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# COUPLED CIRCUITS IN WHICH THE SECONDARY HAS DISTRIBUTED INDUCTANCE AND CAPACITY

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In a previous communication [1] I have discussed the problem of coupled circuits on the assumption that the inductance and capacity of the two circuits are localized; this is equivalent to a system of two degrees of freedom, and such a system oscillates with two distinct frequencies, which were completely determined. It was also shown that such a system has two damping factors. The assumption, however, that the inductance and capacity of the secondary as well as the primary are localized does not give the conditions which correspond to those of wireless telegraphy. In a wireless telegraph system it is the antenna, which has a distributed inductance and capacity, which is the secondary, and this will represent a system of an infinite number of degrees of freedom, and consequently the system will oscillate with an infinite number of frequencies. In this communication it is proposed to investigate such a system, in order to see whether the two fundamental frequencies, as obtained in the previous paper, will be in any way modified by the assumption that the secondary has distributed inductance and capacity, and also whether the other frequencies besides the two fundamental ones will in any way influence the results as previously obtained.

It was shown in my previous paper that the resistance does not influence materially the frequency constants, and since in this problem we are concerned with frequency constants only we will, in order to simplify the mathematical treatment of the problem, neglect the resistance entirely.

[1] This Bulletin, 5, p. 511; 1909.

Let us denote by $L_1$, $C_1$ the inductance and capacity of the primary circuit, $L$ and $C$ the inductance and capacity per unit length of the antenna, and $L_2$ and $C_2$ the total inductance and total capacity of the antenna. If $l$ is the length of the antenna, then $Ll = L_2$ and $Cl = C_2$. We shall also designate by $L_0$ the lumped inductance at the end of the antenna, which is inductively connected with the primary, and by $M$ the mutual inductance.

If, now, $V_1$, $I_1$ and $V_2$, $I_2$ are the potentials and currents of the primary and secondary, respectively, at any instant of time, then we shall have for the primary circuit the following equations:

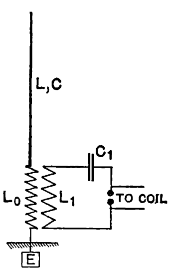



FIG. 1.

$$\left. \begin{aligned} L_1 \frac{dI_1}{dt} + V_1 + M \frac{dI_2}{dt} &= o \\ C_1 \frac{dV_1}{dt} &= I_1 \end{aligned} \right\} \qquad (1)$$

From which we obtain,

$$L_1 C_1 \frac{d^2 I_1}{dt^2} + I_1 + M C_1 \frac{d^2 I_2}{dt^2} = o \qquad (2)$$

For the antenna we have the following relations:

$$\left. \begin{aligned} L \frac{dI_2}{dt} &= -\frac{dV_2}{ds} \\ C \frac{dV_2}{dt} &= -\frac{dI_2}{ds} \end{aligned} \right\} \qquad (3)$$

From these two equations we derive the equation of propagation, which is as follows:

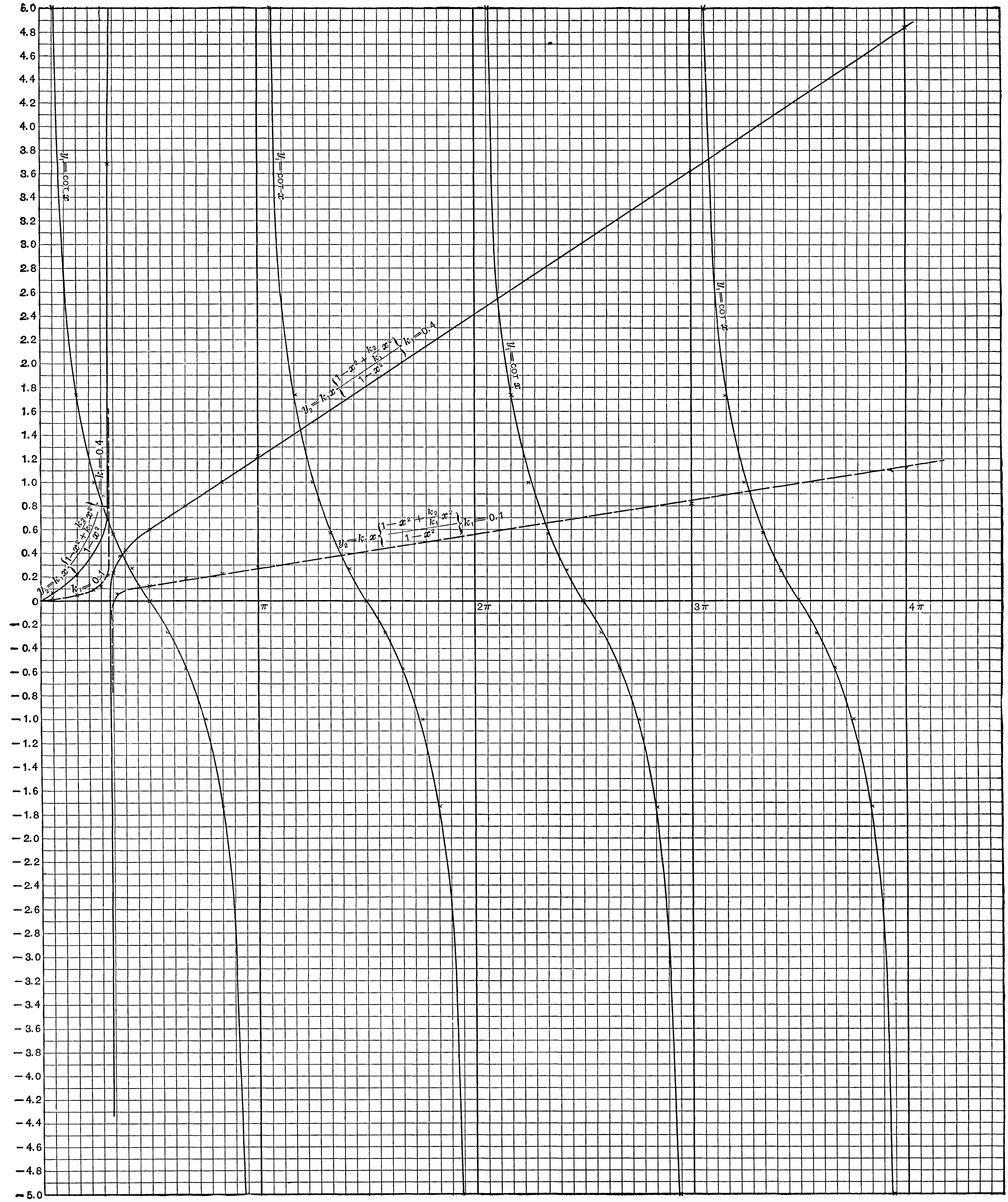
$$LC \frac{d^2 I_2}{dt^2} = \frac{d^2 I_2}{ds^2} \qquad (4)$$

The solution of equation (4) is

$$I_2 = \{A \cos \mu s + B \sin \mu s\} e^{\lambda t} \qquad (5)$$

where

$$\lambda^2 = -\frac{\mu^2}{LC} \qquad (6)$$

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From (5) and the second equation of (3) we can get the expression for $V_2$, which is,

$$V_2 = \frac{\mu}{\lambda C}\left\{A \sin \mu s - B \cos \mu s\right\} e^{\lambda t} \tag{7}$$

We shall also assume that,

$$I_1 = De^{\lambda t} \tag{8}$$

The constants, $A$, $B$, $D$, must be determined so as to satisfy equation (2), and also the boundary conditions, which are as follows: When

$$\left.\begin{array}{ll} s = l, & I_2 = o \\ s = o, & V_2 + L_0\dfrac{dI_2}{dt} + M\dfrac{dI_1}{dt} = o \end{array}\right\} \tag{9}$$

In introducing the value of $I_2$ in (2) and (9) we must bear in mind it is the value of $I_2$ as given by (5) when we put in that equation $s = o$—that is, $I_2 = Ae^{\lambda t}$. Using this value in equations (2) and (9) we get the following:

$$\left.\begin{array}{r} (L_1C_1\lambda^2 + 1)D + MC_1\lambda^2 A = o \\ -\dfrac{\mu}{\lambda C}B + L_0\lambda A + M\lambda D = o \end{array}\right\} \tag{10}$$

Eliminating $\frac{D}{A}$ from these two equations we get,

$$\frac{\mu}{\lambda C}\frac{B}{A} = L_0\lambda + M\lambda\frac{D}{A}$$

$$= L_0\lambda - \frac{M^2C_1\lambda^3}{L_1C_1\lambda^2 + 1}$$

$$= \frac{L_0L_1C_1\lambda^2 + L_0 - M^2C_1\lambda^2}{L_1C_1\lambda^2 + 1}\lambda \tag{11}$$

From the first equation of condition, (9) we have

$$A \cos \mu l + B \sin \mu l = o$$

and

$$\frac{B}{A} = -\cot \mu l \tag{12}$$

Eliminating $\frac{B}{A}$ from (11) and (12) we obtain the following:

$$-\frac{\mu}{\lambda^2 C}\cot \mu l = \frac{L_0 L_1 C_1 \lambda^2 + L_0 - M^2 C_1 \lambda^2}{1 + L_1 C_1 \lambda^2}$$

Replacing $\lambda^2$ by its value as given by equation (6) we get:

$$\cot \mu l = \frac{\mu}{L} \frac{L_0 - L_0 \frac{L_1 C_1}{LC}\mu^2 + M^2 \frac{C_1}{LC}\mu^2}{1 - \frac{L_1 C_1}{LC}\mu^2}$$

$$= \mu l \frac{L_0}{L_2} \frac{1 - \frac{L_1 C_1}{L_2 C_2}\mu^2 l^2 + \frac{M^2 C_1}{L_0 L_2 C_2}\mu^2 l^2}{1 - \frac{L_1 C_1}{L_2 C_2}\mu^2 l^2} \tag{13}$$

Let us assume that $L_1 C_1 = L_2 C_2$. This will be close to the condition of resonance if $L_0$ is small, and let us also put for brevity $\mu l = x$, then we get:

$$\cot x = \frac{L_0}{L_2} x \frac{1 - x^2 + \frac{M^2}{L_0 L_1} x^2}{1 - x^2} \tag{14}$$

If we designate $\frac{L_0}{L_2}$ by $k_1$ and $\frac{M^2}{L_1 L_2}$ by $k_2$, equation (14) can be written in the following form:

$$\cot x = k_1 x \left\{ \frac{1 - x^2 + \frac{k_2}{k_1} x^2}{1 - x^2} \right\} \tag{15}$$

This being a transcendental equation it has an infinite number of roots. If we know the numerical values of $k_1$ and $k_2$ we can determine the roots graphically by plotting the two curves

$$y_1 = \cot x \text{ and } y_2 = k_1 x \left\{ \frac{1 - x^2 + \frac{k_2}{k_1} x^2}{1 - x^2} \right\}$$

The points of intersection of these curves will give the roots of the equation. In plotting these curves the coefficient of coupling is assumed to be one-tenth, and therefore $k_2$, which is the square of the coefficient of coupling, is 0.01. For $k_1$ two different values, 0.25 and 0.1, have been taken. The full-line curve in the graph corresponds to the value $k_1 = 0.25$ and the dotted curve corresponds to the value $k_1 = 0.1$.

In examining the graphs we see that for $k_1 = 0.25$ the roots have the values,

$$58°, 68°, 216°, 380°, 556°$$

For $k_1 = 0.1$ the roots have the values,

$$58°, 82°, 290°, 420°, 590°$$

From equation (6) we see that the frequencies are proportional to $\mu$ that is to the numbers given above. The first two roots represent the two fundamental frequencies, and the other roots represent the harmonics. From these values we see that in one case the frequency of the first harmonic is about three and a half times the fundamental, and in the other case the frequency of the first harmonic is about four times the frequency of the fundamental. In either case, however, if the system is arranged to be in resonance with the fundamental frequency, the upper harmonics will not have any influence in the working of the system.

It remains yet to see whether the ratio of the two fundamental frequencies obtained here are in agreement with the results that are obtained on the assumption that the inductance and capacity are localized in both circuits. It can be easily shown that on the assumption that the inductance and capacity are localized in both circuits, the frequency constants are given by the following expression.[2]

$$\lambda^2 = \frac{L_1C_1 + L_2C_2 \pm \sqrt{4M^2C_1C_2 + (L_1C_1 - L_2C_2)^2}}{2(L_1C_1L_2C_2 - M^2C_1C_2)} \qquad (16)$$

and

$$\frac{\lambda^2_1}{\lambda^2_2} = \frac{L_1C_1 + L_2C_2 - \sqrt{4M^2C_1C_2 + (L_1C_1 - L_2C_2)^2}}{L_1C_1 + L_2C_2 + \sqrt{4M^2C_1C_2 + (L_1C_1 - L_2C_2)^2}} \qquad (17)$$

[2] See Fleming Principles of Wave Telegraphy, p. 210.

where $L_2$ and $C_2$ are the total inductance and capacity of the secondary. In the notation of this paper we have to replace $L_2$ by $L_2+L_0$ and then make $L_2C_2=L_1C_1$. We then get:

$$\frac{\lambda^2_1}{\lambda_2^2}=\frac{L_1C_1+L_2C_2+L_0C_2-\sqrt{4M^2C_1C_2+L_0^2C_2^2}}{L_1C_1+L_2C_2+L_0C_2+\sqrt{4M^2C_1C_2+L_0^2C_2^2}} \tag{18}$$

$$\frac{\lambda_1^2}{\lambda_2^2}=\frac{L_1C_1\left\{2+\frac{L_0}{L_2}-\sqrt{4\frac{M^2}{L_1L_2}+\frac{L_0^2}{L_2^2}}\right\}}{L_1C_1\left\{2+\frac{L_0}{L_2}+\sqrt{4\frac{M^2}{L_1L_2}+\frac{L_0^2}{L_1^2}}\right\}}=\frac{2+k_1-\sqrt{4k_2+k_1^2}}{2+k_1+\sqrt{4k_2+k_1^2}} \tag{19}$$

For the case of $k_1=0.25$ and using for $k_2$ the above value 0.01, we get:

$$\frac{\lambda_1^2}{\lambda_2^2}=\frac{2.25-\sqrt{.04+0.0625}}{2.25+\sqrt{.04+0.0625}}=0.751$$

and

$$\frac{\lambda_1}{\lambda_2}=0.86$$

The ratio of these two values as obtained from the curves are

$$\frac{\lambda_1}{\lambda_2}=\frac{58}{68}=0.85$$

For the case $k_1=0.1$ we get:

$$\frac{\lambda_1}{\lambda_2}=0.9$$

and the ratio of these two values as taken from the curves are

$$\frac{\lambda_1}{\lambda_2}=\frac{58}{80}=0.72$$

From these results it would seem that the greater the ratio of the localized inductance, which is in series with the antenna, is to the inductance of the antenna, the more closely will the ratios of the two frequencies as given by (17) approach the actual conditions which take place in a wireless station.

It is self evident, of course, that by substituting the values of the first two roots, as determined graphically, in equation (6), the values of the frequency constants thus obtained will be approximately the same as those calculated by equation (16).

Now from equation (6) we have,

$$i\lambda = \frac{\mu l}{\sqrt{L_2 C_2}}$$

The first two roots which give the values of $\mu l$ are 58° and 68° or 1.012 and 1.186 radians.

Therefore, we get

$$\lambda_1 = \frac{1.012}{\sqrt{L_2 C_2}} \text{ and } \lambda_2 = \frac{1.186}{\sqrt{L_2 C_2}}$$

From equation (16) we get:

$$\lambda^2 = \frac{2L_2C_2 + L_0C_2 \pm \sqrt{4M^2C_1C_2 + L_0^2C_2^2}}{2\{L_2^2C_2^2 + L_0C_1L_2C_2 - M^2C_1C_2\}}$$

$$= \frac{2 + \frac{L_0}{L_2} \pm \sqrt{4\frac{M^2}{L_1L_2} + \frac{L_0^2}{L_2^2}}}{2L_2C_2\left\{1 + \frac{L_0}{L_2} - \frac{M^2}{L_1L_2}\right\}} = \frac{2 + k_1 \pm \sqrt{4k_2 + k_1^2}}{2L_2C_2\{1 + k_1 - k_2\}}$$

Using the corresponding values of $k_1 = 0.25$ and $k_2 = 0.01$ we get:

$$\lambda_1 = \frac{0.88}{\sqrt{L_2 C_2}} \text{ and } \lambda_2 = \frac{1.02}{\sqrt{L_2 C_2}} \tag{20}$$

It is seen from (19) and (20) that the values of the frequencies as computed by the two different methods differ by thirteen and eighteen per cent respectively. Increasing the ratio of $\frac{L_0}{L_2}$ will bring the values of the frequencies as computed by the two methods more closely together.

WASHINGTON, July 29, 1909.