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May 1996

Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098
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The purpose of this report is to outline an approach to the numerical construction of statistically efficient estimators for linear functionals in emission tomography (ET) that is more efficient than the approach used in [Kur97]. For the sake of brevity, we will assume familiarity with the notation and material in [Kur97].

1 Statistically Efficient Estimators for Linear Functionals in ET

In [Kur97], we described the construction of statistically efficient estimators for linear functionals in ET and the relationship of the efficient estimator to the standard estimator based on the FB algorithm. We found that the standard FB estimator for the linear functional $\int \phi(x)f(x)\,dx$ was equivalent to the linear estimator generated by the observation-space function

$$\psi_{FB} \equiv HR\phi,$$

where $H$ is the ramp-filter operator and $R$ denotes the Radon transform. Also, the efficient estimator at $f_0$ was the linear estimator generated by the observation-space
2 CONSTRUCTION OF PROJECTION OPERATORS

function

\[ \psi_E = p_{N(R^*)}\psi_{FB} \]  

\[ = \psi_{FB} - p_{N(R^*)}\psi_{FB}, \]  

where \( p_{N(R^*)} \) and \( p_{N(R^*)}^\perp \) are the projection operators onto the null space of the back-projection operator \( R^* \) and its orthogonal complement in \( L^2(C, R_{f_0}) \), respectively.

Given the generating functions \( \psi_E \) and \( \psi_{FB} \), it is straightforward to compute the variances of the resulting estimators using proposition 3.4 in [Kur97]. The main technical obstacles are the calculation of the generating functions \( \psi_E \) and \( \psi_{FB} \). While we have been able to generate results for some special cases, the methods that we have used to date are too cumbersome for general use. This report is largely devoted to outlining an alternative approach to the calculation of these functions for a reasonably broad class of scenarios.

2 Construction of Projection Operators

We now discuss concrete representations of the projection operators in equations 1 and 2. Recall from linear algebra that it is easy to compute the projection of a vector onto a subspace given an orthonormal basis for the subspace. The coefficients of the projection of the vector with respect to the orthonormal basis are just the inner products of the vector with the basis vectors. Thus if \( \{v_1, \ldots, v_m\} \) is a set of orthonormal vectors in \( \mathbb{R}^n \), i.e., the norm \( ||v_i|| = 1 \) for \( i = 1, \ldots, m \) and the inner product \( \langle v_i, v_j \rangle = 0 \) for \( i \neq j \), then the projection of a vector \( v \in \mathbb{R}^n \) onto the subspace \( V \) spanned by \( \{v_1, \ldots, v_m\} \) is given by

\[ p_V v = \sum_{i=1}^{m} \langle v, v_i \rangle v_i. \]

Similar ideas apply to the construction of projection operators in \( L^2(C, R_{f_0}) \), where the inner product of \( \psi \) and \( \psi' \) is now defined by

\[ \langle \psi, \psi' \rangle_{L^2(C, R_{f_0})} = \int_C \psi(l)\psi'(l)R_{f_0}(l) dl. \]
The integral over $\mathbb{L}$ on the right side of equation 3 is defined explicitly by

$$\int_C \psi(l)\psi'(l)Rf_0(l) \, dl \equiv \pi^{-1} \int_0^\pi \int_{-1}^1 \psi(\theta, s)\psi'((\theta, s)Rf_0(\theta, s) \, ds \, d\theta,$$

where $\theta$ and $s$ are the usual orientation and signed distance from the origin coordinates on $\mathbb{L}$, respectively. Thus if $B$ and $B^\perp$ are orthonormal bases for $N(R^*)$ and $N(R^*)^\perp$, respectively,

$$p_{N(R^*)}\psi = \sum_{b \in B} \langle \psi, b \rangle_{L^2(C, Rf_0)} b,$$

and

$$p_{N(R^*)^\perp}\psi = \sum_{b \in B^\perp} \langle \psi, b \rangle_{L^2(C, Rf_0)} b.$$

The only real difference between $\mathbb{R}^n$ and $L^2(C, Rf_0)$ is that, at least in theory, the orthonormal bases $B$ and $B^\perp$ are infinite sets. In practice, of course, it is generally necessary to terminate the expansions after a finite number of terms.

In practice, one is often required to construct projection operators from a basis which is not orthonormal. Suppose $\{v_1, \ldots, v_m\}$ is a set of linearly independent, but not necessarily orthonormal, vectors in $\mathbb{R}^n$. Let $V$ denote the $n \times m$ matrix whose columns are $v_1, \ldots, v_m$. Then the projection of a vector $v \in \mathbb{R}^n$ onto the column space of $V$ is given by

$$p_v v = V(V^T V)^{-1} V^T v.$$

We see that the coefficients of $p_v v$ with respect to the basis $\{v_1, \ldots, v_m\}$ are again expressible in terms of inner products, albeit in a more complicated form. In $L^2(C, Rf_0)$, analogous formulas apply by substituting the appropriate inner product.

### 2.1 The Uniform Distribution

For simplicity, we first consider the case where the underlying image $f$ is the uniform distribution on $D$, i.e.,

$$f = f_u \equiv \pi^{-1}.$$
Then orthonormal bases for $N(R^*)$ and $N(R^*)^\perp$ can be given explicitly [Kur97, sec. 5]. Let

$$U_m(\cos \theta) \equiv \frac{\sin[(m+1)\theta]}{\sin \theta}$$

(4)

denote the Chebyshev polynomial of the second kind of order $m$. (The first few are the polynomials $1, 2s, 4s^2 - 1, \text{and } 8s^3 - 4s$.) A orthonormal basis for $N(R^*)^\perp$ in $L^2(C, Rf_u)$ is given by the functions $U_m(s)$ with $m = 0, 2, 4, \ldots$ and $\sqrt{2}U_m \cos(l\theta)$ and $\sqrt{2}U_m \sin(l\theta)$ with $l = 1, 2, 3, \ldots$ and $m = l, l + 2, l + 4, \ldots$. A orthonormal basis for $N(R^*)$ in $L^2(C, Rf_u)$ is given by the functions $\sqrt{2}U_m \cos(l\theta)$ and $\sqrt{2}U_m \sin(l\theta)$ with $l = 1, 2, 3, \ldots$ and $m = l \mod 2, l \mod 2 + 2, \ldots, l - 2$. $Rf_u$ is given explicitly by

$$Rf_u \equiv g_u = 2\pi^{-1}\sqrt{1 - s^2}.$$

Thus to compute $\psi_E$ we need to compute inner products of the form

$$\langle U_m(s) \sin(l\theta), \psi_{FB} \rangle_{L^2(C,g_u)} = 2\pi^{-2} \int_0^\pi \int_{-1}^1 U_m(s) \sin(l\theta) F\phi(\theta, s) \sqrt{1 - s^2} \, ds \, d\theta$$

(5)

and

$$\langle U_m(s) \cos(l\theta), \psi_{FB} \rangle_{L^2(C,g_u)} = 2\pi^{-2} \int_0^\pi \int_{-1}^1 U_m(s) \cos(l\theta) F\phi(\theta, s) \sqrt{1 - s^2} \, ds \, d\theta.$$ 

(6)

An important consequence of this result is that when $\phi$ is a radial function, the linear estimator generated by $\psi_{FB}$ is efficient at $f_u$. The reason is that $F\phi$ is then independent of $\theta$ and symmetric in $s$. These properties imply that the inner products of $\psi_{FB}$ with the basis functions of the form $U_m(s) \cos(l\theta)$ and $U_m(s) \sin(l\theta)$ with $l \geq 1$ vanish.

### 2.2 The General Case

We now consider the general case where $f_0 \neq f_u$. Note that $N(R^*)$ is defined independently of $f_0$ and thus the basis for it given in the preceding subsection is still a basis. It will not, however, generally be an orthonormal basis in $L^2(C, Rf_0)$. 
Similar considerations apply to finding an orthonormal basis for $\mathcal{R}(R_{f_0})$. The functions in $\mathcal{R}(R_{f_0})$ are of the form $Rf/R_{f_0}$. The functions in $\mathcal{R}(R_{f_u})$ are of the form $Rf/R_{f_u}$. Thus a basis for $\mathcal{R}(R_{f_0})$ is given by the functions of the form $gR_{f_u}/R_{f_0}$ with $g$ a basis function for $\mathcal{R}(R_{f_u})$.

3 Numerical Evaluation of Inner Products

We have seen that the computation of projection operators largely reduces to the computation of inner products in $L^2(C, R_{f_0})$ and that these inner products amount to two-dimensional integrals. Unfortunately, closed form expressions for these integrals are not generally available, even when $\phi$ and $f_0$ are fairly elementary functions. Thus the evaluation of these integrals generally must be done numerically. This is a fairly computationally intensive operation. Realistic problems would require the evaluation of thousands of such integrals. In the special case where $\phi$ is symmetric, $F\phi$ can be written as a function of $s$ alone. This reduces the problem considerably to one-dimensional integrals, but again closed form expressions are not usually available. In large part, the difficulty of obtaining closed form solutions has to do with the nature of the map from $\phi$ to its observation-space representation $F\phi$. Even if $\phi$ is a fairly mundane function, $F\phi$ tends to have a complex closed-form expression, if one is available at all. Moreover, $F\phi$ tends to be more oscillatory than $\phi$, which impedes the numerical evaluation of integrals.

4 Analytic Evaluation of Inner Products

While we have computed inner products by numerical evaluation of one and two-dimensional integrals in a number of special cases [Kur97, sec. 6], it is rather cumbersome for extensive use. We are therefore proposing to proceed along a different tack. The idea is to expand $\phi$ and $f_0$ in terms of a certain orthonormal basis for the unit
disk $D$. It turns out that, for these basis functions, the necessary integrals can be evaluated analytically. Thus, once $\phi$ and $f_0$ are expanded in terms of these functions, the required inner products can be computed quite efficiently. The remaining difficulty is the determination of the coefficients of $\phi$ and $f_0$ in the expansions. This amounts to the evaluation of the inner products of $\phi$ and $f_0$ with the basis functions. In general, this involves the evaluation of two-dimensional integral on the image space. However, the situation here is not bad as it was in the observation space and can substantially mitigated by a judicious choice of $\phi$ and $f_0$. For example, call a function $f$ on $\mathbb{R}^2$ separable if it can be written as a product of the form

$$f(r, \phi) = f_s(r)f_t(\phi),$$

where $r$ and $\phi$ are polar coordinates on $\mathbb{R}^2$. As will be seen below, the basis functions in question are separable. Thus if $\phi$ and $f_0$ are separable, then the necessary inner products reduce to the product of two one-dimensional integrals. With an appropriate choice of $\phi$ and $f_0$, these one-dimensional integrals can often be evaluated analytically. Taking $\phi$ and $f_0$ to be the sum of separable functions permits the evaluation of estimator efficiency over a broad range of conditions.

5 Technical Details

The key to our approach is the expansion of $\phi$ and $f_0$ is terms of a basis consisting of the products of so-called Zernike polynomials and sine and cosine harmonics. We let $Z_{m}^{l}$ denote the Zernike polynomial of order $l$ and degree $m$. A discussion of these polynomials may be found in [Dea83, sec. 7.6]. The set of functions of the form

$$b_{l,m} = (m + 1)^{1/2}Z_{m}^{l}(r)e^{il\phi}$$

with $l \in \mathbb{Z}$ and $m \in l + 2\mathbb{N}$ ($\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and nonnegative integers, respectively) are an orthonormal basis for the unit disk $D$ with respect to
the inner product
\[ \langle \phi, \phi' \rangle_{L^2(D, f_u)} = \pi^{-1} \int_D \bar{\phi}(x) \phi'(x) \, dx \]
(the bar denotes complex conjugate) [JS90, p. 257]. We expand \( \phi \) and \( f_0 \) in terms of these basis functions:
\[
\begin{align*}
    f_0(re^{i\phi}) &= \sum_{l \in \mathbb{Z}} \sum_{m \in |l|+2N} c_{l,m}(m + 1)^{-1/2} Z^{|l|}_m(r)e^{il\phi} \\
    \phi(re^{i\phi}) &= \sum_{l \in \mathbb{Z}} \sum_{m \in |l|+2N} \gamma_{l,m}(m + 1)^{1/2} Z^{|l|}_m(r)e^{il\phi}.
\end{align*}
\]
It can then be shown [Dea83, sec. 7.6] that the observation density is
\[
R f_0(\theta, s) \equiv g_0(\theta, s) = g_u \sum_{l \in \mathbb{Z}} \sum_{m \in |l|+2N} c_{l,m}(m + 1)^{-1/2} U_m(s)e^{il\phi}
\]
and that an observation-space representation of this functional is given by
\[
\psi_{FB}(\theta, s) = \sum_{l \in \mathbb{Z}} \sum_{m \in |l|+2N} \gamma_{l,m}(m + 1)^{1/2} U_m(s)e^{il\phi}.
\]
We will now compute \( p_{N(R^*)}g_{l,m}\psi_{FB} \). By equation 2 and the fact that
\[
p_{N(R^*)}g_{l,m}\psi_{FB} = \psi_{FB} - p_{N(R^*)}g_{l,m}\psi_{FB},
\]
it suffices to compute \( p_{N(R^*)}g_{l,m} \) for each \( l, m \) with \( m \in |l|+2N \). Define
\[
g_{l,m}(\theta, s) \equiv U_m(s)e^{il\theta}
\]
We thus need to figure out how to compute \( p_{N(R^*)}g_{l,m} \) for a given \( l, m \) with \( m \in |l|+2N \). This amounts to computing inner products of the form
\[
\langle g_{l,m}, g'_{l,m'} \rangle_{L^2(C_{g0})}
\]
with \( g_{l,m} \in B_u \) and \( g'_{l,m'} \in B_u^\perp \) as well as inner products of the form
\[
\langle g_{l',m''}, g_{l',m'} \rangle_{L^2(C_{g0})}
\]
with \( g_{l',m''}, g_{l',m'} \in B_u^\perp \). The following proposition shows that these inner products can be evaluated analytically.
Proposition 1 Suppose $g_{l',m'}, g_{l'',m''} \in B_u \cup B'_u$. Then

$$
\langle g_{l',m'}, g_{l'',m''} \rangle_{L^2(C, \phi_0)} = \sum_{m = \max(|m'' - m'|, |l'' - l'|)}^{m'' + m'} (m + 1)^{-1/2} c_{l'' - l', m}.
$$

Proof. Then, using lemma 1 below,

$$
\langle g_{l',m'}, g_{l'',m''} \rangle_{L^2(C, \phi_0)}
\begin{align*}
&= \pi^{-1} \sum_{l \in \mathbb{Z}} \sum_{m \in |l| + 2N} c_{l,m} (m + 1)^{-1/2} \int_0^{\pi} e^{i(l''-l'+l)\theta} d\theta \\
&\times \int_{-1}^{1} U_{m'}(s)U_{m''}(s)U_{m}(s)g_u(s) \, ds
\end{align*}
\begin{align*}
&= \pi^{-1} \sum_{l \in l'' + l' + 2\mathbb{Z}} \sum_{m \in |l| + 2N} c_{l,m} (m + 1)^{-1/2} \int_0^{\pi} e^{i(l''-l'+l)\theta} d\theta \\
&\times \int_{-1}^{1} U_{m'}(s)U_{m''}(s)U_{m}(s)g_u(s) \, ds
\end{align*}
\begin{align*}
&= \sum_{m \in |l'' - l'| + 2\mathbb{N}} c_{l'' - l', m} (m + 1)^{-1/2} \int_{-1}^{1} U_{m_1}(s)U_{m_2}(s)U_{m}(s)g_u(s) \, ds
\end{align*}
\begin{align*}
&= \sum_{m = \max(|m'' - m'|, |l'' - l'|)}^{m'' + m'} c_{l'' - l', m} (m + 1)^{-1/2}.
\end{align*}

(The second equality follows since, by lemma 1 below, $\int_{-1}^{1} U_{m'}(s)U_{m''}(s)U_{m}(s)\rho_u(s) \, ds = 0$ if $m' + m'' + m$ is odd. Since $m' + m'' + l' + l'' \in 2\mathbb{Z}$, $m' + m''$ and $l' + l''$ have the same parity, i.e., they are either both odd or both even. Now $m' + m'' + m$ is even only if $m_1 + m_2$ has the same parity as $m$, hence $m$ must have the same parity as $l' + l''$. This gives the second equality. Also note that $l \in l' + l'' + 2\mathbb{Z}$ implies $l' - l'' + l \in l' + l'' + 2\mathbb{Z}$.) This proves the result for $l' \leq l''$. If $l' > l''$, the result for $l' \leq l''$ gives

$$
\langle g_{l'',m''}, g_{l',m'} \rangle_{L^2(C, \phi_0)} = \langle g_{l'',m''}, \overline{g_{l',m'}} \rangle_{L^2(C, \phi_0)}
\begin{align*}
&= \sum_{m = \max(|m'' - m'|, |l'' - l'|)}^{m'' + m'} \overline{c_{l'' - l', m}}
\end{align*}
\begin{align*}
&= \sum_{m = \max(|m'' - m'|, |l'' - l'|)}^{m'' + m'} c_{l'' - l', m}.
\end{align*} \quad \blacksquare
Lemma 1 Suppose \( j, k, m \in \mathbb{N} \). Then
\[
\int_{-1}^{1} U_j(s)U_k(s)U_m(s)\rho_u(s)\,ds = \begin{cases} 
1 & \text{if } j + k + m \text{ is even and } |k - j| \leq m \leq j + k \\
0 & \text{otherwise}
\end{cases}
\]

Proof. The result for \( j + k + m \) odd follows at once from the oddness of the integrand, so in the remainder of the proof we assume \( j + k + m \) is even. First suppose that \( j \leq k \). Using the identity
\[
U_j(s)U_k(s) = \frac{T_{k-j}(x) - T_{j+k+2}(x)}{2(1 - s^2)}
\]
[S087, eq. 22:5:9], which is valid for \( j \leq k \), we have
\[
\int_{-1}^{1} U_j(s)U_k(s)U_m(s)\rho_u(s)\,ds = \frac{1}{\pi} \int_{-1}^{1} \frac{T_{k-j}(s) - T_{k+j+2}(s)}{\sqrt{1 - s^2}} U_m(s)\,ds.
\]
If \( m \geq j + k + 2 \), then, using the identities
\[
T_n(s)U_m(s) = \frac{1}{2} [U_{m+n}(s) + U_{m-n}(s)],
\]
[S087, eq. 22:5:10], which is valid for \( n \leq m \), and
\[
\int_{-1}^{1} \frac{U_n(s)\,ds}{\sqrt{1 - s^2}} = \pi
\]
[EMOT54, eq. 16.1.33], which is valid for even \( n \),
\[
\int_{-1}^{1} U_j(s)U_k(s)U_m(s)\rho_u(s)\,ds
\]
\[= \frac{1}{2\pi} \int_{-1}^{1} \frac{U_{m+k-j}(s) + U_{m-j+k}(s) - [U_{m+k+j+2}(s) + U_{m-k-j-2}(s)]}{\sqrt{1 - s^2}} \,ds
\]
\[= 0.
\]
If \( j + k \geq m \geq k - j \), then, using the identity given by equation 7 and the identity
\[
T_n(s)U_m(s) = \frac{1}{2} [U_{m+n}(s) + U_{n-m-2}(s)],
\]
[S087, eq. 22:5:10], which is valid for \( n \geq m + 2 \),
\[
\int_{-1}^{1} U_j(s)U_k(s)U_m(s)\rho_u(s)\,ds
\]
\[= \frac{1}{2\pi} \int_{-1}^{1} \frac{U_{m+k-j}(s) + U_{m-j+k}(s) - [U_{m+k+j+2}(s) - U_{k+j-m}(s)]}{\sqrt{1 - s^2}} \,ds
\]
\[= 1.
\]
If $k - j - 2 \geq m$, then, using the identity given by equation 8,

$$
\int_{-1}^{1} U_j(s)U_k(s)U_m(s)\rho_u(s)\,ds
= \frac{1}{2\pi} \int_{-1}^{1} \frac{U_{m+k-j}(s) - U_{k-j-m-2}(s) - [U_{m+k+j+2}(s) - U_{k+j-m}(s)]}{\sqrt{1 - s^2}}\,ds
= 0. \,
$$

Acknowledgements

This work was supported in part by the Director, Office of Energy Research, Office of Health and Environmental Research, Medical Applications and Biophysical Research Division of the U.S. Department of Energy under contract No. DE-AC03-76SF00098 and in part by National Heart, Lung, and Blood Institute of the U.S. Department of Health and Human Services under grants T32-HL07367, P01-HL25840, and R01-HL50663.

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