1. Introduction

This lecture introduces the nonaccelerator-specialist to the motion of charged particles in a Storage Ring. The topics of discussion are restricted to the linear and nonlinear dynamics of a single particle in the transverse plane, i.e., the plane perpendicular to the direction of motion. The major omissions for a complete review of accelerator theory, for which a considerable literature exists, are the energy and phase oscillations (1). Other important accelerator physics aspects not treated here are the collective instabilities (2), the role of synchrotron radiation in electron storage rings (3), scattering processes (4), and beam-beam effects in colliding beam facilities (5).

Much of the discussion that follows applies equally well to relativistic electron, proton, or ion synchrotrons. In this narrative, we refer to the particle as electron.

After a broad overview, the magnetic forces acting on the electrons and the associated differential equations of motion are discussed. Solutions of the equations are given without derivation; the method of solution is outlined, and references for deeper studies are given. In this paper, the word electron is used to signify electron or positron. The dynamics of a single particle are not affected by the sign of its charge when the magnetic field direction is changed accordingly.

2. Overview of Electron Dynamics in a Storage Ring

2.1. The Storage Ring

A storage ring accumulates and stores electrons that have been pre-accelerated and transported from an Injection System. The electrons are injected and stored in packets called bunches, which are held together in the direction of motion by the bunching effect of the radio-frequency system. The latter also provides the energy lost by radiation and, if acceleration is needed, the energy gain required by the particles to keep in step with the magnetic field.
The electrons circulate inside a doughnut-shaped chamber, in which a high vacuum is maintained, delimited by metallic walls. The chamber is surrounded by magnets alternating with empty, or drift, spaces. The magnets curve the electron trajectories (dipole field) and keep them close together in the plane perpendicular to the direction of motion (quadrupole field).

2.2. Collective and Individual Motion, Frame of Reference.

Figure 1 gives simplified top and cross section views of electron bunches, frozen in time, circulating in a vacuum chamber surrounded by magnets. The picture is much out of proportion. The circumference of a ring may vary from a few hundred meters (synchrotron light sources) to several tens of kilometers (see the LEP storage ring at CERN). The bunch length is on the order of centimeters or smaller. Tens to hundreds of bunches may circulate in a storage ring.

The motion of the electrons is described in a reference system with an azimuthal axis tangent to the orbit, and the transverse horizontal (x) and vertical (y) coordinates, lying in the plane perpendicular to the orbit (indicated in Fig. 2). The word orbit refers to an electron path that closes on itself after one revolution around the accelerator; thus, it is always a closed orbit. When this orbit describes the motion in an ideal lattice without magnetic imperfections or misalignment, this closed orbit is also the ideal orbit. The word trajectory is used to describe an oscillation around the closed orbit, called a betatron oscillation, which does not close after one revolution. The azimuthal coordinate s is the independent variable, and is the distance along the orbit from a reference point s=0. Lattice designers and orbit scientists spend much of their time studying the functions x(s), x'(s) = dx/ds, y(s), and y'(s).

The trajectory of an individual electron in a storage ring is qualitatively shown in Fig. 2. The betatron oscillations take place in both the horizontal and vertical planes. The orbit has horizontal and vertical components, which is defined by the azimuthal and horizontal/vertical axis. In an accelerator with neither vertical bends nor magnetic errors or misalignments, the orbit lies in the horizontal plane (x-s in Fig. 2), and the vertical closed orbit is zero everywhere. This ideal situation does not occur in practice, and there is always an orbit component in the vertical plane.
Figure 1. Simplified top and cross section views of electron bunches circulating in a storage ring.

Figure 2. Descriptive view of the closed orbit and a betatron oscillation. The open section is meant to emphasize that the betatron oscillation is not closed.
A large number of electrons (10^{10} or more per bunch) oscillate around a closed orbit with all possible phases and amplitudes. The amplitudes are within a given range defined by the transverse size of the vacuum chamber or by the maximum stable amplitude.

Not all the particles in a bunch have the same energy, but fall within a distribution of energies. To each energy, there corresponds a closed orbit, around which off-energy particles execute betatron oscillations.

2.3. *Lattice Definition*

The lattice of a storage ring is defined to be the sequence of magnetic lenses designed to insure that electrons circulate for a period of several hours, corresponding to billions of revolutions, while maintaining the appropriately small dimensions of the beam.

The magnetic properties of the lattice, together with the electron energy, determine the transverse size and divergence of the beam.

3. Equations of Motion and Solution

3.1. *Basic Magnetic Elements in a Lattice*

The basic lattice of a storage ring consists of a sequence of dipole (bending) and quadrupole (focusing or defocusing) magnets joined by field-free regions, or drift spaces. The sequence closes on itself to allow the electrons repeated revolutions around a determined reference orbit and within a confined region around this orbit.

The *bending magnets* are characterized by a magnetic field that is perpendicular to the direction of motion and is uniform in the region occupied by the beam. A bending magnet causes a charged particle to follow a circular trajectory along its length. Straight trajectories are joined by sections of circles. The bending magnets are positioned in such a way that there exists a trajectory that is a closed curve which satisfies given geometrical constraints. An electron on this trajectory repeats its motion every revolution. This trajectory is the *closed orbit*. In the absence of magnetic, alignment, and other imperfections, it is also the ideal (or design, or reference) orbit.

If the bending magnets were the only elements of the lattice, particles with spatial coordinates different from those of the ideal orbit would move progressively away from this orbit. Since a beam of electrons contains a distribution of particles that have different positions and angles, as well as energies, eventually the whole beam would spread out and be lost. For this reason, focusing elements are required to
keep together this collection of particles with different coordinates. These elements are quadrupole magnets, and they are characterized by a magnetic field whose components are linear functions of the \( x \) and \( y \) coordinates. (Dipole and quadrupole fields may coexist in a single combined function magnet. These magnets have found applications in the most recent third generation light sources, like the ALS, ELETTRA, and the SRRC light source. For the sake of clarity, we treat the dipole and quadrupole magnets separately.)

The components of the magnetic field of the basic lattice components, dipoles, and quadrupoles, are:

\[
\begin{align*}
B_y &= B_0, \\
B_x &= 0, \
\text{for dipoles, and} \\
B_y &= Gx, \\
B_x &= Gy,
\end{align*}
\]

(1)

where \( B_0 \) and \( G \) are constant. In a quadrupole, the field is zero at \( x = y = 0 \). This point defines the magnetic axis in the azimuthal direction. A quadrupole for which a particle away from the magnetic axis is deflected back onto (away from) it is called focusing (defocusing). As a consequence of Maxwell’s equations, a quadrupole field that is horizontally focusing is vertically defocusing, and vice versa. A sequence of focusing and defocusing quadrupoles, appropriately designed, can focus the strong focusing (6), on which all modern synchrotrons are based.

3.2. The Synchronous Orbit

For a given bending field, there is one value of the electron energy for which the particle follows the ideal orbit. We call this the synchronous energy and the particle the synchronous particle. The energy is given by the expression equating the centrifugal force to the Lorentz force:

\[
E_0 = ecB_0\rho_0,
\]

(2)

where \( E_0 \) is the electron energy, \( e \) the electron charge, \( c \) the speed of light in a vacuum, \( B_0 \) the bending field, and \( \rho_0 \) the radius of curvature in the field of the dipole magnets. (Recall that we are using the ultrarelativistic approximation \( E = cP \), where \( P \) is the particle momentum.)

In a more commonly used, form, Eq. (2) can be written as
(2a) \[ E_0[GeV] = 0.3B[T]\rho_0[m], \]

where GeV, T, and m denote giga-electron-volt, Tesla, and meter, respectively.

3.3. Equations of the Synchronous Orbit and their Solutions

An electron that, at a given initial azimuthal position \( s_0 \), has the same energy as the synchronous particle, but is displaced (in position or angle in the transverse coordinates \( x, x', y, y' \)) with respect to the ideal orbit, executes betatron oscillations around this orbit. These oscillations occur in the horizontal and vertical plane, and are defined by the following differential equations of motion:

\[
\begin{align*}
x'' + K_x(s)x &= 0, \\
y'' + K_y(s)y &= 0.
\end{align*}
\]

(3)

The focusing strengths \( K_{x,y}(s) \) are proportional to the quadrupole fields (focusing or defocusing) and also include relatively small effects of the dipoles not discussed here (7). For the derivation of Eq. 3, see, for instance, Ref. 6.

Because the magnets have constant fields along the direction of motion, these functions are dichotic (\( K_{x,y}(s) = \text{constant} = 0 \) in magnet-free regions, or \( \neq 0 \) in a magnetic field). Equation (3) was solved in the original, classical paper, in which the principles of strong focusing were described (6).

\[
\begin{align*}
x(s) &= \sqrt{\varepsilon_x \beta_x(s)} \cos[\phi_x(s) + \phi_{0x}], \\
y(s) &= \sqrt{\varepsilon_y \beta_y(s)} \cos[\phi_y(s) + \phi_{0y}].
\end{align*}
\]

(4)

Here, \( x \) and \( y \) are the transverse displacements from the closed orbit defined earlier. The meaning of the constants \( \varepsilon_x \) and \( \varepsilon_y \) is discussed in Section 3.5, together with the functions \( \beta_x \) and \( \beta_y \). The betatron phases \( \phi_x \) and \( \phi_y \) are functions of the distance \( s \) along the closed orbit, and \( \phi_{OX} \) and \( \phi_{OY} \) are the initial phases. The \( \phi_{x,y} \) are given by

\[
\phi_{x,y} = \int_{0}^{s} \frac{ds}{\beta_{x,y}(s)}.
\]

(5)

The motion is a pseudoharmonic oscillator, with instantaneous amplitudes proportional to the square root of the \( \beta \)-functions and instantaneous wavelength \( \lambda_{x,y}(s) = 2\pi \beta_{x,y}(s) \).
3.4. The $\beta$-Function

The reader who has been exposed to accelerator terminology will have heard the term $\beta$-function used often. It was seen in Eq. 4 that these functions (horizontal and vertical) are related to the maximum amplitude of the oscillations at a given location $s$:

$$x, y_{\text{max}}(s) = \sqrt{\varepsilon_{x,y} \beta_{x,y}(s)}.$$  \hfill (6)

Similarly, the maximum angle of the oscillation at a location $s$ is given by

$$x', y'_{\text{max}}(s) = \sqrt{\varepsilon_{x,y}/\beta_{x,y}(s)}.$$  \hfill (7)

The $\beta$-functions are periodic in $s$ and follow the periodicity of the lattice. Together with the constants $\varepsilon_{x,y}$, they determine the maximum amplitude of the betatron oscillations. The units in use are meter-radian for $\varepsilon_{x,y}$ and meter/radian for the $\beta$-functions.

3.5. The Emittance

Figure 3 shows the locus of all possible positions and angles ($x, x'$ or $y, y'$) of a particle that is going around the accelerator, as it would be monitored by an observer placed at an azimuth $s$. All the points fall in an ellipse whose area, it can be shown, is equal to the constant $\varepsilon_{x,y}$ multiplied by $\pi$. The shape and orientation of the ellipse changes as a function of $s$. In an optical system without acceleration, emission of radiation, collective effects, or horizontal-vertical coupling, $\varepsilon_{x,y}$ remains constant as the particle revolves around the accelerator (a consequence of Liouville's theorem). In reality, this constant is perturbed by radiation emission and acceleration.

When the ellipse represents the motion of that particle in a bunch with the highest value of $\varepsilon_{x,y}$ is the emittance of the beam. Its importance is immediately recognized: multiplying by the value of the $\beta$-function at a given location and taking the square root (Eq. 6) gives the value of the maximum amplitude of the oscillations in the beam (there are other factors that contribute to the beam transverse dimensions).
In an electron storage ring, the distribution of betatron oscillation amplitudes is Gaussian; it is normal practice to define the emittance of the beam as the values of the constants $\varepsilon_{x,y}$ that are related to the standard deviation of the distribution of amplitudes and angular divergences. The relationships are given by the expressions [derived from Eqs. (6) and (7)],

\[ \sigma_x = \sqrt{\varepsilon_x \beta_x}, \quad \sigma_y = \sqrt{\varepsilon_y \beta_y}, \]

\[ \sigma'_x = \sqrt{\varepsilon_x / \beta_x}, \quad \sigma'_y = \sqrt{\varepsilon_y / \beta_y}, \]

where $\sigma_{x,y}$ and $\sigma'_{x,y}$ are the distribution standard deviations of position and angle.

3.6. Tunes and Resonances

3.6.1. Definition of tunes

The numbers of horizontal and vertical betatron oscillations per ring revolution are called the tunes, and are denoted by the symbols $v_x$ and $v_y$. From Eq. (5), the tunes are given by
\[ v_{x,y} = \frac{1}{2\pi} \int_{0}^{C} \frac{ds}{\beta_{x,y}(s)} \]

(9)

The integral is extended to the entire lattice length C. The tunes play an important role in the stability of the motion. Sometimes the symbol \( Q_{x,y} \) is used for the tunes.

3.6.2. Survey and magnetic imperfections, linear and nonlinear resonances

The analysis of the motion shows (8) that there are certain values of the tunes that potentially threaten the stability of the motion. Considering for a moment the transverse plane only (i.e. neglecting energy oscillations), they are those that satisfy the relationship

\[ mv_x \pm n v_y = p, \]

(10)

where \( m \), \( n \), and \( p \) are integer values. Equation (10) expresses the phenomenon that, if there are magnetic perturbations in the accelerator (unavoidable), the perturbing effect (colloquially called the kick in accelerator jargon) can add up at each revolution, causing the amplitude of the oscillations to grow. For this to happen, the numerical relationship of Eq. (10) must be satisfied; otherwise, the perturbations tend to cancel each other over a sufficiently large number of turns. Since \( m \) and \( n \) can take any integer values, it appears very difficult to find a pair of tunes values that escape Eq. (10). Fortunately, the perturbing effect becomes weaker and weaker as the order of the resonance, defined as the sum of \( |m| \) and \( |n| \), becomes larger. In general, in electron accelerators, there is no need to worry about resonances for which \( |m| + |n| > 5 \). This is because radiation damping tends to neutralize the resonant amplitude growth when it and the damping rates are of the same order. In proton and ion machines, where radiation damping is very small, resonances of order 5 and higher may play a role.

It is not necessary for Eq. (10) to be perfectly satisfied for a magnetic perturbation to be felt. There is a region around a resonance line, defined by Eq. (10), where the trajectory can be perturbed. Fortunately, this band (called stop-band width) becomes narrower the higher the order of the resonance.

3.6.2.1. Linear resonances, orbit and focusing perturbations

The resonances for which \( |n| + |m| \leq 2 \) are driven by linear imperfections in the lattice. The resonances \( v_{x,y} = \text{integer} \) are particularly disruptive. They are driven by magnetic imperfections of the dipole type, by survey imperfections in the transverse locations of the quadrupoles, and by rotational errors.
in the placement of the dipoles. The orbit distortions act like \( \frac{1}{(v_{x,y}^2 - p^2)} \) (where \( p \) is any integer) and, although the tunes are normally set at a respectable distance from an integer value, orbit distortions can be, and are, driven at any tune values. For this reason, dipole correctors are used to correct the orbit distortions and are a necessary part of any accelerator. To reduce the amplitude of the orbit distortions, tight tolerances are set for the random relative variation of the bending field (on the order of a few times \( 10^{-4} \) rms), for the transverse positioning of the quadrupoles (typically 0.10–0.15 mm rms) and for the rotation angle of the bending magnets (0.5–1.0 mrad rms).

Magnetic imperfections of the quadrupole type drive second order resonances. They perturb the \( \beta \)-functions, couple the horizontal and vertical motion (see Section 3.6.2.3), and, if strong enough, may lead to an unstable lattice. The tolerances on the variation of the field gradient \( G \) in Eq. (1)) from quadrupole to quadrupole are specified to limit this effect, and are typically on the order of \( 10^{-3} \).

3.6.2.2. Nonlinear resonances

Those resonances for which \( |n_l| + |m_l| > 2 \) are driven by nonlinear fields, for example the two third integer resonances:

\[
\begin{align*}
3v_x &= p, \\
2v_y \pm v_x &= p. 
\end{align*}
\] (11)

They are driven by sextupoles fields that have the form

\[
\begin{align*}
B_y &= S(x^2 - y^2), \\
B_x &= S2xy, 
\end{align*}
\] (12)

where \( S \) is the sextupole strength, normally expressed in T/m\(^2\). Sextupole magnets are part of any storage ring lattice because they are needed to correct the chromatic aberrations of the quadrupoles.

Higher order resonances are driven by magnetic fields with higher order nonlinearities. For instance, octupole fields (those that have cubic dependence on the displacement, \( B_y (x,0) = \text{constant } x^3 \)) drive fourth order resonances, for which \( |n_l| + |m_l| = 4 \). Decapole fields (quartic dependence on displacement) drive fifth order resonances, and so on. Some resonances require a slight rotation of the magnetic axis in order to be driven, and this often sets the survey tolerances.
At the construction stage the magnet builder requires a set of tolerances from the accelerator physicists for the purity of the magnetic field, which is defined as the relative variation of the field with respect to the ideal value. This is typically on the order of a few times $10^{-4}$.

Two more points need to be mentioned concerning Eq. (10). It can be shown that only the + sign ($\text{sum resonances}$) on the left hand side of the equation leads to indefinite growth in both the horizontal and vertical directions. The – sign resonances ($\text{difference resonances}$) lead to a transfer of oscillation amplitudes from the horizontal into the vertical, and vice versa, but the motion is bounded. The behavior is much like that of a coupled pendulum, with the maximum amplitudes beating between the two directions of transverse motion. Sum resonances are in general much more dangerous.

For a resonance condition to be established, the perturbation (dipoles, quadrupoles, nonlinear fields) must have a $p$-th [integer of Eq. (10)] Fourier component, analyzed as a function of the azimuth, that is nonzero. This is the harmonic that drives the resonance. Linear and nonlinear resonances are corrected by canceling out, with appropriate magnets, the more dangerous harmonics of the field errors.

Figure 4 shows the working diagram of the Advanced Light Source (ALS) storage ring (9). This is a plot of resonance lines defined by Eq. (10), with axes given by $v_x$ and $v_y$. The working point is the point having the tune values as coordinates. The accelerator physicist chooses this working point to be at a suitable distance from resonance lines. It is worthwhile to mention the order of magnitude of the tolerable departure of the tunes from the design (or, in an existing machine, experimentally found, optimum) values. This tolerance varies greatly from storage ring to storage ring, but it could be as tight as 0.001 in tune. Remember that tune values are in the tens of units. Thus, this tolerance is rather tight and is reflected in the high stability required from the power supplies.
Due to the restoring force of the radio-frequency field, particles oscillate in energy, describing *synchrotron oscillations*. The number of oscillations per revolution is denoted by the symbol $v_s$ (synchrotron wave number), and is on the order of 0.01 (100 turns per oscillation period). If the three-dimensional motion is considered (two transverse and one energy variable), then more resonances appear that involve the energy oscillations. The extended numerological condition for resonance is

$$m v_x \pm n v_y \pm k v_z = p.$$  \hspace{1cm} (13)

When Eq. (13) applies, the resonance is called a synchrotron-betatron resonance. It may be driven, for instance, when the value of the dispersion at the location of the radio-frequency accelerating cavities is nonzero.

### 3.6.2.3. Horizontal-vertical coupling

It is important to note that in Eq. (3) horizontal and vertical motions were not coupled; i.e., the horizontal differential equation of motion did not depend on the vertical coordinates, and vice versa. This is
only true in an ideal lattice in which the horizontal and vertical components of the magnetic field are perfectly aligned and in absence of field imperfections [see Eq. (1)]. In practice, a small amount of coupling is always present.

Particularly important is the coupling due to a *rotated quadrupole*, i.e., a quadrupole that, because of survey tolerances, is slightly (on the order of one meter-radian or less) rotated around its magnetic axis. This imperfection excites the coupling resonances \( v_x \pm v_y = p \). The sum resonance must be avoided. Although normally not *fatal*, special attention is required also for the difference resonance \( v_x - v_y = p \). This resonance couples horizontal and vertical motion. In synchrotron light sources, for instance, the vertical beam emittance is only a few percent of the horizontal one, and this resonance may appreciably increase the vertical beam size.

To combat the linear coupling effect, most storage rings are provided with rotated quadrupoles (i.e., quadrupoles that are rotated by 45° around the magnetic axis) placed at strategic positions to cancel the effect of the rotation errors of the lattice quadrupoles.

### 3.7. Off-Energy Particle Motion, Dispersion, Beam Size, and Momentum Compaction

In this section, nonsynchronous orbits, namely those of electrons having energy different from the one defined by Eq. 2, are discussed. We remarked earlier on that the electrons in a bunch follow a distribution of energy, typically a gaussian distribution centered around the synchronous energy.

Four important functions describe the motion of off-energy particles. Two are the *dispersion*, normally denoted by the symbol \( \eta \), and its derivative \( \eta' \) with respect to the independent variable \( s \). The others are the horizontal and vertical *chromaticities*.

#### 3.7.1. The dispersion

If the momentum of a particle changes, the bending radius in the dipoles changes according to Eq. 2, and the closed orbit also changes. A particle whose energy differs from the reference value follows a different orbit. The differential equations of motion (Eq. 3) now becomes:

\[
\begin{align*}
x'' &= K_x(s)x + \frac{1}{\rho_0(s)} \frac{\Delta E}{E_0}, \\
y'' &= K_y(s)y = 0.
\end{align*}
\]
They differ from Eq. (3) by the presence of a driving term in the x-axis and by a small, but important, change in the focusing terms $K_x$ and $K_y$. The latter reflects the fact that a change in energy (denoted as the relative change $\Delta E/E_0$ with respect to the synchronous energy $E_0$) changes the focusing strength of the quadrupoles. The term $\left(1/\rho_0(s)\right)(\Delta E/E_0)$ represents the perturbation introduced by the fact that the energy of the particle does not match the strength of the bending field, $\rho_0(s)$ being the bending radius in the dipoles of the synchronous particle with energy $E_0$. The vertical plane does not have such perturbation, unless vertical bends are present in the lattice. The function $1/\rho_0(s)$ follows the periodicity of the bending magnets. One of the solutions of Eq. 14 is periodic with the lattice periodicity; i.e., satisfies the conditions $x(0) = x(C), x'(0) = x'(C)$, where C is the length of the orbit after one revolution and can be expressed in terms of the dispersion $\eta(s)$ and its derivative $\eta'(s)$ defined as

\begin{align}
  x_c(s) &= \eta(s) \frac{\Delta E}{E_0}, \\
  x'_c(s) &= \eta'(s) \frac{\Delta E}{E_0}.
\end{align}

The dispersion is expressed in units of meters. Its derivative is dimensionless.

The solutions of Eq. 14 are those of the nonhomogeneous and the homogenous forms, the latter given by Eq. 4. In a general form that includes energy deviation and betatron oscillation, the horizontal motion of an electron can be described as the sum of a term which is a periodic function of $s$ and of an oscillatory term:

\begin{align}
  x(s) &= \eta(s) \frac{\Delta E}{E_0} + \sqrt{\varepsilon_x \beta_x(s)} \cos[\phi_x(s) + \phi_{0x}].
\end{align}

The slope, $dx/ds$, is given by

\begin{align}
  x'(s) &= \eta'(s) \frac{\Delta E}{E_0} - \alpha(s) \sqrt{\frac{\varepsilon_x}{\beta_x(s)}} \cos[\phi_x(s) + \phi_{0x}] - \frac{\varepsilon_x}{\sqrt{\beta_x(s)}} \sin[\phi_x(s) + \phi_{0x}],
\end{align}

with

\begin{align}
  \alpha(s) = -\frac{1}{2} \frac{d\beta}{ds}.
\end{align}
3.7.2. The beam size and divergence.

Having introduced a function for the motion of off-energy particles, we are in a position to generalize the beam size and divergence expressed by Eqs. (6) and (7). Those equations ignored the contribution of the spread in energy that is always present in a beam. Like the distribution of betatron amplitudes, the distribution of the energy spread is Gaussian. If \( \langle \Delta E \rangle \) is the root-mean-square of the energy deviation, and, as is normal practice, the emittance \( \varepsilon_x \) also defines the rms of betatron amplitudes, then, since these quantities are uncorrelated, they contribute quadratically to the overall beam size:

\[
\sigma_x = \sqrt{\varepsilon_x \beta_x + \eta^2 \left( \frac{\langle \Delta E \rangle}{E_0} \right)^2},
\]

\[
\sigma'_x = \sqrt{\frac{\varepsilon_x}{\beta_x} + \eta^2 \left( \frac{\langle \Delta E \rangle}{E_0} \right)^2}.
\]

Typically, the relative energy spread is on the order of \( 10^{-3} \), and the dispersion is measured in meters. One meter dispersion gives a contribution of 1 mm to the beam size. For comparison, an emittance of \( 5 \times 10^{-9} \) meter-radians at a location at which \( \beta_x \) is, say, 10 m, gives a beam size of 0.22 mm.

3.7.3. The momentum compaction factor

Let us now introduce a quantity that is of fundamental importance for the longitudinal motion. This parameter is the momentum compaction. It is a measure of how the time taken by the particle to complete one turn in the accelerator varies with energy. In high-energy electron accelerators the velocity of the particle is nearly constant with energy, and the revolution time is determined by the longer (or shorter) path a higher (lower) energy particle has to travel. Only the curved sections contribute to a lengthening of the orbit with energy, and higher energy particles have a larger bending radius. The momentum compaction factor is defined as

\[
\alpha_c = \frac{\Delta T/T_0}{\Delta E/E_0},
\]

(19)

where \( \Delta E \) is the energy difference from the synchronous energy \( E_0 \), and \( T_0 \) is the revolution period of the synchronous particle. The momentum compaction is determined by the properties of the lattice. In fact, it is the average of the dispersion in the bending section divided by the average machine radius (10). The stronger the focusing, the lower this value \( \alpha_c \) is. An approximation often used is \( \alpha_c = 1/\nu_x^2 \). The small
value of the momentum compaction function in synchrotron radiation sources has important implications for the longitudinal motion.

3.8. Chromaticity Correction and Dynamic Aperture

The focusing (or defocusing) action of the quadrupoles is inversely proportional to the particle energy. In analogy with optical lenses, this effect is called a chromatic effect. It leads to a dependence of the tunes on energy. This dependence is measured by the horizontal and vertical chromaticities, \( \xi_x \) and \( \xi_y \):

\[
\Delta v_x = \xi_x \frac{\Delta E}{E_0}, \quad \Delta v_y = \xi_y \frac{\Delta E}{E_0},
\]

where the \( \Delta v_{x,y} \) are the shifts in tunes from those of the synchronous particle and are caused by a change in energy \( \Delta E/E_0 \).

Because the focusing action decreases with energy, the uncorrected chromaticities are negative numbers. Corresponding to a spread in energy within a beam of particles, Equation (20) implies that a spread in tunes follows, and this may have adverse effects if it results in crossing resonance lines. Sextupole magnets, nonlinear elements already introduced in Eq. (12), are used to correct the chromaticities. Most accelerators operate with zero or slightly positive chromaticities. The next section gives a simple treatment of how sextupoles are used to control the chromaticities.

3.8.1. How sextupoles correct the chromaticities

Consider the field of a sextupole magnet [Eq. (12)]:

\[
B_y = S(x^2 - y^2), \quad B_z = 2Sxy.
\]

If a particle is off-energy, its horizontal displacement \( x \) consists of two terms: a betatron oscillation \( x_\beta \) and an orbit shift \( x_E \) [Eq. (16)]. The vertical displacement is a pure betatron oscillation \( y_\beta \). The field seen by the particle can be decomposed into the components of the displacements

\[
B_y = Sx_\beta^2 + 2Sx_Ex_\beta + Sx_E^2 - Sy_\beta^2, \\
B_z = 2Sx_\beta y_\beta + 2Sx_E y_\beta.
\]

The terms in bold in Eq. (21) have the form of a quadrupole field, a field that is linear in the betatron displacements \( x_\beta \) and \( y_\beta \). The strength of the quadrupole is \( 2Sx_E \), and is proportional to the
particle energy via its closed orbit displacement $x_E$. This fact is utilized to offset the (linear) energy dependence of the focusing strength of the quadrupoles. Figure 5 shows the quadratic dependence of the horizontally deflecting field.

Since the horizontal and vertical machine chromaticities are both negative, but the equivalent quadrupole of Eq. (21) has opposite focusing and defocusing effects in the horizontal and vertical axes, two families of sextupoles are required, both placed in dispersive regions. The horizontally correcting sextupoles are located in regions in which the horizontal $\beta$-function is high and the vertical $\beta$-function is low. The converse is true for the vertical chromaticity correcting sextupoles. Since, from Eq. 15, $x_E = \eta(s) \Delta E/E_0$, it is convenient, in order to reduce the sextupoles strength, to place the sextupoles at locations where the dispersion is high.

Equation (21) indicates that, besides the useful terms (in bold characters) that correct the chromaticities, unwanted, nonlinear terms crop-up that perturb the motion. Some storage rings include more than two families of sextupoles, the additional families being used to neutralize some of the resonances create by the unwanted terms of Eq. 21.

3.8.2. The dynamic aperture problem

Sextupoles are nonlinear elements, and, while they correct for the linear part of the chromatic aberrations, they can also disrupt the motion and cause particle loss. Low emittance lattices are characterized by strong sextupoles, and the problem of the dynamic aperture is one of the most important design issues.

The dynamic aperture is defined to be the maximum betatron oscillation that can be sustained in

![Figure 5. Sextupole field and local field gradient for an orbit displaced by $x_E$](image)
the accelerator for a sufficient number of turns. In electron storage rings, the time scale is on the order of the damping time.

The amplitude may be limited by the transverse size of the vacuum chamber (physical aperture), or by the perturbing effect of the nonlinear fields (dynamic aperture). The problem of determining the maximum stable amplitude of the oscillations in the presence of nonlinear perturbations is not amenable to an exact mathematical solution. The dynamic aperture limit is estimated by computer simulation of the motion of the particles in the presence of the nonlinear field of the sextupoles and other perturbing nonlinearities, like those caused by magnetic imperfections. It is desirable to design a lattice and chromaticity correction sextupoles such that the maximum amplitude of the betatron oscillations is determined by the physical aperture of the chamber, and not by the nonlinear perturbations.

The dynamic aperture is often plotted in a graph that depicts the maximum amplitudes of the vertical betatron oscillations that are stable as a function of the maximum stable horizontal amplitudes. Figure 7 shows the dynamic aperture of the Advanced Light Source (ALS), as computed for the Conceptual Design Report (9).

The dynamic aperture is sensitive to the degree of symmetry of an accelerator, high periodicity usually being associated with a larger dynamic aperture. Unfortunately, even in a machine designed with
high periodicity, the regular lattice pattern is broken by magnetic imperfections and orbit errors. Figure 6 shows the dynamic aperture of the ALS lattice in which the only nonlinear elements are the chromaticity sextupoles (without errors curve). The maximum stable amplitudes are reduced when magnet misalignments and field imperfections are included in the computation.

References


6. E. D. Courant and H. S. Snyder, “Theory of the Alternating-Gradient Synchrotron,” Annals of Physics 3, 1–48 (1958). The principle of strong focusing opened the way to a new generation of synchrotrons and storage rings in which the beam size, and thus magnet aperture and cost, could be kept much smaller than in the previous, weak focusing type, accelerators.


