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Super-Galilei Invariant Field Theories in 2+1 Dimensions^{*}

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ABSTRACT

We extend the Galilei group of space-time transformations by gradation, construct interacting field-theoretic representations of this algebra, and show that non-relativistic Super-Chern-Simons theory is a special case. We also study the generalization to matrix valued fields, which are relevant to the formulation of superstring theory as a $1/N_c$ expansion of a field theory. We find that in the matrix case, the field theory is much more restricted by the supersymmetry.

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1. Introduction

Galilean invariance is generally thought of as a low energy approximation to Poincaré invariance, the exact space-time symmetry of relativistic systems. In this point of view Galilei invariant field theories describe the dynamics of low energy, or non-relativistic, systems. There are two main applications for Galilei invariant field theories in this context. One is to consider them as non-relativistic limits ($c \rightarrow \infty$) of corresponding relativistic field theories, and to study generic features of field theory in this simpler setting. This pedagogical approach has been used to exhibit features such as gauge invariance,^[1] triviality and renormalization,^[2] self-dual solitons,^[3] and the conformal anomaly^[4] in non-relativistic field theories, with the occasional hope of learning something useful about the relativistic theories.

The other application of non-relativistic field theories is second-quantization of non-relativistic quantum mechanical systems. This is used in condensed matter systems, which are non-relativistic by nature. Frequently a second-quantized approach can shed some light on a seemingly intractable quantum mechanical problem. A good example is the age old Aharonov-Bohm (AB) scattering problem,^[5] in which a charged particle scatters off an infinitely long and infinitesimally thin solenoid carrying a magnetic flux. The exact solution was discovered by Aharonov and Bohm in 1959. Ignoring this fact for a moment, one is naturally led to using perturbation theory via the Born expansion to get an approximate solution. Curiously, all attempts at reproducing even the lowest order term in a Taylor expansion of the exact solution have failed,^[6] until recently.^[7] In the second-quantized approach of Ref.[7], one is led to a natural resolution of the perturbative puzzle by simply including terms in the action required for consistency of the field theory.

There is an alternative point of view, in which Galilean invariance is relevant for relativistic systems. This happens when relativistic systems are quantized

in light-cone variables. Light-cone coordinates are defined by singling out one of the spatial directions, say x^{D-1} , and letting

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1}).$$

In light-cone quantization the role of time is played by x^+ , so its conjugate momentum p^- is the light-cone Hamiltonian. x^- is the longitudinal coordinate, and x^i , with $i = 1, \dots, D-2$, are the transverse coordinates. In these coordinates a transverse Galilei group in $D-2$ space and one time dimension emerges as a manifest subgroup of the D dimensional Poincaré group. Transverse spatial translations are generated by p^i , time translation is generated by p^- , transverse spatial rotations are generated by the transverse components of the Lorentz tensor M^{ij} , and transverse Galilei boosts are generated by the mixed components M^{+i} . The remaining components of the Lorentz generator M^{-i}, M^{+-} are not part of the Galilei sub-algebra. From the point of view of the Galilei subgroup, the longitudinal momentum p^+ plays the role of Newtonian mass, even though it is a generator of the Poincaré group.

One can imagine systems in which the Poincaré symmetry breaks down to its Galilei subgroup in light-cone variables. These systems are by no means non-relativistic in the usual sense of $c \rightarrow \infty$, but they are still strictly Galilei invariant. One such system is relativistic string quantized in light-cone gauge. Except at $D = 26$, where the full Poincaré invariance is realized, the dynamics of light-cone string are only Galilei invariant. In light-cone gauge p^+ is essentially the length of a piece of string. Replacing it with a discrete variable allows for a description of string as a composite of string bits, obeying Galilei invariant dynamics in $(D-2)+1$ dimensions.^[8-10] The mass of each bit is $p^+ / (\# \text{bits})$, so the total mass is p^+ , as implied by the Galilei algebra. The dynamics of the bits must be such as to give a strong nearest neighbor attraction and weaker non-nearest neighbor interactions, in order for long closed polymers to form. The missing dimension of string, the coordinate x^- , reappears in the limit where its conjugate p^+ becomes continuous, and the discrete polymer becomes a continuous string. The dynamics of the bits are described by a Galilei invariant

field theory. But a nearest neighbor interaction pattern in continuous space clearly cannot be achieved by single component fields. It requires the introduction of a matrix field theory and the use of 't Hooft's $1/N_c$ expansion.^[10] The nearest neighbor interaction will then appear at zeroth order in this expansion. Non-nearest neighbor interactions, which lead to the breaking of a polymer into several polymers, will appear at higher orders in the expansion.

The possibility of extending the Galilei group of transformations by a gradation to a Super-Galilei algebra in 3+1 dimensions was first suggested in Ref.[11]. The author showed that there are two possible superalgebras $\mathcal{S}_1\mathcal{G}$ and $\mathcal{S}_2\mathcal{G}$, where $\mathcal{S}_1\mathcal{G} \subset \mathcal{S}_2\mathcal{G}$. The smaller superalgebra includes a single two-component spinor supercharge Q , and the larger superalgebra includes in addition a second two-component spinor supercharge R . He then proceeded to construct field theoretic representations of the $\mathcal{S}_1\mathcal{G}$ algebra. Ref.[12] explored a particular $\mathcal{S}_2\mathcal{G}$ invariant field theory in 2+1 dimensions, namely non-relativistic Super-Chern-Simons theory. It was actually derived as a non-relativistic limit of relativistic Super-Chern-Simons theory.^[13] The authors showed that the existence of non-relativistic self-dual solitons was guaranteed by supersymmetry. Later^[7] it was also suggested that the conformal anomaly of the non-relativistic Chern-Simons theory vanishes in the supersymmetric case, bringing together the concepts of supersymmetry, self-duality and conformal invariance.

As with the Galilei algebra, the Super-Galilei algebra appears as a subalgebra of the Super-Poincaré algebra in light-cone coordinates. The possibility of constructing Super-Galilei invariant field theories may then lead to a reformulation of superstring theory as a supersymmetric bit theory in one less dimension. We leave this issue for another paper.^[14] In this paper we are interested in constructing $\mathcal{S}_2\mathcal{G}$ invariant field theories in 2+1 dimensions. It is this larger superalgebra that emerges as a subalgebra of the Super-Poincaré algebra, whereas the non-relativistic limit of the Super-Poincaré algebra is just $\mathcal{S}_1\mathcal{G}$. We consider only 2+1 dimensions since that corresponds to four dimensional Super-Poincaré invariance. Critical superstring lives in ten dimensions, and thus the bit formulation should be in 8+1 dimensions. 2+1 dimensional Super-Galilei

invariant field theories serve first of all as toy models for the 8+1 dimensional model we eventually want to construct. In addition, they may also be the basis of a physical four dimensional superstring theory, with compactified dimensions built out of internal degrees of freedom in the bit theory.

In section 2 we present the Super-Galilei algebras $\mathcal{S}_1\mathcal{G}$ and $\mathcal{S}_2\mathcal{G}$ in 2+1 dimensions. In section 3 we construct field theoretic representations for $\mathcal{S}_2\mathcal{G}$ by second-quantizing all the charges and deriving the Hamiltonian from the superalgebra. We also discuss the Super-Fock space and the super-wavefunctions. In section 4 we show that non-relativistic Super-Chern-Simons theory is just a special case of the general Galilei invariant field theory constructed in section 3. In section 5 we construct a Super-Galilei invariant matrix field theory, and discuss singlet Fock states and the $1/N_c$ expansion. This section is a prelude to developing a bit model for superstring in the Green-Schwarz formulation. In the last section we present a brief discussion of our results.

2. The Super-Galilei Algebra in 2+1 Dimensions

The generators of the Galilei group in 2+1 dimensions include a 2 dimensional momentum vector $\mathbf{P}$, a 2 dimensional boost vector $\mathbf{K}$, a planar angular momentum scalar J , and a Hamiltonian H . In addition there is also a number operator M , counting the total number of particles in the system. The only non-vanishing commutators in the algebra are given by

$$\begin{aligned} [P_i, K_j] &= i\delta_{ij}mM & [P_i, J] &= -i\epsilon_{ij}P_j \\ [H, K_i] &= iP_i & [K_i, J] &= -i\epsilon_{ij}K_j. \end{aligned} \quad (2.1)$$

One can extend this algebra by adding a complex odd (fermionic) charge Q , satisfying the following commutator:

$$[Q, J] = \frac{1}{2}Q, \quad (2.2)$$

and its hermitian conjugate counterpart. All other commutators vanish. The graded algebra will close if in addition the following anti-commutators (and

their hermitian conjugates) are satisfied:

$$\begin{aligned}\{Q, Q^\dagger\} &= mM \\ \{Q, Q\} &= 0.\end{aligned}\tag{2.3}$$

The graded algebra given by (2.1)-(2.3) defines the superalgebra $\mathcal{S}_1\mathcal{G}$. The extended Super-Galilei algebra $\mathcal{S}_2\mathcal{G}$ requires an additional supercharge R satisfying the following commutators:

$$\begin{aligned}[R, J] &= -R/2 \\ [R, K^-] &= -iQ,\end{aligned}\tag{2.4}$$

and their hermitian conjugates, with all other commutators vanishing. The $\pm$ components of any real two dimensional vector $\mathbf{V}$ are defined by $V^\pm \equiv V^1 \pm iV^2$. To close the $\mathcal{S}_2\mathcal{G}$ algebra we need the following anti-commutators:

$$\begin{aligned}\{Q, R\} &= \{R, R\} = 0 \\ \{Q, R^\dagger\} &= -P^-/2 \\ \{R, R^\dagger\} &= H/2,\end{aligned}\tag{2.5}$$

and their hermitian conjugates. The algebra given by (2.1)-(2.5) then defines the super-algebra $\mathcal{S}_2\mathcal{G}$. We turn next to a field theoretic representation of this super-algebra.

3. Field Theoretic Representation of $\mathcal{S}_2\mathcal{G}$

The simplest $N = 1$ Galilei supermultiplet in 2+1 dimensions consists of a complex scalar field $\phi(\mathbf{x})$ and a one-component complex Grassmann field $\psi(\mathbf{x})$ corresponding to a spin helicity of $-1/2$ in the plane. The fields satisfy the canonical commutation relations:

$$[\phi(\mathbf{x}), \phi^\dagger(\mathbf{y})] = \{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}).$$

The superalgebra $\mathcal{S}_2\mathcal{G}$ can then be realized with free fields as follows:

$$\begin{aligned}M &= \int d\mathbf{x} [\phi^\dagger(\mathbf{x})\phi(\mathbf{x}) + \psi^\dagger(\mathbf{x})\psi(\mathbf{x})] \\ P^i &= -i \int d\mathbf{x} [\phi^\dagger(\mathbf{x})\partial^i\phi(\mathbf{x}) + \psi^\dagger(\mathbf{x})\partial^i\psi(\mathbf{x})] \\ K^i &= - \int d\mathbf{x} [\phi^\dagger(\mathbf{x})(it\partial^i + mx^i)\phi(\mathbf{x}) + \psi^\dagger(\mathbf{x})(it\partial^i + mx^i)\psi(\mathbf{x})] \\ J &= -i \int d\mathbf{x} [\phi^\dagger(\mathbf{x})(\mathbf{x} \times \nabla)\phi(\mathbf{x}) + \psi^\dagger(\mathbf{x})(\mathbf{x} \times \nabla - \frac{i}{2})\psi(\mathbf{x})] \\ Q &= -i\sqrt{m} \int d\mathbf{x} \psi^\dagger(\mathbf{x})\phi(\mathbf{x}) \\ Q^\dagger &= i\sqrt{m} \int d\mathbf{x} \phi^\dagger(\mathbf{x})\psi(\mathbf{x}) \\ \mathcal{R}^{(0)} &= \frac{1}{2\sqrt{m}} \int d\mathbf{x} \psi^\dagger(\mathbf{x})\partial^+\phi(\mathbf{x}) \\ \mathcal{R}^{(0)\dagger} &= -\frac{1}{2\sqrt{m}} \int d\mathbf{x} \phi^\dagger(\mathbf{x})\partial^-\psi(\mathbf{x}).\end{aligned}\tag{3.1}$$

We use script letters for the second-quantized supercharges to avoid later confusion with their first-quantized counterparts. The Hamiltonian for this field theory is then clearly

$$H^{(0)} = 2\{\mathcal{R}^{(0)}, \mathcal{R}^{(0)\dagger}\} = \frac{1}{2m} \int d\mathbf{x} [|\nabla\phi(\mathbf{x})|^2 + |\nabla\psi(\mathbf{x})|^2].\tag{3.2}$$

To construct an interacting field theory one usually adds higher order terms to the Hamiltonian (or action). To check that the resulting theory is supersymmetric is somewhat cumbersome, it is more convenient to add higher order

terms to the supercharge $\mathcal{R}^{(0)}$ instead. For the resulting theory to be supersymmetric certain conditions on the higher order terms must hold. Let $\mathcal{R}'$ denote the additional terms, then the total supercharge is

$$\mathcal{R} = \mathcal{R}^{(0)} + \mathcal{R}', \quad (3.3)$$

and the total Hamiltonian is given by

$$H = 2\{\mathcal{R}, \mathcal{R}^\dagger\}. \quad (3.4)$$

This supercharge must satisfy the $S_2\mathcal{G}$ algebra, and consequently $\mathcal{R}'$ must satisfy the following relations

$$\begin{aligned} [\mathcal{R}', M] &= 0 & \{\mathcal{R}', Q\} &= 0 \\ [\mathcal{R}', K^\pm] &= 0 & \{\mathcal{R}', Q^\dagger\} &= 0 \\ [\mathcal{R}', J] &= -\frac{1}{2}\mathcal{R}' & 2\{\mathcal{R}', \mathcal{R}^{(0)}\} + \{\mathcal{R}', \mathcal{R}'\} &= 0. \end{aligned} \quad (3.5)$$

The conjugate supercharge $\mathcal{R}'^\dagger$ satisfies similar relations, except the spin is reversed. Invariance under the global $U(1)$ symmetry given by the first commutator implies that $\mathcal{R}'$ has an equal number of creation and annihilation operators. Invariance under Galilei boosts given by the second commutator further restricts this to be so at each point. For simplicity we limit modifications to quartics in the fields. The anti-commutation relations with Q and $Q^\dagger$ then restrict the form of $\mathcal{R}'$ to:

$$\begin{aligned} \mathcal{R}' &\propto \int dx dy V^+(y-x) \psi^\dagger(x) \rho(y) \phi(x) \\ \mathcal{R}'^\dagger &\propto \int dx dy V^-(y-x) \phi^\dagger(x) \rho(y) \psi(x), \end{aligned} \quad (3.6)$$

where $\rho = \phi^\dagger \phi + \psi^\dagger \psi$. Finally, the spin condition restricts the functions $V^\pm$ to be of the following form,

$$\begin{aligned} V^+(x) &= (\partial^1 + i\partial^2) f(|x|) \\ V^-(x) &= (\partial^1 - i\partial^2) f^*(|x|). \end{aligned} \quad (3.7)$$

It is now straightforward to show that the above supercharges satisfy the last

of the conditions in (3.5). The total supercharges can be written concisely as

$$\begin{aligned} \mathcal{R} &= \frac{1}{2\sqrt{m}} \int dx \psi^\dagger(x) \mathcal{D}^+ \phi(x) \\ \mathcal{R}^\dagger &= -\frac{1}{2\sqrt{m}} \int dx \phi^\dagger(x) \mathcal{D}^- \psi(x), \end{aligned} \quad (3.8)$$

where

$$\mathcal{D}^\pm = \partial^\pm - i \int dy V^\pm(y-x) \rho(y). \quad (3.9)$$

The transformation of the component fields under the $S_2\mathcal{G}$ algebra can be read off from (3.1) and (3.8), by taking commutators of the fields with the charges.

The Hamiltonian obtained by anticommuting the supercharges in (3.8) is given by

$$\begin{aligned} H &= \frac{1}{2m} \int dx [|\nabla \phi(x)|^2 + |\nabla \psi(x)|^2] \\ &+ \frac{i}{2m} \int dx dy \left[V^+(y-x) (-\partial^- \phi^\dagger(x) \rho(y) \phi(x) + \psi^\dagger(x) \rho(y) \partial^- \psi(x)) - \text{h.c.} \right] \\ &- \frac{i}{2m} \int dx dy \left[\partial_y^- V^+(y-x) \psi^\dagger(x) \phi^\dagger(y) \phi(x) \psi(y) - \text{h.c.} \right] \\ &+ \frac{1}{2m} \int dx dy dz V^+(y-x) V^-(z-x) [\phi^\dagger(x) \rho(z) \rho(y) \phi(x) + \psi^\dagger(x) \rho(y) \rho(z) \psi(x)]. \end{aligned} \quad (3.10)$$

The above Hamiltonian defines a Super-Galilei ($S_2\mathcal{G}$) invariant quantum field theory. The Fock space of this field theory consists of bosonic and fermionic states created by $\phi^\dagger$ and $\psi^\dagger$, respectively. The two creation operators can be collected into a single superfield,

$$\Phi^\dagger(x, \theta) = \phi^\dagger(x) + \psi^\dagger(x) \theta,$$

where θ is an anti-commuting c-number. This field creates a single "superparticle". Multi-superparticle states are created by acting on the vacuum with

several superfields,

$$|\Psi\rangle = \int \prod_{k=1}^M (d^2 x_k d\theta_k) \Phi^\dagger(x_1 \theta_1) \cdots \Phi^\dagger(x_M \theta_M) |0\rangle \Psi(x_1 \theta_1, \dots, x_M \theta_M). \quad (3.11)$$

The super-wavefunction Ψ is composed of component wave functions, each of which describes a well defined number of bosons and a well defined number of fermions.

By acting on the state $|\Psi\rangle$ with the generators of $S_2\mathcal{G}$, one can derive the first-quantized representations of these generators which act on the super-wavefunction. In particular, the supercharges are given by:

$$\begin{aligned} Q &= -i\sqrt{m} \sum_{k=1}^M \frac{\partial}{\partial \theta_k}, \quad Q^\dagger = i\sqrt{m} \sum_{k=1}^M \theta_k \\ R &= \frac{1}{2\sqrt{m}} \sum_{k=1}^M \left[\partial_k^+ - i \sum_{l \neq k} V^+(x_l - x_k) \right] \frac{\partial}{\partial \theta_k} \\ R^\dagger &= -\frac{1}{2\sqrt{m}} \sum_{k=1}^M \left[\partial_k^- - i \sum_{l \neq k} V^-(x_l - x_k) \right] \theta_k. \end{aligned} \quad (3.12)$$

By anti-commuting R and $R^\dagger$, or equivalently using the quantum field equations of motion,

$$i\partial_t \Phi^\dagger = [\Phi^\dagger, H],$$

in (3.11), we arrive at the first-quantized form of the Hamiltonian and the Schrödinger equation for the super-wavefunction,

$$\begin{aligned} i\partial_t \Psi &= \left\{ -\frac{1}{2m} \sum_k \nabla_k^2 + \frac{i}{m} \sum_{k,l \neq k} V(x_l - x_k) \cdot \nabla_k \right. \\ &\quad + \frac{i}{2m} \sum_{n,k,l \neq k} \left[\partial_n^+ V^-(x_l - x_k) \frac{\partial}{\partial \theta_n} \theta_k + \partial_n^- V^+(x_l - x_k) \theta_n \frac{\partial}{\partial \theta_k} \right] \\ &\quad \left. + \frac{1}{2m} \sum_{k,l \neq k, n \neq k} V^-(x_l - x_k) V^+(x_n - x_k) \right\} \Psi, \end{aligned} \quad (3.13)$$

where $\mathbf{V} = (\text{Re}V^\pm, \pm \text{Im}V^\pm)$.

4. Non-relativistic Super-Chern-Simons Theory

We begin by using (3.9) and some integration by parts to rewrite the Hamiltonian in the following suggestive manner,

$$\begin{aligned} H &= \frac{1}{2m} \int d\mathbf{x} [|\mathcal{D}^+ \phi(\mathbf{x})|^2 + |\mathcal{D}^+ \psi(\mathbf{x})|^2] \\ &\quad + \frac{1}{m} \int d\mathbf{x} d\mathbf{y} \nabla_y \times \mathbf{V}(\mathbf{y} - \mathbf{x}) [\psi^\dagger(\mathbf{x}) \phi^\dagger(\mathbf{y}) \psi(\mathbf{y}) \phi(\mathbf{x}) - \psi^\dagger(\mathbf{x}) \rho(\mathbf{y}) \psi(\mathbf{x})]. \end{aligned} \quad (4.1)$$

Note that for the special choice*

$$\mathbf{V}(\mathbf{x}) = \alpha \nabla \times \ln |\mathbf{x}|,$$

with α an arbitrary real constant, one gets $\nabla \times \mathbf{V}(\mathbf{x}) = -2\pi\alpha\delta(\mathbf{x})$, and only the top part of (4.1) remains. Such a theory is characterized by static classical configurations (solitons) obeying a first order (self-dual) differential equation,

$$\mathcal{D}^+ \phi = \mathcal{D}^+ \psi = 0. \quad (4.2)$$

We will have more to say about this later.

As the notation suggests, we can interpret $\mathcal{D}$ as a covariant derivative, $\mathcal{D} = \nabla - ie\mathbf{A}$, where $\mathbf{A}$ is a "background abelian gauge field" given by

$$\mathbf{A} = \frac{1}{e} \int d\mathbf{y} \mathbf{V}(\mathbf{y} - \mathbf{x}) \rho(\mathbf{y}). \quad (4.3)$$

The following identities

$$\begin{aligned} \int d\mathbf{x} |\mathcal{D}^+ \phi|^2 &= \int d\mathbf{x} [|\mathcal{D}\phi|^2 + \int d\mathbf{y} \nabla_y \times \mathbf{V}(\mathbf{y} - \mathbf{x}) \phi^\dagger(\mathbf{x}) \rho(\mathbf{y}) \phi(\mathbf{x})] \\ \int d\mathbf{x} |\mathcal{D}^+ \psi|^2 &= \int d\mathbf{x} [|\mathcal{D}\psi|^2 + \int d\mathbf{y} \nabla_y \times \mathbf{V}(\mathbf{y} - \mathbf{x}) \psi^\dagger(\mathbf{x}) \rho(\mathbf{y}) \psi(\mathbf{x})], \end{aligned} \quad (4.4)$$

then allow us to express the Hamiltonian as a minimal coupling of the matter

* For vectors in the plane the cross product is defined by $\mathbf{V} \times \mathbf{U} = \epsilon^{ij} V^i U^j$, and the curl is defined by $\nabla \times \mathbf{V} = \epsilon^{ij} \partial^i V^j$. The curl of a scalar is defined by $(\nabla \times S)^i = \epsilon^{ij} \partial^j S$. This quasi three dimensional vector notation makes sense because, in dimensional reduction from 3 dimensions, the "3" component is an $SO(2)$ scalar. Thus $\nabla \times \mathbf{V}$ has only a "3" component and is a scalar. Similarly $\nabla \times S$ for an $SO(2)$ scalar function S is to be thought of as the curl of a 3-vector with only 3rd component non-vanishing and equal to $S(x^1, x^2)$.

to the "gauge field" plus additional matter coupling terms,

$$H = \frac{1}{2m} \int dx [|\mathcal{D}\phi|^2 + |\mathcal{D}\psi|^2] + \frac{1}{2m} \int dx dy [\nabla_y \times \mathbf{V}(y - \mathbf{x})] \\ \times \left[: (|\phi(\mathbf{x})|^2 - |\psi(\mathbf{x})|^2) \rho(y) : + 2\psi^\dagger(\mathbf{x}) \phi^\dagger(y) \psi(y) \phi(\mathbf{x}) \right]. \quad (4.5)$$

For a specific choice of the vector function $\mathbf{V}(\mathbf{x})$ the Hamiltonian (4.5) can be derived by solving the Gauss' law constraint of a particular Super-Galilei invariant *gauge theory*, namely non-relativistic Super-Chern-Simons theory. This is the only known example of a Super-Galilei invariant gauge theory.^[12] Let us review the construction of this theory. Chern-Simons theory coupled to nonrelativistic bosons and fermions is described by the following action:

$$S_{CS} = \int d^3x \left[\frac{\kappa}{2} \partial_t \mathbf{A} \times \mathbf{A} - \kappa A^0 B + \phi^\dagger \left(i\mathcal{D}_t + \frac{\mathcal{D}^2}{2m} \right) \phi + \psi^\dagger \left(i\mathcal{D}_t + \frac{\mathcal{D}^2}{2m} \right) \psi \right. \\ \left. - \frac{e}{2m} B |\psi|^2 + \lambda_1 |\phi|^4 + \lambda_2 |\phi|^2 |\psi|^2 \right], \quad (4.6)$$

where the time component of the covariant derivative is given by $\mathcal{D}_t = \partial_t + ieA^0$. The Pauli interaction term has been explicitly included, as well as two additional matter coupling terms with coupling constants λ_1, λ_2 . The theory possesses the $\mathcal{S}_2\mathcal{G}$ Super-Galilei symmetry for the following values of the coupling constants:

$$\lambda_1 = -\frac{e^2}{2m\kappa}, \quad \lambda_2 = 3\lambda_1. \quad (4.7)$$

Note that the last three terms in (4.6) differ by minus signs from the same terms in Ref.[12], since our convention for the helicity of the fermion is opposite to theirs. The normal ordered Hamiltonian derived from (4.6) is given by

$$H_{CS} = \frac{1}{2m} \int dx [|\mathcal{D}\phi|^2 + |\mathcal{D}\psi|^2] \\ + e : B |\psi|^2 : - 2m\lambda_1 : |\phi|^4 : - 2m\lambda_2 |\phi|^2 |\psi|^2. \quad (4.8)$$

This Hamiltonian is accompanied by the Gauss' Law constraint, derived by

varying (4.6) with respect to A_0 ,

$$B = -\frac{e}{\kappa} (\phi^\dagger \phi + \psi^\dagger \psi). \quad (4.9)$$

The solution of this constraint in Coulomb gauge is given by

$$\mathbf{A}(\mathbf{x}) = -\frac{e}{\kappa} \int dy [\nabla_y \times \ln |\mathbf{y} - \mathbf{x}|] \rho(y). \quad (4.10)$$

Consequently the Super-Chern-Simons Hamiltonian H_{CS} agrees with the Hamiltonian in Eq.(4.5) for $\mathbf{V}(\mathbf{x}) = -(e^2/\kappa) \nabla \times \ln |\mathbf{x}|$.

Interestingly, this is the same vector function for which the Hamiltonian had a self-dual form. In fact the non-relativistic Super-Chern-Simons theory does indeed possess self-dual solitons,^[12] which are generalizations of the non-relativistic Chern-Simons solitons discovered by Jackiw and Pi.^[3] In the purely bosonic theory, self-duality was imposed by hand, by adding a contact interaction term of appropriate strength. In the supersymmetric case self-duality is automatic. The connection between supersymmetry and self-duality in relativistic field theories has been known for quite a while.^[15] The above analysis indicates that there is a connection between supersymmetry and self-duality in some Galilei invariant (non-relativistic) theories as well.

In addition to $\mathcal{S}_2\mathcal{G}$ invariance, the non-relativistic Super-Chern-Simons theory (4.6) also possesses an $SO(2,1)$ conformal invariance.^[12] Such a Galilean conformal symmetry is usually broken by quantum mechanical anomalies,^[4,7] but it turns out that supersymmetry guarantees that it is anomaly free.^[7] Thus supersymmetry, self-duality and conformal invariance co-exist at the particular point in parameter space given by (4.7). If the parameters λ_1, λ_2 are changed, not only would it spoil supersymmetry, but also self-duality and conformal invariance.

5. Matrix Valued Fields

As one of the motivations for studying Super-Galilei invariant field theories, we mentioned that superstrings and their interactions may be a consequence of a $1/N_c$ expansion of a Super-Galilei invariant unitary matrix field theory. In a separate paper^[14] we construct a field theory that gives the free superstring. The point-like objects (bits) of the field theory carry two color indices, and are created by the $N_c \times N_c$ matrix valued fields $\phi^\dagger(\mathbf{x})_\alpha^\beta$ and $\psi^\dagger(\mathbf{x})_\alpha^\beta$. The canonical commutators of the matrix fields are given by

$$[\phi(\mathbf{x})_\alpha^\beta, \phi^\dagger(\mathbf{y})_\gamma^\delta] = \{\psi(\mathbf{x})_\alpha^\beta, \psi^\dagger(\mathbf{y})_\gamma^\delta\} = \delta(\mathbf{x} - \mathbf{y}) \delta_\alpha^\delta \delta_\gamma^\beta. \quad (5.1)$$

In addition to Super-Galilean invariance (either $S_1\mathcal{G}$ or $S_2\mathcal{G}$) the field theory is required to have a global $U(N_c)$ symmetry. The fields are matrices transforming in the adjoint representation of $U(N_c)$, and the terms in the action, or Hamiltonian, involve traces of products of matrices. The generalization of the free $S_2\mathcal{G}$ charges (3.1) to matrix fields is straightforward: simply elevate the fields to matrix fields, understanding all products as matrix products, and take the trace. This is true for any operator which is quadratic in the fields. The free Hamiltonian is then just the trace of (3.2).

For products of more than two fields the matrix ordering (color routing) is important, since different orderings (routings) can lead to different traces. By the cyclicity of the trace, there are $(n-1)!$ ways to order n matrix fields inside a trace. In particular there are thirty six possibilities for the interaction term in the supercharge $\mathcal{R}$. However the $S_2\mathcal{G}$ algebra is only satisfied for some of them, and only for a special choice of the function $V^+(\mathbf{x})$. Consider for example the following supercharge:

$$\mathcal{R}'_1 = \frac{-i}{2N_c\sqrt{m}} \int d\mathbf{x} d\mathbf{y} V^+(\mathbf{y} - \mathbf{x}) : \text{Tr} [\psi^\dagger(\mathbf{x}) \phi(\mathbf{x}) \rho(\mathbf{y})] :, \quad (5.2)$$

where $\rho_\alpha^\beta = [\phi^\dagger \phi + \psi^\dagger \psi]_\alpha^\beta$, and $V^+(\mathbf{x}) = \partial^+ f(|\mathbf{x}|)$ as before. It satisfies all but

the last equation in (3.5),

$$2\{\mathcal{R}'_1, \mathcal{R}^{(0)}\} + \{\mathcal{R}'_1, \mathcal{R}'_1\} = \frac{-1}{2mN_c^2} \int d\mathbf{x} d\mathbf{y} d\mathbf{z} : \text{Tr} [\rho(\mathbf{x}) \psi^\dagger(\mathbf{y}) \phi(\mathbf{y}) \psi^\dagger(\mathbf{z}) \phi(\mathbf{z})] : \\ \times [V^+(\mathbf{x} - \mathbf{z}) V^+(\mathbf{z} - \mathbf{y}) + V^+(\mathbf{x} - \mathbf{y}) V^+(\mathbf{y} - \mathbf{z}) + V^+(\mathbf{z} - \mathbf{x}) V^+(\mathbf{x} - \mathbf{y})], \quad (5.3)$$

which vanishes only when the function $V^+(\mathbf{x})$ satisfies

$$V^+(\mathbf{x} - \mathbf{z}) V^+(\mathbf{z} - \mathbf{y}) + V^+(\mathbf{x} - \mathbf{y}) V^+(\mathbf{y} - \mathbf{z}) + V^+(\mathbf{z} - \mathbf{x}) V^+(\mathbf{x} - \mathbf{y}) = 0.$$

When combined with the constraint (3.7), $V^+(\mathbf{x}) = \partial^+ f(|\mathbf{x}|)$, the solution to the above condition is

$$V^+(\mathbf{x}) = \alpha \partial^+ \ln |\mathbf{x}|, \quad (5.4)$$

where α is an arbitrary complex number, $\alpha = \alpha_1 - i\alpha_2$. In the field theory of section 3 there was no restriction on $V^+(\mathbf{x})$ other than (3.7). The requirement of $S_2\mathcal{G}$ supersymmetry in the matrix field theory restricts this function much more. The Hamiltonian is again found by anti-commuting the total supercharge $\mathcal{R}$ with its hermitian conjugate. We refrain from presenting its explicit form due to its length.

The Fock space of this theory consists of states transforming in various representations of $U(N_c)$. As we are primarily interested in applying matrix field theories to a reformulation of superstring theory, let us restrict our discussion to the singlet states given by products of matrix traces of products of creation operators acting on the vacuum. Single trace states are defined as

$$|\Psi\rangle = \int \prod_{k=1}^M (d^2 x_k d\theta_k) \text{Tr} [\Phi^\dagger(\mathbf{x}_1 \theta_1) \cdots \Phi^\dagger(\mathbf{x}_M \theta_M)] |0\rangle \Psi(\mathbf{x}_1 \theta_1, \dots, \mathbf{x}_M \theta_M), \quad (5.5)$$

where Ψ is the wavefunction describing a closed chain of M bits in a first-quantized formalism. Acting on this state with the supercharge $\mathcal{R}$ one finds that the trace structure is altered, and thus it cannot be an energy eigenstate. Singlet operators like $\mathcal{Q}$ and $\mathcal{R}$ relate singlet states to other singlet states, so

a single chain can in general break into several chains. One-body operators always preserve the number of traces, so a state of the form (5.5) is changed to a state of the same form by $\mathcal{Q}$ and $\mathcal{R}^{(0)}$. Two-body operators such as $\mathcal{R}'$ can change the number of traces by one. However, if the matrix ordering in a two-body operator is such that the creation operators are consecutive there will be terms in which the number of traces doesn't change, and they will get multiplied by a factor of N_c . To see how this happens consider for simplicity a single component matrix creation operator $a^\dagger(x)_\alpha^\beta$, and let Ω_2 be a single trace 2-body operator with consecutive creation operators,

$$\Omega_2 = \frac{1}{N_c} \int dx dy V(y-x) \text{Tr}[a^\dagger(x)a^\dagger(y)a(y)a(x)]. \quad (5.6)$$

Applying this operator to the singlet Fock state $|M\rangle = \text{Tr}[a^\dagger(x_1) \cdots a^\dagger(x_M)]|0\rangle$, gives after one contraction

$$\begin{aligned} \Omega_2 |\psi\rangle &= \frac{1}{N_c} \int dy \sum_k V(y-x_k) \cdot \\ &\cdot \text{Tr}[a^\dagger(x_k)a^\dagger(y)a(y)a^\dagger(x_{k+1}) \cdots a^\dagger(x_M)a^\dagger(x_1) \cdots a^\dagger(x_{k-1})]|0\rangle. \end{aligned}$$

To continue the evaluation we note that it matters crucially which creation operator the last remaining $a(y)$ contracts against. The contraction with $a^\dagger(x_{k+1})$ produces a factor of $\sum_\alpha \delta_\alpha^\alpha = N_c$. All other contractions fail to provide this factor. Thus in the limit $N_c \rightarrow \infty$

$$\Omega_2 \text{Tr}[a^\dagger(x_1) \cdots a^\dagger(x_M)]|0\rangle \rightarrow \sum_{k=1}^M V(x_{k+1}-x_k) \text{Tr}[a^\dagger(x_1) \cdots a^\dagger(x_M)]|0\rangle. \quad (5.7)$$

Note that only a nearest neighbor interaction survives once we take the limit $N_c \rightarrow \infty$. This is precisely what we require of a non-interacting polymer chain of bits. The other contractions change the trace structure of the state, giving $1/N_c$ times a state with two traces. Thus $1/N_c$ corrections allow a closed polymer chain to rearrange its bonds and transform to two closed polymer chains.

The matrix ordering of the supercharge $\mathcal{R}'_1$ in (5.2) is such that there are no consecutive annihilation operators, and the same will be true of the Hamiltonian. A nearest neighbor interaction pattern will thus not be established in the limit $N_c \rightarrow \infty$. We therefore seek other possibilities for $\mathcal{R}'$ to remedy this situation. Consider the following ordering:

$$\mathcal{R}'_2 = \frac{-i}{2N_c\sqrt{m}} \int dx dy V^+(y-x) : \text{Tr}[\psi^\dagger(x)\rho(y)\phi(x)] :. \quad (5.8)$$

The above contains consecutive annihilation operators, but fails to anti-commute with $\mathcal{Q}^\dagger$ for any non-trivial function $V^+(x)$,

$$\{\mathcal{R}'_2, \mathcal{Q}^\dagger\} = \frac{1}{2N_c} \int dx dy V^+(y-x) : \text{Tr}[(\phi(x)\phi^\dagger(x) - \psi(x)\psi^\dagger(x))\rho(y)] : \neq 0. \quad (5.9)$$

What is needed is a more complicated ordering than $\mathcal{R}'_1$ or $\mathcal{R}'_2$, which contains consecutive annihilation operators and satisfies the entire $\mathcal{S}_2\mathcal{G}$ algebra. The following combination^{*}

$$\mathcal{R}' = \frac{-i}{2N_c\sqrt{m}} \int dx dy V^+(y-x) : \text{Tr}\left[\left([\phi^\dagger(y), \phi(y)] + \{\psi^\dagger(y), \psi(y)\}\right)[\psi^\dagger(x), \phi(x)]\right] :, \quad (5.10)$$

contains consecutive annihilation operators and satisfies all but the last equation in (3.5),

$$\begin{aligned} 2\{\mathcal{R}', \mathcal{R}^{(0)}\} + \{\mathcal{R}', \mathcal{R}'\} &= \frac{-1}{4mN_c^2} \int dx dy dz \\ &\times : \text{Tr}\left[\left([\phi^\dagger(x), \phi(x)] + \{\psi^\dagger(x), \psi(x)\}\right)[\psi^\dagger(y), \phi(y)][\psi^\dagger(z), \phi(z)]\right] : \\ &\times \left[V^+(x-z)V^+(z-y) + V^+(x-y)V^+(y-z) + V^+(z-x)V^+(x-y)\right], \end{aligned} \quad (5.11)$$

which again vanishes only for $V^+(x) = \alpha\partial^+ \ln|x|$.

^{*} The commutators and anti-commutator above refer only to matrix ordering (color routing), whereas normal ordering refers to the operator elements of the matrices.

The first-quantized representations of the supercharges are obtained by acting on the state $|\Psi\rangle$ and taking the limit $N_c \rightarrow \infty$,

$$\begin{aligned} Q &= -i\sqrt{m} \sum_{k=1}^M \frac{\partial}{\partial \theta_k} \quad , \quad Q^\dagger = i\sqrt{m} \sum_{k=1}^M \theta_k \\ R &= \frac{1}{2\sqrt{m}} \sum_{k=1}^M \left[\partial_k^+ + i(V^+(\mathbf{x}_{k-1} - \mathbf{x}_k) - V^+(\mathbf{x}_k - \mathbf{x}_{k+1})) \right] \frac{\partial}{\partial \theta_k} \\ R^\dagger &= -\frac{1}{2\sqrt{m}} \sum_{k=1}^M \left[\partial_k^- + i(V^-(\mathbf{x}_{k-1} - \mathbf{x}_k) - V^-(\mathbf{x}_k - \mathbf{x}_{k+1})) \right] \theta_k \end{aligned} \quad (5.12)$$

where $V^-(\mathbf{x}) = \alpha^* \partial^- \ln |\mathbf{x}|$. The difference between these and the supercharges in the non-matrix case (3.12) is that the two body terms in R and $R^\dagger$ include only nearest neighbor interactions. By taking the anti-commutator of R and $R^\dagger$ we arrive at the first-quantized Hamiltonian:

$$\begin{aligned} H &= -\frac{1}{2m} \sum_{k=1}^M \nabla_k^2 + \frac{i}{m} \sum_{k=1}^M \mathbf{V}(\mathbf{x}_k - \mathbf{x}_{k+1}) \cdot (\nabla_k - \nabla_{k+1}) \\ &+ \frac{i}{2m} \sum_{k=1}^M \left\{ 2\partial_k^+ V^-(\mathbf{x}_k - \mathbf{x}_{k+1}) \right. \\ &\quad \left. - [\partial_k^+ V^-(\mathbf{x}_k - \mathbf{x}_{k+1}) - \partial_k^- V^+(\mathbf{x}_k - \mathbf{x}_{k+1})] (\theta_{k+1} - \theta_k) \left(\frac{\partial}{\partial \theta_{k+1}} - \frac{\partial}{\partial \theta_k} \right) \right\} \\ &+ \frac{1}{2m} \sum_{k=1}^M \left\{ 2|V(\mathbf{x}_k - \mathbf{x}_{k+1})|^2 \right. \\ &\quad \left. - V^+(\mathbf{x}_{k-1} - \mathbf{x}_k) V^-(\mathbf{x}_k - \mathbf{x}_{k+1}) - V^+(\mathbf{x}_k - \mathbf{x}_{k+1}) V^-(\mathbf{x}_{k+1} - \mathbf{x}_k) \right\} \end{aligned} \quad (5.13)$$

Recall that $V^\pm = V^1 \pm iV^2$ and $\alpha = \alpha_1 - i\alpha_2$, therefore

$$\mathbf{V}(\mathbf{x}) = \alpha_1 \nabla \ln |\mathbf{x}| + \alpha_2 \nabla \times \ln |\mathbf{x}|. \quad (5.14)$$

The above Hamiltonian with the vector function (5.14) describes supersymmetric dynamics of bits which are ordered around a loop. This does not yet imply that this loop is in any sense a physical bound chain (see Figure 1). That

Figure 1. a) Particles ordered around a loop. b) A bound chain of particles.

question must be answered by studying the bound states of the system, if they exist.

As a first step let us concentrate on a small part of the loop consisting of only two particles[†] (see Figure 2). This would correspond to a single link in the chain if the two particles were bound.

Figure 2. A two particle link.

The corresponding piece of the super-wavefunction transforms in the adjoint

[†] The dynamics of a piece of the polymer loop with any number K of bits can be precisely obtained from the second quantized theory by applying the various singlet dynamical variables to a non-singlet Fock state of the form

$$|\Psi_\alpha^\beta\rangle = \int \prod_{k=1}^K (d^2 x_k d\theta_k) [\Phi^\dagger(\mathbf{x}_1 \theta_1) \cdots \Phi^\dagger(\mathbf{x}_K \theta_K)]_\beta^\alpha |0\rangle \Psi_\alpha^\beta(\mathbf{x}_1 \theta_1, \dots, \mathbf{x}_K \theta_K)$$

and taking the large N_c limit. If such a sector showed a bound state we could call it a piece of string.

representation of $U(N_c)$ and has four components,

$$\begin{aligned}\Psi_\alpha^\beta(\mathbf{x}_1, \mathbf{x}_2, \theta_2) = & u_1(\mathbf{x}_1, \mathbf{x}_2) + (\theta_1 + \theta_2)u_2(\mathbf{x}_1, \mathbf{x}_2) \\ & + (\theta_1 - \theta_2)u_3(\mathbf{x}_1, \mathbf{x}_2) + \theta_1\theta_2u_4(\mathbf{x}_1, \mathbf{x}_2),\end{aligned}$$

corresponding respectively to the boson-boson, two boson-fermion, and fermion-fermion wave functions. The matrix indices have been dropped from the component wave functions since the dynamics are $U(N_c)$ invariant and will not affect them. The part of the Hamiltonian relevant for a single link in the chain is given by

$$H_{\text{link}} = -\frac{\nabla^2}{m} + \frac{1}{mr^2} \left[2i\alpha_1 \mathbf{r} \cdot \nabla - 2i\alpha_2 \mathbf{r} \times \nabla + |\alpha|^2 \right] + \frac{2\pi i}{m} (\alpha_1 \pm i\alpha_2) \delta^{(2)}(\mathbf{r}), \quad (5.15)$$

where $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ and $\nabla = (\nabla_1 - \nabla_2)/2$. The upper sign in the coefficient of the δ -function holds for u_1 and u_2 , and the lower sign holds for u_3 and u_4 . Note that the two terms in the Hamiltonian proportional to α_1 are not separately hermitian, but their sum is, due to the identity

$$\nabla \cdot \frac{\mathbf{r}}{r^2} = 2\pi \delta^{(2)}(\mathbf{r}). \quad (5.16)$$

The above Hamiltonian contains no dimensionful parameters (other than the mass), and therefore implies classically scale-invariant dynamics. It appears therefore that a bound state of finite energy is precluded. However we know from the simpler problem of the δ -function potential,^[16] that regularization of the contact interaction and an interpretation of the coupling constant as a bare parameter which depends on the regulator can yield a bound state of finite energy depending on the regulator. To analyze the problem at hand further requires a similar regularization. We would like to choose a regularization that makes the Schrödinger equation simplest to analyze. One such regularization is to replace the δ -function at the origin by a δ -function at radius R ,^[17]

$$\delta^{(2)}(\mathbf{r}) \rightarrow \frac{1}{2\pi R} \delta(r - R),$$

so that in the limit $R \rightarrow 0$ they are equal. To ensure hermiticity of the regularized Hamiltonian we also need to regularize the $\mathbf{r} \cdot \nabla/r^2$ term. To do so we

make the following replacement:

$$\frac{1}{r^2} \rightarrow \frac{\theta(r - R)}{r^2},$$

where $\theta(r - R)$ is the step function. We choose to make this replacement for all the terms, since it will simplify the analysis considerably. The regularized Hamiltonian in radial coordinates is then given by

$$\begin{aligned}H_{\text{link}} = & -\frac{1}{m} \left\{ \frac{\partial^2}{\partial r^2} + [1 - 2i\alpha_1 \theta(r - R)] \frac{1}{r} \frac{\partial}{\partial r} \right. \\ & \left. + \frac{1}{r^2} \left[\left(\frac{\partial}{\partial \varphi} + i\theta(r - R)\alpha_2 \right)^2 - \theta(r - R)\alpha_1^2 \right] - \frac{i(\alpha_1 \pm i\alpha_2)}{R} \delta(r - R) \right\}. \quad (5.17)\end{aligned}$$

The regularization we propose corresponds to regularizing $\mathbf{V}(\mathbf{r})$ by replacing it with $\theta(r - R)\mathbf{V}(\mathbf{r})$. Since this can be done already at the level of the supercharges (5.12), the above regularized Hamiltonian is given by an anti-commutator of a regularized supercharge and its conjugate, and is therefore a positive definite operator. Thus a negative energy bound state should not exist. A positive energy bound state is not possible since the potential vanishes at infinity. To see this explicitly we solve the Schrödinger equation:

$$\begin{aligned}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + k^2 \right] u_n(r, \varphi) &= 0 \quad r < R \\ \left[\frac{\partial^2}{\partial r^2} + (1 - 2i\alpha_1) \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left[\left(\frac{\partial}{\partial \varphi} + i\alpha_2 \right)^2 - \alpha_1^2 \right] + k^2 \right] u_n(r, \varphi) &= 0 \quad r > R.\end{aligned} \quad (5.18)$$

The parameter α_1 can be eliminated from the second equation by redefining the outer wave function, resulting in the following equations for the inner and outer wave functions

$$\begin{aligned}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + k^2 \right] u_n(r, \varphi) &= 0 \quad r < R \\ \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \varphi} + i\alpha_2 \right)^2 + k^2 \right] r^{-i\alpha_1} u_n(r, \varphi) &= 0 \quad r > R.\end{aligned} \quad (5.19)$$

Defining the radial wave functions by

$$u_n(r, \varphi) = e^{il\varphi} \chi_n(r),$$

with l an arbitrary integer labeling the angular momentum, the jump condition

on the logarithmic derivatives imposed by the δ -function is given by:

$$\frac{\chi'_n(R+\epsilon)}{\chi_n(R)} - \frac{\chi'_n(R-\epsilon)}{\chi_n(R)} = \frac{i(\alpha_1 \pm i\alpha_2)}{R}. \quad (5.20)$$

For negative energy solutions we define $B \equiv -k^2 > 0$. Regularity at the origin and normalizability implies the following form for the negative energy solution:

$$\chi_n(r) = \begin{cases} AI_{|l|}(\sqrt{BR}) & \text{for } r < R \\ Cr^{i\alpha_1} K_{|l+\alpha_2|}(\sqrt{BR}) & \text{for } r > R, \end{cases} \quad (5.21)$$

where the constants A and C are determined by the continuity condition at $r = R$ and by normalization. The jump condition (5.20) then gives the following:

$$\frac{K'_{|l+\alpha_2|}(\sqrt{BR})}{K_{|l+\alpha_2|}(\sqrt{BR})} - \frac{I'_{|l|}(\sqrt{BR})}{I_{|l|}(\sqrt{BR})} = \mp \frac{\alpha_2}{\sqrt{BR}}. \quad (5.22)$$

Using the following recurrence relations for the modified Bessel functions,

$$\begin{aligned} K'_\nu(z) &= -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z) \\ I'_\nu(z) &= I_{\nu+1}(z) + \frac{\nu}{z} I_\nu(z), \end{aligned} \quad (5.23)$$

gives the condition

$$\sqrt{BR} \left[\frac{K_{|l+\alpha_2|-1}(\sqrt{BR})}{K_{|l+\alpha_2|}(\sqrt{BR})} + \frac{I_{|l|+1}(\sqrt{BR})}{I_{|l|}(\sqrt{BR})} \right] = \pm \alpha_2 - |l| - |l + \alpha_2|. \quad (5.24)$$

Since the functions $K_\nu(z)$ and $I_\nu(z)$ with $\nu > -1$ are positive for $z > 0$ the left hand side of the equation is positive for $\sqrt{BR} > 0$. When $\sqrt{BR} = 0$ the left hand side vanishes. The right hand side is clearly negative or zero, since

$$\pm \alpha_2 - |l| \leq |l + \alpha_2| \quad \text{for all } l.$$

Consequently there is no solution except when the above inequality is saturated, in which case the bound state energy vanishes. Since the two particles comprising a link in the chain cannot bind, a closed chain will not form. This is therefore, as we expected, an unsatisfactory model for describing discretized superstring.

6. Discussion

We have presented field theoretic representations of the full $(\mathcal{S}_2\mathcal{G})$ Super-Galilei algebra of space-time transformations, both with single component and matrix valued fields. In the first case we showed that non-relativistic Super-Chern-Simons theory emerges as a special case of our model. The matrix field theory is motivated by the discretized light-cone superstring, but fails to be a satisfactory model since closed polymer chains, which become strings in the continuum limit, do not form. The two-body Hamiltonian is positive definite, which precludes any negative energy bound states. For zero energy or positive energy bound states to exist, we must have a potential energy which is positive and non-vanishing at infinite separation. In a separate paper we present an $\mathcal{S}_1\mathcal{G}$ invariant matrix field theory which achieves a satisfactory free superstring limit, because it employs a harmonic potential between string bits. Although the large N_c limit of that model had the full $\mathcal{S}_2\mathcal{G}$ invariance, the symmetry was broken to $\mathcal{S}_1\mathcal{G}$ at finite N_c , and $\mathcal{S}_1\mathcal{G}$ invariance is not sufficient to force the correct superstring interactions. Thus the ultimate goal should be to build a satisfactory string bit theory with the full $\mathcal{S}_2\mathcal{G}$ Super-Galilei symmetry at all values of N_c .

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