Approximation of Attractors and Applications

Final Report

for Period 5/1/92 to 4/30/95

ROGER TEMAM

INDIANA UNIVERSITY

Institute for Scientific Computing

& Applied Mathematics

618 East Third Street #108

Bloomington, Indiana 47405

October 23, 1995

Prepared for:

THE U.S. DEPARTMENT OF ENERGY

AGREEMENT NO. DE-FG02-92ER25120

MASTER

RECEIVED
MAR 18 1996
OSTI

Approximation of Attractors and Applications

DOE Grant No. DE-FG02-92ER25120

Final Project Report

TABLE OF CONTENTS

1. Project Summary	2
2. List of publications acknowledging the grant	3
3. Description of the results	5
3.1. Navier-Stokes equations and turbulence	5
3.2. Approximation of attractors	6
3.3. Climatology problems	8

1. PROJECT SUMMARY

From the Project Proposal

Our aim in this project is to explore what can be learned about turbulence from the theoretical and computational viewpoints, using the dynamical systems approach to turbulence. Some part of the effort will be also devoted towards global climate problems.

According to Dynamical Systems theory the permanent regime for a turbulent flow is mathematically represented by an attractor that is a compact set in the phase space. It is hoped that the study of this attractor can improve our understanding of the physics of turbulent flows. It is also likely that the study of the approximation of this attractor can lead to efficient new algorithms for the numerical simulation of turbulence and some significant results have been already obtained in this direction.

The concept of Approximate Inertial Manifolds (AIM) yields simple smooth manifolds that approximate the attractor. By projecting the fluid mechanics equations (or similar dissipative evolution equations) on such manifolds we obtain a new type of numerical algorithms called the Nonlinear Galerkin methods. Several types of Nonlinear Galerkin methods can be derived depending on the AIM which is used and on the discretization method used for the equation (spectral methods, finite elements, finite differences, wavelets). The utilization of the simplest AIM with a pseudo spectral discretization has produced a very stable and efficient algorithm which allows reduction of computing time of about 50%. We intend to develop the study of other forms of the Nonlinear Galerkin method which seem to be very promising. In the case of finite difference discretizations this is done by using the newly introduced concept of incremental unknowns (IU method).

On the theoretical side we intend to study more particularly the equations of meteorology and oceanography and to study the connection between inertial manifolds and the concept of slow manifolds of meteorology which seem to be a particular case of inertial manifold.

2. LIST OF PUBLICATIONS ACKNOWLEDGING THE GRANT

- [CT] M. Chen and R. Temam, Nonlinear Galerkin Method with multilevel incremental unknowns, in *Contributions in Numerical Mathematics*, R.P. Agarwal Ed., World Scientific Series in Applicable Analysis, Singapore, 1993.
- [DDT] A. Debussche, T. Dubois and R. Temam, The Nonlinear Galerkin Method: A multiscale method applied to the simulation of homogeneous turbulent flows, ICASE Preprint series, 1993, and *Theoretical and Computational Fluid Dynamics*, 7, 4, 1995, 279-315.
- [DJT] T. Dubois, F. Jauberteau and R. Temam, Solution of the incompressible Navier-Stokes equations by the Nonlinear Galerkin Method, *J. Scientif. Comp.*, 8, No. 2, 1993, 213-240.
- [DT] T. Dubois and R. Temam, Separation of scales in turbulence using the nonlinear Galerkin method, to appear in *Advances in Computational Fluid Dynamics*, W.G. Habashi and M. Hafez eds, Gordon and Breach 1993.
- [DT] T. Dubois and R. Temam, The nonlinear Galerkin method applied to the simulation of turbulence in a channel flow, in Proceeding, 13th ICNMF, Rome 1992, M. Napolitano & F. Sabetta Springer-Verlag, Lecture Notes in Physics 1993.
- [FMT] C. Foias, O. Manley, R. Temam, Iterated Approximate Inertial Manifolds for Navier-Stokes Equations in 2D, *J. Math. Anal. Appl.*, 178, No. 2, 1993, 567-583.
- [FT] C. Foias and R. Temam, Approximation of attractors by algebraic or analytic sets, *SIAM J. Math. Anal.*, 25, 5 1994, p 1269-1302.
- [GT] D. Gottlieb and R. Temam, Implementation of the Nonlinear Galerkin Method with Pseudo-spectral (collocation) discretizations, *Applied Num. Math.*, 12, 1993, 119-134.
- [JTX1] M.S. Jolly, R. Temam and C. Xiong, Convergence of a chaotic attractor with increased spatial resolution of the Ginzburg-Landau Equation, *Chaos, Solitons and Fractals*, 5, 1995, 1833-1845.
- [JTX2] M.S. Jolly, R. Temam and C. Xiong, An application of approximate inertial manifolds to a weakly damped nonlinear Schrödinger equation, *Num. Func. Anal. & Opt.*, 7, 1995, 923-937.
- [JX] M.S. Jolly and C. Xiong, On computing the long-time solutions of the 2-D Navier-Stokes equations, *Theoretical and Computational Fluid Dynamics*, 7, 1995, 261-278.
- [LTW1] J.L. Lions, R. Temam and S. Wang, On the equations of the large-scale ocean, *Nonlinearity*, 5, 1992, 1007-1053.
- [LTW2] J. L. Lions, R. Temam and S. Wang, Models of the coupled atmosphere and ocean, in Computational Mechanics Advances, J.T. Oden ed., 1, 1993, 5-119.

- [LTW3] J. L. Lions, R. Temam and S. Wang, Mathematical study of the coupled models of atmosphere and ocean (CAO III), *J. Math. Pures Appl.*, **74**, 1995, 105-163.
- [ST] J. Shen and R. Temam, Nonlinear Galerkin Method using Chebyshev and Legendre Polynomials I. The One Dimensional Case, *SIAM J. Num. Anal.*, Vol. **32**, 1, 1995.

3. DESCRIPTION OF THE RESULTS

Three directions of research have been explored as part of this project:

3.1. Navier–Stokes Equations and Turbulence.

The understanding of the equations of fluid mechanics and turbulence as well as the development of efficient numerical codes for the solution of these equations are challenging problems of considerable importance in particular in industry or for the studies in meteorology or global climate change. We have conducted several projects in this direction.

(a) Parallel Computing

We have written a number of performant CFD codes for *parallel computers*, namely the CM5 at Minneapolis (M. Jolly and C. Xiong) and the Crays at PSC (T. Dubois, F. Jauberteau and R. Temam).

(b) Algorithm Development: Multi-Resolution Methods

Whatever the performance of the next generation of parallel computers, they still will not be able to resolve all the scales of fully turbulent flows. A better understanding of the physics of turbulence and the development of effective new algorithms are necessary.

The algorithms that we have developed as part of this project are:

- The nonlinear Galerkin method for spectral methods and finite elements discretizations
- The Incremental Unknowns method, for finite difference discretizations.

These are *multiresolution methods* which are based on the decomposition of the unknown u into two or more arrays of different sizes:

$$\begin{cases} u = y + z, \\ \text{or} \\ u = y + z_1 + \cdots + z_r, \end{cases} \quad (1)$$

the crucial point being to treat differently arrays of different orders of magnitudes.

Important results have been obtained in this direction. In particular we have implemented the nonlinear Galerkin method for three-dimensional turbulent flows [DT], [DJT], [DDT] and we have reached more than 80% gain in computing time, i.e. we have shown that we can run codes five times faster than the best existing codes (in 3D-spectral). These performances are due to a better understanding and substantial improvements to the nonlinear Galerkin method.

3.2. Approximation of Attractors.

On the theoretical side, important results have been obtained in [FT]. This article has been recently reviewed as a **Featured Review** in the Mathematical Reviews.

On the computational side, the long range goal is the construction of an effective algorithm which should allow the integration of a large class of dissipative PDE's, including the 2D Navier-Stokes equations, with a small number of parameters, that is, by reducing the equations to differential algebraic equations of the form

$$\frac{d}{dt} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = F(y, z), G(y, z) = 0, \quad (2)$$

where n is a small number and F, G are polynomials in y and z . At the present stage we are developing general procedures for finding adequate polynomials F, G . The basic idea is the following: If G is very small on the *global* attractor of the equation and if the differential of G is invertible in directions “transversal” to the attractor, then G will have zeros near the attractor wherever those two properties hold on the attractor. In this way the integration of (2) will automatically take place in the vicinity of the attractor of the equation, although that attractor is unknown. C. Foias and R. Temam introduced in [FT] a general method for constructing three types of such polynomials (denoted I_N, J_N and K_N) for any equation with a structure similar to the Navier-Stokes equation. Before applying that method to

the Navier-Stokes equations we decided to do very thorough numerical tests on two much simpler cases, namely the Lorenz system and the Kuramoto-Sivashinsky equation.

In the case of the Lorenz system the gap in the eigenvalues is relatively small, forcing us to consider very large degree polynomials. It is encouraging that computations can be done in such a stable manner that the zero set continues to approach the attractor, even for *polynomials of degree as high as 1000*. From a practical point of view, however, we would hope that for the dissipative PDE case, where the gap can be made arbitrarily large, the convergence would be faster. Extensive numerical studies of the Kuramoto-Sivashinsky equation (KSE) indicate that it has a 3-D global attractor over a parameter range which includes a number of interesting global bifurcations.

The computation was carried out over data representing three main subsets of the global attractor at $\alpha = 33$ for a six-mode Galerkin approximation to the KSE with 2π -periodic (and odd) boundary conditions

$$\frac{\partial u}{\partial t} + 4 \frac{\partial^4 u}{\partial x^4} + \alpha \left[\frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} \right] = 0.$$

Several relevant quantities were averaged over all u in each data set. We took $m = 3$, and used a fixed truncation up through the $2m$ in all computations of the nonlinear term. This is done for efficiency, as the cost of dealiasing would grow exponentially with the degree n .

One data set, shown in figure 1a, represents a limit cycle after a several period doublings. We varied the local tolerance of the integrator (again the backward difference code, ODESSA), and observed that the value of the polynomial on the computed periodic solution decreases as the degree is increased up to a critical degree, after which the value grows [see figure 2a.] This is consistent with the theory, as the polynomial is known to diverge off of the attractor. The greater critical degrees for the finer tolerances is also to be expected from the continuity of the polynomial. Figures 2b and 2c indicate that for n as small as five, the zero set is within 10^{-6} of the attractor, when both the zero set and the attractor are computed with at least as much accuracy. This distance dips as low as 10^{-10} for $n = 8$

and an integrator tolerance of 10^{-11} . At $n = 8$, however, our convergence criteria results in a zero set which is only accurate to within 10^{-8} . Since we have determined the accuracy of the zero set by the value of the polynomial, rather than the actual error in phase space, this serves only as an indication of how close the true zero set is to the attractor.

The other two data sets, shown in figure 1b, represent 2-D unstable manifolds of steady states, constructed as described in the section 2(b). The steady states are the zero solution, and the so called (unstable) bimodal solution of the KSE. In these two cases, for the experiments we conducted, the integration error was insignificant compared to the error due to the initial linear approximation of the manifolds, which, in turn, was insignificant compared to the interpolation error.

The ultimate goal would be to incorporate the algebraic approximation into some sort of integration scheme, as is done for explicitly defined AIMs in nonlinear Galerkin methods. At first glance this seems difficult since the resulting scheme must approximate a differential algebraic equation, rather than an ODE. If it can be done effectively, however, the potential exists for tremendous reduction in phase space, since one can improve the accuracy by increasing the degree of the polynomial rather than the dimension of the AIM.

3.3. Climatology Problems.

J.L. Lions, R. Temam and S. Wang have started a collaboration on Computational Climatology problems and have written a number of articles on this subject (see [LTW1,2,3]). In that series of articles, the following questions have been addressed:

- Formulation and mathematical theory of the primitive equations of the atmosphere.
- Formulation and mathematical theory of the primitive equations of the ocean [LTW1].
- Models of the coupled atmosphere and ocean (CAOI), [LTW2].
- Numerical analysis of the coupled atmosphere and ocean models (CAOII).
- Mathematical theory of the equations of coupled atmosphere and oceans (CAOIII), [LTW3].

This material was presented in the invited plenary lecture of J.L. Lions at the 3rd ICIAM meeting in Hambourg, July 1995.

We think that mathematicians can contribute substantially to these problems in particular, in the long range, for the development of better algorithms, the **computational problems being daunting**. Unfortunately, although outstanding mathematicians have been involved in meteorology since the early developments of this science, the present level of collaboration between mathematicians and specialists of climate has not reached the level of intertwining that we observe between mathematics and engineering or between some segment of physics and mathematics. Nevertheless during this project we have worked at bridging the gaps, and we are optimistic for the long range.

Figure 1 KSE, $\alpha=33$, limit cycle after several period doublings

a_1 , a_2 , a_3 are the amplitudes of the first three modes.

Figure 2

The origin is in the middle of the pentagon. The edge of the manifold on top is near the limit cycle in figure 1a.

Figure 3

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.