n)$$