10
9-30-96 JS①

LBNL-38801
UC-405

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

**Analytic Evaluation of Statistical Projection
Operators for Emission Tomography**

A. Kuruc

May 1996

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED *29*

MASTER

Donner Laboratory

**Biology &
Medicine
Division**

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. Neither the United States Government nor any agency thereof, nor The Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or The Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or The Regents of the University of California and shall not be used for advertising or product endorsement purposes.

Available to DOE and DOE Contractors
from the Office of Scientific and Technical Information
P.O. Box 62, Oak Ridge, TN 37831
Prices available from (615) 576-8401

Available to the public from the
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, VA 22161

Lawrence Berkeley Laboratory is an equal opportunity employer.

LBL-38801
UC-405

Analytic Evaluation of Statistical Projection Operators for Emission Tomography

Alvin Kuruc

Life Sciences Division
Ernest Orlando Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

May 1996

This work was supported in part by the Director, Office of Energy Research, Office of Health and Environmental Research, Medical Applications and Biophysical Research Division, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098, and the National Heart, Lung, and Blood Institute of the U.S. Department of Health and Human Services under Grants T32-HL07367, P01-HL25804, and R01-HL50663.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Analytic Evaluation of Statistical Projection Operators for Emission Tomography

Alvin Kuruc

May 6, 1996

The purpose of this report is to outline an approach to the numerical construction of statistically efficient estimators for linear functionals in emission tomography (ET) that is more efficient than the approach used in [Kur97]. For the sake of brevity, we will assume familiarity with the notation and material in [Kur97].

1 Statistically Efficient Estimators for Linear Functionals in ET

In [Kur97], we described the construction of statistically efficient estimators for linear functionals in ET and the relationship of the efficient estimator to the standard estimator based on the FB algorithm. We found that the standard FB estimator for the linear functional $\int \phi(x)f(x)dx$ was equivalent to the linear estimator generated by the observation-space function

$$\psi_{\text{FB}} \equiv HR\phi,$$

where H is the ramp-filter operator and R denotes the Radon transform. Also, the efficient estimator at f_0 was the linear estimator generated by the observation-space

function

$$\psi_E = p_{N(R^*)^\perp} \psi_{FB} \quad (1)$$

$$= \psi_{FB} - p_{N(R^*)} \psi_{FB}, \quad (2)$$

where $p_{N(R^*)}$ and $p_{N(R^*)^\perp}$ are the projection operators onto the null space of the back-projection operator R^* and its orthogonal complement in $L^2(C, Rf_0)$, respectively.

Given the generating functions ψ_E and ψ_{FB} , it is straightforward to compute the variances of the resulting estimators using proposition 3.4 in [Kur97]. The main technical obstacles are the calculation of the generating functions ψ_E and ψ_{FB} . While we have been able to generate results for some special cases, the methods that we have used to date are too cumbersome for general use. This report is largely devoted to outlining an alternative approach to the calculation of these functions for a reasonably broad class of scenarios.

2 Construction of Projection Operators

We now discuss concrete representations of the projection operators in equations 1 and 2. Recall from linear algebra that it is easy to compute the projection of a vector onto a subspace given an orthonormal basis for the subspace. The coefficients of the projection of the vector with respect to the orthonormal basis are just the inner products of the vector with the basis vectors. Thus if $\{v_1, \dots, v_m\}$ is a set of orthonormal vectors in $\mathbb{R}^n$, i.e., the norm $\|v_i\| = 1$ for $i = 1, \dots, m$ and the inner product $\langle v_i, v_j \rangle = 0$ for $i \neq j$, then the projection of a vector $v \in \mathbb{R}^n$ onto the subspace V spanned by $\{v_1, \dots, v_m\}$ is given by

$$p_V v = \sum_{i=1}^m \langle v_i, v \rangle v_i.$$

Similar ideas apply to the construction of projection operators in $L^2(C, Rf_0)$, where the inner product of ψ and ψ' is now defined by

$$\langle \psi, \psi' \rangle_{L^2(C, Rf_0)} \equiv \int_C \psi(l) \psi'(l) Rf_0(l) dl. \quad (3)$$

The integral over $\mathbb{L}$ on the right side of equation 3 is defined explicitly by

$$\int_C \psi(l)\psi'(l)Rf_0(l)dl \equiv \pi^{-1} \int_0^\pi \int_{-1}^1 \psi(\theta,s)\psi'(\theta,s)Rf_0(\theta,s)dsd\theta,$$

where θ and s are the usual orientation and signed distance from the origin coordinates on $\mathbb{L}$, respectively. Thus if B and $B^\perp$ are orthonormal bases for $N(R^*)$ and $N(R^*)^\perp$, respectively,

$$p_{N(R^*)}\psi = \sum_{b \in B} \langle \psi, b \rangle_{L^2(C, Rf_0)} b$$

and

$$p_{N(R^*)^\perp}\psi = \sum_{b \in B^\perp} \langle \psi, b \rangle_{L^2(C, Rf_0)} b.$$

The only real difference between $\mathbb{R}^n$ and $L^2(C, Rf_0)$ is that, at least in theory, the orthonormal bases B and $B^\perp$ are infinite sets. In practice, of course, it is generally necessary to terminate the expansions after a finite number of terms.

In practice, one is often required to construct projection operators from a basis which is *not* orthonormal. Suppose $\{v_1, \dots, v_m\}$ is a set of linearly independent, but not necessarily orthonormal, vectors in $\mathbb{R}^n$. Let V denote the $n \times m$ matrix whose columns are $v_1, \dots, v_m$. Then the projection of a vector $v \in \mathbb{R}^n$ onto the column space of V is given by

$$p_V v = V(V^T V)^{-1} V^T v.$$

We see that the coefficients of $p_V v$ with respect to the basis $\{v_1, \dots, v_m\}$ are again expressible in terms of inner products, albeit in a more complicated form. In $L^2(C, Rf_0)$, analogous formulas apply by substituting the appropriate inner product.

2.1 The Uniform Distribution

For simplicity, we first consider the case where the underlying image f is the uniform distribution on D , i.e.,

$$\begin{aligned} f &= f_u \\ &\equiv \pi^{-1}. \end{aligned}$$

Then orthonormal bases for $N(R^*)$ and $N(R^*)^\perp$ can be given explicitly [Kur97, sec. 5].

Let

$$U_m(\cos \theta) \equiv \frac{\sin[(m+1)\theta]}{\sin \theta} \quad (4)$$

denote the Chebyshev polynomial of the second kind of order m . (The first few are the polynomials 1 , $2s$, $4s^2 - 1$, and $8s^3 - 4s$.) A orthonormal basis for $N(R^*)^\perp$ in $L^2(C, Rf_u)$ is given by the functions $U_m(s)$ with $m = 0, 2, 4, \dots$ and $\sqrt{2}U_m \cos(l\theta)$ and $\sqrt{2}U_m \sin(l\theta)$ with $l = 1, 2, 3, \dots$ and $m = l, l+2, l+4, \dots$. A orthonormal basis for $N(R^*)$ in $L^2(C, Rf_u)$ is given by the functions $\sqrt{2}U_m \cos(l\theta)$ and $\sqrt{2}U_m \sin(l\theta)$ with $l = 1, 2, 3, \dots$ and $m = l \bmod 2, l \bmod 2 + 2, \dots, l-2$. Rf_u is given explicitly by

$$\begin{aligned} Rf_u &\equiv g_u \\ &= 2\pi^{-1}\sqrt{1-s^2}. \end{aligned}$$

Thus to compute ψ_E we need to compute inner products of the form

$$\langle U_m(s) \sin(l\theta), \psi_{FB} \rangle_{L^2(C, g_u)} = 2\pi^{-2} \int_0^\pi \int_{-1}^1 U_m(s) \sin(l\theta) F\phi(\theta, s) \sqrt{1-s^2} ds d\theta \quad (5)$$

and

$$\langle U_m(s) \cos(l\theta), \psi_{FB} \rangle_{L^2(C, g_u)} = 2\pi^{-2} \int_0^\pi \int_{-1}^1 U_m(s) \cos(l\theta) F\phi(\theta, s) \sqrt{1-s^2} ds d\theta. \quad (6)$$

An important consequence of this result is that when ϕ is a radial function, the linear estimator generated by ψ_{FB} is efficient at f_u . The reason is that $F\phi$ is then independent of θ and symmetric in s . These properties imply that the inner products of ψ_{FB} with the basis functions of the form $U_m(s) \cos(l\theta)$ and $U_m(s) \sin(l\theta)$ with $l \geq 1$ vanish.

2.2 The General Case

We now consider the general case where $f_0 \neq f_u$. Note that $\mathcal{N}(R^*)$ is defined independently of f_0 and thus the basis for it given in the preceding subsection is still a basis. It will not, however, generally be an *orthonormal* basis in $L^2(C, Rf_0)$.

Similar considerations apply to finding an orthonormal basis for $\mathcal{R}(R_{f_0})$. The functions in $\mathcal{R}(R_{f_0})$ are of the form Rf/R_{f_0} . The functions in $\mathcal{R}(R_{f_u})$ are of the form Rf/R_{f_u} . Thus a basis for $\mathcal{R}(R_{f_0})$ is given by the functions of the form $g \frac{Rf_u}{R_{f_0}}$ with g a basis function for $\mathcal{R}(R_{f_u})$.

3 Numerical Evaluation of Inner Products

We have seen that the computation of projection operators largely reduces to the computation of inner products in $L^2(C, R_{f_0})$ and that these inner products amount to two-dimensional integrals. Unfortunately, closed form expressions for these integrals are not generally available, even when ϕ and f_0 are fairly elementary functions. Thus the evaluation of these integrals generally must be done numerically. This is a fairly computationally intensive operation. Realistic problems would require the evaluation of thousands of such integrals. In the special case where ϕ is symmetric, $F\phi$ can be written as a function of s alone. This reduces the problem considerably to one-dimensional integrals, but again closed form expressions are not usually available. In large part, the difficulty of obtaining closed form solutions has to do with the nature of the map from ϕ to its observation-space representation $F\phi$. Even if ϕ is a fairly mundane function, $F\phi$ tends to have a complex closed-form expression, if one is available at all. Moreover, $F\phi$ tends to be more oscillatory than ϕ , which impedes the numerical evaluation of integrals.

4 Analytic Evaluation of Inner Products

While we have computed inner products by numerical evaluation of one and two-dimensional integrals in a number of special cases [Kur97, sec. 6], it is rather cumbersome for extensive use. We are therefore proposing to proceed along a different tack. The idea is to expand ϕ and f_0 in terms of a certain orthonormal basis for the unit

disk D . It turns out that, for these basis functions, the necessary integrals can be evaluated analytically. Thus, once ϕ and f_0 are expanded in terms of these functions, the required inner products can be computed quite efficiently. The remaining difficulty is the determination of the coefficients of ϕ and f_0 in the expansions. This amounts to the evaluation of the inner products of ϕ and f_0 with the basis functions. In general, this involves the evaluation of two-dimensional integral on the image space. However, the situation here is not bad as it was in the observation space and can substantially be mitigated by a judicious choice of ϕ and f_0 . For example, call a function f on $\mathbb{R}^2$ *separable* if it can be written as a product of the form

$$f(r, \phi) = f_r(r) f_t(\phi),$$

where r and ϕ are polar coordinates on $\mathbb{R}^2$. As will be seen below, the basis functions in question are separable. Thus if ϕ and f_0 are separable, then the necessary inner products reduce to the product of two one-dimensional integrals. With an appropriate choice of ϕ and f_0 , these one-dimensional integrals can often be evaluated analytically. Taking ϕ and f_0 to be the sum of separable functions permits the evaluation of estimator efficiency over a broad range of conditions.

5 Technical Details

The key to our approach is the expansion of ϕ and f_0 in terms of a basis consisting of the products of so-called Zernike polynomials and sine and cosine harmonics. We let Z_m^l denote the Zernike polynomial of order l and degree m . A discussion of these polynomials may be found in [Dea83, sec. 7.6]. The set of functions of the form

$$b_{l,m} \equiv (m+1)^{1/2} Z_m^{|l|}(r) e^{il\phi}$$

with $l \in \mathbb{Z}$ and $m \in l + 2\mathbb{N}$ ($\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and nonnegative integers, respectively) are an orthonormal basis for the unit disk D with respect to

the inner product

$$\langle \phi, \phi' \rangle_{L^2(D, f_u)} \equiv \pi^{-1} \int_D \bar{\phi}(x) \phi'(x) dx$$

(the bar denotes complex conjugate) [JS90, p. 257]. We expand ϕ and f_0 in terms of these basis functions:

$$f_0(re^{i\phi}) = \sum_{l \in \mathbb{Z}} \sum_{m \in |l| + 2\mathbb{N}} c_{l,m} (m+1)^{-1/2} Z_m^{(l)}(r) e^{il\phi}$$

$$\phi(re^{i\phi}) = \sum_{l \in \mathbb{Z}} \sum_{m \in |l| + 2\mathbb{N}} \gamma_{l,m} (m+1)^{1/2} Z_m^{(l)}(r) e^{il\phi}.$$

It can then be shown [Dea83, sec. 7.6] that the observation density is

$$Rf_0(\theta, s) \equiv g_0(\theta, s)$$

$$= g_u \sum_{l \in \mathbb{Z}} \sum_{m \in |l| + 2\mathbb{N}} c_{l,m} (m+1)^{-1/2} U_m(s) e^{il\theta}$$

and that an observation-space representation of this functional is given by

$$\psi_{\text{FB}}(\theta, s) = \sum_{l \in \mathbb{Z}} \sum_{m \in |l| + 2\mathbb{N}} \gamma_{l,m} (m+1)^{1/2} U_m(s) e^{il\theta}.$$

We will now compute $p_{N(R^*)} g_{l,m} \psi_{\text{FB}}$. By equation 2 and the fact that

$$p_{N(R^*)} g_{l,m} \psi_{\text{FB}} = \psi_{\text{FB}} - p_{N(R^*)} g_{l,m} \psi_{\text{FB}},$$

it suffices to compute $p_{N(R^*)} g_{l,m}$ for each l, m with $m \in |l| + 2\mathbb{N}$. Define

$$g_{l,m}(\theta, s) \equiv U_m(s) e^{il\theta}$$

We thus need to figure out how to compute $p_{N(R^*)} g_{l,m}$ for a given l, m with $m \in |l| + 2\mathbb{N}$.

This amounts to computing inner products of the form

$$\langle g_{l,m}, g_{l',m'} \rangle_{L^2(C, g_0)}$$

with $g_{l,m} \in B_u$ and $g_{l',m'} \in B_u^\perp$ as well as inner products of the form

$$\langle g_{l'',m''}, g_{l',m'} \rangle_{L^2(C, g_0)}$$

with $g_{l'',m''}, g_{l',m'} \in B_u^\perp$. The following proposition shows that these inner products can be evaluated analytically.

Proposition 1 Suppose $g_{l',m'}, g_{l'',m''} \in B_u \cup B'_u$. Then

$$\langle g_{l'',m''}, g_{l',m'} \rangle_{L^2(C,g_0)} = \sum_{m=\max(|m''-m'|, |l''-l'|), 2}^{m''+m'} (m+1)^{-1/2} c_{l''-l',m}.$$

Proof. Then, using lemma 1 below,

$$\begin{aligned} & \langle g_{l'',m''} | g_{l',m'} \rangle_{L^2(C,g_0)} \\ &= \pi^{-1} \sum_{l \in \mathbb{Z}} \sum_{m \in |l|+2\mathbb{N}} c_{l,m} (m+1)^{-1/2} \int_0^\pi e^{i(l'-l''+l)\theta} d\theta \\ & \quad \times \int_{-1}^1 U_{m'}(s) U_{m''}(s) U_m(s) g_u(s) ds \\ &= \pi^{-1} \sum_{l \in l'+l''+2\mathbb{Z}} \sum_{m \in |l|+2\mathbb{N}} c_{l,m} (m+1)^{-1/2} \int_0^\pi e^{i(l'-l''+l)\theta} d\theta \\ & \quad \times \int_{-1}^1 U_{m'}(s) U_{m''}(s) U_m(s) g_u(s) ds \\ &= \sum_{m \in |l''-l'|+2\mathbb{N}} c_{l''-l',m} (m+1)^{-1/2} \int_{-1}^1 U_{m_1}(s) U_{m_2}(s) U_m(s) g_u(s) ds \\ &= \sum_{m=\max(|m''-m'|, |l''-l'|), 2}^{m''+m'} c_{l''-l',m} (m+1)^{-1/2}. \end{aligned}$$

(The second equality follows since, by lemma 1 below, $\int_{-1}^1 U_{m'}(s) U_{m''}(s) U_m(s) \rho_u(s) ds = 0$ if $m' + m'' + m$ is odd. Since $m' + m'' + l' + l'' \in 2\mathbb{Z}$, $m' + m''$ and $l' + l''$ have the same parity, i.e., they are either both odd or both even. Now $m' + m'' + m$ is even only if $m_1 + m_2$ has the same parity as m , hence m must have the same parity as $l' + l''$. This gives the second equality. Also note that $l \in l' + l'' + 2\mathbb{Z}$ implies $l' - l'' + l \in l' + l'' + 2\mathbb{Z}$.) This proves the result for $l' \leq l''$. If $l' > l''$, the result for $l' \leq l''$ gives

$$\begin{aligned} \langle g_{l'',m''}, g_{l',m'} \rangle_{L^2(C,g_0)} &= \langle g_{l'',m''}, \overline{g_{l',m'}} \rangle_{L^2(C,g_0)} \\ &= \sum_{m=\max(|m'-m''|, |l'-l''|), 2}^{m'+m''} \overline{c_{l'-l'',m}} \\ &= \sum_{m=\max(|m''-m'|, |l''-l'|), 2}^{m''+m'} c_{l''-l',m}. \quad \blacksquare \end{aligned}$$

Lemma 1 Suppose $j, k, m \in \mathbb{N}$. Then

$$\int_{-1}^1 U_j(s)U_k(s)U_m(s)\rho_u(s)ds = \begin{cases} 1 & \text{if } j+k+m \text{ is even and } |k-j| \leq m \leq j+k \\ 0 & \text{otherwise} \end{cases}$$

Proof. The result for $j+k+m$ odd follows at once from the oddness of the integrand, so in the remainder of the proof we assume $j+k+m$ is even. First suppose that $j \leq k$. Using the identity

$$U_j(s)U_k(s) = \frac{T_{k-j}(x) - T_{j+k+2}(x)}{2(1-s^2)}$$

[SO87, eq. 22:5:9], which is valid for $j \leq k$, we have

$$\int_{-1}^1 U_j(s)U_k(s)U_m(s)\rho_u(s)ds = \frac{1}{\pi} \int_{-1}^1 \frac{T_{k-j}(s) - T_{j+k+2}(s)}{\sqrt{1-s^2}} U_m(s)ds.$$

If $m \geq j+k+2$, then, using the identities

$$T_n(s)U_m(s) = \frac{1}{2}[U_{m+n}(s) + U_{m-n}(s)], \quad (7)$$

[SO87, eq. 22:5:10], which is valid for $n \leq m$, and

$$\int_{-1}^1 \frac{U_n(s)ds}{\sqrt{1-s^2}} = \pi$$

[EMOT54, eq. 16.1.33], which is valid for even n ,

$$\begin{aligned} & \int_{-1}^1 U_j(s)U_k(s)U_m(s)\rho_u(s)ds \\ &= \frac{1}{2\pi} \int_{-1}^1 \frac{U_{m+k-j}(s) + U_{m-j+k}(s) - [U_{m+k+j+2}(s) + U_{m-k-j-2}(s)]}{\sqrt{1-s^2}} ds \\ &= 0. \end{aligned}$$

If $j+k \geq m \geq k-j$, then, using the identity given by equation 7 and the identity

$$T_n(s)U_m(s) = \frac{1}{2}[U_{m+n}(s) + U_{n-m-2}(s)], \quad (8)$$

[SO87, eq. 22:5:10], which is valid for $n \geq m+2$,

$$\begin{aligned} & \int_{-1}^1 U_j(s)U_k(s)U_m(s)\rho_u(s)ds \\ &= \frac{1}{2\pi} \int_{-1}^1 \frac{U_{m+k-j}(s) + U_{m-j+k}(s) - [U_{m+k+j+2}(s) - U_{k+j-m}(s)]}{\sqrt{1-s^2}} ds \\ &= 1. \end{aligned}$$

If $k - j - 2 \geq m$, then, using the identity given by equation 8

$$\begin{aligned} & \int_{-1}^1 U_j(s) U_k(s) U_m(s) \rho_u(s) ds \\ &= \frac{1}{2\pi} \int_{-1}^1 \frac{U_{m+k-j}(s) - U_{k-j-m-2}(s) - [U_{m+k+j+2}(s) - U_{k+j-m}(s)]}{\sqrt{1-s^2}} ds \\ &= 0. \quad \blacksquare \end{aligned}$$

Acknowledgements

This work was supported in part by the Director, Office of Energy Research, Office of Health and Environmental Research, Medical Applications and Biophysical Research Division of the U.S. Department of Energy under contract No. DE-AC03-76SF00098 and in part by National Heart, Lung, and Blood Institute of the U.S. Department of Health and Human Services under grants T32-HL07367, P01-HL25840, and R01-HL50663.

References

- [Dea83] Stanley R. Deans. *The Radon Transform and Some of Its Applications*. John Wiley & Sons, New York, 1983.
- [EMOT54] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Tables of Integral Transforms*, volume 1. McGraw-Hill, New York, 1954.
- [JS90] Iain M. Johnstone and Bernard W. Silverman. Speed of estimation in positron emission tomography and related inverse problems. *Ann. Statist.*, 18:251–280, 1990.
- [Kur97] A. Kuruc. Efficient estimation of linear functionals in emission tomography. *SIAM J. Appl. Math.*, 1997. In press.

- [SO87] Jerome Spanier and Keith B. Oldham. *An Atlas of Functions*. Hemisphere Publishing Corp., Washington, 1987.