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High Temperature Fracture and Fatigue of Ceramics

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1.0 Introduction

This report covers work done in the first year of our new contract "High Temperature Fracture and Fatigue of Ceramics," which commenced in August, 1995 as a follow-on from our prior contract "Mechanisms of Mechanical Fatigue in Ceramics." Our activities have consisted mainly of studies of the failure of fibrous ceramic matrix composites (CMCs) at high temperature; with a little fundamental work on the role of stress redistribution in the statistics of fracture and cracking in the presence of viscous fluids.

2.0 High Temperature Fracture of CMCs

We are now generating remarkably detailed *in situ* observations of cracking in woven SiC/SiC laminates at 1150°C [1]. In conjunction with high temperature fracture data from other laboratories (e.g., [2]), we are forming a consistent picture for the first time of the failure sequence under creep conditions. In SiC/SiC, the first nonlinearity to set in as temperature rises in an inert environment is fiber creep. At the temperature of our tests, the matrix remains essentially elastic; and, since the fiber creep rate is strongly stress dependent, fiber creep occurs mainly in fibers that bridge matrix cracks and therefore have unusually high loads. The experiments show subcritical matrix crack growth at fixed load, with one matrix becoming dominant. We believe this crack growth results from loss of crack tip shielding as the bridging fibers creep. Cracks that would remain arrested and benign (in many applications) in a rate independent material grow subcritically to become through cracks. Through cracks are often fatal in themselves; and their presence also accelerates fiber failure.

The experiments are confirming the validity of our modeling approach, which out of our impatience had moved a little ahead of experiment! Our bridged crack codes were quickly adapted to the problem of creeping fibers (rate dependent bridging tractions). The subcritical crack growth problem has been analyzed in detail by a combination of numerical and analytical methods; regimes of stable and unstable crack growth have been mapped out (stable growth usually being preferred for damage tolerant design); and some important transitions in failure mode have been predicted [3,4]. We have solved the problem of a dominant matrix crack emanating from a notch in a standard specimen as well as the problem of natural crack initiation in a typical 0/90° laminate [4].

One of the essential results of our experimental and theoretical work is that the steady state matrix cracking stress, first derived by Aveston, Cooper, and Kelly and a key material design parameter at room temperature, is no longer a safe limiting stress at elevated temperature. Subcritical crack growth will allow failure at much lower stresses, given sufficient time. In engineering practice, there will be a new material parameter, which will probably take the form of a single rate constant associated with the softening of bridging tractions due to fiber creep, which will lead to predictions of time to failure as a function of load. This has so far been demonstrated only for SiC/SiC, but it is likely to

be a general result for all composites in which the fibers are polycrystalline: polycrystalline fibers must be fine-grained to be strong; and fine-grained fibers will creep.

3.0 Cyclic Fatigue Effects

We have also begun to acquire evidence of the interaction of mechanical fatigue loading and ultimate failure after subcritical crack growth. Figure 1 shows fracture surfaces formed when subcritical matrix crack growth at 1150°C loaded fibers to the point where they failed by brittle fracture, destroying the specimen. Where the matrix crack had been formed by pre-fatigue at room temperature, all fibers failed on the fracture plane (Figure 1a). There was no pullout. Across the matrix crack formed by subcritical crack growth, there was extensive fiber pullout (Figure 1b). Thus pre-fatigue weakened the fibers at the pre-fatigue matrix crack plane (which became the final failure plane), presumably by attrition. This has very important consequences for toughness and high temperature failure mechanisms. Our next phase of work will include studies of the interaction of high temperature and cyclic loading effects.

(a)

(b)

Figure 1. Comparison of high temperature fracture surfaces where the matrix crack had been formed (a) in pre-fatigue at room temperature and (b) during subsequent subcritical crack growth at high temperature. (SiC/SiC plain weave laminate.)

4.0 Statistics of Fracture of Unidirectional CMCs

In collaboration with Professor Bob McMeeking of UCSB and our jointly supervised student Mike McGlockton, we are completing a fundamental study of how the brittleness of a unidirectional CMC is affected by the statistics of flaws and the redistribution of load around fiber failures [5]. We have developed a new finite element model which solves all aspects of this problem in a physically realistic way for the first time. The same model is also capable of modeling textile composites, which may be the way of the future for ceramic composites.

5.0 References

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4. B. N. Cox and D. B. Marshall, "Crack Initiation in Fiber Reinforced Brittle Laminates," *J. Amer. Ceram. Soc.*, in press.
5. M. A. McGlockton, R. M. McMeeking, and B. N. Cox, "A 3-Dimensional Finite Element model for Assessing Unidirectional CMC Strength," submitted to *Acta Mater.*

6.0 Collaborations and Other Activities

Our collaboration with Professor Bob McMeeking has been described above. We are now collaborating with Dr. Frank Zok, also of UCSB, to write an invited review of recent progress in understanding the toughness of CMCs.

We are also sustaining our collaboration with Professor Reiner Dauskardt of Stanford University. His student Keith Yi is currently using our bridged crack codes to analyze crack growth in the presence of a viscous crack-filling fluid. We are addressing water in glass to begin with, but the same mechanics are believed to apply to high temperature fatigue and fracture in many CMCs containing glassy phases.

We continue a small collaboration with Professor Nasr Ghoniem of UCLA on damage in CMCs at high temperature. Professor Ghoniem supplied us with the SiC/SiC sample we have been testing. We are planning to have one of his graduate students work here in our laboratory to assist in the high temperature experiments.

Invited papers on our work were presented at American Ceramic Society meetings in New Orleans and Indianapolis.

7.0 Cumulative List of Publications under This Contract

1. B.N. Cox and D.B. Marshall, "The Determination of Crack Bridging Forces," *Int. J. Fracture* **49**, 159-76 (1991).
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4. B.N. Cox and D.B. Marshall, "Crack Bridging in the Fatigue of Fibrous Composites," *Fatigue Fract. Enging. Mater. Struct.* **14**, 847-61 (1991).
5. B.N. Cox and C.S. Lo, "Load Ratio, Notch, and Scale Effects for Bridged Cracks in Fibrous Composites," *Acta Met. Mater.* **40**, 69-80 (1992).
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13. M. R. Begley, B. N. Cox, and R. M. McMeeking, "Time Dependent Crack Growth in Ceramic Matrix Composites with Creeping Fibers," *Acta Metallurgica et Materialia*, in press.

14. B.N. Cox, "Scaling for Bridged Cracks," *Mechanics of Materials*, **15** (1993) 87-98.
15. Edited book: *Fatigue of Advanced Materials*, ed. R.O. Ritchie, R.H. Dauskardt, and B.N. Cox, Proc. Engng Foundation Int. Conf., Santa Barbara, January 1991 (MCE Publ., Birmingham, UK, 1991).
16. B. N. Cox and L. R. F. Rose, "A Self-Consistent Approximation for Crack Bridging by Elastic/Perfectly Plastic Ligaments," submitted to *Mechanics of Materials*.
17. B. N. Cox and L. R. F. Rose, "Quasi-Steady State Solutions for Time Dependent Bridged Cracks," in preparation.
18. D. R. Mumm, W. L. Morris, M. S. Dadkhah, and B. N. Cox, "High Temperature Crack Growth in Ceramic Composites," *Thermal and Mechanical Test Methods and Behaviour of Continuous-Fiber Ceramic composites*, ASTM STP 1309, ed. M. G. Jenkins, S. T. Gonczy, E. Lara-Curzio, N. E. Ashbaugh, L. P. Zawada (American Society for Testing and Materials, Philadelphia, 1996).
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Edited book:

Fatigue of Advanced Materials, ed. R. O. Ritchie, R. H. Dauskardt, and B. N. Cox, Proc. Engng Foundation Int. Conf., Santa Barbara, January 1991 (MCE Publishing, Birmingham, UK, 1991).

8.0 Publications in this Reporting Period

The work in references 1, 3, 4, and 5 was executed primarily in this reporting period. Reprints are appended. (An earlier version of Ref. 4 was included in our last report. This, the final version, contains significant enhancements.)

Appendix

“High Temperature Crack Growth in Ceramic Composites,” by D. R. Mumm, W. L. Morris, M. S. Dadkhah, and B. N. Cox

“A Model for Creep Rupture in Ceramic Matrix Composites,” by R. M. McMeeking, M. R. Begley, and B. N. Cox

“Crack Initiation in Fiber Reinforced Brittle Laminates,” by B. N. Cox and D. B. Marshall

“A 3-Dimensional Finite Element Model for Assessing Unidirectional CMC Strength,” by M. A. McGlockton, R. M. McMeeking, and B. N. Cox

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HIGH TEMPERATURE CRACK GROWTH IN CERAMIC COMPOSITES

REFERENCE: Mumm, D. R., Morris, W. L., Dadkhah, M. S., and Cox, B. N., "High Temperature Crack Growth in Ceramic Composites," Thermal and Mechanical Test Methods and Behavior of Continuous-Fiber Ceramic Composites, ASTM STP 1309, Michael G. Jenkins, Stephen T. Gonczy, Edgar Lara-Curzio, Noel E. Ashbaugh, Larry P. Zawada, Eds., American Society for Testing and Materials, Philadelphia, 1996.

ABSTRACT: An *in situ* experimental technique is described which allows high resolution, high sensitivity determination of displacements and full-field strains during high temperature mechanical testing. The technique is used to investigate elevated temperature crack growth in SiC/SiC_f composites. At 1150°C, the reinforcing fibers have a higher creep susceptibility than the matrix. Fiber creep leads to relaxation of crack bridging tractions, resulting in subcritical crack growth. Differential image analysis is used to measure the crack opening displacement profile $u(x)$ of an advancing, bridged crack. With appropriate modeling, such data can be used to determine the traction law, from which the mechanics of cracking and failure may be determined.

KEYWORDS: crack growth, bridging, differential image analysis, creep, composites

Fiber reinforcement has been demonstrated to be an effective means of toughening intrinsically brittle materials, such as ceramics and intermetallics. A principal source of the improved toughness is crack bridging by fibers left intact in the wake of an advancing crack [1,2]. If crack bridging is the primary toughening mechanism, the force/displacement behavior of the bridging fibers determines the crack shielding. The mechanics of cracking and failure can be determined fully provided the magnitude and distribution of the crack bridging tractions are known [3]. Numerous models have been developed expressing the crack bridging tractions, p , as a function of the crack opening displacement, $u(x)$, at a position, x , in the crack wake. Cox and Marshall [3] have developed a method for deducing the function $p(u)$ from experimental measurements of the full crack opening

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profile $u(x)$ of a single crack. For an assessment of both the functional form and quantitative validity of bridged crack models for brittle-matrix composites, precise crack opening profile measurements are needed. However, few such measurements have been reported.

Analysis of bridging tractions from crack opening profiles requires an accuracy in displacement measurements of ~ 1 percent of the opening at the notch root or the crack mouth [3]. For a composite such as $\text{SiC}_w/\text{Al}_2\text{O}_3$ loaded to the critical applied stress intensity, the maximum opening of a $100\text{ }\mu\text{m}$ crack may be on the order of $1\text{ }\mu\text{m}$. The required measurement accuracy is then $\sim 100\text{ }\text{\AA}$. This is beyond the limits of conventional techniques for strain and displacement measurement, but within the capabilities of stereoscopy [4,5] or related automated displacement mapping techniques [6,7] based upon differential image analysis. The present article addresses the need for critical measurements of crack opening profiles during mechanical testing of ceramic composites, by describing an experimental technique that allows quantitative measurement of crack opening displacement profiles for bridged cracks in brittle-matrix composites.

At high temperature, crack bridging tractions may be affected by one or more of the following: (1) creep of the composite constituents, (2) viscous deformation of interphase layers and grain boundary phases, (3) relief of residual stresses due to thermal expansion mismatch, and (4) environmental attack, e.g. oxidation of the fiber-matrix interface. Although the temperature or time dependence of the crack opening profile must be accounted for, understanding the mechanics of crack growth and failure remains a question of determining the crack bridging tractions from measurements of $u(x)$.

The purpose of this work is to develop a high temperature test system that is capable of providing optical images of suitable quality for differential image analysis and thence to provide $u(x)$ data of sufficient resolution for assessment of models accounting for temperature-dependent and time-dependent crack bridging in CMCs. The system described in this work is used to investigate an important damage mechanism for high temperature ceramic composites, where crack growth occurs owing to a systematic degradation of bridging tractions with creep relaxation of crack bridging fibers. At 1150°C , stable crack growth is observed in SiC/SiC_f composites under constant load conditions. Changes in the crack opening displacement profile are measured as a function of time.

STRAIN MEASUREMENTS VIA DIFFERENTIAL IMAGE ANALYSIS

Image Acquisition and Analysis

Strain measurement using differential image analysis requires high quality images of the specimen surface contrast during various stages of a test procedure. Sensitive measurements of surface strains are made through comparison of a pair of optical micrographs, one taken before a change in the material takes place (the reference) and the other taken after. For example, a pair of micrographs may be taken before and after specimen loading, or the micrographs may be from different times of a constant load creep experiment. Strains or crack openings are measured by determining the relative changes in position of surface contrast features. The relative displacements may be measured by comparing pairs of micrographs using a stereoviewer, where in-plane displacements are perceived as changes in height [4,5]. Stereoimaging takes advantage of the exceptional

sensitivity of the human visual system to small changes in apparent depth [4,5]. A crack appears as a sharp ledge viewed from above. Quantitative displacement measurements are made by comparison of the perceived surface profile with a calibrated travelling spot [8]. Since the observed features are essentially unchanged from one image to the next, other than in precise position, the accuracy of measurement exceeds the point-to-point resolution of the microscope used to record the images [6]. Relative displacements of $\pm 5 \mu\text{m}$ can be determined on high quality optical micrographs. This corresponds to a displacement sensitivity of $\sim 100 \text{ nm}$ on micrographs of only 50X magnification, i.e. over a field of view of $\sim 2 \text{ mm}$ on the sample.

Manual methods of differential image analysis (stereoscopy) suffer from being tedious; considerable time can be spent analyzing displacements or strains from a single pair of micrographs. Furthermore, only the displacements parallel to the axis of the stereoviewer are measured. To obtain the orthogonal component of the relative displacement required for accurate determination of shear strains, both micrographs must be rotated exactly 90° , and the displacements must be measured from precisely the same reference position. Such limitations can be alleviated by automating the differential image analysis through digital image processing techniques [6,7] making use of cross-correlation procedures. In addition to improving the speed at which measurements may be obtained, automated image correlation is a powerful technique for determination of shear strains, as both components of the in-plane displacement are measured simultaneously. In this work, manual stereoscopic image analysis is used as displacements are only sought in one direction and only a small number of measurements are made per pair of micrographs.

Specimen Preparation

The strain measurement technique relies on imaging contrast that occurs on the surface of the specimen. A specimen surface with a dense, random distribution of highly contrasting features is ideal. In many cases, sufficient contrast exists on the specimen surface naturally. Material inhomogeneity or the microstructure can provide useful contrast. A highly polished sample is not always optimal. For instance, with the composites discussed in the following part of this paper, a highly polished surface shows large, matrix-rich regions with no contrast. Strain measurements can not be obtained from such regions. In other materials, the scale of the microstructure or other natural features providing contrast is too fine (or, possibly too coarse) to provide useful contrast at the desired magnification. In such cases, intentional contamination, surface decoration, or even fine scratching of the specimen surface can be introduced. For accurate strain measurement during high temperature testing, the features providing contrast must be thermally stable. The specimen surface must be flat (relative to the depth of field of the viewing instrument at the given magnification) so that the area over which measurements are to be made is in focus, and stays in focus with deformation of the specimen.

EXPERIMENTAL APPARATUS

The apparatus developed for quantitative investigation of high temperature crack growth has four independent systems to allow: (1) acquisition of high resolution images; (2) precise control of loading, deformation and fracture; (3) accurate temperature control;

and (4) atmosphere control. The system is built around a standard benchtop-sized servo-hydraulic test frame with associated electronics (servo-controller and signal conditioners). Images are acquired using an optical microscope attached to a high-stiffness, three-axis positioning system. To control vibrations transmitted from the surroundings, the loading system and optical microscope are rigidly fixed to a common plate and attached to a large mass sitting upon pneumatic isolators. A means of securing the camera assembly was devised so that inserting and removing polaroid film does not disturb positioning or focal conditions, while still allowing fine adjustments of the microscope for initial focussing.

The specimen is heated with a compact, custom-built furnace with a flat Kanthal™ heating element designed to allow viewing of the specimen while minimizing thermal gradients. The furnace includes a quartz window which is partially mirrored to reflect much of the radiant energy. Sample temperature is determined with attached thermocouples. The loading system includes water-cooled grips. The optical microscope incorporates a custom co-axial light source with a fiber-optic cable and high intensity xenon lamp. The resulting intense illumination overpowers the thermal radiation of the sample, providing high quality images with consistent contrast, regardless of temperature. The current system includes an enclosure so that testing may be done in air or with a slight overpressure of inert gas, although there are no technical barriers to more elaborate atmosphere control. The ability to control the test environment can be a significant advantage over *in situ* techniques making use of an SEM, where all testing is done in vacuum, because high temperature failure mechanisms involving environmental degradation (e.g. oxidation of the fiber matrix interface) may be directly investigated.

In examining potential sources of error, the magnification uniformity for this system has been analyzed over the available magnification range of 6.3X - 32X. Analysis of control micrographs, where the only change made is the position of the imaging system, shows that with positioning errors as large as 100 μm on the sample surface, errors in the measured strains are well below the measurement resolution.

HIGH TEMPERATURE CRACK GROWTH IN SiC/SiC_f COMPOSITES

At high temperature, brittle matrix composites are subject to a variety of cracking and damage modes, depending on the relative creep susceptibilities of the fiber and matrix [9,10]. In many ceramic composites, fine-grained fibers exhibit creep behavior while the matrix remains elastic at service temperatures. Such materials are susceptible to crack growth at high temperature owing to an interactive process involving fiber creep and matrix cracking. Creep relaxation of bridging fibers leads to crack propagation as the shielding effect is reduced and stress transfer to the matrix occurs. With crack propagation, new fibers are brought into the bridging zone and fiber stresses increase, advancing creep relaxation effects [11]. Materials exhibiting this behavior include SiC/SiC_f composites reinforced with Nicalon fibers.

Experimental work reported by Henager and Jones [12] showed that under constant load conditions, steady state cracking is observed with SiC/SiC_f composites at 1100°C. The reported crack growth rates were calculated on the basis of compliance change, assuming all changes were due entirely to crack growth. However, compliance changes continuously with relaxation of bridging tractions even in the absence of crack

propagation. The *in situ* technique described in the previous sections is used to investigate time-dependent crack growth in SiC/SiC_f composites at 1150°C. The technique allows direct measurement of crack length and crack opening displacement profiles, so that changes may be directly correlated with test parameters: the load, temperature, and time.

Materials and Specimen Preparation

Composites of Nicalon SiC fiber reinforced CVI β -SiC, with reinforcement in the form of plain weave 0/90° cloth layers, were tested. The as-supplied eight-ply composites were approximately 3.5 mm thick. Compact-tension (CT) specimens were machined from a 200 mm x 200 mm plate in accordance with ASTM Standard Test Method for Measurement of Fatigue Crack Growth Rates (E647 – 93), with specimen dimensions of $W = 31.75$ mm and $B = 3.00$ mm. The specimens were notched to a depth of 13.40 mm ($a_0/W = 0.42$), with a half chevron notch present 2.24 mm farther into the material. A Knoop indent (5 kg applied load) was placed at the tip of the chevron notched region to further facilitate the initiation of a sharp crack. The sample surfaces were polished flat to a 25 μ m finish. Additional polishing reduced contrast across the specimen surface.

A pertinent review of available creep data for CVD SiC and Nicalon SiC fibers is given by El-Azab and Ghoniem [13]. Using data reported by Gulden and Driscoll [14], who measured the creep characteristics of CVD SiC, and the results of DiCarlo and Morscher [15], where creep strains for Nicalon fibers were measured over the temperature range 1000°C – 1500°C, the relative creep rates of the composite constituents are compared. It is shown that at a temperature of 1150°C, the creep rate of Nicalon fibers is more than five orders of magnitude greater than that of CVD SiC. Therefore, in this work, creep effects are only considered for the bridging Nicalon fibers.

Test Procedures

Specimens were precracked at room temperature by fatigue loading at 5 Hz, at a stress intensity range of $\Delta K = 10.6$ MPa $\sqrt{m}$ and an R ratio of 0.1. Sharp cracks were observed to initiate at the tip of the Knoop indent, growing approximately half the distance to the end of the chevron notch. The samples were unloaded before heating to the test temperature of 1150°C. Both precracking and testing were done in a flowing argon atmosphere to minimize environmental effects (e.g. interfacial oxidation).

Observations and measurements made during testing of a single specimen are as follows. The specimen was held at an applied stress intensity of 0.1 MPa $\sqrt{m}$ while heating to 1150°C. After stabilizing at the desired test temperature, reference micrographs were obtained before the load was increased monotonically. Loading was interrupted at 350 N and 700 N for image acquisition. The load was held at 700 N to acquire images over ~200 min. The specimen was then loaded to 800 N and held for over 110 hours, until failure occurred with creep crack growth. All optical images were obtained at 30X magnification.

RESULTS AND DISCUSSION

The ability to map displacements with high precision allowed detection of damage that would not have been observable by direct imaging. An example is shown in Fig. 1.

With monotonic loading to 350N (less than half the maximum precracking load, $K_{app} \sim 5.0 \text{ MPa}\sqrt{\text{m}}$), partial opening of the fatigue precrack was observed, see Fig. 1a. Continued loading, above the maximum precracking load, to 800N ($K_{app} \sim 13.3 \text{ MPa}\sqrt{\text{m}}$) caused subcritical growth of the initial crack. Secondary cracks developed to the sides of the main crack, indicating the distributed nature of the damage occurring with monotonic loading (Fig. 1b). While holding at constant load, the main crack continued to open and extend, presumably because of creep relaxation of crack bridging fibers (Fig. 1c,d). The secondary cracks, and significant portions of the main crack, could not be observed in individual micrographs as the resolution is insufficient. Such cracks were only detected through differential image analysis, where a crack appears as a discontinuity in the differential displacement of contrast features.

The orientations of surface ply fiber tows are indicated in Fig. 1a (0° tows are parallel to the loading axis). A pore resulting from incomplete infiltration during CVD

Figure 1 — Evolution of damage with loading at 1150°C: (a) partial opening of the fatigue precrack at 350N, (b) crack extension and secondary cracking observed immediately upon loading to 800N, and after holding at constant load for (c) 706 minutes and (d) 4232 minutes. Width of field 3.61 mm.

processing is also noted. The precrack initiated within a matrix-rich region between tows and deflected at the edge of a 0° tow. Secondary cracking during monotonic loading (Fig. 1b) occurred parallel to the fibers within 90° tows. The main crack extended perpendicular to the 0° fibers, eventually linking up with one of the secondary cracks. The crack paths appear to correlate with geometrical features of the fiber architecture visible in the surface ply. However, far-field deformations indicate that cracks illustrated in Fig. 1 are present as through-thickness features, rather than as surface cracks. Furthermore, although the overall $0^\circ/90^\circ$ orientation of fiber tows is maintained, the individual plies are randomly offset relative to one another, so that 0° tows in one ply may be aligned with 0° tows, 90° tows, matrix rich regions, or pores in other plies. Therefore, a developing crack encounters microstructural features other than those noted on the specimen surface, and the overall crack growth is not correlated with any particular microstructural feature.

Crack opening displacements measured at points along the main and secondary cracks immediately following monotonic loading to 800N are shown in Fig. 2. The measured crack profiles are qualitatively consistent with expectations for bridged edge and center cracks. More importantly, the crack opening displacement at the notch root is more than double that measured anywhere along either of the side cracks. Stress in the bridging fibers is proportional to the local crack opening displacement, and the rate of creep will, therefore, be greater for bridging fibers at the notch root. Growth of the crack emanating from the notch root occurs while attendant changes in the secondary cracks are not observed.

To investigate the effects of fiber creep independently of crack growth, the initial

Figure 2 — Crack opening displacements measured immediately after loading to 800N at 1150°C . Width of field = 3.61 mm.

loading of the specimen was interrupted at a level (700N) just below that used for precracking. At this load, bridging fibers were likely to be stressed above the creep threshold, so that creep relaxation would occur while holding at constant load. Moreover, slight degradation of the bridging tractions could occur without attaining a critical condition for crack advance. Crack opening displacement profiles were determined from micrographs taken immediately after the loading interruption and after holding at constant load for 184 min. The results are plotted in Fig. 3. Changes in the crack opening of $< 0.5 \mu\text{m}$ were resolved. Although the crack tip position remained unchanged over this time period, the crack opening increased with time, the largest changes occurring nearest the notch where the bridging tractions were highest and the fibers were most susceptible to creep.

Figure 3 — Evolution of the crack opening displacement profile with creep relaxation of bridging fibers, under constant load conditions at 700N.

Similar measurements were made while holding at 800N load. Changes in the crack opening displacement profile and the crack length were characterized over a period of ~110 hours, at which point matrix crack propagation occurred unstably across the remainder of the specimen. The results are summarized in Fig. 4. The profile measurements show a continuously increasing crack opening displacement, again owing to stress

Figure 4 — Evolution of the crack opening displacement profile with creep relaxation of bridging fibers, under constant load conditions at 800N. Relaxation of bridging stresses leads to subcritical crack propagation.

relaxation in the bridging fibers. Crack growth occurred intermittently, presumably because of local variations in the crack resistance with material heterogeneity (matrix rich regions and local variations in the fiber volume fraction). Measurements such as those plotted in Fig. 4, obtained as a function of temperature and time, provide $u(x)$ data for direct comparison to theoretical predictions of crack growth at high temperature.

CONCLUSIONS

An *in situ* technique permitting high resolution, high sensitivity strain measurements during elevated-temperature mechanical testing of brittle-matrix composites is discussed. The technique has general utility for investigation of damage and failure mechanisms for high temperature, high performance structural materials. The technique is applied to study crack growth in a ceramic composite at 1150°C, where the reinforcing fibers are subject to creep. Crack opening displacements are determined as a function of position behind the crack tip, providing data for a direct evaluation of the mechanics of crack bridging at high temperature. At constant applied load, crack growth can be corre-

lated with degradation of the crack bridging tractions, measured as an increased crack opening displacement along the length of the bridged crack. Such data can be used with appropriate models to determine the traction law during various stages of crack growth. The measurements and observations presented demonstrate the utility of this technique for investigation of high temperature crack growth and failure mechanisms for CMCs.

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A MODEL FOR CREEP RUPTURE IN CERAMIC MATRIX COMPOSITES

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Abstract. A model for crack growth in ceramic matrix composites with creeping fibers has been developed using a time dependent bridging law to account for the effect of fibers bridging a matrix crack. Time dependent crack growth was predicted when the matrix crack growth occurs at a critical stress intensity factor at the matrix crack tip. Crack growth rates are presented as a function of crack length and time. After the crack has grown completely across a laminate, life is controlled by the time to rupture exposed creeping fibers. A transition is expected from life dominated by matrix crack growth at low stress to life dominated by fiber creep rupture after crack growth at higher stress.

1. INTRODUCTION

The creep rupture behavior of ceramic matrix composites (CMCs) is most harmful when the fibers exhibit creep behavior and the matrix is elastic [1,2]. Stress relaxation in the creeping fibers causes load to be shed to the matrix, increasing the likelihood of matrix cracking. Matrix cracks will cause additional loading of the fibers, advancing creep and possibly leading to creep rupture of the fibers. The decaying bridging tractions of creeping fibers cause time dependent growth of matrix cracks, due to the reduction of the shielding effect of the fibers. A model for this time dependent crack growth is presented here.

We consider the growth of matrix cracks from naturally occurring flaws in unnotched laminated CMCs. In laminated

CMCs subjected to increasing stress in the 0° fiber direction, flaws grow into tunnelling cracks, which appear first in the 90° layers [3]. (See Figure 1.) Since the tunnelling cracks propagate through the 90° plies unstably, this phase of crack growth is not affected by creep. Subsequently, crack growth will proceed stably into the adjacent 0° layers, with increasing stress required to increase the size of the bridged region [4]. When the fibers creep, however, the bridging tractions will decay by relaxation and the crack will continue to grow through the 0° layers. The time to rupture will depend strongly on how long it takes matrix cracks to completely penetrate the 0° layers. Once they have done so, cracks will grow unstably across the specimen until the material is held together only by creeping fibers.

intensity factor is evaluated using the standard weight function relationship for crack surface tractions on a finite crack in an infinite body [6]. If the crack tip stress intensity factor equals or exceeds the matrix toughness at the end of the increment, Dt is too large; if K_{tip} is smaller than the matrix toughness, Dt is too small. Iteration is pursued varying Dt until $K_{tip} = K_c$ to some specified precision after the crack growth increment. Finally, the crack growth rate $v = da/dt$ is computed by interpolating the computed crack length versus time curve with polynomials over a few increments.

4. RESULTS

The predicted crack velocity as a function of crack length for several notch sizes and load levels is shown in Figure 3. The curves shown in Figure 3 represent a universal set of results for composites exhibiting the behavior outlined in the introduction, given that the fiber and matrix have identical elastic properties. All relevant combinations of material property constants are contained in the normalizations. The term a_m in the normalization of velocity and crack length is

$$a_m = \frac{\pi E \sigma_{mc} \lambda}{4} \quad (5)$$

where σ_{mc} is the matrix cracking stress [10] given by

$$\sigma_{mc} = \left[\frac{12 E f^2 \tau \Gamma}{D(1-f)} \right]^{1/2} \quad (6)$$

and Γ is the matrix toughness. The applied stress is σ_a . The generalizations of the normalizing parameters when fiber and matrix have different elastic properties is given in [6].

The starting point for the velocity curves in Figure 3 is the crack length corresponding to the condition $K_{tip} = K_c$ for a given applied load level, which creates an initial bridge with a length dependent on the applied load [3,4,6].

Increasing the applied load increases the initial bridged crack length. Thus, velocity curves with higher loads will start further along the crack length axis. The initial crack velocities are high because creep allows the initially high fiber bridging stresses to decay rapidly. The relatively rapid crack growth at the beginning causes fiber bridging stresses to rebuild elastically and the crack slows down. After the initial deceleration, the crack growth rate steadily increases as the crack lengthens and the crack accelerates monotonically thereafter. For large crack lengths (beyond the deceleration transient), the crack velocity appears to be asymptotically independent of initial crack size. The bridging traction profile is found to be very similar for different initial crack sizes in this regime and depends only on current crack size and applied load. Crack acceleration in the long crack domain also declines with crack length and increasing applied stress. However the logarithmic scale on the ordinate of Figure 3 minimizes the appearance of increases in velocity.

Integration of the reciprocal of the velocity shown in Figure 3 with respect to crack length yields the crack length as a function of time. Typical results are shown in Figure 4 for one initial crack size and several load levels. As expected from Figure 3, the time to a given crack length depends strongly on the load level. Figure 4 also indicates that the deceleration transient sometimes dominates the time to extend the crack to significant lengths. For instance, for stress $\sigma_a/\sigma_{mc} = 0.4$, a deep narrow dip in Figure 3 corresponds to the long times taken to achieve modest extensions of the crack as shown in Figure 4. In a laminated composite as shown in Figure 1 with plies of equal thickness, the critical crack length is $a = 3a_0$; that is, crack growth across the first 0° ply. For all the cases shown in Figure 3, crack growth up to this critical length is dominated by the deceleration transient.

5. DISCUSSION

The rupture time of a composite can be considered to have two components. The first component is the time required to grow matrix cracks across the 0° plies. All such plies will be affected by matrix

Figure 3. Crack velocity as a function of crack length.

Figure 4. Time needed to grow the matrix crack to various lengths. The absolute time is the normalized time multiplied by $1/\beta$.

cracks growing across them and they will either link up or a given crack will propagate unstably across all zero degree plies. This component of the rupture time can be estimated directly from the results shown in Figure 4. The second component of the rupture time of the composite is the time required to cause creep rupture of the fibers holding the remaining intact pieces of material together. This contribution to the time to rupture, t_r , is estimated by considering fibers loaded uniformly. This gives [6]

$$t_r = \frac{f \epsilon_c}{B \sigma_a (1 + \epsilon_c)} \quad (7)$$

where ϵ_c is the critical true strain for rupture. For Nicalon SiC fibers, a critical strain of $\epsilon_c = 1\%$ has been suggested [13]. Use of this critical strain with $B = 6.51 \times 10^{-16} \text{ (Pa s)}^{-1}$ (determined for a temperature of 1200 C from the creep curves in [13]), $\sigma_a = 150 \text{ MPa}$, and $f = 1/3$, Eq. (7) predicts the approximate creep rupture time for the fibers as $t_r \approx 9.4$ hours.

This creep rupture time of the fibers can be compared directly with results presented in Figure 4. For the case of a common SiC/SiC composite, the values used above correspond to the case where $\sigma_a/\sigma_{mc} = 0.6$. For the case where $a_0/a_m = 2$ and B is the same as above, the time needed to penetrate the first 0° ply (crack length $a/a_m = 6$) is $t \approx 8.2$ hours. Thus stable matrix crack growth and creep rupture of the fibers after crack growth is complete take similar amounts of time in this case.

The initial crack velocity varies very strongly with applied load as shown by Fig. 3. On the other hand, t_r , which is inversely proportional to σ_a has a weaker dependence. Therefore, a transition will occur from a lifetime dominated by creep-controlled crack growth at lower applied loads to a lifetime dominated by creep rupture after crack growth at higher loads.

6. CLOSURE

Solutions have been presented for the problem of a crack growing through a brittle matrix when it is bridged by creeping fibers. The crack velocity is characterized by a transient deceleration, during which the stress in fibers near the edge of the unbridged segment of the crack falls rapidly through creep. Subsequently acceleration occurs during which the velocity appears to be most strongly influenced by fibers near the crack tip.

The solutions presented are pertinent to predicting the creep-rupture life of unnotched laminated ceramic matrix composites at high temperature. If life is defined by the rupture of the laminate, then lifetime is dominated by creep-controlled matrix crack growth at stresses well below the rate-independent matrix cracking stress and by fiber rupture following crack growth at higher stresses.

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Crack Initiation in Fiber Reinforced Brittle Laminates

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Abstract

In fiber reinforced brittle laminates, crack growth under monotonic tension generally consists of crack tunneling along the weaker ply (usually the 90° ply) followed by plane strain crack growth through the adjacent, more resistant plies (the 0° plies). In this paper, the details of this transition in crack mode are examined. The tunneling crack configuration is generalized to allow the crack to penetrate the 0° ply during tunneling. The effects of crack bridging in the 0° plies on the energetics of tunneling are computed numerically for general cases and combined with analytical results for certain limits. The nature of the transition from tunneling to plane strain cracking is found to depend on the ratio of the toughnesses of the 90° and 0° plies. Implications for laminate design are discussed.

1. INTRODUCTION

Under loads aligned with the 0° fibers, the first matrix cracking observed in $0/90^\circ$ brittle matrix, fiber reinforced laminates is usually multiple cracking in the 90° plies.¹⁻³ The 90° plies, i.e., the transversely loaded plies, tend to be weaker because weak fiber matrix interfaces offer easy fracture paths and residual stresses in the matrix often act in the same sense as the applied load.⁴

A schematic of the most popular paradigm of crack initiation in the 90° plies is reproduced from Ref. 3 in Fig. 1. The cracks are believed to grow from small flaws (bounded in every direction) in the 90° plies, of diameter less than or perhaps comparable to the 90° ply width. The flaws propagate first as tunneling cracks,⁵ as shown in Fig. 1, drawn along the 90° ply by its relative weakness.^{5,6}

When the tunneling crack is infinitely long, further loading will cause it to spread as a plane strain crack into the neighboring 0° plies. As the plane strain crack grows, it will be bridged by intact 0° fibers: further growth becomes the very familiar problem of a bridged plane strain crack in a brittle matrix fibrous composite.⁷⁻¹⁰

In the first analysis of the energetics of the tunneling problem,⁶ the tunneling cracks were considered to propagate along the 90° ply without penetrating the zero degree plies at all. Consistently, the phase of plane strain crack growth has been assumed to begin with a crack of width equal to the 90° ply width, the result of tunneling.¹¹ More generally, the tunneling crack should not always be expected to be confined to the 90° ply. Under the right conditions, it might propagate at a lower stress if its front extends into the 0° plies. Indeed, this possibility has already been foreshadowed in models of cracks tunneling through surface films along a front that penetrates the substrate.¹²

This paper reexamines the energetics of tunneling cracks in fiber reinforced laminates and presents the details of the transition of a tunneling crack into a plane strain crack.

2. TUNNELING CRACKS IN LAMINATES

2.1 Steady State Tunneling Cracks

In confining laminar geometries, cracks can tunnel as illustrated in Fig. 2 along one of the layers that has lower fracture toughness or is subject to tensile residual stress.

As the ratio l/a of the length of the tunnel to its width increases, a steady state condition is approached, for which the strain energy release rate takes a constant value, G_{ss} , independent of l . The wake of the steady-state tunnel crack is a plane strain crack, with strain energy release rate G_p tending to drive sideways expansion. Whether or not the crack penetrates the adjacent layers as it tunnels under steady-state conditions depends on the relative values of G_{ss} and G_p and the fracture energies Γ_1 and Γ_2 of the layers.

The energetics of steady-state tunneling can be readily evaluated by hypothetically allowing the length of the tunnel to increase by (i) removing a strip of material (of unit thickness in the y direction) from far ahead of the tunnel crack, (ii) forming a plane-strain crack of width $2a$ in the strip, and (iii) joining the strip into the wake of the tunnel crack. Defining U as the change in energy of the system during the cracking step (i.e., changes in strain energy in the strip and potential energy of the loading system), the strain energy release rate driving tunneling is

$$G_{ss} = -U/2a \quad (1)$$

The strain energy release rate driving sideways expansion of the wake into layer 2 (Fig. 2) is, by definition,

$$G_p = \frac{1}{2} \frac{dU}{da} \quad (2)$$

Crack growth occurs whenever G_{ss} or G_p exceeds the corresponding fracture energy.

When both layers are homogeneous, perfectly brittle materials with identical elastic properties and are subject to uniform applied stress σ_a , the energy U is given by

$$U = -\pi \sigma_a^2 a^2 / \bar{E} \quad (3)$$

where $\bar{E} = E/(1-\nu)$ is the plane strain Young's modulus. In this case, $G_p = 2G_{ss}$ and therefore a tunnel crack will be confined to layer 1 if and only if $\Gamma_2 > 2\Gamma_1$. If this condition is not satisfied, the crack penetrates the adjacent layer at a lower stress than is required for tunneling and, since in this simple case G_p is an increasing function of a , the crack would grow sideways unstably: the tunnel crack would not form.

If, on the other hand, G_p decreases as the plane strain crack grows into the adjacent layer, a tunnel crack penetrating partly into the layer could form. This situation can arise in systems with inhomogeneous residual stresses (as in the surface films alluded

to above) or if the adjacent layers contain reinforcements that bridge the crack, as in fiber reinforced laminates.

2.2 Approach to Steady State

Ho and Suo⁵ showed that the stress required to extend a tunnel crack as a function of the ratio l/a falls rapidly from initially high values to the steady-state value, which is reached for $l/a \approx 2$. Therefore, the steady state is a lower bound for the critical stress; but as long as flaws are available of initial size comparable to the layer thickness, the actual critical tunneling stress will not be far above the bound. For example, the critical stress for a penny crack of diameter equal to the layer width, which satisfies

$$G_{ss} = \left(\frac{4}{\pi}\right) \sigma^2 a \bar{E} = \Gamma_1 \quad (4a)$$

is only ~10% larger than that for a steady-state tunnel crack confined to the layer, which is given by

$$G_{ss} = \left(\frac{\pi}{2}\right) \sigma^2 a \bar{E} = \Gamma_1 \quad (4b)$$

3. MODEL REPRESENTATION OF THE LAMINATE

Possible histories of crack initiation will be built up from solutions to the plane strain crack system of Fig. 3a. A 90° ply of width $2h_{90}$ is bounded by two 0° plies, which are each considered to be semi-infinite. (Some remarks on what will happen if the 0° plies are finite appear in a later section.) Both the 0° and 90° plies are modeled as homogeneous, isotropic elastica with identical elastic constants. This of course will only be exact if the fibers and matrix have equal elastic constants. Nevertheless, finite element calculations of Xia and Hutchinson¹¹ and Xia et al⁶ show that solutions to a variety of related crack problems are remarkably insensitive to differences in the elastic constants of the fibers and matrix. They compared exact results for an orthotropic laminate with results for a homogeneous isotropic body in which Young's modulus was set equal to the modulus of the laminate in a direction parallel to either set of fibers and Poisson's ratio to the value implied by the concomitant out-of-plane component of the laminate strain. For cross ply laminates with equal ply thicknesses and with modulus ratios as large as 5, the stress intensity factors and compliance changes associated either with tunnel cracks contained completely within the 90° plies or with unbridged plane-strain cracks penetrating the 0° plies do not differ for the laminate and its homogenized dual by more than about 8%.

Where the crack exists in the 0° plies, crack bridging tractions of magnitude p are imposed by the fibers according to the constitutive law

$$p = \beta u^{1/2} \quad (5a)$$

where $2u$ is the total crack opening displacement. From energy considerations in a simple micromechanical model of sliding fibers ⁹

$$\beta = \left[\frac{4\tau f^2 E_f E^2}{R (1-f)^2 E_m^2} \right]^{1/2} \quad (5b)$$

where f is the fiber volume fraction; E_f and E_m are Young's moduli for the fibers and matrix; $E = fE_f + (1-f) E_m$; R is the fiber radius; and τ is a uniform friction stress coupling the fibers and the matrix.

The cracks under consideration are matrix cracks and the matrix is assumed to be perfectly brittle. Thus crack propagation occurs when the strain energy release rate attains a critical value, which is a material constant. The critical value will be denoted Γ_0 for cracks propagating in the 0° plies. Since a fraction f of the fracture surfaces of matrix cracks in the 0° plies is occupied by unbroken fibers

$$\Gamma_0 = (1-f) \Gamma_m \quad (6)$$

where Γ_m is the fracture energy for the matrix alone.

The critical energy release rate for crack propagation in the 90° ply will be written

$$\Gamma_{90} = \eta \Gamma_0 \quad (7)$$

If the crack in the 90° ply lies entirely within the matrix, which it might, and it forms a flat surface, which would be more difficult, then $\eta = 1/(1-f)$; and $1 \leq \eta < \infty$. However, the ratio of toughnesses could easily have values less than unity. For example, if tunneling cracks form in the 90° plies by linking together fiber-matrix interface cracks in a composite with negligible interface toughness, then η could be arbitrarily small. In the following, the entire range $0 \leq \eta \leq \infty$ will be considered.

4. EXTENSION OF THE PLANE STRAIN WAKE CRACK

Figure 4 shows the critical stress, σ_p , required to grow the partially bridged plane strain crack of Fig. 3(a) when bridging is governed by the constitutive law of Eq. (5). These curves were obtained by solving for the crack opening profile and the distribution of bridging tractions self consistently; evaluating the stress intensity factor, K , by a Green's function calculation; and applying the fracture criterion of $G_p = \Gamma_0$ with the definition $G_p = K^2/\bar{E}$. (Similar results have been published previously, computed via various techniques. All solutions presented here were generated using the integral equation formulation and numerical methods described in Ref. 13.)

The stresses and crack lengths in Fig. 4 are plotted in dimensionless form, normalized by the parameters:

$$\sigma_1 = [(3/2) \beta^2 \Gamma_0]^{1/3} \quad (8)$$

$$a_m = \frac{\pi \bar{E}}{4} \left(\frac{3 \Gamma_0}{2} \right)^{1/3} \beta^{-4/3} \quad (9)$$

The stress σ_1 is the matrix cracking stress for a unidirectional composite in the steady state limit implied by the seminal work of by Aveston, Cooper, and Kelly.¹⁴ When the crack is sufficiently large, the unbridged 90° ply has negligible effect on the crack tip and σ_p approaches the constant value σ_1 . The bridging length scale, a_m , provides a measure of the length of the crack that must be bridged before the effects of bridging become dominant.^{7,10,15}

With these normalizations, the only other parameter affecting the critical stress-crack length relation is the ratio of unbridged crack length, h_{90} , to the bridging length scale a_m . The curves shown in Fig. 4 correspond to representative values of this ratio. When $h_{90}/a_m = 8/3\pi^2$, the value of σ_p at $a = h_{90}$ (unbridged crack spanning the 90° ply) is equal to σ_1 . For larger values of h_{90}/a_m , $\sigma_p(a)$ is a monotonically increasing function and stable crack growth can be observed. For smaller values of h_{90}/a_m , $\sigma_p(a)$ is a decreasing function for nearly all crack lengths and crack growth is unstable.*

* There is a small domain at the beginning of the plane strain crack resistance curve for $h_{90}/a_m = 0.2$ in Fig. 4 within which $d\sigma_p/da > 0$. This unimportant quirk is discussed more fully in [10].

5. EXTENSION OF THE TUNNELING CRACK

5.1 The Mechanics of Bridged Tunneling Cracks

The steady-state strain energy release rate for the partially bridged tunnel crack of Fig. 3(b) is given by Eq. (1) with the energy U being that of the plane strain bridged crack of Fig. 3(a). The general result for a bridged crack, derived in Appendix A and given earlier in a different but equivalent form by Chan et al¹⁶, is[†]

$$G_{ss} = -\frac{2}{a} \int_0^{\sigma_a} \int_0^a \sigma \frac{\partial u[(\sigma_a - \sigma), x]}{\partial \sigma} dx d\sigma, \quad (10)$$

where $u[(\sigma_a - \sigma), x]$ is the crack opening displacement at position x within a crack whose surfaces are subjected to the sum of the uniform applied load σ_a ; a uniform fictitious traction, $-\sigma$; and the distribution, $p(x)$, of bridging tractions, determined self-consistently according to Eq. (4) or some other constitutive law. In the absence of the bridging tractions, i.e., for a linear elastic body, $u[(\sigma_a - \sigma), x]$ is proportional to $(\sigma_a - \sigma)$ and Eq. (10) reduces to the result given previously by Ho and Suo:⁵

$$G_{ss} = \frac{1}{a} \int_0^a \sigma_a(x) u_a(x) dx. \quad (11)$$

where $u_a(x)$ is the crack opening displacement under the applied load σ_a .

The critical stress for tunneling crack propagation, σ_t , is given by equating G_{ss} to the fracture energy. For the partially bridged tunnel crack of Fig. 3(b)

$$G_{ss}(\sigma_a = \sigma_t) = \frac{h_{90}}{a} \Gamma_{90} + \left(1 - \frac{h_{90}}{a}\right) \Gamma_0 \quad (12)$$

Equations (10) and (12) can be conveniently normalized using the same parameters, σ_1 and a_m , used for the plane strain crack. With a bar denoting normalized quantities, these equations can be combined and written as

[†] This result also applies in the presence of uniform residual stress, with σ_a being replaced by the sum of σ_a and the residual stress. A more general result valid in the presence of non-uniform residual stresses is given in Appendix A.

$$\int_0^{\sigma_t/\sigma_1} \int_0^{a/a_m} \bar{\sigma} \frac{\partial \bar{u}}{\partial \bar{\sigma}} d\bar{x} d\bar{\sigma} = -\frac{1}{3} \left[(\eta-1) h_{90}/a_m + a/a_m \right] \quad (13)$$

where $\bar{u} \equiv u[(\sigma_a - \sigma), x]/u_1$ and $p(u_1) = \sigma_1$ for the function $p(u)$ of Eq. (5a). Thus the relation between the normalized critical tunnel stress and crack width depends only on the two parameters h_{90}/a_m and η .

5.2. The Tunneling Crack Stress as a Function of Crack Width

The critical tunneling stresses were obtained from Eq. (13) using the same self-consistent crack opening profiles used to determine the plane strain wake crack growth. Typical families of curves are shown in Fig. 5, superimposed on curves for the critical stress, $\sigma_p(a)$, for plane strain cracks.

When the width, a , of a tunneling crack becomes very large, the unbridged interval within the 90° ply must again become inconsequential; and the energetics of the tunneling crack propagating in the direction y must become identical to those of the very large plane strain crack propagating in the direction x . Thus $\sigma_t \rightarrow \sigma_1$ as $a \rightarrow \infty$.

For each value of h_{90}/a_m there is an intermediate range of toughness ratios, η , for which the curve $\sigma_t(a)$ possesses an extremum. The locus of the extremum for different values of η coincides with the curve $\sigma_p(a)$. This is a general result, valid for *any* bridging law, which can be proven analytically as follows (the result was first proven by Beuth¹⁷ in the context of tunneling cracks in surface films). From Eqs. (1) and (12) the critical tunneling condition can be written in terms of the energy U :

$$U(a, \sigma_t) = 2 [h_{90} \Gamma_{90} + (a - h_{90}) \Gamma_0] \quad . \quad (14)$$

Consider the total derivative

$$\frac{dU}{da} = \frac{\partial U}{\partial a} \bigg|_{\sigma_a=\sigma_t} + \frac{\partial U}{\partial \sigma_t} \frac{\partial \sigma_t}{\partial a} = 2\Gamma_0 \quad . \quad (15)$$

At any extremum of the tunneling stress curve, $\partial \sigma_t / \partial a = 0$ and Eq. (15) reduces to

$$\left. \frac{\partial U}{\partial a} \right|_{\sigma_a = \sigma_t} = 2\Gamma_0 \quad (16)$$

This is simply the condition for growth of the plane strain wake crack. Thus, as long as an extremum exists, it must coincide with a point on the curve $\sigma_p(a)$ for the plane strain crack.

For 90° plies that are sufficiently wide that $\sigma_p(a)$ is an increasing function (i.e., $h_{90}/a_m > 8/3\pi^2$), three domains of the relative toughness, η , may be distinguished in Fig. 5. Domain I: At the lowest values of η , $\sigma_t(a)$ is a monotonically increasing function which falls below and never intersects the plane strain crack curve, $\sigma_p(a)$. Domain II: At intermediate values of η , $\sigma_t(a)$ possesses a minimum at which it intersects $\sigma_p(a)$. Domain III: At high values of η , $\sigma_t(a)$ is monotonically decreasing, is bounded below by σ_1 , and therefore once again never intersects $\sigma_p(a)$. The first domain covers the interval $(0 \leq \eta \leq 1/2)$, the upper limit being determined by the condition $\sigma_t = \sigma_p$ at $a = h_{90}$ i.e., equality of Eqs. (1) and (2) for unbridged tunneling and plane strain cracks. The second domain covers the interval $(1/2 \leq \eta \leq \eta_c)$, where the upper bound η_c must be computed numerically. The third domain occupies $\eta_c < \eta < \infty$.

For small 90° ply widths, $h_{90}/a_m \leq 8/3\pi^2$, the extremum in $\sigma_t(a)$ is a maximum occurring at $\sigma_p > \sigma_1$. The extremum still coincides with the plane strain crack curve, $\sigma_p(a)$; but since $d\sigma_p/da < 0$ nearly always (e.g., the case $h_{90}/a_m = 0.2$ in Fig. 4), crack growth is unstable and a transition from tunneling to plane strain cracks would not be seen experimentally.

6. CRACK PROPAGATION SEQUENCES FOR LAMINATES

The above results, especially Fig. 5, allow the possible sequence of events as a finite flaw propagates into a plane strain crack to be specified. Crack initiation will be assumed here to begin with a preexisting flaw in or overlapping the 90° ply. The flaw is finite in all directions and comparable in diameter to the 90° ply width. In this case the tunneling stress is close to the steady-state value (see Sect. 2.2) and the width of the tunneling crack will be determined approximately by the position of the minimum of the steady-state tunneling stress curve, $\sigma_t(a)$.

The possibilities for crack initiation depend upon the 90° ply width, h_{90}/a_m , and the toughness value, η , as shown on the failure map of Fig. 6. The responses in the three domains are as follows.

Domain I, $\eta \leq 1/2$: For low toughness ratios, i.e., a relatively weak 90° ply, the minimum tunneling stress occurs at $a = h_{90}$. Therefore, the tunneling crack will indeed be confined to the 90° ply, just as assumed in earlier work^{5,6,11}. With further increase in the applied stress after tunneling propagation, the crack will advance into the 0° plies as a plane strain crack, following the curve $\sigma_p(a)$ for the given value of h_{90}/a_m . For $\eta < 1/2$, this curve begins at a higher stress than the tunneling stress. Therefore, as the applied load is increased, unstable popping in of tunneling cracks will be followed first by an interval free of crack growth; and then stable broadening of cracks in plane strain.

Domain II, $1/2 \leq \eta \leq \eta_c$: For intermediate toughness ratios, the minimum in $\sigma_t(a)$ occurs at some $a_t > h_{90}$. Therefore, when tunneling occurs at the corresponding stress, $\sigma_t^{(min)}$, the crack front will already penetrate the 0° plies to a distance $a_t - h_{90}$. Because the minimum of $\sigma_t(a)$ lies on the plane strain cracking curve, $\sigma_p(a)$, the applied load will then have precisely the value required for the commencement of stable growth into the 0° plies, following the curve $\sigma_p(a)$. There will be no hiatus in cracking as the applied load increases.

Loci of constant relative tunnel width, a_t/h_{90} , in the space $(\eta, h_{90}/a_m)$ are plotted for Domain II in Fig. 6. For given 90° ply width, the tunnel width increases with increasing toughness ratio, from unity at $\Gamma_{90}/\Gamma_0 = 1/2$, to unbounded values as the upper limit of Domain II, $\Gamma_{90}/\Gamma_0 = \eta_c$, is approached.

The normalized stress, $\sigma_t^{(min)}/\sigma_1$, required for tunneling in Domain II depends only on the toughness ratio η and the 90° ply width, h_{90}/a_m . It is shown for representative cases in Fig. 7. The normalized stress ranges from the value for an unbridged tunneling crack confined entirely to the 90° ply when $\eta = 1/2$

$$\sigma_t^{(min)}/\sigma_1 = \frac{2}{\pi} \sqrt{\frac{2a_m}{3h_{90}}} \quad (\eta = 1/2) \quad (17)$$

(derived from Eqs. (1), (3), (8), and (9) with $G_{ss} \rightarrow \Gamma_{90}$)[†]; to unity when $\eta = \eta_c$.

[†] Equation (17) is identical to $\sigma_t^{(min)} = \left[\frac{2\Gamma_{90}E}{\pi a} \right]^{1/2}$, as required by Eqs. (1) and (3) with $G_{ss} \rightarrow \Gamma_{90}$ and $\sigma_a \rightarrow \sigma_t^{(min)}$.

Throughout Domain II, assuming the tunneling crack is confined to the 90° ply would lead to an overestimate of the applied stress for the onset of tunneling, to wit $\sigma_t (a = a_0)$ instead of $\sigma_t^{(min)}$. However, the difference is large only for $\eta \gg 1$ (Fig. 7), which is expected to be very uncommon (see below). On the other hand, the width of the first observable stable cracks, i.e., a_t , can be much larger than the ply width, h_{90} , as indicated on Fig. 6. Whether $a_t = h_{90}$ or $a_t > h_{90}$ might prove to be a sensitive test of whether $\eta > 1/2$.

Domain III, $\eta > \eta_c$: If the 90° ply has very high relative toughness, then tunneling will not occur. Instead, the flaw will grow unstably in all directions simultaneously.

Following the creation of a plane strain crack, the tunneling mode of propagation might appear in one further manner. Consider the crack geometry of Fig. 8: a fluctuation in material strength has caused the plane strain crack front to bulge out at one point. The question arises: will the rest of the crack front catch up by advancing uniformly in the direction x ; or will a uniform front be restored by the propagation of an incremental tunneling crack from the bulge (as indicated by the dashed curve and arrow in Fig. 8)? Hutchinson and Suo¹⁸ have shown by comparing the energy release rates for these two processes (a very simple calculation) that they should both require the same increment in external stress. Whichever occurs has no import for crack growth.

7. FURTHER CONSIDERATIONS OF LAMINA DIMENSIONS

If the 0° ply has finite width $2h_0$, the possibility exists that the computed tunneling crack half-width, a_t , of Fig. 6 will exceed $h_{90} + 2h_0$. The newly formed crack front would then lie in the next 90° ply, where there are no bridging fibers. Subsequent plane strain crack growth would then be unstable. Therefore, the initiation process as a whole would be unstable, leading immediately to cracks that span the whole specimen. Distinct tunneling and plane strain cracking phases would not be observed.

In the failure map of Fig. 6, the loci of constant relative tunnel width, a_t/h_{90} , define the upper limit for stable tunnel cracking in laminates with the corresponding ply width, $(h_{90} + 2h_0)/h_{90} = a_t/h_{90}$. For the most common case of equal 0° and 90° ply widths, the upper limit ($a_t/h_{90} = 3$) does not differ greatly from the case of an infinite 0° ply width. Since the critical values rise almost linearly with h_{90} (Appendix B), a coarse laminate structure, when everything else is equal, is less prone to catastrophic cracking during initiation than a fine laminate structure.

Many current ceramic matrix composites are not simple laminates of unidirectional plies, but rather laminates in which each ply consists of plain woven fibers. Nevertheless, the woven tows or yarns commonly have high cross-sectional aspect ratios, typically 3-10. In this case, the results of this paper can probably still be applied. A plausible tunneling crack configuration for a woven composite loaded along the warp direction (equivalent to the z axis of Fig. 1) is shown in Fig. 9.¹⁹ The tunneling crack could certainly propagate in the weft tow until it spans the width of the adjacent warp tows; and it might just as well follow the same weft tow to the ends of the specimen, lapping into successive neighboring warp tows as it goes.

8. CRACK STABILITY AND COMPOSITE PROPERTIES

The location of a particular composite on the failure map of Fig. 6 depends on the relative fracture energies Γ_{90}/Γ_0 and the parameters that determine the bridging length scale a_m (Eqs. 5(b) and (9)). The bridging length scale for ceramic composites typically falls within the range 50-1000 μm ¹⁰, while ply thicknesses are typically several hundred microns. Therefore, the range of h_{90}/a_m shown in Fig. 6 would include most ceramic composites of interest.

The toughness ratio is more difficult to specify because of the sensitivity of Γ_{90} to microstructural properties such as fiber arrangement, interface strength, and residual stresses (see next section). Nevertheless, a crude estimate may be obtained from simple geometrical considerations. If the matrix material is the same in the 0° and 90° plies and the fiber matrix interface toughness is negligible, which is usually true in ceramic composites, then the toughness ratio estimated for a hexagonal fiber array loaded normal to one of the close-packed rows of fibers, with a fracture surface of minimum energy formed by linking interface cracks with the shortest possible segments of matrix crack, is

$$\Gamma_{90}/\Gamma_0 = \frac{1 - \sqrt{\frac{2\sqrt{3}f}{\pi}}}{1-f} \quad (18)$$

This very rough guide to Γ_{90}/Γ_0 is shown in Fig. 10. It suggests that Γ_{90}/Γ_0 is always less than unity and is sensitive to the volume fraction of fibers. If the interface toughness were nonzero, then $\Gamma_{90}/\Gamma_0 > 1$ becomes possible; but even then values larger than two are not realistic in laminates containing identical matrix material in the 0° and 90° plies. However, larger values are possible in hybrid laminates in which the 90° plies are replaced by a different material.^{20,21}

To provide a reference point in Fig. 6, the location of a particular composite consisting of Nicalon fibers in a glass ceramic matrix has been marked. The following microstructural properties were used in evaluating a_m : $E_f = 200$ GPa, $E_m = 100$ GPa, $f = 0.4$, $R = 8$ μm , $\tau = 2$ MPa and $\Gamma_m = 10$ J/m². The 90° ply width was taken as 400 μm , while the toughness ratio was taken from Eq. (18). The height of the hatched rectangle represents the uncertainty in Γ_{90} .

The relative positions of other composites can be readily estimated from the dependence of a_m on the microstructural parameters, as indicated in Table 1. The arrows emanating from the Nicalon/glass-ceramic location in Fig. 6 indicate the shifts that would result from reducing any of the parameters f , R , τ and Γ_m by a factor of two.

9. RESIDUAL STRESSES

When the fibers and matrix have different coefficients of thermal expansion, residual stresses arise upon cooling from the processing temperature. In analyzing their effects on crack initiation and the stability of crack growth, it is helpful to view the residual stress as the sum of two components: (1) the composite stresses that would develop in an isolated ply; and (2) the interlaminar stresses associated with anisotropic thermal expansion of the plies. The composite stresses exhibit strong spatial variations on the scale of the fibers, whereas the interlaminar stresses are uniform within individual plies.

The composite residual stresses will be modeled as modifying the driving force for crack growth. The matrix residual stresses will be modeled as modifying the ply toughnesses, Γ_0 and Γ_{90} , or the constitutive law of bridging. This resolution is consistent with the formulation of sections 2-6.

Because residual stresses are inhomogeneous in laminates, to predict their effect quantitatively it is necessary to recompute the self-consistent crack profiles and calculate strain energy release rates via an expression similar to Eq. (10) (modified to include the residual stress fields - see Appendix A). For brevity, a qualitative survey of anticipated trends is offered here, with details left to the interested reader.

9.1 Composite Residual Stresses.

The long range residual stresses in the composite associated with expansion anisotropy are uniform within plies but change discontinuously at ply boundaries. For the

crack system of Fig. 1, their most important component is that in the z direction, i.e., parallel to the applied load. Denote these composite stresses $\sigma_z^{(90)}$ and $\sigma_z^{(0)}$ in the 90° and 0° plies respectively. For equilibrium,

$$h_{90}\sigma_z^{(90)} + h_0\sigma_z^{(0)} = 0 \quad (20)$$

If the matrix has the greater thermal expansion, then $\sigma_z^{(90)}$ will be tensile and $\sigma_z^{(0)}$ compressive. For cracks of width $a = h_{90}$ (i.e., cracks confined to the 90° plies), both the tunneling stress, $\sigma_t(a)$, and the plane strain cracking stress, $\sigma_p(a)$, will be lowered by a stress equal to $\sigma_z^{(90)}$. (If they would become negative, then spontaneous cracking accompanies cooldown from processing.) For cracks of width $a > h_{90}$, the critical stresses will be lowered by a lesser amount, since the effect of the composite residual stresses in the 90° ply diminishes as a increases and the effect of the compressive residual stress in the 0° ply tends to increase. This trend enhances the stability of plane strain crack growth. The critical normalized 90° ply width, h_{90}/a_m , for stable plane strain crack growth will become less than $8/3\pi^2$. For a given 90° ply width satisfying $h_{90}/a_m > 8/3\pi^2$, the plane strain crack curve will exhibit an increased slope. The minimum 0° ply width required to avoid catastrophic cracking following initiation by tunneling will decrease for fixed toughness ratio η in Domain II.

If the matrix has the lower thermal expansion, both $\sigma_t(a)$ and $\sigma_p(a)$ will be raised by the composite residual stresses; the minimum normalized ply width for stable plane strain cracking will increase above $8/3\pi^2$; the slope of the plane strain crack curve for stable cases will be lowered; and wider 0° plies will be required to contain tunneling cracks in Domain II.

9.2 Matrix Residual Stresses in the Fiber Direction

The matrix residual stress, $\sigma_z^{(m)}$, in the 0° ply can be modeled as a modification to the bridging tractions.²² Equation (5a) is simply replaced by

$$p = -(E/E_m) \sigma_z^{(m)} + \beta u^{1/2} \quad (21)$$

Thus if the matrix has the greater thermal expansion, $\sigma_z^{(m)}$ is positive (tensile) and the matrix residual stress acts as an opening force on the fracture surfaces. Since the opening stress acts only in the 0° ply, it has no effect on the critical stress for a tunneling or plane

strain crack confined to the 90° ply. For crack widths $a > h_{90}$, the opening stress lowers both of the critical stresses σ_t and σ_p by an amount that increases with a . For very large values of a , both σ_t and σ_p approach a new limit, $\sigma_1 - (E/E_m) \sigma_z^{(m)}$, where σ_1 is the limiting stress of Eq. (8). The slope of the plane strain cracking curve is lowered. The critical stress for crack initiation (tunneling) in Domain II is lowered; and the propensity for catastrophic cracking rather than stable plane strain cracking after initiation by tunneling is increased.

If the matrix has the lower thermal expansion, the matrix residual stress, $\sigma_z^{(m)}$, acts as a closing bridging traction and all trends are reversed.

9.3 Local Residual Stresses Normal to the Fiber Direction

Residual stresses also have significant components normal to the fiber direction. If the matrix has the greater thermal expansion, compressive radial stresses exist in each fiber, while the immediately peripheral matrix bears compressive radial stresses and tensile hoop stresses.

When a crack propagates in the 90° ply in the orientation of Fig. 1, these components of residual stress are substantially relaxed over each fracture surface for a depth comparable to the fiber diameter. The recovered strain energy can be modeled as a decrease in the effective 90° ply toughness, Γ_{90} . Thus these residual stresses influence the results of this paper partly by lowering the toughness ratio, η .

The radial stress component also influences the friction stress, τ , the two being approximately proportional in the absence of interfacial roughness. The effects of changing τ on initiation and crack stability have already been discussed.

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Appendix A: Strain Energy Release Rate for a Bridged Tunneling Crack

(a) No Residual Stresses

The energetics of steady-state tunneling may be evaluated by relating a strip of material of unit thickness far ahead of the tunneling crack of Fig. 3(b) and a similar strip in the far crack wake via the reversible thermodynamic cycle illustrated in Fig. A1. The initial state (a) (ahead of the crack) is crack free and subject to the applied stress σ_a . Traction is applied to the prospective crack surface to cancel the stress σ_a , a cut is made along this surface, and the system is relaxed (by applying equal and opposite tractions to the crack surface) to give the state (b) (crack wake). Equating the fracture energy of the newly created crack surface to the change, U , in mechanical energy of the system during relaxation gives the condition for growth of the tunneling crack, as expressed by Eqs. (1) and (12). The cycle is completed by applying a negative uniform stress, $-\sigma$, to the crack surfaces (state (c)), and increasing the magnitude of σ from zero to σ_a to close the crack and restore state (a). The work, W , done by σ during this step is equal to $-U$.

To evaluate W , let σ in (c) increase incrementally, with corresponding change $\delta u(x)$ in the crack opening profile. The work done by σ during this increment is

$$\delta W = -2 \int_{-a}^a \sigma \delta u(x) dx \quad (A1)$$

The total work done in closing the crack is given by integrating (A1):

$$W = -4 \int_0^{\sigma_a} \int_0^a \sigma \frac{\partial u}{\partial \sigma} dx d\sigma \quad (A2)$$

where $u = u[(\sigma_a - \sigma), x]$ is the crack opening displacement, as determined by the applied tractions $(\sigma_a - \sigma)$, and the constitutive response of the material, which, for the case of bridged cracks, includes the effect of bridging tractions $p(x)$. Therefore, the strain energy release rate is given by

$$G_{ss} = -\frac{U}{2a} = \frac{W}{2a} = -\frac{2}{a} \int_0^{\sigma_a} \int_0^a \sigma \frac{\partial u[(\sigma_a - \sigma), x]}{\partial \sigma} dx d\sigma \quad (A3)$$

Equation (A3) can also be expressed as

$$G_{ss} = \frac{2}{a} \int_0^{\sigma_a} \int_0^a u(\sigma, x) dx d\sigma$$

which was given previously by Chan et al.¹⁶

(b) Composites with Residual Stress

In the presence of residual stresses, additional tractions are required to cancel the stress in the matrix in state (a) of Fig. A1 and in the final step of the cycle to close the crack surfaces. If the residual stress is uniform over the crack, these tractions are accounted for simply by replacing σ_a in Fig. (A3) by the sum of σ_a and the residual stress component, σ^R , normal to the crack. If the residual stress is non-uniform (as it is in fiber reinforced laminates), the closure stress σ is also non uniform and may be written

$$\sigma = \zeta (\sigma_a + \sigma^R(x)) \quad , \quad (A4)$$

where ζ is a parameter that increases from zero to unity in restoring the original state (a) of Fig. A1. The crack opening displacement is a function of the surface tractions and position within the crack:

$$u = u[(\sigma_a + \sigma^R(x)) (1-\zeta), x] \quad . \quad (A5)$$

With these changes, the strain energy release rate becomes

$$\begin{aligned} G_{ss} &= \frac{2}{a} \int_0^a \int_0^1 [\sigma_a + \sigma^R(x)] \zeta \frac{\partial u}{\partial \zeta} d\zeta dx \quad . \\ &= \frac{2}{a} \int_0^a \int_0^1 [\sigma_a + \sigma^R(x)] u(\zeta', x) dx d\zeta' \end{aligned} \quad (A6)$$

In fiber reinforced laminates, the additional tractions $\sigma^R(x)$ consists of two components. One is due to the interply residual stresses associated with the anisotropic thermal expansion of the plies

$$\sigma^P(x) = \begin{cases} \sigma_z^{(90)} & x < h_{90} \\ -\frac{h_{90}}{h_0} \sigma_z^{(90)} & x > h_{90} \end{cases} \quad (A7)$$

The other is due to the residual stresses that would develop in isolated plies. Within the continuum framework of the present analysis, these intraply stresses average to zero over any portion of the crack plane and their effect on cracking can be modeled as a change in the bridging traction law.¹⁹ For 0° plies with the bridging law of Eq 5(a), this change amounts simply to the addition of a stress $-(E/E_m) \sigma_z^{(m)}$, as given by Eq. (21), where $\sigma_z^{(m)}$ is the residual axial stress in the matrix. (This additional bridging stress is equal to the remotely applied stress that would be needed to reduce the stress in the matrix to zero.) Since this additional bridging stress is present where the crack is closed in Fig. A1, the net additional closure traction, $\sigma^R(x)$, within the 0° ply is the sum of $\sigma^P(x)$ and $(E/E_m) \sigma_z^{(m)}$.

The residual stresses in the 90° ply can in principle be treated in a similar manner. However, since the corresponding bridging tractions are zero over most of the opening displacements of interest (they are only non zero for very small displacements), the effect of the residual stresses amounts to a constant energy per unit area of crack (independent of applied stress) which can be most conveniently treated as a reduction in toughness of the 90° ply.

Appendix B. Asymptote for the Critical Toughness Ratio, η_c .

When the 90° ply width, $2h_{90}$, is very large, an asymptote can be found analytically for the critical toughness ratio, η_c , above which stable plane strain cracking cannot occur. The critical toughness ratio corresponds to the limit ($a \rightarrow \infty$, $\sigma_p \rightarrow \sigma_1$) on the plane strain crack R-curve. The crack configuration in this limit can be separated for large h_{90} into two weakly interacting parts. For $|x| > h_{90}$, the steady state matrix crack configuration prevails, except in a small domain near $x = h_{90}$, which has diminishing effect as $a \rightarrow \infty$. For $|x| < h_{90}$, the crack profile for large h_{90} must become very close to that of an unbridged crack of width $2h_{90}$, because the opening becomes much greater over most of $|x| < h_{90}$ than the opening in most of the bridged region, $|x| > h_{90}$. Since the steady state crack configuration in $|x| > h_{90}$ will lead to $\sigma_t = \sigma_1$, the whole tunneling crack will achieve $\sigma_t \rightarrow \sigma_1$ if the energetics of the unbridged region $|x| < h_{90}$, would lead to $\sigma_t = \sigma_1$ when the unbridged region is considered independently. Thus from Eqs. (1) and (3) (with $\sigma_a \rightarrow \sigma_1$ and $G_{ss} \rightarrow \Gamma_{90}$)

$$\Gamma_{90} = \frac{\pi h_{90}}{2\bar{E}} \sigma_1^2 \quad (B.1)$$

Using Eqs. (8) and (9), this leads to

$$\Gamma_{90} = \frac{3\pi^2}{16} \Gamma_0 h_{90}/a_m \quad (B.2)$$

and thus

$$\eta_c \approx \frac{3\pi^2}{16} \frac{h_{90}}{a_m} \quad (h_{90} \gg a_m) \quad (B.3)$$

This asymptote is reasonably accurate for $h_{90}/a_m > 20$ (relative error < 10%). At lower values of h_{90} , it is too low by a factor of up to two. The ply width, h_{90} , must be very large before the bridged region has the same effect as uncracked material on the profile of the crack in the 90° ply.

Table 1. Dependence of a_m and σ_1 on Material Parameters for the Bridging Law of Eq. (5). (From Eqs. (5), (6), (8) and (9).)

Friction Stress	$a_m \sim \tau^{2/3}$	$\sigma_1 \sim \tau^{-1/3}$
Moduli	$a_m \sim \left(\frac{E_m^4 \bar{E}^3}{E_f^2 E^4} \right)^{1/3}$	$\sigma_1 \sim \left(\frac{E_m^4 \bar{E}^3}{E_f^2 E^4} \right)^{1/3}$
Fiber Radius ^(a)	$a_m \sim R^{2/3}$	$\sigma_1 \sim R^{-1/3}$
Fiber Volume Fraction ^(b)	$a_m \sim (1-f)^{1/3} \left(\frac{1-f}{f} \right)^{4/3}$	$\sigma_1 \sim \left(\frac{f^2}{1-f} \right)^{1/3}$
Matrix Toughness	$a_m \sim \Gamma_m^{1/3}$	$\sigma_1 \sim \Gamma_m^{1/3}$

(a) at constant fiber volume fraction.

(b) at constant fiber radius; including factor (1-f) from substituting Eq. (6) into Eqs. (8) and (9).

Figure Captions

1. Crack evolution in the 90° plies of a 0/90° laminate loaded in tension parallel to the 0° fibers (from [3]).
2. A tunneling crack in a confining laminar geometry.
3. (a) A plane strain crack spanning a 90° ply and penetrating into two semi-infinite 0° plies. The crack is fully bridged in the 0° plies.

(b) Plan view of a crack tunneling mainly through a 90° ply along the y axis of Fig. 1.
4. Critical stress $\sigma_p(a)$ for partially bridged plane strain cracks growing from 90° plies (unbridged flaws) of various widths h_{90} .
5. Variation of tunneling stress (fine curves) with crack width for various values of the toughness ratio, η . Cases shown: 90° ply width (a) $h_{90} = 0.2 a_m$; (b) $h_{90} = a_m$; and (c) $h_{90} = 10 a_m$. In each case, the tunneling stress is shown for $\eta = 0, 1/11, 1/5, 1/3, 1/2, 5/7, 1, 7/5, 2, 3, 5$, and 11. The heavy curves show the plane strain cracking stress.
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A 3-DIMENSIONAL FINITE ELEMENT MODEL FOR ASSESSING
UNIDIRECTIONAL CMC STRENGTH

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Abstract

A model for the behavior of unidirectional fiber reinforced ceramic matrix materials with weak interfaces is presented as an assemblage of coarse 3-D effective medium finite elements and 1-D spring (fiber) elements. The model is used to simulate the stochastic failure process leading to ultimate stress and the transition to pullout controlled extension. Brittle cracking of the matrix is assumed to limit the axial load sustained by the matrix, not damage the fibers, but allows the matrix to retain shear stiffness. However, fiber element fracture nuclei consistent with the Weibull distribution fail stochastically with increasing load. Fiber-matrix load transfer at fiber failure sites is governed by uniform shear strength representing frictional constraint at unbonded interfaces. For fiber and matrix with identical moduli, the model predicts the peak stress sustained by the composite is independent of the interfacial shear stress τ . The results are shown to be consistent with sparsely populated fiber damage prior to peak loads in which stress concentrations on neighboring fibers continuously decrease as a function of τ/σ . Localization is shown to dominate the post peak pullout except possibly for very large bundles or composites. This implies work of fracture derived from the $p(u)$ relationship is lower for bundles than for large composites and can lead to brittle behavior for bundles of tows in woven composites or if non-interacting bundles form in composites.

1.0 Introduction

The strength and toughness of ceramic matrix composites (CMC) exhibits a strong dependence upon the mechanical properties of the fiber-matrix interface. Under load, the dominance of the interface necessitates bounds on interface properties to control fracture

and load transfer near flaws. Damage in the composite commences with flaws within the matrix, since the failure strain of the matrix is usually much less than that of the reinforcing fibers. Under tensile loads, the matrix develops cracks which propagate across the composite and which may cause the fibers to fail as the crack tip approaches. For intact fibers to bridge the matrix cracks, the interface debond energy must be sufficiently low to prevent matrix cracks from penetrating the fibers and the fiber stresses in the crack wake must be less than the fiber strength. If fibers survive an impinging matrix crack but fail in the crack wake, the composite will exhibit brittle catastrophic failure at loads much less than the fiber strength. On the other hand, if fibers survive indefinitely in the crack wake at matrix cracking stress levels, the strength of the composite is determined by the distribution of flaws in the fibers and by the influence of τ and composite elastic properties on the redistribution of load from failed fibers. The interfacial shear stress which develops at debonded interfaces governs load transfer between the fibers and the matrix, with low τ leading to gradual load transfers (over a length L_D) and low stress concentrations in neighboring fibers. When L_D is much less than the gauge length of the composite, multiple fiber failures are expected along the length of the composite and its strength is governed by the weakest link in a chain of fiber bundles (of length L_D) simultaneously supporting the composite load. Higher τ diminishes the load transfer length of failed fibers and increases the stress concentration in neighboring fibers.

To analyze the failure of CMCs, analytical and computational models are required which account for random flaw populations in the constituents, initiation and accumulation of constituent damage, and the subsequent stress redistribution surrounding regions of

damage. Accurate results from such models depend upon the use of micromechanical models of the failure processes that are properly founded on experimental observations. Although the failure processes dictating the longitudinal mechanical behavior of unidirectional, laminated, or woven composite architectures are complex, the critical mechanisms associated with tough behavior can be illustrated by considering a standard tensile test of a unidirectional CMC.

The schematic of the macroscopic stress-strain behavior of a typical tough unidirectional CMC at room temperature is illustrated in Fig. 1. When the load is applied along the fiber axis and the macroscopic stress remains below a critical level (pt A), the stress-strain curve is linear, with the slope (the composite modulus) being well approximated in terms of the fiber and matrix moduli by the isostrain rule of mixtures. Between the initial critical value (pt. A) and a saturation stress (pt. B), increasing the stress causes multiple matrix cracks to propagate across the cross-section of the specimen without significant damage to the fibers. Eventually the entire specimen becomes saturated with matrix cracks of somewhat uniform longitudinal spacing and most of any further load on the specimen is borne by the fibers. As a result, the tangent modulus above the saturation stress is very close to the volume fraction of fibers f multiplied by the elastic modulus of the fibers E_f .

At elevated levels of stress, flaws within the fibers begin to fail. Load is transferred from failed fiber segments, via the fiber-matrix interfacial shear stress, to neighboring intact fibers. Such load transfers may or may not be fairly localized. If high stress concentrations are localized or fiber strengths are very narrowly distributed,

neighboring fibers may fail and result in correlated fiber damage prior to ultimate failure. If stress concentrations are more global than local or if fiber strengths are sufficiently broadly distributed, the next fiber to fail will be uncorrelated with existing fiber breaks, causing damage which is randomly distributed throughout the composite prior to ultimate failure. When the external load is increased monotonically and the level of accumulated fiber damage is such that an increase in the load causes the remaining intact fibers to fail, the composite is supporting its ultimate load. Whether a given composite, as a function of material properties and microstructure, achieves an ultimate load which maximizes the utilization of the available fiber strength in a damage tolerant manner or fails in a brittle manner at a small fraction of the available fiber strength has yet to be fully understood. However, some aspects of the functional dependence of strength on material properties have been investigated and will be reviewed next.

Research in the 1980s and early 1990s (Marshall and Ritter,87, Evans,90) identified the energy absorbing processes of damage accumulation as those which were responsible for the toughness of brittle matrix composites such as CMCs. The dominant mechanisms were matrix cracking, debonding or delamination along the fiber-matrix interface, fiber bridging of matrix cracks, and the existence of interfacial shear stress during the slip of fractured fiber segments relative to the matrix. Several micromechanics models for predicting the initiation and evolution of the mechanisms have been formulated to exhibit their dependence on the properties of the fiber-matrix interface (Kerans, 89, Evans, 90,91). Aveston, Cooper and Kelly (ACK) developed an expression for the initial matrix cracking stress as a function of fiber volume fraction, constituent properties, and interfacial shear stress. The model is based on a balance between the release of strain

energy and the potential energy of the loads during matrix cracking and dissipation of energy in frictional slip and cracking. Modifications to the ACK analysis which address initial matrix flaw sizes, thermal residual stress, and the evolution of multiple matrix crack spacing have been proposed in the literature (Marshall, Cox and Evans 85, Budiansky, Hutchinson and Evans 86, Zok and Spearing 92). In these analyses there are two common assumptions for the fiber matrix interface. The fibers are unbonded and constrained within the matrix by combinations of thermal expansion mismatch strains and friction; or weak bonds at the fiber-matrix interface are debonded by high stresses associated with the passage of matrix cracks through the specimen at the fiber-matrix interface. In the latter, roughness on the debonded surfaces causes frictional resistance, perhaps also enhanced by thermal expansion mismatches. Since strong interface bonds and high interfacial friction in unbonded interfaces will cause matrix cracks to propagate into the fibers and fail them, CMCs are usually designed to have weakly bonded or unbonded interfaces and low interface friction. Hence, after multiple matrix cracking, slip of the fibers relative to the matrix is constrained by relatively low friction at the fiber-matrix interface. Although its longitudinal strength is limited by matrix cracking, the cracked matrix retains the ability to support shear stress transmitted from the fiber through the fiber-matrix interface. It follows that above the matrix cracking saturation stress, matrix slabs separated by matrix cracks are primarily agents keeping the bundle of fibers together and transferring load in shear from broken fibers to intact fibers.

If unbroken fibers redistribute load equally, then the composite is said to exhibit global load sharing (GLS). This is thought to occur when interfacial shear strength is low (He, Evans, Curtin). The composite ultimate strength is then a competition between stress

redistribution and damage localization near broken fibers and the breadth of the fiber strength distribution. The failure of a composite in GLS as the fibers break is controlled by a Poisson process in which uncorrelated fractures eventually saturate the fiber system. Analytical models for this phenomenon should give an upper bound for the composite strength (Curtin, Hui, et. al). CMCs with low fiber-matrix interfacial shear stress can have tensile strengths which approach the prediction of the GLS model (Curtin, 91).

On the other hand, high interfacial shear strength is thought to give rise to local load sharing (LLS) in which the load transferred from a single broken fiber is shared by only a few neighboring fibers. As a result, there is a detectable stress concentration in these fibers compared to the global set, which can enhance their likelihood of failure. This means that further damage can cluster around previous breaks with detrimental effects on the strength and ductility of the composite material. Since the failure process involves dependent probabilities of failure in groups of fibers, it is much more difficult to analyze, though there are notable models for it (Hedgepeth, Scop and Argon, Schwietert & Steif).

2.0 Models

The problem of analyzing the failure of brittle matrix fiber reinforced composites is complex and simplifying assumptions are typically made to reach a tractable formulation which realistically represents the fiber failure statistics and stress redistribution mechanics. The simplest approaches ignore the existence of the matrix altogether, implying that unbroken fibers share the load equally and all broken fibers are unloaded (Peirce, Daniels, Coleman, Smith, Phoenix&Taylor). These assumptions result in dry bundle estimates for composite strength, typified by the dry bundle strength of fibers with a Weibull

distribution(Daniels, Coleman). Advances beyond the elementary dry bundle strength were made by analyzing bundles of fibers in which the effective length for stress recovery in broken fibers determined the length of the fiber bundle, although broken fibers remained unloaded (Gucer & Gurland, Rosen). Attempts were also made to include stress concentration effects (HVD, Zweben & Rosen, Scop & Argon) into bundle models. However, the complexity of the failure process prevented the analyses from modelling the full fiber failure process.

Recent models incorporate uncorrelated fiber failures and reloading of broken fibers by low magnitude frictional load transfer at the fiber matrix interface. In the single filament composite paradigm proposed by Curtin to analyze composite strength, the fragmentation and stress distribution of a single filament embedded in a matrix with large strain to failure is used to predict the behavior of a composite with a large number of equivalent fibers. This is consistent with low τ at the fiber matrix interface producing negligible stress concentrations in neighboring fibers and thereby global load sharing. When low stress concentrations are assumed, this concept is the basis for several models of CMC stress strain behavior (Curtin, Phoenix, Neumeister, Hui) in which the bundle length is proportional to the the stress recovery zone size.

The global load sharing assumption, and consequently the Poisson process for fiber failure, can be justified for low fiber-matrix interfacial shear tractions. However, the combination of constituent properties and the range of interfacial shear stresses over which the global load sharing assumption remains valid have yet to be determined. A model which captures the scale of stress redistribution and damage accumulation as a

function of material parameters is the objective of the 3-dimensional finite element model described in this paper.

3.0 The Model's Assumptions

A number of assumptions underlie the model developed in this paper. They concern the manner in which the brittle matrix fails, the matrix load bearing capacity, the mechanism of fiber failure, and the transfer of load from broken fibers to the surrounding matrix.

3.1 Matrix Failure

Cracking of the matrix as depicted in Fig. 1 is assumed to occur after a single valued, predetermined matrix longitudinal failure strain. Multiple matrix cracks are assumed to occur and saturate prior to any fiber failure. The process of cracking renders the matrix ineffective at supporting increased axial stress. The spacing of matrix cracks at saturation is assumed to be small compared to the size of finite elements used in the analysis and the separation of the fiber breaks.

The threshold composite stress for an isolated matrix crack to grow completely across the matrix in the absence of thermal residual stresses, as introduced by (ACK), is

$$\sigma_c^{mc} = E_c \left(\frac{6\tau\Gamma_m f^2 E_f}{E_c(1-f)E_m^2 R} \right)^{1/3} \quad (1)$$

in which E is the longitudinal Young's Modulus, τ is the constant shear stress at the interface during relative fiber-matrix slip, f is fiber volume fraction, R is the fiber radius, Γ is the fracture energy for the matrix, and subscripts c , f , m are used for the composite,

fiber, and matrix properties, respectively. Complete matrix cracking will occur spontaneously at the ACK level if there are sufficient initial flaws within the matrix which exceed a critical size (MCE, McCartney). In the finite element model, sufficient flaws are assumed to exist in the matrix so that matrix cracking always occurs deterministically at the ACK level. A shear lag analysis gives the distance from the matrix crack, d , over which interfacial slip occurs between the fiber and matrix:

$$\frac{d}{R} = \frac{\sigma_c^{mc} E_m (1-f)}{2E_c f\tau} \quad (2)$$

Evolution of the matrix crack spacing with increasing stress is governed by the energetics of matrix cracks interacting through interfacial slip zones and by the distribution of matrix flaw sizes. Analytical models and Monte Carlo simulations indicate the average matrix crack spacing saturates at approximately $1.3\sigma_c^{mc}$. At this stage, the longitudinal matrix stresses, equal to zero at the matrix cracks, are controlled by the transfer of load from the fibers through the interfacial shear stresses. Since the average spacing of the matrix cracks is twice the slip distance, any further straining increases the stress on the fibers but cannot transfer any further load to the matrix (see, Fig. 2). This overall behavior has been confirmed in experiments (Beyerle, et. al. 92). Consequently, with finite elements much larger than d , the matrix longitudinal behavior can be represented by a material constitutive law which involves saturation at a critical stress σ_c taken to be representative of σ_c^{mc} . Below σ_c , the matrix is linear elastic and when the longitudinal stress reaches σ_c , the matrix yields in a perfectly plastic manner. This choice approximates the initiation and saturation of matrix cracking to be occurring simultaneously.

The process of matrix cracking (yielding) is assumed to leave the transverse and shear properties of the matrix unaffected. The effective longitudinal elastic shear modulus of the matrix can be adjusted to account for the matrix cracking. However, the ability of the matrix to transmit longitudinal shear stresses is retained. Thus loads shed from fibers which have broken can be transmitted in shear through the matrix. Matrix properties assumed in the model are summarized in Appendix A.

3.2 Fiber Failure

Fibers are assumed to break when the longitudinal stress reaches a critical level controlled by pre-existing flaws. A statistical distribution of strength prevails within the fibers and a representative value is given to each fiber element based on its length.

The statistical representation of fiber flaw distributions can take a number of forms. A two parameter Weibull distribution for dependence on size and stress is assumed here to govern the fiber strengths in the composite and the statistical distribution of the strengths of the fiber elements in the model is required to be consistent with it. Thus, for a fiber element of length L , which is subjected to an axial stress σ , the expected number of flaws which fail at a stress equal to or less than σ is

$$\Phi(L, \sigma) = \frac{L}{L_0} \left(\frac{\sigma}{\sigma_0} \right)^m \quad (3)$$

where the Weibull parameters σ_0 , L_0 and m are the reference stress, reference gauge length, and Weibull modulus respectively. It should be noted that other models, such as 3 parameter distributions, can be used to represent the statistics of fiber strengths. This

would merely change the details of the model, perhaps improving it. However, the principles of the approach and the primary conclusions would not be changed.

3.3 Load Transfer

Once fibers in the composite fail, each fiber break is accompanied by regions of slip on either side of the break which transfer load from the broken fiber via the matrix (Fig. 2a). With a constant interfacial shear τ , load transfer from the broken fiber segment through the matrix ceases when the fiber stress has recovered to the nominal level of fiber stress, $\sigma_f^\infty = E_f \varepsilon$, at a distance

$$L_T(\sigma_f^\infty) = \frac{\sigma_f^\infty R}{2\tau} \quad (4)$$

from the break. At distances (x) less than $L_T(\sigma_f^\infty)$ from a break, the fiber stress is

$$\sigma_f(x) = \frac{2\tau}{R} x \quad (5)$$

and will not increase with the external load. This is complementary to the stress behavior in the matrix between cracks which has similar features but on a smaller length scale d (see Fig. 2). Therefore, at distances greater than $L_T(\sigma_f^\infty)$ from the break, the fiber stress is unaffected by the presence of the break and is assumed to have an average stress equal to the nominal level $\sigma_f^\infty = E_f \varepsilon$, where ε is the longitudinal strain. At all values of x along the fiber, perturbations to the fiber stress profile caused by matrix damage are negligible.

4.0 Finite Element Method

The finite element program is a special version of the Binary Model program described elsewhere (Cox et al.). The special version of the program developed to simulate the behavior of unidirectional fiber reinforced ceramic matrix composites is based on the 3-dimensional geometry shown in Fig. 3. The code is a finite deformation updated Lagrangian finite element method which accounts for finite strains (see Appendix B). A cuboid of material of dimension $N_x a_x$ by $N_y a_y$ by $N_z a_z$ is analyzed with N_y fibers along one side and N_z fibers along the other. The finite element nodes coincide with the fibers so that there are N_x fiber elements along the length of the cuboid. The fiber elements are 1-dimensional units between the nodes and prior to failure are simple uniaxial elastic springs. Their stiffness is chosen to represent a fiber with a finite cross-section; i.e. $E_f \pi R^2 / a_x$ where a_x is the element length. Upon fiber failure, the fiber segment stiffness is eliminated from the finite element model.

The space among the nodes is filled with cuboidal effective medium elements which have 8 nodes per element. The purpose of these elements is to simulate the phenomena not represented by the fiber elements. Thus they represent the composite properties in the transverse direction and roughly speaking, the matrix properties in the longitudinal direction. The stiffness matrix of the finite element equations is that of the effective medium elements plus the spring stiffnesses of the fiber elements connected to the nodes along the x-direction.

The effective medium elements have anisotropic properties. In this paper, it is assumed that the matrix and fiber elasticities are identical and isotropic with Poisson's ratio of 0.22 (the case of contrast between the fiber and matrix elasticity will be treated in

a separate paper). As described in Appendix B, the choice of Poisson's ratio is not significant, since the longitudinal response of the composite material after matrix cracking is of primary interest. Prior to matrix cracking, the elastic properties (in the form summarized in Appendix A) of the effective medium are given by

$$E_x = (1 - f)E \quad (6)$$

where E is the modulus in the fiber direction, $E = E_f = E_m$, and

$$E_y = E_z = (1 - f)E / (1 - f + fv^2) \quad (7)$$

where E_y and E_z are the moduli in the directions transverse to the fibers. The Poisson's ratios in Cartesian coordinates are

$$v_{xy} = v_{xz} = v, \quad (8)$$

$$v_{yx} = v_{zx} = v / (1 - f + fv^2), \quad (9)$$

$$v_{yz} = v_{zy} = (v - fv - fv^2) / (1 - f + fv^2). \quad (10)$$

The 3 effective shear moduli are all equal to $E/2(1+v)$. This scheme ensures that the anisotropic elastic 3-d effective medium elements filling the entire space of the model possess a symmetric stiffness matrix. Moreover, when the effective medium elements act in conjunction with the 1-d fiber elements, they produce a composite behavior which is isotropic with Young's modulus E and Poisson's ratio v . It should be noted, however, that in this case the fiber arrangement is assumed to have the same spacing in both the y and z directions (i.e. $a_y = a_z$) and the constraint

$$f = \pi R^2 / a_y a_z \quad (11)$$

must be respected in all calculations.

4.1 Matrix Failure

The effective medium elements are elastic until a critical value of σ_{xx} is reached. The critical value is $\sigma_c = E_x \varepsilon_c$ and because of the form of the axial modulus, σ_c is equal to a fraction of the matrix cracking stress σ_c^{mc} , $\sigma_c = (1 - f)\sigma_c^{mc}$. A uniform deterministic value of σ_c^{mc} is chosen so that $d \ll a_x$. After the initiation of cracking the stress σ_{xx} in the effective medium is constrained to be equal to $r\sigma_c$ where $0 \leq r \leq 1$. With multiple matrix cracks, a value of $r = 1/2$ might be expected (Fig. 2). Since $r\sigma_c$ is assumed to be much less than the composite strength, r is taken to be 0. This, though perhaps unrealistic, is consistent with choosing $d \ll a_x$ (see Fig. 2) and is a convenient way of ensuring matrix stresses remain negligible compared to fiber stresses during fiber failure.

4.2 Fiber Failure

There are N_x by N_y by N_z fiber elements embedded within the effective medium connecting the nodes along the x-axis to form initially intact long fibers. Each fiber element between 2 nodes behaves as an elastic perfectly brittle solid in which a single flaw determines its ultimate strength, Q^U . The flaw strength is a random variable chosen by the Monte Carlo method from the selected Weibull distribution (see Appendix C). Consider next a set of initially intact fiber elements in a single fiber which is subjected to a uniform axial load Q . When Q first exceeds the strength of a particular element, that

element fails and the program simulates the debonded, slipping fiber elements on either side of the break to represent the situation shown in Fig. 2. This is done in the computer program in the following manner.

The size of the slipping region is first identified. The break is assumed to take place at a preassigned random location in the failing element. Shear lag theory predicts the slip zone should extend a distance $L_D = Q/2\pi R\tau$ on either side of the break. The number of slipping elements to one side of the broken element is N_D where

$$N_D + \alpha_0 \leq L_D/a_x \quad (12)$$

where $0 \leq \alpha_0 \leq 1$ is the randomly assigned location of the break within the undeformed element. Thus discretization in the finite element code gives rise to a slip length that is always less than or equal to the exact slip length for a given isolated break.

Since the stress in the broken fiber element and the slipping fiber element will no longer change as the strain is increased, these elements are eliminated from the finite element model (i.e. their stiffness contributions are removed). This process is illustrated schematically in Fig. 4. The effect of the missing fiber segments on the matrix and remaining fiber segments (see Fig. 4) is then reintroduced by the imposition of equivalent nodal forces (see Appendix D for details of calculations) to represent the traction τ of the fiber acting on the matrix and the presence of the stressed fiber termination at the end of the missing zone of elements. Thus, nodal forces equivalent to $2\pi R a_x \tau$ acting away from the break are placed on nodes within the zone of missing elements (see Fig. 4). The exceptions are the nodes adjacent to the break and the nodes at the ends of the zone of

missing elements. Nodes adjacent to breaks account for the position of breaks relative to effective medium nodes, denoted αa_x , by applying nodal forces away from the break equivalent to $2\pi R\tau(1/2 + \alpha)a_x$. (see “fiber pullout” below for updating α). Nodes at the ends of the zone of missing elements are subjected to a nodal force acting towards the break equivalent to $2\pi R\tau(N_D + \alpha)a_x$. These nodal forces, acting on the effective medium, represent the transmission of load from the broken fiber end into the composite material, and ensure that the retained fiber segments away from the break are in equilibrium. If a slip zone is calculated to project beyond the end of the model, it is introduced in a periodic fashion into the other end of the model. In this way, a periodic system is simulated. As failure progresses, the changes in these forces which account for changes in element size (i.e. $l_x \neq a_x$) during deformation are outlined in Appendix D.

As the strain is increased, the load on the fiber Q will usually increase, requiring slip zone length increases. This adjustment is made in the finite element program, with the number of debonded elements ($2N_D + 1$) being increased accordingly. In some cases, the slip zones of 2 nearby breaks in a single fiber can meet each other either through growth or when a new break occurs just outside the slip zone of a previous break. In this case, the fiber segments of both slip zones are removed and replaced with equivalent nodal forces pointing away from the break to which they are nearest, since the intersection of the slip zones will be half way between the 2 cracks.

4.3 Fiber Pullout

Until all of the fibers in the model have broken, it is possible for increased strain to increase the load in an intact fiber element. However, as can be deduced from Fig. 5, the

equilibrium load on a plane cut through the model can no longer increase when all of the fibers have broken and the slip lengths from fiber breaks on *all fibers* overlap on that plane. We use this condition in the FE model to determine when pullout is possible and to determine the location of the cross-section capable of supporting the lowest load. This cross-section will become the fracture plane for ultimate failure.

By equilibrium, the total load on any other cross-section must equal that in the ultimate fracture plane which therefore limits the supportable load for the entire finite element model. As the load on the ultimate fracture plane falls during pullout, elastic unloading occurs at other cross-sections. Fiber pullout is determined by the distribution of fiber breaks relative to the location of the ultimate fracture plane.

When a fiber break terminates a slipping fiber segment on the right (i.e. at the right end of the slip zone), the exact position of the fiber break is determined by adding

$$\Delta u = \frac{\tau(N_D + \alpha_0)^2 a_x^2}{RE_f} \quad (13)$$

to the displacement of the node at the left hand end of the slip zone for that break (see Fig. 5). Since the code follows the updated Lagrangian procedure, the nodal positions are regularly updated and the position of the node adjacent to the left end of the slip zone, x_L is easily found. The relative position of the break from an effective medium node, αa_x , is determined from the nearest effective medium node with x , denoted x_{near} , less than the position of the break,

$$\alpha a_x = x_L + (N_D + \alpha_0)a_x + \Delta u - x_{near} . \quad (14)$$

When a fiber break is at the left hand end of a fiber segment, α_0 is replaced by $1-\alpha_0$ and Δu is subtracted from the displacement of the node at the right hand end of the slip zone for that break. Accounting for sign changes and making appropriate replacements for x_L , α_0 , and N_D in (14), yields α_{ax} for a fiber break at the left hand end of a fiber segment. As the relative displacement reduces α_{ax} , the equivalent forces from the tractions applied to the effective medium elements adjacent to the break are gradually reduced until they disappear as the fiber ends displace past the element's nodes.

As fiber pullout occurs, the axial load on the finite element model will generally diminish and the lengths of slip zones at fiber breaks diminish in the calculation. However it is assumed for simplicity that this is accomplished without the formation of reverse slip zones. The length of the slip zone is simply reduced to the value it had after monotonic loading to the current stress level. Any potential effects of load cycling are ignored. In the pullout zone, this simple approach is probably correct because reverse slip will usually be overcome by the pullout process. Hysteresis associated with reversal effects at breaks not in the pullout zone are believed to have a negligible effect on overall composite behavior.

Pullout is ultimately achieved in the code after a plane with all fibers slipping across it has been created due to all fibers failing. This cross-section is assumed to be a matrix crack which opens to form the ultimate failure plane and therefore transmits no axial or shear tractions between the pieces of the finite element model on either side of the failure plane. When a fiber end has passed this plane, it is considered to have pulled out

and is no longer connecting together the two pieces of the finite element model. Failure will finally occur when all fibers have pulled out.

4.4 Finite Element Algorithm

As noted previously, the FEM code is an updated Lagrangian program and solves for changes (increments) in nodal displacement at nodes due to changes (increments) in the total nodal forces arising from applied tractions, applied displacements, and fiber slip zones. The code allows precise control over all aspects of the elastic deformation and failure mechanisms as outlined above. Since the transverse behavior of the composite is not a concern, all transverse displacements are taken to be zero. Thus only axial degrees of freedom remain, providing savings in computational time compared to truly 3-dimensional calculations. Further savings are obtained from the fact that the stiffness of the model does not change rapidly as damage occurs, so the stiffness is recalculated only after significant amounts of new damage or when errors have become significant. Choosing to restrict the transverse displacements implies axial stresses and displacements can't cause transverse displacements, so $v_{xy} = v_{yx} = v_{xz} = v_{zx} = 0$ is imposed during calculations for consistency with the constrained transverse displacements. The other effective medium elastic properties remain as defined above so that the axial Young's modulus E_x and axial shear moduli G_{xy} and G_{xz} of the effective medium continue to duplicate the desired axial properties of the composite when acting in concert with the fiber elements. Other details of the Finite Element Method appear in Appendix B.

The boundary conditions can be imposed in terms of displacements or tractions. With reference to Fig. 3, when displacement boundary conditions are used, the nodes on

the left and right end planes are given a specified uniform displacement in the x-direction. These are steadily incremented to increase the axial strain of the composite material. When traction boundary conditions are used, equal nodal force increments are applied to the nodes on the end planes to put the model in tension. The nodal force increments are controlled to ensure monotonically increasing longitudinal strain even after peak load.

At the beginning of each simulation, the number and sizes of the effective medium elements, the number and diameter of the fiber segments, the elastic moduli, and the statistical distribution of the strengths, Q^U , of each fiber segment are defined. Displacement or load control is chosen along with termination conditions such as total axial elongation, percentage drop from a peak load, number of fibers failed, number of fibers which are overstressed, or whether a pullout plane has formed as in Fig. 5. The function of the simulation algorithm, whether using load control or displacement control, is to calculate and produce an approximate strain increment for the composite which causes failure of 1 fiber element during that strain increment (see Fig. 6). Using displacement control, the original length of the composite is used to calculate the next uniform increment of the end plane displacement. Using load control, the macroscopic stiffness of the composite is estimated from the ratio of the current load and the total strain. The product of this stiffness and the estimated strain increment is used as the incremental force which is added to the end planes to achieve the desired strain increment. Since fiber breaks or slip zones may occur at the end planes and the matrix is assumed to carry little axial load, the incremental force is divided evenly amongst the nodes attached to fibers which do not have damage on *either* end plane.

The algorithm uses the new FEM stresses in the fiber elements resulting from the incremental loads to update the slip zones of all broken elements prior to seeking the fiber element which exceeded its value of Q^U by the largest margin. If the fiber element which was stressed near its strength prior to the increment is consumed by the growth of a slip zone, the algorithm begins again by calculating a new strain increment for the next fiber element to break. Otherwise, the algorithm breaks and creates a slip zone for the element which exceeds Q^U by the largest amount. This may lead to the failure of more fiber elements, so the algorithm redistributes the load lost from the broken fiber and its slip zone without applying new incremental forces or displacements to the endplanes. This sequence of load redistributions from fiber breaks continues until no more fiber elements are overstressed. Thereafter, unless a termination condition is met, the algorithm begins again by calculating the next strain increment. During the execution of the algorithm the incremental changes in the damage to the fibers, the load on the composite, and the location of the failure plane are recorded for post process analysis.

5.0 RESULTS

5.1 Isolated Single Fiber Break Simulations

When a fiber breaks in a composite the load carried by the broken fiber is transferred to neighboring fibers through fiber-matrix interfacial shear. This process was analyzed by computing the stress concentration in neighboring fibers resulting from a single isolated fiber break. The results were obtained with a 9x9 array of fibers with 39 elements per fiber, i.e. $N_Y = N_Z = 9$ and $N_X = 39$. The fiber elements were assumed to have uniform elastic properties and dimensions with $E_f = 400$ GPa and a fiber radius of 7 μm .

The matrix was assumed to have no transverse or axial stiffness so the only nonzero

material constant of the effective medium elements is the shear modulus $G = \frac{E_f}{2(1+\nu)}$,

where ν is set 0.22. All fiber elements in the model had an infinite strength except the twentieth element on the central fiber which had no strength. Calculations were carried out for 32 different values of τ up to σ_f^* , where σ_f^* is the fixed final value nominal fiber stress (i.e. in unbroken fibers far from the break). The fiber volume fractions are 30% and 60% which enters via the effective medium element cross-sectional dimensions according to eq. (11) with $a_y = a_z$.

To obtain accurate results, it is necessary for the slip zone on either side of the break to traverse several fiber elements. In the simulations carried out, the fiber element length was chosen to be

$$a_x = \frac{R\sigma_f^*}{19\tau}. \quad (14)$$

This choice means that the slip length on one side of the break is 9.5 times the element length.

Some simulations were carried out with uniform displacements applied to the ends of the model to produce extensional strain. From these calculations, the ratio of the largest stress in the 4 fibers which are nearest neighbors to the broken fiber element to the nominal stress σ_f^* are plotted in Fig. 7, (marked displacement control). The (surprising) result is that the stress concentration falls from a peak value as τ is increased above $0.2\sigma_f^*$. The falling stress concentration is due to the uniform end plane displacements

which constrain the fiber element displacements. If the end planes were free to displace under the same nominal load, the reduction in stiffness caused by the broken and slipping fiber elements would cause same additional displacement beyond the applied uniform displacements. Therefore, as the endplanes are moved closer to the break (i.e. high τ cases), they will limit the redistribution of stress.

When load control is used for the calculations (each end plane node attached to a fiber is loaded with an equal load $\sigma_f^\infty \pi R^2$), the nominally flat endplanes are free to distort when fibers break and debond. Thus the load on the end planes is maintained by a combination of effective medium shear and axial fiber element loads in the distorted configuration determined by the FEM solution. Lacking end plane displacement constraints, the load controlled results are more realistic and there is an asymptotic increase in the stress concentration as τ is increased. The asymptotic limit is 1.12 for the stress concentration which is effectively reached at $\tau = \sigma_f^\infty$. This limit should be compared with the result from Hedgepeth and Van Dyke which is 1.146. They developed an approximate model of load transfer from broken fibers in an infinitely large square array for the case of fibers strongly bonded to the matrix. In their model, direct transfer of load from the broken fiber to its 4 nearest neighbors was assumed, so their stress elevation is probably an upper bound since incorporating more neighbors directly should reduce the average load in the nearest neighbors. In contrast to the HVD case, when the interface debonds and slips near fiber breaks, load transfer becomes a function of the interfacial shear stress and material properties. However, as interfacial shear stress is increased, nearest neighbor load transfer should approach the HVD results, although there will be

discrepancies due to the assumptions in the HVD model and the coarseness of the FEM mesh. Since the finite element method attempts to solve the field equations of elasticity, load transfer includes potentially all fibers in the model and should be lower than that found in HVD.

5.2 Multiple Scattered Fiber Breaks

Consider a failure process in which scattered fiber breaks occur randomly throughout the composite over a range of stresses. The foregoing discussion of isolated fiber breaks implies the highest stress concentrations occur at the lowest loads and diminish continuously as the external load increases. Therefore, the range of stress over which random fiber failures occur determines the local stress concentration in the neighborhood of isolated breaks and the tendency to localize damage. The range of fiber stresses over which failure is expected can be estimated using (3) and global load sharing assumptions found in (Curtin). For a fixed number of fibers, the stress expected when the

first fibers begin to fail is approximately $\sigma_f^\infty \equiv \left(\frac{1}{N_Y N_Z} \right)^{\frac{1}{m}} \sigma_R$, where $\sigma_R = \left(\frac{\sigma_0^m L_0 \tau}{R} \right)^{\frac{1}{m+1}}$.

The stress in unbroken fibers near the ultimate fiber stress is $\sigma_f^U = \left(\frac{2}{2+m} \right)^{\frac{1}{m+1}} \sigma_R$. For

fixed τ , the ratio of τ to these two stresses establishes a range of dimensionless interfacial shear stress for isolated fiber failure stress concentrations in the composite. If the range of this ratio is such that all of the stress concentrations occur in or near the 12% asymptotic region, a fixed stress concentration ratio can be selected for fiber failures and fast algorithms developed (Zhou & Curtin, Phoenix, etc) to analyze the statistics of composite

failure. However, as illustrated in Fig. 7 with identical fiber/matrix moduli, the asymptotic region is not reached until the ratio τ/σ is near 1. Fig. 8 plots the highest value of the dimensionless interfacial shear achieved when m is varied from 2 to 10 and τ is varied from 20 MPa to 160 MPa for material properties similar to those found in CMCs. With 225 fibers in the bundle, the dimensionless interfacial shear stress is always less than about 0.2 and is near the limit where load controlled simulations and displacement based simulations begin to diverge. Thus constrained, the stress concentrations are nearly linear functions of applied stress and are grossly simplified in models which approximate them as constants.

5.3 End Plane Displacement Control Bundle Simulations

Results from Monte Carlo simulations with square arrays of fibers are reported for the range of Weibull moduli and interfacial shear stresses discussed above. The results were obtained with 11x11 arrays and 15x15 arrays of fibers with 12 elements representing each fiber. The fiber and matrix properties remain the same as those identified in the isolated fiber break case, except all fiber element strengths are randomly drawn from the Weibull distribution. The matrix is assumed to fail at zero strain and retains no axial load carrying capacity ($\epsilon = r = 0$). Fiber elements fail randomly with increasing strain, so increments (5%) in the accumulated fiber damage, the ratio of the sum of the number of broken and slipping fiber elements to the total number of fiber elements, controls recalculation of the model stiffness. The transverse effective medium element dimensions are fixed by taking $f = 0.3$ in equation 11 with $a_y = a_z$. Paralleling the method used with isolated fiber breaks, the length of model elements, a_x , is equated to 1/9 of the expected

maximum load transfer length $R\sigma_f^u/2\tau$.

In the following sections the execution of the model algorithms is illustrated using two simulations with $m = 2$ and $m = 10$. The evolution of damage in the composite is examined in terms of local stress concentrations, the formation of the ultimate fracture plane, and the resulting pullout distributions at the ultimate fracture plane. Distinctions are drawn between the two extreme cases when $m = 2$ and $m = 10$. The section concludes with a presentation of statistical averages from the simulations which illustrate the behavior of fiber bundles.

5.3.1 Low ($m = 2$) Weibull Modulus Simulation

The simulation output depicted in Fig. 9a & 9b are typical of simulations with the Weibull modulus of the fibers set to $m = 2$ and the remaining fiber properties are σ_0 1500 MPa, $L_0 = 25$ mm, and $R = 7\mu\text{m}$. The plotted curve in Fig. 9a is the computed composite stress from the model algorithm as it successively breaks the weakest fiber element segments by increasing composite strain (recall matrix cracking occurs at zero strain in the model). The curves in Fig. 9b are snapshots of the fiber damage in the composite taken at strains denoted by vertical lines in Fig. 9a. Each curve is the ratio of broken fiber elements in each plane to the number of unbroken elements in each plane when the indicated percentage of fibers have experienced at least one break. From the equations above the occurrence of the first fiber break at 0.14% strain is near the expected first fiber failure strain of 0.12% when $\tau = 20$ MPa. From the first break until 40% of the fibers have at least one failure, the stress strain curve is nearly linear. Likewise, in the same range of fiber failures, the underlying fraction of broken fiber elements depicted in Fig. 9b is well

distributed in the composite. The damage increases in small increments to an average of 4% on each plane and is bounded by extremes of 3% and 6%. On average each broken fiber is surrounded by 17 to 33 unbroken fibers. Thus, most fiber breaks are insulated from each other and produce decreasing stress concentrations in neighboring fibers governed by the ratio of τ/σ . Beyond the 40% damage level, fiber failures begin to localize into bands concentrated near the ends of the model and the first signs of significant nonlinearity in the stress-strain curve appear. As the peak load is approached at 64% failed fibers, the gap between the extremes of the damage distribution has doubled from 3% to 6%. However, all damaged planes still have fewer than 10% failed fiber elements. The post peak behavior is governed by a relatively undamaged stiff interior region surrounded by increasingly damaged and compliant regions on each end of the model where the ultimate fracture plane forms. The increased damage in these compliant regions tends to result in multiple overstressed fibers following individual fiber failures (usually less than 2 neighbors per failure). Fiber failure and the generation of overstressed fibers in the simulations is coincident with the large, rapid drops in the composite stress near 1.5% and 2% strain.

The uniform distribution of breaks which develops during the quasi-linear portion of the stress-strain curve creates long fiber fragments which pullout (see Fig. 10) and transmit large loads between the fractured sections of the model. Damage tolerance is enhanced by the prolonged, gradual reduction of the load from the peak load. In fact, simple estimates of the work of fracture indicate the contribution to the work of fracture between the peak stress and the last break at 2.2% strain is equivalent to the contribution to the work of fracture required to reach the peak load. The contribution to the work of

fracture due to pullout is governed by the average distance of breaks from the ultimate fracture plane. The average is $2.04 a_x$, which compares well with $2.25 a_x$ expected from a completely Poisson fiber failure process over a 9 element load transfer length. The combination of low stress concentrations as governed by τ/σ and the broad distribution in strengths implied by $m = 2$ are responsible for this behavior.

5.3.2 High ($m = 10$) Weibull Modulus Simulation

Figures 11a and 11b are typical of simulations with the Weibull modulus set to $m = 10$. With $\tau = 20$ MPa, the strain to the actual first fiber failure and the predicted failure strain of 0.33% are nearly identical. However, the quasi-linear portion of the stress-strain curve now extends to the peak stress with far fewer fiber failures (14%) prior to the peak load. The fiber breaks are distributed throughout the model prior to the peak with the fraction of fiber elements failed in a plane ranging from 0% to 4%. These breaks are more isolated than the $m = 2$ case, so the decreasing stress concentrations in neighbors is a function of τ/σ . Beyond the peak, the appearance of overstressed fibers with each additional break is immediate. Initially 3 to 4 neighbors become overstressed, but the wave of ensuing failures eventually causes most of the remaining fibers (~ 30 at most) to become overstressed in a localized band near the ultimate fracture plane. This causes nearly 80% of the fiber breaks to occur within ± 1 element of the ultimate fracture plane. Likewise, the abundance of overstressed fibers causes the large drop from the peak and minimizes the contribution to the work of fracture from the peak to the last fiber break. The work of fracture accorded to pullout is reduced since the average pullout length of $0.86 a_x$ (see Fig. 12) is significantly less than the $2.25 a_x$ expected from a Poisson failure

process. Even with low stress concentrations up to the peak, the uniformity of the fiber strength distribution implied by $m = 10$ causes significant localization after the peak load is reached.

5.4 Monte Carlo Simulations and Statistical Strength/Pullout Behavior

Monte Carlo simulations for 11x11 and 15x15 square arrays of fibers are summarized for the cases where τ is varied from 20 to 160 MPa in increments of 20 MPa and even Weibull moduli (m) between 2 and 10 are selected. Each simulation consists of a model similar to those described in detail above except each simulation is assigned a new fiber element flaw population. For comparison with the work of others, the results are normalized by $f = 0.3$, σ_R , and $\epsilon_R = \sigma_R/E_f$.

For each value of m , the ultimate composite stress σ_c^U from the 11x11 and 15x15 Monte Carlo simulations is plotted against τ in the normalized form, $\sigma_c^U / f\sigma_R$. In Fig. 13 and Fig 14, the normalized stresses lie in distinct bands. However, none of the plots implies a correlation between τ and the normalized ultimate stress exists. This seems counter-intuitive, until it is recalled that stress concentrations from breaks are governed by τ/σ and are low and decreasing throughout the simulations. It is possible that the effect of increasing τ and increasing the number of fibers in the models will lead to a correlation between τ and ultimate stress. However, since the range of τ/σ must be near 0.5 before high stress concentrations (~ 1.1) occur, the number of fibers for this to occur in the model is on the order of $(\sigma_R/2\tau)^m$. This is 1000 fibers for $m = 2$ and $\tau = 160$ MPa and increases if m is increased or τ is decreased. The number of fibers, when combined with

the length of the model, does not imply a large composite volume. However, the composite volume is much larger than that heretofore considered for stress concentration effects in models where the nature of the nonlinear variation in the stress concentration is ignored. As shown in a separate publication, the effect of increasing τ becomes more apparent as the ratio of the fiber stiffness to the matrix stiffness increases.

Since the results appear to be independent of τ , the simulation data for each bundle size and value of m are combined for the purposes of fitting a bivariate normal distribution. The normalized ultimate stress $\sigma_c^U / f\sigma_R$ and the normalized strain to ultimate stress $\epsilon_c^U / \epsilon_R$ are assumed to have normal distributions so that their bivariate distribution can be characterised by the variate means, standard deviations, and correlation coefficient. These five parameters are estimated using the combined data and are tabulated using standard statistical methods (Hogg and Craig, Press, et. al.). The results are shown in Fig. 15 and Fig. 16 for the 11x11 and 15x15 models respectively. In both cases, the normalized strength increases monotonically with m , but the normalized strain to ultimate stress decreases with increasing m up to about $m = 6$. Beyond $m = 6$, the average normalized strain increases slightly and is on the order of 0.83 to 0.85 for both 11x11 and 15x15 data. The correlation coefficients are between 0.4 and 0.9 for all data sets, but don't follow a well defined curve. Nevertheless, the correlation coefficients tend to rise as the Weibull modulus falls. Likewise, the standard deviations shown in Figures 15 and 16, whether for stress or strain, are generally decreasing functions of m .

The pullout distributions have been converted to equivalent composite stress by equating the fiber pullout length with the frictional tractions exerted on the fibers by τ .

The normalized composite stress and normalized strain at which pullout controls the simulation stress are plotted in Figures 17 and 18. Unlike with the normalized ultimate stress data, increasing the Weibull modulus decreased the magnitude of the normalized pullout stress and usually decreased the normalized strain to the commencement of pullout. The variance of both the pullout stress and the pullout strain decreases with increasing m . The decrease in the magnitude of the pullout stress as a function of m reduces the average pullout stress from approximately 50% of the normalized ultimate stress at $m = 2$ to 25% of the ultimate stress at $m = 10$.

6.0 Discussion

The data from the simulations are in general agreement with analytical results postulated for fiber bundles under global load sharing conditions. Curtin's analytical result for the stress and strain to failure under global load sharing conditions provides a convenient comparison which is plotted on Figures 15a and 16a. For $m \geq 6$, the simulation mean and the ellipse formed by the standard deviations encloses Curtin's estimate, although Curtin's estimate of the ultimate stress understates the bundle strength. For $m \leq 4$, Curtin's estimates are always outside the ellipse and under estimate the mean bundle strength by 5% to 8% and the bundle strain to peak by 5% to 15%. The increase in the composite stress above those expected from Curtin's approximation result from the shadowing of defects by the creation of slip zones and the overlap of slip zones which become significant at low Weibull moduli (Neumeister, Phoenix and Raj) within the model.

In contrast, all of the normalized pullout stresses from the simulations are below a simple estimate of the the pullout stress which is depicted on Figure 17 as the dashed line. For fiber failures controlled by a Poisson process prior to pullout, the ultimate fracture plane should have equal numbers of fiber failures on each side of the plane. The dashed line is the pullout stress expected when the average distance of a break from the ultimate fracture plane is 1/4 of the ultimate load transfer length δ_U . Even though all of the simulations begin with uniform damage distributed throughout the composite (see Fig. 9b and 11b), only the wide distribution in strength implied by $m = 2$ is able to preclude significant localization of the fiber breaks near the fracture plane in the 11x11 simulations. In the 15x15 simulations, the size of the model has increased the average pullout stress at almost all values of m , with both $m = 2$ and $m = 4$ now $\geq 73\%$ of the Poisson process value. The increase in pullout stress with increasing size should continue until more of the isolated fiber failures begin to occur in the asymptotic stress concentration regime with 1000 or more fibers (note however this critical size diminishes when the elastic moduli ratios differ from 1).

Contributions to the work of fracture can be determined from the stress-strain curves resulting from combining ultimate stress and pullout data from the simulations. The work of fracture exhibits a strong dependence on m and is illustrated in Fig. 19, where the shape of prospective average normalized stress strain curves is plotted through the 11x11 data. Increasing the model size shifts the peak down and the pullout stress up towards the Poisson process limit, albeit slowly.

7.0 Conclusion

A model for the behavior of unidirectional fiber reinforced ceramic matrix materials with weak interfaces was presented as an assemblage of coarse 3-D effective medium finite elements and 1-D spring (fiber) elements. The model is used to simulate the mechanical response of the composite material. It was found that high ratios of τ/σ must be present in the simulations to create high stress concentrations and local load sharing conditions. When the ratio of the elastic moduli of the fiber and matrix are identical, for the range of Weibull moduli and the range of interfacial shear stresses simulated in this work, τ/σ was restricted to values less than 0.2 by the statistics of the flaw population. Under these conditions, the simulations were governed by isolated fiber breaks which were surrounded by 15 or more unbroken neighbors on average. The stress concentrations of the isolated fiber breaks were constrained and decreased continuously under load, so that global load sharing conditions prevailed for bundles with 121 or 225 fibers. To reach the asymptotic plateau of the stress concentration profiles, the volume of fiber bundles required in simulations was determined to be on the order of at least 1000 fibers within the load transfer length. Even though the stress concentrations were low and the breaks prior to the peak stress were well distributed in the model simulations, the pullout stresses were below those for a Poisson process, indicating localization will dominate the post peak pullout except for large bundles or composites. This also implies work of fracture is lower for bundles than for large composites and can lead to brittle behavior for bundles of tows in woven composites or if non-interacting bundles form in composites.

Appendix A

Elastic Properties of Model Elements without Damage

Take a single effective medium element with dimensions a_x , a_y , and a_z . Since fiber elements are collocated along the edges of effective medium elements, embed a single fiber element of length a_x in the center of the effective medium element in the x-axis direction. Assign elastic properties to the effective medium and fiber element which replicate the known elastic properties of an equivalent volume of the composite. Defining the symmetric terms $s_{ij} = \nu_{ij}/E_i = s_{ji} = \nu_{ji}/E_j$, the elastic properties of an effective medium element are represented by the compliance matrices

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} = \begin{bmatrix} 1/E_x & -s_{xy} & -s_{xz} \\ -s_{xy} & 1/E_y & -s_{yz} \\ -s_{xz} & -s_{yz} & 1/E_z \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix}$$

and

(A1)

$$\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} 1/G_{xy} & 0 & 0 \\ 0 & 1/G_{yz} & 0 \\ 0 & 0 & 1/G_{zx} \end{bmatrix} \begin{Bmatrix} \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}.$$

The elastic properties of the fiber element are $Q = k \Delta l / l_0$, where Q is the force in the fiber element, Δl the change in length of the element, and l_0 its original length. The elastic properties of the composite are represented by the compliance matrices

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -s_{12} & -s_{13} \\ -s_{12} & 1/E_2 & -s_{23} \\ -s_{13} & -s_{23} & 1/E_3 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix}$$

and

$$\begin{Bmatrix} \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \begin{bmatrix} 1/G_{12} & 0 & 0 \\ 0 & 1/G_{23} & 0 \\ 0 & 0 & 1/G_{31} \end{bmatrix} \begin{Bmatrix} \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{Bmatrix}.$$

The behavior of the composite and effective medium is assumed to be transversely isotropic and carry over directly in shear so that

$$G_{xy} = G_{zx} = G_{12} = G_{31},$$

$$G_{yz} = G_{23},$$

$$E_y = E_z,$$

$$s_{xy} = s_{xz},$$

$$E_2 = E_3,$$

$$s_{12} = s_{13},$$

leaving only five undetermined constants, E_x , E_y , s_{xy} , and s_{yz} , and k which are functions of the known composite properties.

Applying identical normal stresses separately in the x direction and then in the y direction on the composite volume and model elements leads to eight equations relating the stresses and strains. Assuming the behavior of the composite and the model elements are the same reduces the number of equations to four in terms of the known composite properties and the remaining unknown constants. If it is further assumed that $k = E_t A_t$,

where E_f is the fiber axial Young's modulus and A_f the cross-sectional area of a fiber, the solution of the four equations yields

$$E_x = E_1 - fE_f,$$

$$E_y = E_2 \left(\frac{E_x}{E_x + fs_{12}^2 E_1 E_2 E_f} \right),$$

$$s_{xy} = s_{12} \frac{E_1}{E_x}, \text{ and}$$

$$s_{yz} = s_{23} - fs_{12}^2 E_1 E_f / E_x.$$

The stiffness matrix $[C]$ of Appendix A results from inversion of the effective medium compliance matrices.

Elastic Properties of Damaged Effective Medium Elements

When the effective medium element simulates matrix damage, it is assumed that changes in the axial strain are no longer purely elastic and that increments in strain do not produce increments in axial stress. This constraint is combined with (A1) to modify the stiffness terms so that axial and transverse behavior is consistent with a loss of load carrying capacity in the x direction. Thus the change in axial elastic strains is given by

$$\Delta \epsilon_x = -s_{xy} \Delta \sigma_y - s_{yz} \Delta \sigma_z \quad (A2)$$

and changes in the transverse strains are related to changes in transverse stresses by

$$\begin{Bmatrix} \Delta \epsilon_y \\ \Delta \epsilon_z \end{Bmatrix} = \begin{bmatrix} 1/E_y & -s_{yz} \\ -s_{yz} & 1/E_z \end{bmatrix} \begin{Bmatrix} \Delta \sigma_y \\ \Delta \sigma_z \end{Bmatrix}. \quad (A3)$$

Inversion of eq. (A3) and using $E_y = E_z$ the stiffness matrix [C] is modified to agree with

$$\begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_y \\ \Delta\sigma_z \end{Bmatrix} = \frac{1}{1-\nu_{yz}^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & E_y & \nu_{yz} \\ 0 & \nu_{yz} & E_y \end{bmatrix} \begin{Bmatrix} \Delta\epsilon_x \\ \Delta\epsilon_y \\ \Delta\epsilon_z \end{Bmatrix} \quad (\text{A4})$$

When only axial displacements are allowed in a simulation, this modification implies that the choices for Poisson's ratios and transverse Young's modulus E_y no longer affect the finite element solution after the effective medium element has failed.

Appendix B

Finite Element Method

The code is an updated Lagrangian finite deformation formulation first described in a paper by McMeeking and Rice which transforms a conventional small strain finite element program into one capable of arbitrary finite deformations. The program is based on a mesh which represents the current configuration of the composite and a corresponding incremental virtual work statement,

$$\int_{V^R} \dot{\mathcal{E}}_j \delta \left(\frac{\partial v_j}{\partial X_i} \right) dV^R + \sum \mathcal{Q}_j \delta v_j = \int_{S^R} \dot{T}_i \delta v_i dS^R. \quad (B1)$$

The regions V^R and S^R are taken as the reference configuration of the model and X is the position of a material point in the reference configuration. $\mathbf{t}$ is the nominal stress tensor of the effective medium elements which produces the force per unit reference surface area $\mathbf{T} = \mathbf{n} \cdot \mathbf{t}$ acting on the surface dS with unit outward normal $\mathbf{n}$. Q_j is a component of the force acting at the end of a fiber element. The superposed dot on a symbol indicates rate and δv is a virtual velocity compatible with the prescribed displacements on $S^R - S_T^R$. In (McMeeking and Rice) the current configuration is selected as the reference configuration so that the time rate of change of the nominal stress $\mathbf{t}$ is related to the current Cauchy stress $\boldsymbol{\sigma}$ and the Jaumann rate of Kirchhoff stress $\boldsymbol{\tau}^*$ through

$$\dot{\mathcal{E}}_{ij} = \tau_{ij}^* - \sigma_{kj} D_{ik} - \sigma_{ik} D_{kj} + \sigma_{ik} L_{jk}, \quad (B2)$$

where $\boldsymbol{\tau} = \mathbf{J}\boldsymbol{\sigma}$ relates the Cauchy stress $\boldsymbol{\sigma}$ to the Kirchhoff stress $\boldsymbol{\tau}$. $\mathbf{L}$ and the rate of deformation tensor $\mathbf{D}$ are defined in the current configuration by $L_{ij} = \partial v_j / \partial x_i = v_{j,i}$ and

$D_{ij} = \frac{1}{2}(L_{ij} + L_{ji})$ for material points located at the position vector $\mathbf{x}$. $J = dV/dV^R$ (the Jacobian) is the ratio of the current volume to the reference volume (i.e. $J = 1$ initially).

Eq. (B1) and eq. (B2) are the basis for the finite element method when $\mathbf{v}$, $\mathbf{L}$, and $\mathbf{D}$ are related to the vector of nodal rates $\{\psi\}$ through matrices $[\mathbf{N}]$, $[\mathbf{B1}]$, and $[\mathbf{B}]$

$$\begin{aligned} v_i &= N_{in} \psi_n, \\ L_{(ij)} &= B1_{(ij)n} \psi_n \\ B1_{(ij)n} &= N_{in,j} \\ D_{(ij)} &= B_{(ij)n} \psi_n \\ B_{(ij)n} &= \frac{1}{2}(B1_{(ij)n} + B1_{(ji)n}). \end{aligned} \tag{B3}$$

Juxtaposed subscripts within parentheses, e.g. (ij) , are treated as an index which runs from 1 to 9 as i and j vary from 1 to 3 so that $[\mathbf{B1}]$ and $[\mathbf{B}]$ become 2 dimensional arrays and $\mathbf{L}$, $\mathbf{D}$, $\boldsymbol{\tau}$, and $\boldsymbol{\sigma}$ become vectors. The complete finite element equations are

$$([K]_E^{EM} + [K]_E^Q + [K]_s^{EM} + [K]_s^{Qr})\{\psi\} = \{F\}, \tag{B4}$$

where the effective medium stiffness $[K]_E^{EM}$ and fiber stiffness $[K]_E^Q$ are those commonly used in small strain elasticity programs

$$[K]_E^{EM} = \int_{V^R} [B]^T [C] [B] dV^R,$$

$$[K]_E^Q = \frac{k}{l_0} \begin{bmatrix} [nn] & -[nn] \\ -[nn] & [nn] \end{bmatrix} \quad (B5)$$

$$[nn]_{ij} = n_i n_j \text{ (} n_i \text{ is a direction cosine of the fiber element)}$$

The (m,n) entry in the initial effective medium stress stiffness $[K]_s^{EM}$ and the initial fiber stress stiffness $[K]_s^Q$ matrices are defined as

$$[K]_s^{EM} = \int_{V^R} B l_{(ji)m} \sigma_{(ik)} B l_{(jk)n} - 2 B_{(ij)m} \sigma_{(jk)} B_{(ki)n} dV^R$$

$$[K]_s^Q = \frac{Q}{l} \begin{bmatrix} [I - nn] & -[I - nn] \\ -[I - nn] & [I - nn] \end{bmatrix} \quad (B6)$$

$$[I - nn]_{ij} = \delta_{ij} - [nn]_{ij}.$$

The load rate vector is defined as

$$\{\dot{P}\} = \int_{S^R} [N]^T \dot{\tau} dS^R. \quad (B7)$$

Specification of eq. (B5) and eq. (B6) is completed by the constitutive relations for the effective medium and fiber elements which relate the material stiffness $[C]$, τ^* , the fiber element stiffness k , and the fiber load Q to material deformation,

$$\{\tau^*\} = [C]\{D\},$$

$$Q = k \left(\frac{l - l_0}{l_0} \right), \quad (B8)$$

where l is the current fiber element length, and l_0 is the original length of the fiber element.

Using the incremental finite element solution with eq. (B2) and eq. (B3) provides the rate of change in the nominal stress $\mathbf{t}$. This is used with the relationship between Kirchhoff stress $\boldsymbol{\tau}$ and nominal stress $\mathbf{t}$, $\tau_{ij} = \partial x_i / \partial X_k t_{kj}$, to compute the rate of change in the Cauchy stress $\bar{\boldsymbol{\sigma}}$. $\mathbf{Q}$ is updated using eq. (B8).

Appendix C

Generating Random Fiber Element Strengths

The strength of a fiber element is selected based on the fundamental transformation law of probabilities. Given a random variable x which is uniformly distributed between 0 and 1 and a fiber strength $Q^U(\sigma)$ with cumulative probability of failure distribution $P_f(\sigma)$, the distribution of $Q^U(x)$ is determined by inverting

$$x = P_f(\sigma),$$

where

$$\sigma = Q/\pi R^2$$

is the average stress in the fiber. For fiber elements of length a_x , the 2 parameter Weibull distribution $P_f(\sigma)$ with Weibull modulus m , reference stress σ_0 , and reference length L_0 is defined as

$$P_f(\sigma) = 1 - \exp\left[-\frac{a_x}{L_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right].$$

Thus, using a uniform random number generator for x_i which is contained in the open interval (0,1) produces a sequence of random flaw strengths

$$Q(x_i) = \pi R^2 \left[\frac{L_0 \sigma_0^m}{a_x} \ln\left(\frac{1}{1-x_i}\right) \right]^{\frac{1}{m}}$$

which are distributed according to the Weibull distribution.

Appendix D

Equivalent Traction Forces

Referring to Fig. D1, a distributed load per unit length (line load) is applied to a portion of the element boundary between the nodes labeled 1 and 2 by broken fiber segments which slip past the neighboring effective medium elements. In the figure, one such line load is considered which partially covers the element boundary. The equivalent nodal forces which are contributed based on eq. A8 from broken slipping fibers are required. Assuming linear interpolation of displacements between nodes 1 and 2, the nodal forces F_1 and F_2 applied in the direction shown and which are consistent with the virtual work of the line load are

$$F_1 = ql(1/2 - \zeta_b + \zeta_b^2/2),$$

$$F_2 = ql(1/2 - \zeta_b^2/2),$$

where the magnitude of the line load is $q = 2\pi R\tau$, l is the current length of the element, $\zeta = (x - x_1)/l$ is normalized position along the element with respect to node 1, and the normalized location of the fiber break is $\zeta_b = (x_b - x_1)/l$. For nodes in the slip zone not adjacent to a fiber segment containing a break, $\zeta_b = 0$ and $F_1 = F_2 = ql/2$. For effective medium elements not near the final failure plane $l \equiv a_x$, however for elements near the final failure plane, $l \neq a_x$, so the equivalent force calculations use l instead of a_x in computing the equivalent nodal forces F_1 and F_2 .

Ceramic Matrix Composite Load Deflection Curve

Broken Fiber in Composite

Unidirectional Composite and FEM Mesh

Fiber Element Failure and Load Transfer

Discontinuous solid curve represents approximation to $Q(x)$ over elements 1-4.

Element 5 remains elastic since it isn't fully contained in L

Equilibrium is ensured if fictitious nonlinear springs causing discontinuities in approximation are assigned forces equivalent to interface shear tractions.

Fiber with single break

Removed fiber elements

Equivalent remaining tractions and forces in FEM model

Simulation Algorithm

DOFs at each node

Typical Element in Model,
Spring elements not shown

Displacement Increment

Load Increment

Hedgepeth & Van Dyke Simulation

$G_f / G_m = 1$

Range of Dimensionless Shear values for first fiber break

Dimensionless interfacial shear stress as a function of Weibull modulus m , interfacial shear stress τ , and stress in intact fiber of composite when first fiber is expected to break ($N_f = 225$).

Stress/Strain Curve for an 11x11 Array of Fibers

$R = 7 \mu\text{m}$, $E_f = 400 \text{ GPa}$, $m = 2$, $\sigma_0 = 1500 \text{ MPa}$, $L_0 = 25 \text{ mm}$
 $\tau = 20 \text{ MPa}$

Fiber Break Damage Histogram for 11x11 Array of Fibers

$R = 7 \mu\text{m}$, $E_f = 400 \text{ GPa}$, $m = 2$, $\sigma_0 = 1500 \text{ MPa}$, $L_0 = 25 \text{ mm}$
 $\tau = 20 \text{ MPa}$

During an increment of load, each enclosed polygon represents the incremental increase in the fraction of breaks over the length of the composite model.

Pullout Distribution for an 11x11 Array of Fibers

Stress/Strain Curve for an 11x11 Array of Fibers

$R = 7 \mu\text{m}$, $E_f = 400 \text{ GPa}$, $m = 10$, $\sigma_0 = 1500 \text{ MPa}$, $L_0 = 25 \text{ mm}$
 $\tau = 20 \text{ MPa}$

Fiber Break Damage Histogram for 11x11 Array of Fibers

$R = 7 \mu\text{m}$, $E_f = 400 \text{ GPa}$, $m = 10$, $\sigma_0 = 1500 \text{ MPa}$, $L_0 = 25 \text{ mm}$
 $\tau = 20 \text{ MPa}$

During an increment of load, each enclosed polygon represents the incremental increase in the fraction of breaks over the length of the composite model.

Pullout Distribution for an 11x11 Array of fibers

Correlation of Ultimate Stress & τ 11 by 11

Correlation of Ultimate Stress & τ 15 by 15

Comparison of Simulations and Analytical Models 11 by 11, Gf/Gm = 1

Correlation Coefficient versus m 11x11

Comparison of Simulations and Analytical Models 15 by 15

Correlation Coefficient versus m 15x15

Pullout Stress in Simulations 11 by 11, $G_f/G_m = 1$

Pullout Stress in Simulations 15 by 15, $G_f/G_m = 1$

Prospective Normalized Stress-Strain Curves

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