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**NON LINEAR HYDROTECTONIC PHENOMENA: PART I-
FLUID FLOW IN OPEN FRACTURES UNDER DYNAMICAL
STRESS LOADING**

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**NON LINEAR HYDROTECTONIC PHENOMENA: PART I - FLUID FLOW IN
OPEN FRACTURES UNDER DYNAMICAL STRESS LOADING**

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Non-Linear Hydrotectonic Phenomena: Part I - Fluid Flow in Open Fractures Under Dynamical Stress Loading

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Introduction

A fractured solid under stress loading (or unloading) can be viewed as behaving macroscopically as a medium with internal, hidden, degrees of freedom, wherein changes in fracture geometry (*i.e.* opening, closing and extension) and flow of fluid and gas within fractures will produce major changes in stresses and strains within the solid. Likewise, the flow process within fractures will be strongly coupled to deformation within the solid through boundary conditions on the fracture surfaces. The effects in the solid can, in part, be phenomenologically represented as inelastic or plastic processes in the macroscopic view. However, there are clearly phenomena associated with fracture growth and open fracture fluid flows that produce effects that can not be described using ordinary inelastic phenomenology. This is evident from the fact that a variety of energy release phenomena can occur, including seismic emissions of previously stored strain energy due to fracture growth, release of dissolved gas from fluids in the fractures resulting in enhanced buoyancy and subsequent energetic flows of gas and fluids through the fracture system which can produce rapid extension of old fractures and the creation of new ones. Additionally, the flows will be modulated by the opening and closing of fractures due to deformation in the solid, so that the flow process is strongly coupled to dynamical processes in the surrounding solid matrix, some of which are induced by the flow itself.

In studying such highly interactive, non-linear fluid-gas-solid systems, and attempting to quantitatively describe them with some degree of generality and accuracy, one approach is to first break down this complicated non-linear problem into its basic elemental parts. Here the idea would be to solve component problems in sufficient generality so that they can be combined together in a way that allows

the complete interactive phenomena to be represented, at least in some well defined degree of approximation.

Clearly in the case of interest here, one elemental phenomena is flow of a fluid or gas (or a fluid-gas mixture) in a fracture. In order that this flow problem be of relevance in the present context however, the boundaries of the fracture must be assumed to move, since in the fractured solid the fracture geometry responds to changes of strain in the solid matrix as well as to changes of fluid pressure due to the flow. This coupling of the fluid flow in a fracture with deformation in the solid is through boundary conditions of continuity of traction and continuity of the normal component of the particle velocity on the fracture surfaces. In the usual case the fracture boundaries move at a rate equal to the normal component of the particle velocity of the solid at the fracture surface. Thus, by considering fluid flow in a narrow fracture with arbitrarily moving boundaries, we can focus on basic flow solutions in a "simple" variable aperture fracture where the movement of a boundary is considered an unknown function of time.

To link the flow to the deformation of the surrounding solid matrix, one would equate the normal particle velocity in the solid to the rate of boundary movement and also equate fluid-solid tractions across this interface. As a first approximation the dynamical response of the solid could be computed using macroscopically averaged material properties for the composite. Then, as an approximation, the amount (and sense) of boundary movement in fractures would be computed from the dilatation in the composite; that is from the divergence of the particle displacement field at fracture locations. This gives $\delta \approx \delta_0 (1 + \nabla \cdot \mathbf{u})^{1/2}$ if δ is the aperture of a narrow fracture at time t and δ_0 its initial value, at $t = 0$. Here $\mathbf{u}$ is the displacement in the composite and would approximately account for (passive) interactions between fractures under stress loading. A compatible approximation for traction continuity would be to equate the change in fluid pressure ($p^{(l)}$) from its equilibrium value to the product of the bulk modulus of the composite (k_c) times the dilatation ($\nabla \cdot \mathbf{u}$) at the fracture boundary. Note, in this regard, that if we consider changes in stresses in the solid to occur much more rapidly than the flow in

fractures can change pressures, then the initial value of $p^{(1)}$ is given by the equilibrium value of the dilatation in the solid composite following an earthquake or some other tectonic event.

If the relation $p^{(1)} \approx k_c \nabla \cdot \mathbf{u}$ is used and combined with the previous expression for the aperture function, δ , then the continuity conditions can be satisfied to first order by requiring $\delta = \delta_0 (1 + p^{(1)} / k_c)^{1/2}$. Boundary movement specified in this way will satisfy boundary conditions and cause the fracture to open or close as the pressure in the fluid increases ($p^{(1)} > 0$) or decreases ($p^{(1)} < 0$) from the ambient value during the flow.

Thus, for the fluid the entire effect of the surrounding solid, and the deformational changes within it, can be approximated by the movement of the fracture boundary surface in the manner described. On the other hand the changes of pressure in the fluid and boundary movement associated with the flow will also manifest themselves in the surrounding solid matrix through the boundary conditions at the fracture surfaces. As a higher order approximation for the estimate of secondary deformation in the solid due to fluid flow and changes of pressure in the fracture, one could employ dislocation equivalents in the solid, where imposed displacement offsets and stress discontinuities could be used to represent the effect of fluid pressure changes and fracture boundary movement due to the flow. Thus, a second order approximation for the composite response could involve imposing dislocation equivalents at the fracture locations to produce the effects of coupling between flow of fluids in the fracture and induced, secondary, deformation in the solid. In this study we will focus on the fluid flow problems and defer detailed consideration of the higher order approximations of the composite behavior for subsequent studies.

Beyond anticipating the solid-fluid interactive features inherent in this physical system, and accounting for them in the elemental flow solutions, it is also necessary to account for the fact that fractures in a solid would ordinarily occur as an intersecting network, where orientations of fractures are widely distributed and where intersections occur at essentially all scale lengths. Thus, individual fracture lengths will be distributed and intersections will result in bifurcations in flow patterns that

must be taken into account. The approach that can be taken is to use integral mass, momentum and energy balance methods (*e.g.* Brodkey, 1967) at the intersections to obtain a balance of conserved quantities entering and leaving the intersection zones. In this case elemental solutions in fracture regions removed from the intersection zones would be linked across the zone using the integral balance method, much in the way two asymptotic solutions are joined. In this approach the elemental solution for flow in a fracture would be taken to be a solution in a fracture with no specified end conditions, that is in a fracture of unbounded length. Such a solution would be a good approximation at distances removed from intersection zones. Solutions for flow in elemental fractures could then be linked by choosing free parameters in the solutions to satisfy equations arising from mass, momentum and energy balance across fracture intersections. In this way linkages could be developed through an ensemble of intersecting fractures.

The analysis to follow addresses the most fundamental problem in the program of development of a quantitative model for hydrotectonic phenomena, that is flow in a dynamically deforming fracture. The approach that will be taken is to consider a class of flow behavior that is non-turbulent and has properties of a time dependent laminar flow but includes effects of a time and spatially varying fracture aperture. The essential simplifications that will allow this problem to be successfully attacked and solved analytically involve making use of the knowledge that this type of flow will be characterized by particle velocity derivatives in the direction normal to the fracture boundaries that are much larger than in the other directions and also that the functional dependence of field variables (*e.g.* pressure and particle velocity) can be accurately expressed, within the narrow fracture, as a power series in the direction along the normal to the fracture boundaries. These restrictions and approximations result in closed form solutions that can be used with the integral balance method described earlier to obtain a representation of flows in networks of fractures. Therefore, while restricted to transient laminar flows, this elemental solution will allow us to formulate a general representation for relatively uniform "late-time" flow in a fractured system that accounts for nonlinear fluid-solid interactions.

Basic Flow Relations and Constraints

We consider the fluid (or fluid and gas mixture in some cases) to be isotropic. The general equation expressing the conservation of momentum is:†

$$\rho \left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right] = \frac{\partial T_{ij}}{\partial x_j} + \rho f_i$$

where T_{ij} is the stress tensor and f_i the body force density. Here (eg. Landau and Lifshitz, 1959):

$$T_{ij} = -p\delta_{ij} + V_{ij}$$

Where p is the pressure and V_{ij} the viscosity tensor. The viscosity tensor must depend on the spatial derivatives of the velocity field and must vanish under rigid rotation and translation of the fluid. The most general second order tensor with such properties is:

$$V_{ij} = a \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right] + b \frac{\partial v_k}{\partial x_k} \delta_{ij}$$

where a and b are independent of the fluid velocity. (Also, isotropy of the fluid is assumed.) The equation for V_{ij} may be rewritten and the constants a and b can be redefined in terms of the usual viscosity coefficients, that is the relation can be rewritten as:

$$V_{ij} = \eta \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_k}{\partial x_k} \right] + \zeta \delta_{ij} \frac{\partial v_k}{\partial x_k}$$

where the term in brackets vanishes when contracted on i and j . Here η is the shear viscosity and ζ the bulk viscosity. It can be shown that:

$$\eta > 0 \text{ and } \zeta > 0.$$

Now, the equation of motion becomes

$$\rho \left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right] = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left\{ \eta \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_k}{\partial x_k} \right] \right\} + \frac{\partial}{\partial x_i} \left[\zeta \frac{\partial v_k}{\partial x_k} \right] + \rho f_i$$

In general η and ζ are functions of temperature and pressure, so their spatial derivatives are non-zero. However, to a good approximation they can be taken to be nearly constant in the fluid over a

† The summation convention on repeated indices is used throughout.

reasonably extended spatial region. In this case we get:

$$\rho \left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right] = - \frac{\partial p}{\partial x_i} + \eta \left\{ \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i} \left[\frac{\partial v_j}{\partial x_j} \right] - \frac{2}{3} \frac{\partial}{\partial x_i} \left[\frac{\partial v_k}{\partial x_k} \right] \right\} + \zeta \frac{\partial}{\partial x_i} \left[\frac{\partial v_k}{\partial x_k} \right] + \rho f_i$$

Therefore:

$$\rho \left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right] = - \frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j} + (\zeta + \eta/3) \frac{\partial}{\partial x_i} \left[\frac{\partial v_k}{\partial x_k} \right] + \rho f_i$$

Since:

$$\nabla^2 v_i \equiv \frac{\partial^2 v_i}{\partial x_j \partial x_j} ; \quad \nabla \cdot \mathbf{v} \equiv \frac{\partial v_k}{\partial x_k}$$

then, in vector form:

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \eta \nabla^2 \mathbf{v} + (\zeta + \eta/3) \nabla (\nabla \cdot \mathbf{v}) + \rho \mathbf{f} \quad (1a.)$$

If the fluid has a bulk viscosity, ζ , that is small relative to the shear viscosity, η , then

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \eta \left[\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right] + \rho \mathbf{f} \quad (1b.)$$

If the fluid is incompressible, so $\nabla \cdot \mathbf{v} = 0$ (and $\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} = \frac{d\rho}{dt} = 0$), then

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \eta \nabla^2 \mathbf{v} + \rho \mathbf{f} \quad (1c.)$$

which is usually called the Navier-Stokes equation.

Laminar Flow in a Fracture

The geometry for the flow to be considered is indicated in Figure 1. The flow of interest will be quasi-laminar in that the velocity gradients in the x_2 direction dominate over those in the x_1 , x_3 directions. Therefore:

$$\left| \frac{\partial v_i}{\partial x_2} \right| \gg \left[\left| \frac{\partial v_i}{\partial x_1} \right|, \left| \frac{\partial v_i}{\partial x_3} \right| \right] ; \quad i = 1, 2, 3 \quad (2.)$$

and, as well, the change in gradient in the x_2 direction dominates, so:

$$(3.) \quad \left| \frac{\partial^2 v_1}{\partial x_2^2} \right| \gg \left| \frac{\partial^2 v_1}{\partial x_1^2} \right|, \quad \left| \frac{\partial^2 v_1}{\partial x_2^2} \right| \gg \left| \frac{\partial^2 v_1}{\partial x_3^2} \right|; \quad i = 1, 2, 3 \dots (3)$$

In component form, the equation (1.a) becomes, for $i = 1$ and setting $b_i \equiv p f_i$:

$$\frac{\partial p}{\partial x_1} = \eta \left[\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2} \right] + \left(\zeta + \frac{\eta}{3} \right) \frac{\partial}{\partial x_1} \left[\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right] - p \left[v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} \right] - p \frac{\partial}{\partial t} + b_1$$

Or, after neglecting the smaller terms, according to equations (2) and (3):

$$(4.) \quad \frac{\partial p}{\partial x_1} = \eta \frac{\partial^2 v_1}{\partial x_2^2} - p v_2 \frac{\partial v_1}{\partial x_2} - p \frac{\partial}{\partial t} + b_1$$

The x_2 component equation becomes:

$$\frac{\partial p}{\partial x_2} = \eta \frac{\partial^2 v_1}{\partial x_2^2} + \left(\zeta + \frac{\eta}{3} \right) \frac{\partial^2 v_2}{\partial x_2^2} - p v_2 \frac{\partial v_2}{\partial x_2} - p \frac{\partial}{\partial t} + b_2$$

or,

$$(5.) \quad \frac{\partial p}{\partial x_2} = (4/3 \eta + \zeta) \frac{\partial^2 v_2}{\partial x_2^2} - p v_2 \frac{\partial v_2}{\partial x_2} - p \frac{\partial}{\partial t} + b_2$$

The x_3 component equation becomes:

$$(6.) \quad \frac{\partial p}{\partial x_3} = \eta \frac{\partial^2 v_3}{\partial x_3^2} - p v_2 \frac{\partial v_3}{\partial x_3} - p \frac{\partial}{\partial t} + b_3$$

For the case when ζ is negligible, as in (1-b.) we have:

$$\begin{aligned} \frac{\partial p}{\partial x_1} &= \eta \frac{\partial^2 v_1}{\partial x_2^2} - p v_2 \frac{\partial v_1}{\partial x_2} - p \frac{\partial}{\partial t} + b_1 \\ \frac{\partial p}{\partial x_2} &= 4/3 \eta \frac{\partial^2 v_2}{\partial x_2^2} - p v_2 \frac{\partial v_2}{\partial x_2} - p \frac{\partial}{\partial t} + b_2 \\ \frac{\partial p}{\partial x_3} &= \eta \frac{\partial^2 v_3}{\partial x_3^2} - p v_2 \frac{\partial v_3}{\partial x_3} - p \frac{\partial}{\partial t} + b_3 \end{aligned}$$

When the fluid is incompressible, as in (1-c.), then:

$$\frac{\partial p}{\partial x_i} = \eta \frac{\partial^2 v_i}{\partial x_2^2} - \rho v_2 \frac{\partial v_i}{\partial x_2} - \rho \frac{\partial v_i}{\partial t} + b_i, \quad i = 1, 2, 3$$

The general form of all these equations is the same. That is we can write them all in the form:

$$\frac{\partial p}{\partial x_k} = \mu^{(k)} \frac{\partial^2 v_k}{\partial x_2^2} - \rho v_2 \frac{\partial v_k}{\partial x_2} - \rho \frac{\partial v_k}{\partial t} + b_k; \quad k = 1, 2, 3$$

where

$$\mu^{(1)} = \mu^{(3)} = \eta \text{ in all cases, and}$$

$$\mu^{(2)} = 4/3 \cdot \eta + \zeta \text{ (in the general case)}$$

$$\mu^{(2)} = 4/3 \cdot \eta \text{ (when } \zeta \ll \eta \text{)}$$

$$\mu^{(2)} = \eta \text{ (when } \nabla \cdot \mathbf{v} = 0 \text{)}$$

Zeroth and first order approximations.

The "zeroth order" approximation corresponds to the equilibrium situation where $\mathbf{v} \equiv 0$. In the first order approximation we take:

$$|v_2^{(1)}| \ll |v_1^{(1)}|, \quad |v_2^{(1)}| \ll |v_3^{(1)}| \quad (8.)$$

and the flow under consideration becomes strictly laminar when v_2 is neglected in (7.). For a steady state laminar flow, then $\partial v_i / \partial t = 0$ also, and this will constitute the first order approximation.

In order to develop zeroth order, first order and higher order approximations for the flow, we consider the fluid variables ($\mathbf{v}$ and p) to be expandable in the forms:

$$\begin{aligned} \mathbf{v} &= \mathbf{v}^{(0)} + \mathbf{v}^{(1)} + \mathbf{v}^{(2)} + \dots \\ p &= p^{(0)} + p^{(1)} + p^{(2)} + \dots \end{aligned} \quad (9.)$$

Where $\mathbf{v}^{(0)}$ and $p^{(0)}$ refer to the equilibrium values and $\mathbf{v}^{(1)}$ and $p^{(1)}$ to the steady state laminar flow values of velocity and pressure. As indicated above, $\mathbf{v}^{(0)} \equiv 0$ (by definition of an equilibrium state) and for laminar flow there is negligible flow in the x_2 direction ($v_2^{(1)} = 0$). Further, when the flow is steady state, then $\partial v_i / \partial t = 0$. Under these definitions for the zeroth and first order fields, the equations (7.) give, with b_i representing gravity forces only:

$$\frac{\partial p^{(0)}}{\partial x_i} = b_i^{(0)} = \rho^{(0)} g^{(0)} \cdot \hat{e}_i \quad i = 1, 2, 3 \quad (10.)$$

since $v = 0$ by definition.

For the first order steady state laminar flow approximation:

$$\left[\frac{\partial p^{(0)}}{\partial x_i} + \frac{\partial p^{(1)}}{\partial x_i} \right] = \mu^{(i)} \frac{\partial^2 v_i}{\partial x_2^2} - \rho v_2 \frac{\partial v_i}{\partial x_2} + \rho g \cdot \hat{e}_i$$

where, $\rho = \rho^{(0)} + \rho^{(1)}$, $g = g^{(0)} + g^{(1)}$, $v_i = v_i^{(0)} + v_i^{(1)} = v_i^{(1)}$, $i = 1, 2, 3$; to first order. In view of equation (8.), we have:

$$\frac{\partial p^{(1)}}{\partial x_i} = \eta \frac{\partial^2 v_i^{(1)}}{\partial x_2^2} - \rho^{(0)} v_2^{(1)} \frac{\partial v_i^{(1)}}{\partial x_2} + \rho^{(1)} g^{(0)} \cdot \hat{e}_i$$

Finally if $v_2^{(1)} = 0$ for the laminar first order case, then the component equations are:

$$\frac{\partial p^{(1)}}{\partial x_1} = \eta \frac{\partial^2 v_1^{(1)}}{\partial x_2^2} + \rho^{(1)} g^{(0)} \cdot \hat{e}_1$$

$$\frac{\partial p^{(1)}}{\partial x_2} = \rho^{(1)} g^{(0)} \cdot \hat{e}_2$$

$$\frac{\partial p^{(1)}}{\partial x_3} = \eta \frac{\partial^2 v_3^{(1)}}{\partial x_2^2} + \rho^{(1)} g^{(0)} \cdot \hat{e}_3$$

or setting

$$g_i^{(0)} \equiv g^{(0)} \cdot \hat{e}_i$$

then:

$$\left. \begin{aligned} \frac{\partial p^{(1)}}{\partial x_j} &= \eta \frac{\partial^2 v_j}{\partial x_2^2} + \rho^{(1)} g_j^{(0)} : j = 1, 3 \\ \frac{\partial p^{(1)}}{\partial x_j} &= \rho^{(1)} g_2^{(0)} \end{aligned} \right\} \quad (11.)$$

Solutions for the First Order Approximation in the Steady State

To the first order consider both pressure driven (pumped) flows and thermally driven (convected) flows in a gravity field. Thus, the terms $\rho^{(1)} b_i^{(0)}$, with $j = 1, 2, 3$, will not be neglected in (9.), except as special cases. (e.g. $g_2^{(0)} = 0$ for fractures aligned with the gravity field or if, for narrow fractures, the

approximation $g_2^{(0)} \approx 0$ is used.)

Consequently, take v_i to be of the form:

$$v_i = f(x_2) \frac{\partial p^{(1)}}{\partial x_i} + h(x_2) \frac{\partial T^{(1)}}{\partial x_i} + k_i(x_1, x_2, x_3); \quad i = 1, 3 \quad (12.)$$

where we also take the temperature change across the aperture of the fracture to be small and uniform,

so:

$$\frac{\partial T^{(1)}}{\partial x_2} \approx 0 \dots \quad (13.)$$

Now using (12.) in the first relation in (11.) gives for $i = 1, 3$:

$$\frac{\partial p^{(1)}}{\partial x_i} = \eta \frac{\partial}{\partial x_2} \left\{ \frac{\partial f}{\partial x_2} \frac{\partial p^{(1)}}{\partial x_i} + f \frac{\partial^2 p^{(1)}}{\partial x_2 \partial x_i} \right\} + \eta \frac{\partial T^{(1)}}{\partial x_i} \left[\frac{\partial^2 h}{\partial x_2^2} \right] + \eta \frac{\partial^2 k_i}{\partial x_2^2} + \rho^{(1)} g_i^{(0)} \quad (14.)$$

From the second equation in (11.) however, note that:

$$\frac{\partial^2 p^{(1)}}{\partial x_2 \partial x_i} = \frac{\partial}{\partial x_i} \left[\frac{\partial p^{(1)}}{\partial x_2} \right] = \frac{\partial}{\partial x_i} \left[\rho^{(1)} g_2^{(0)} \right]$$

Taking $g^{(0)}$ to be nearly constant (over the extent of the fracture segment under consideration), then

$$\frac{\partial^2 p^{(1)}}{\partial x_2 \partial x_i} = g_2^{(0)} \frac{\partial \rho^{(1)}}{\partial x_i} + \rho^{(1)} \frac{\partial g_2^{(0)}}{\partial x_i} \approx g_2^{(0)} \frac{\partial \rho^{(1)}}{\partial x_i}$$

when

$$\left| \frac{1}{\rho^{(1)}} \frac{\partial \rho^{(1)}}{\partial x_i} \right| \gg \left| \frac{1}{g_2^{(0)}} \frac{\partial g_2^{(0)}}{\partial x_i} \right|$$

provided $\rho^{(1)} \neq 0$ and $g_2^{(0)} \neq 0$. (Note that if either $\rho^{(1)}$ or $g_2^{(0)}$ are zero, then $\partial^2 p^{(1)} / \partial x_2 \partial x_i = 0$.) Under these conditions:

$$\frac{\partial^2 p^{(1)}}{\partial x_2 \partial x_i} = g_2^{(0)} \frac{\partial \rho^{(1)}}{\partial x_i}$$

and equation (14.) becomes:

$$\begin{aligned} \frac{\partial p^{(1)}}{\partial x_i} = \eta \left[\frac{\partial^2 f}{\partial x_2^2} \frac{\partial p^{(1)}}{\partial x_i} + 2 \frac{\partial f}{\partial x_2} \frac{\partial \rho^{(1)}}{\partial x_i} g_2^{(0)} + f \frac{\partial^2 \rho^{(1)}}{\partial x_i \partial x_2} g_2^{(0)} \right] \\ + \eta \frac{\partial^2 h}{\partial x_2^2} \left[\frac{\partial T^{(1)}}{\partial x_i} \right] + \eta \frac{\partial^2 k_i}{\partial x_2^2} + \rho^{(1)} g_i^{(0)} \end{aligned} \quad (15.)$$

Consider, however, that because of the existence of an equation of state, we can write:

$$\rho = \rho(p, T) \quad (16.)$$

Consequently we have:

$$\delta\rho = \left[\frac{\partial\rho}{\partial p} \right]_T \delta p + \left[\frac{\partial\rho}{\partial T} \right]_p \delta T \quad (17a.)$$

Since $\rho = \rho^{(0)} + \delta\rho$, $p = p^{(0)} + \delta p$ and $T = T^{(0)} + \delta T$, where $\delta\rho$, etc. are small quantities and $\rho^{(0)}$, etc. are equilibrium values, then to a good approximation $\rho^{(1)} = \delta\rho$, $p^{(1)} = \delta p$ and $T^{(1)} = \delta T$ and (17a.) can be rewritten as

$$\rho^{(1)} = \left[\frac{\partial\rho}{\partial p} \right]_T p^{(1)} + \left[\frac{\partial\rho}{\partial T} \right]_p T^{(1)} \equiv a_1 p^{(1)} + a_2 T^{(1)} \quad (17b.)$$

Here a_1 and a_2 are constants of the material (isothermal coefficient of compressibility and isobaric coefficient of thermal expansion) where

$$a_1 \equiv \left[\frac{\partial\rho}{\partial p} \right]_T ; a_2 \equiv \left[\frac{\partial\rho}{\partial T} \right]_p$$

and are both evaluated at $T^{(0)}$ and $p^{(0)}$.

Now, noting that by use of (17b.) we have that:

$$\frac{\partial\rho^{(1)}}{\partial x_i \partial x_2} = \frac{\partial}{\partial x_i} \left[\frac{\partial}{\partial x_2} \left(a_1 p^{(1)} + a_2 T^{(1)} \right) \right] \approx \frac{\partial}{\partial x_i} \left[a_1 \frac{\partial p^{(1)}}{\partial x_2} \right]$$

since $\partial T^{(1)}/\partial x_2 \approx 0$ from equation (13.). Therefore, using the second equation in (11) again,

$$\frac{\partial^2 \rho^{(1)}}{\partial x_i \partial x_2} \approx a_1 \frac{\partial}{\partial x_i} \left[\rho^{(1)} g_2^{(0)} \right] \approx a_1 g_2^{(0)} \frac{\partial \rho^{(1)}}{\partial x_i}$$

Thus equation (15.) becomes:

$$\frac{\partial \rho^{(1)}}{\partial x_i} = \eta \left[\frac{\partial^2 f}{\partial x_2^2} \frac{\partial p^{(1)}}{\partial x_i} + \left\{ 2 \frac{\partial f}{\partial x_2} + a_1 f \right\} \frac{\partial \rho^{(1)}}{\partial x_i} g_2^{(0)} \right] + \eta \frac{\partial^2 h}{\partial x_2^2} \left[\frac{\partial T^{(1)}}{\partial x_i} \right] + \eta \frac{\partial^2 k_i}{\partial x_2^2} + \rho^{(1)} g_i^{(0)} ; i = 1, 3$$

Again expressing $\rho^{(1)}$ in terms of $T^{(1)}$ and $p^{(1)}$, this becomes,

$$\frac{\partial \rho^{(1)}}{\partial x_i} = \eta \left[\frac{\partial^2 f}{\partial x_2^2} + \left\{ 2 \frac{\partial f}{\partial x_2} + a_1 g_2^{(0)} f \right\} a_1 g_2^{(0)} \right] \left[\frac{\partial p^{(1)}}{\partial x_i} \right] + \eta \left[\frac{\partial^2 h}{\partial x_2^2} + \left\{ 2 \frac{\partial f}{\partial x_2} + a_1 g_2^{(0)} f \right\} a_2 g_2^{(0)} \right] \left[\frac{\partial T^{(1)}}{\partial x_i} \right]$$

$$+ \eta \frac{\partial^2 k_i}{\partial x_2^2} + \rho^{(1)} g_i^{(0)} \quad (18.)$$

or

$$\left[\frac{\partial^2 f}{\partial x_2^2} + 2a_1 g_2^{(0)} \frac{\partial f}{\partial x_2} + (a_1 g_2^{(0)})^2 f - \frac{1}{\eta} \right] \frac{\partial p^{(1)}}{\partial x_i} + \left[\frac{\partial^2 h}{\partial x_2^2} + 2a_2 g_2^{(0)} \frac{\partial f}{\partial x_2} + a_1 a_2 (g_2^{(0)})^2 f \right] \frac{\partial T^{(1)}}{\partial x_i} + \left[\frac{\partial^2 k_i}{\partial x_2^2} + \frac{\rho^{(1)} g_i^{(0)}}{\eta} \right] = 0 \quad (19.)$$

Now we must chose the functions f , h and k_i so as to satisfy this equation for $i = 1, 3$. Consider first the final bracketed term involving k_i . We can choose $k_i(x_1, x_2, x_3)$ to cause this quantity to vanish, that is take k_i such that:

$$\frac{\partial^2 k_i}{\partial x_2^2} + \frac{\rho^{(1)} g_i^{(0)}}{\eta} = 0 \quad (20.)$$

Let:

$$k_i(x_1, x_2, x_3) = w_i(x_1, x_3) \left[\gamma_2 x_2^2 + \gamma_1 x_2 + \gamma_0 \right] \quad (21.)$$

Then:

$$\frac{\partial^2 k_i}{\partial x_2^2} = 2\gamma_2 w_i = - \frac{\rho^{(1)} g_i^{(0)}}{\eta}$$

Consequently

$$k_i(x_1, x_2, x_3) = - \frac{\rho^{(1)} g_i^{(0)}}{2\eta} \left[x_2^2 + \left[\frac{\gamma_1}{\gamma_2} \right] x_2 + \left[\frac{\gamma_0}{\gamma_2} \right] \right] \quad (22.)$$

Next it is necessary to find the functions f and h such that both the first and second bracketed relations in (19.) also vanish. That is, we require:

$$\left. \begin{aligned} \frac{\partial^2 f}{\partial x_2^2} + 2a_1 g_2^{(0)} \frac{\partial f}{\partial x_2} + (a_1 g_2^{(0)})^2 f &= \frac{1}{\eta} \\ \frac{\partial^2 h}{\partial x_2^2} + 2a_1 g_2^{(0)} \frac{\partial f}{\partial x_2} + a_1 a_2 (g_2^{(0)})^2 f &= 0 \end{aligned} \right\} \quad (23.)$$

Or, letting

$$c \equiv a_1 g_2^{(0)} \text{ and } b \equiv a_2 g_2^{(0)} \quad (24.)$$

then we need to have

$$\left. \begin{aligned} \frac{\partial^2 f}{\partial x_2^2} + 2c \frac{\partial f}{\partial x_2} + c^2 f &= \frac{1}{\eta} \\ \frac{\partial^2 h}{\partial x_2^2} + 2b \frac{\partial f}{\partial x_2} + b c f &= 0 \end{aligned} \right\} \quad (25.)$$

Multiplying the second equation by c/b and subtracting from the first gives:

$$\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{c}{b} \right] \frac{\partial^2 h}{\partial x_2^2} = \frac{\partial^2}{\partial x_2^2} \left[f - \left[\frac{a_1}{a_2} \right] h \right] = \frac{1}{\eta}$$

where $c/b = a_1/a_2$ from (24.). Taking

$$F(x_2) \equiv f(x_2) - \left[\frac{a_1}{a_2} \right] h(x_2)$$

then we have:

$$\frac{\partial^2 F}{\partial x_2^2} = \frac{1}{\eta}$$

now taking

$$F(x_2) = \alpha_2 x_2^2 + \alpha_1 x_2 + \alpha_0$$

then

$$\frac{\partial^2 F}{\partial x_2^2} = 2\alpha_2 = \frac{1}{\eta} ; \quad \alpha_2 = \frac{1}{2\eta}$$

Consequently

$$F(x_2) = \frac{1}{2\eta} x_2^2 + \alpha_1 x_2 + \alpha_0 = f - \left[\frac{a_1}{a_2} \right] h$$

and, for $a_1 \neq 0$,

$$h(x_2) = \left[\frac{a_2}{a_1} \right] \left[f(x_2) - \frac{1}{2\eta} x_2^2 - \alpha_1 x_2 - \alpha_0 \right] \quad (26.)$$

For $h(x_2)$ of this form, the second equation in (25.) becomes the same as the first equation. Therefore, we must find $f(x_2)$ such that

$$\frac{\partial^2 f}{\partial x_2^2} + 2c \frac{\partial f}{\partial x_2} + c^2 f = \frac{1}{\eta}$$

Taking:

$$f(x_2) = \beta_2 x_2^2 + \beta_1 x_2 + \beta_0$$

produces

$$2\beta_2 + 2c(2\beta_2 x_2 + \beta_1) + c^2(\beta_2 x_2^2 + \beta_1 x_2 + \beta_0) = \frac{1}{\eta}$$

Here η is regarded as constant, so we must have:

$$\beta_2 = 0 \text{ and } \beta_1 = 0$$

while

$$\beta_0 = \frac{1}{c^2 \eta}$$

is required. Thus

$$\left. \begin{aligned} f(x_2) &= \frac{1}{c^2 \eta} \\ h(x_2) &= -\left[\frac{a_2}{a_1}\right] \left[\frac{1}{2\eta} x_2^2 + \alpha_1 x_2 + \left[\alpha_0 - \frac{1}{c^2 \eta} \right] \right] \end{aligned} \right\} \quad (27.)$$

The velocity components are therefore, assuming $c \neq 0$:

$$\begin{aligned} v_i &= \left[\frac{1}{c^2 \eta} \right] \frac{\partial p^{(1)}}{\partial x_i} - \left[\frac{a_2}{a_1} \right] \left[\frac{x_2^2}{2\eta} + \alpha_1 x_2 + \left[\alpha_0 - \frac{1}{c^2 \eta} \right] \right] \frac{\partial T^{(1)}}{\partial x_i} \\ &\quad - \frac{\rho^{(1)} g_i^{(0)}}{2\eta} \left[x_2^2 + \left[\frac{\gamma_1}{\gamma_2} \right] x_2 + \left[\frac{\gamma_0}{\gamma_2} \right] \right] \end{aligned} \quad (28.)$$

for $i = 1, 3$; and where $v_2 = 0$.

Alternate Solution Forms and Special Cases for Steady State Flows

The solution in (28.) can be put in alternate form by making use of the equation of state relationships between $p^{(1)}$, $p^{(1)}$ and $T^{(1)}$. That is, from (17b.)

$$p^{(1)} = a_1 p^{(1)} + a_2 T^{(1)}$$

then

$$\frac{\partial p^{(1)}}{\partial x_i} = a_1 \frac{\partial p^{(1)}}{\partial x_i} + a_2 \frac{\partial T^{(1)}}{\partial x_i}$$

Using this in (28.) to eliminate $\frac{\partial p^{(1)}}{\partial x_i}$ gives:

$$v_i = \left[\frac{1}{\eta a_1 c^2} \right] \frac{\partial p^{(1)}}{\partial x_i} - \left[\frac{a_2}{a_1} \right] \left[\frac{x_2^2}{2\eta} + \alpha_1 x_2 + \alpha_0 \right] \frac{\partial T^{(1)}}{\partial x_i} - \rho^{(1)} g_i^{(0)} \left[x_2^2 + \left[\frac{\gamma_1}{\gamma_2} \right] x_2 + \left[\frac{\gamma_0}{\gamma_2} \right] \right] \quad (29a.)$$

Further, eliminating $\frac{\partial T^{(1)}}{\partial x_i}$ gives:

$$v_i = \left[\frac{x_2^2}{2\eta} + \alpha_1 x_2 + \alpha_0 \right] \frac{\partial p^{(1)}}{\partial x_i} - \left[\frac{1}{a_1} \right] \left[\frac{x_2^2}{2\eta} + \alpha_1 x_2 + \left[\alpha_0 - \frac{1}{\eta c^2} \right] \right] \frac{\partial p^{(1)}}{\partial x_i} - \frac{\rho^{(1)} g_i^{(0)}}{2\eta} \left[x_2^2 + \frac{\gamma_1}{\gamma_2} x_2 + \frac{\gamma_0}{\gamma_2} \right] \quad (29b.)$$

The special cases of importance involve the cases in which the gravity force component in the x_2 direction can be neglected. Consider the case when $g_2^{(0)}$ is small and so $\rho^{(1)} g_2^{(0)}$ can be neglected in the first order approximation. (This occurs, for example, when the fractures are aligned, or nearly aligned, with the gravity field, so that $\hat{e}_2$ is normal to $g^{(0)}$ or when the derivatives $\partial p^{(1)}/\partial x_i$ are small.) In this case the equation (11.) becomes:

$$\left. \begin{aligned} \frac{\partial p^{(1)}}{\partial x_j} &= \eta \frac{\partial^2 v_j}{\partial x_2^2} + \rho^{(1)} g_j^{(0)}, \quad j = 1, 3 \\ \frac{\partial p^{(1)}}{\partial x_2} &= 0 \end{aligned} \right\} \quad (30.)$$

Here again taking the solution for v_i to have the form as in (12.), that is:

$$v_i = f(x_2) \frac{\partial p^{(1)}}{\partial x_i} + h x_2 \frac{\partial T^{(1)}}{\partial x_i} + k_i (x_1, x_2, x_3)$$

we get for (30.):

$$\frac{\partial p^{(1)}}{\partial x_i} = \eta \left[\frac{\partial^2 f}{\partial x_2^2} \right] \frac{\partial p^{(1)}}{\partial x_i} + \eta \left[\frac{\partial^2 h}{\partial x_2^2} \right] \frac{\partial T^{(1)}}{\partial x_i} + \eta \frac{\partial^2 k_i}{\partial x_2^2} + \rho^{(1)} g_i^{(0)} \quad (31.)$$

This same result is produced from eqn. (15.) when the spatial derivatives of $\rho^{(1)}$ are neglected relative to $p^{(1)}$ pressure derivatives. Again we can take k_i to be such that:

$$\eta \frac{\partial^2 k_i}{\partial x_2^2} = -\rho^{(1)} g_i^{(0)} ; i = 1, 3$$

Thus, with

$$k_i = w_i(x_1, x_3) [\gamma_2 x_2^2 + \gamma_1 x_2 + \gamma_0]$$

as in equation (21.) under similar circumstances, we get:

$$k_i = -\frac{\rho^{(1)} g_i^{(0)}}{2\eta} \left[x_2^2 + \left(\frac{\gamma_0}{\gamma_2} \right) x_2 + \left(\frac{\gamma_0}{\gamma_2} \right) \right] \quad (32.)$$

Likewise, as before we require the coefficient factors multiplying the temperature and pressure gradients to vanish. That is:

$$\left\{ \begin{array}{l} \left[\frac{\partial^2 f}{\partial x_2^2} - \frac{1}{\eta} \right] \frac{\partial p^{(1)}}{\partial x_i} = 0 \\ \left[\frac{\partial^2 h}{\partial x_2^2} \right] \frac{\partial T^{(1)}}{\partial x_i} = 0 \end{array} \right. \quad (33.)$$

These conditions are satisfied by:

$$\begin{aligned} f(x_2) &= \frac{x_2^2}{2\eta} + \alpha'_1 x_2 + \alpha'_0 \\ h(x_2) &= \beta'_1 x_2 + \beta'_0 \end{aligned}$$

Consequently the solution in this case is, from (12.):

$$v_i^{(1)} = \left[\frac{x_2^2}{2\eta} + \alpha'_1 x_2 + \alpha'_0 \right] \frac{\partial p^{(1)}}{\partial x_i} + [\beta'_1 x_2 + \beta'_0] \frac{\partial T^{(1)}}{\partial x_i} - \frac{\rho^{(1)} g_i^{(0)}}{2\eta} \left[x_2^2 + \left(\frac{\gamma_1}{\gamma_2} \right) x_2 + \left(\frac{\gamma_0}{\gamma_2} \right) \right] \quad (34.)$$

Here again the relation $\rho^{(1)} = a_1 p^{(1)} + a_2 T^{(1)}$ can be used to generate alternate expressions of this result.

Boundary Conditions: Application to Steady State Laminar Flows

There are only two required conditions at the boundaries of a fracture in a solid; these are that the tractions are continuous (conservation of momentum across the boundary) and that the normal component of the particle velocity is continuous (conservation of mass across the boundary). The assumption here is that the boundary moves with the particles. These conditions require:

$$[[T_{ij} n_j]]_B = 0 \quad \text{and} \quad [[v_i n_i]]_B = 0 \quad (35.)$$

where n_j denotes the j th component of the normal vector at the boundary surface and the double bracket $[[\phi]]_B$ denotes the difference in the quantity ϕ across the boundary B ; that is:

$$[[\phi]]_B = \phi(B^+) - \phi(B^-)$$

In the steady state approximation the boundary is considered fixed and in the first order approximation $v_2 = 0$, so that the second equation in (35.) is automatically satisfied. (The normal particle velocity at the boundary is zero on both sides of B .)

Ordinarily, the velocities v_1 and v_3 are required to be zero at the boundary. (See for example Landau and Lifshitz, 1959). This is based on a physical premise, namely that if the fluid is viscous and the solid rigid (the particles on the solid boundary do not move) then the fluid adheres to the solid at the boundary and all velocities vanish there. This seems to give good agreement with flow tests but is, in fact, not required by the conservation relations (boundary conditions) in (35.).

If this condition ($v_1 = v_3 = 0$ at $x_2 = \pm\delta$) is nevertheless applied to the results of the previous sections, then only the special case in which $\rho^{(1)} g_2^{(0)}$ is neglected in the equations, as expressed in equation (30), can satisfy these conditions in the first order steady state approximation. In this case (34) is the solution and we have:

At $x_2 = \delta$; for $i = 1, 3$

$$\left[\frac{(\delta)^2}{2\eta} + \alpha'_1 \delta + \alpha'_0 \right] \frac{\partial p^{(1)}}{\partial x_i} + \left[\beta'_1 \delta + \beta'_0 \right] \frac{\partial T_1}{\partial x_i} - \frac{\rho^{(1)} g_i^{(0)}}{2\eta} \left[(\delta)^2 + \left[\frac{\gamma_1}{\gamma_2} \right] \delta + \frac{\gamma_0}{\gamma_2} \right] = 0$$

At $x_2 = -\delta$, for $i = 1, 3$:

$$\left[\frac{(-\delta)^2}{2\eta} + \alpha'_1 (-\delta) + \alpha'_0 \right] \frac{\partial p^{(1)}}{\partial x_i} + \left[-\beta'_1 \delta + \beta'_0 \right] \frac{\partial T^{(1)}}{\partial x_i} - \frac{\rho^{(1)} g_i^{(0)}}{2\eta} \left[(-\delta)^2 - \left[\frac{\gamma_1}{\gamma_2} \right] \delta + \frac{\gamma_0}{\gamma_2} \right] = 0$$

These equations can only be satisfied if:

$$\begin{aligned} \alpha'_1 &= 0, \quad \beta'_1 = 0, \quad \beta'_0 = 0, \quad \gamma_1 = 0 \\ \alpha'_0 &= -\frac{\delta^2}{2\eta}, \quad \frac{\gamma_0}{\gamma_2} = -\delta^2 \end{aligned}$$

The solution is , in this case:

$$v_i = \frac{1}{2\eta} \left[x_2^2 - \delta^2 \right] \left\{ \frac{\partial p^{(1)}}{\partial x_i} - \rho^{(1)} g_i^{(0)} \right\} \quad (36.)$$

with $i = 1, 3$ and $v_2 = 0$. Here the term involving the density perturbation $\rho^{(1)}$ can correspond to a buoyancy effect ($\rho^{(1)} < 0$), or since $\rho^{(1)} = a_2 T^{(1)}$, it can correspond to a thermal convection effect.

On the other hand, it is reasonable to also consider flows that involve slip at the boundaries of the fracture and a non-rigid solid boundary. In this case the solid is coupled to the fluid through the boundary condition $[[T_{ij} n_j]]_B = 0$ and the solution forms in (28.) and (29a,b) are all possible, so long as $a_1 \neq 0$. If, however, density gradients are small compared to pressure and gravitational (buoyancy) effects, then the solution has the same general form as that in (37.), that is:

$$v_i = \left[\frac{x_2^2}{2\eta} + \alpha_1 x_2 + \alpha_0 \right] \frac{\partial p^{(1)}}{\partial x_i} - \frac{\rho^{(1)} g_i^{(0)}}{2\eta} \left[x_2^2 + \left[\frac{\gamma_1}{\gamma_2} \right] x_2 + \frac{\gamma_0}{\gamma_2} \right] \quad (37.)$$

but now with the constants $\alpha_1, \alpha_0, \gamma_1/\gamma_2, \gamma_0/\gamma_2$ constrained by the condition of traction continuity at $x_2 = \pm\delta$.

Finally, it is worth noting that neither (36) nor (37) depends on the temperature gradients, although they can be alternatively expressed in terms of both temperature and density gradients, using

the equation of state.

Time dependent Flows with Moving Fracture Boundaries

If we consider the case in which flow in a fracture is driven by the movement of the fracture boundary, in response to a changing stress state in the solid, then we cannot take $v_2 = 0$. Further, in order to calculate the response of the fluid to movement of the boundary it is necessary to consider the time dependent problem. However, we can still approximate the flow as quasi-laminar, that is we can still assume that the gradients in the x_2 direction are much larger than those in the x_1 or x_3 directions. In this case equations (2.) and (3.) still apply and we therefore consider the equations (7.), which are:

$$\frac{\partial p}{\partial x_k} = \mu^{(k)} \frac{\partial^2 v_k}{\partial x_2^2} - \rho v_k \frac{\partial v_k}{\partial x_2} - \rho \frac{\partial v_k}{\partial t} + b_k ; \quad k = 1, 2, 3 \quad (38.)$$

Where $\mu^{(1)} = \mu^{(3)} \equiv \eta$ and $\mu^{(2)} = 4/3\eta + \zeta$ in general, and in special cases, $\mu^{(2)} = 4/3\eta$ when $\zeta \ll \eta$, or $\mu^{(2)} = \eta$ when $\nabla \cdot \mathbf{v} = 0$.

We can again use an approximation for the $k = 2$ equation by noting that if the boundaries at $x_2 = \pm \delta$ move with the particles then, with $x_2 = \delta(x_1, x_3, t)$ as the fracture boundary equation, $v_2 = (\partial\delta/\partial t)$ at $x_2 = \delta$ and similarly for v_2 at $x_2 = -\delta$. Further, since only an expansion or contraction of the fracture aperture affects the flow (local rigid translations or rotations of the fracture produce no deformation) then we can require:

$$v_2|_{x_2=\delta} = -v_2|_{x_2=-\delta}$$

Thus v_2 will be an odd function of x_2 . Using a power law expansion in x_2 for v_2 gives:

$$v_2 = a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3$$

The requirement of an odd function gives $a_0 = a_2 = 0$, so $v_2 = a_1 x_2 + a_3 x_2^3$ to third order. Since we require

$$v_2|_{x_2=\delta} = \frac{\partial\delta}{\partial t} \equiv \delta_t$$

then

$$v_2|_{x_2=\delta} = a_1\delta + a_3\delta^3 = \delta_t$$

or

$$a_1 = \left[\frac{\delta_t}{\delta} \right] - a_3\delta^2; \quad \delta \equiv \delta(x_1, x_3, t)$$

Therefore:

$$v_2 = \left[\frac{\delta_t}{\delta} \right] x_2 + a_3 (x_2^2 - \delta^2) x_2; \quad -\delta \leq x_2 \leq +\delta \quad (39a.)$$

as a third order approximation that should be quite accurate over the small width of fractures.

Now, if the fracture surface function given by $x_2 = \delta(x_1, x_3, t)$ is expressed as a separable function in time and space; so:

$$\delta(x_1, x_3, t) = \Delta(x_1, x_3) s(t)$$

then

$$(\delta_t / \delta) = s_t(t) / s(t)$$

so that the ratio δ_t / δ is a function of time only. Now, if we use

$$v_2 = \left[\frac{\delta_t}{\delta} \right] x_2; \quad -\delta \leq x_2 \leq +\delta \quad (39b.)$$

corresponding to the second order approximation for v_2 , then the quasi-laminar flow inequalities for v_2 , in equations (2.), are met. That is, the inequality

$$\left| \frac{\partial v_2}{\partial x_2} \right| \gg \left[\left| \frac{\partial v_2}{\partial x_1} \right|, \left| \frac{\partial v_2}{\partial x_3} \right| \right]$$

becomes equivalent to

$$|s_t(t)/s(t)| \gg 0$$

Indeed, these inequalities are too strong as conditions since $\partial v_2^{(1)}/\partial x_1 = \partial v_2^{(1)}/\partial x_3 = 0$ are obviously sufficient (but not necessary) to reduce the equations of motion in the manner required. Further, in this

case the conditions in (3) are replaced by

$$\frac{\partial^2 v_2}{\partial x_i^2} = 0 ; i = 1, 2, 3$$

which is sufficient, but not necessary, to give the quasi-laminar approximation to the equations of motion. Therefore with

$$v_2(x_1, x_2, x_3, t) = \left[\frac{\delta_t}{\delta} \right] x_2 = \left[\frac{s_t(t)}{s(t)} \right] x_2 \quad (39c.)$$

$$\delta(x_1, x_3, t) = \Delta(x_1, x_3) s(t)$$

then the equations in (38.) are still appropriate. For v_2 as in (39c.), we get

$$\frac{\partial v_2}{\partial x_2} = \left[\frac{\delta_t}{\delta} \right], \quad \frac{\partial^2 v_2}{\partial x_2^2} = 0$$

and

$$\frac{\partial v_2}{\partial t} = \left[\frac{\delta_{tt}}{\delta} - \left(\frac{\delta_t}{\delta} \right)^2 \right] x_2$$

Now the equation for v_2 is, from (38.):

$$\frac{\partial p^{(1)}}{\partial x_2} = -\rho \left[v_2^{(1)} \frac{\partial v_2^{(1)}}{\partial x_2} + \frac{\partial v_2^{(1)}}{\partial t} \right] + \rho^{(1)} g_2^{(0)} \quad (40.)$$

Where the zeroth order (equilibrium) equation

$$\frac{\partial p^{(0)}}{\partial x_2} = \rho^{(0)} g_2^{(0)}$$

has been used and where $p^{(1)} = p - p^{(0)}$, $\rho^{(1)} = \rho - \rho^{(0)}$ and $v_2 = v_2^{(0)} + v_2^{(1)}$, with $v_2^{(0)} = 0$. (Consequently, v_2 in equations (38) and (39.) is the same as $v_2^{(1)}$ in (40.).)

Using the relations between the boundary surface movement in (39.) and v_2 gives, from (40.):

$$\frac{\partial p^{(1)}}{\partial x_2} = \rho^{(1)} g_2^{(0)} - \rho \left[\left[\frac{\delta_t}{\delta} x_2 \right] \left[\frac{\delta_t}{\delta} \right] + \frac{\delta_{tt}}{\delta} x_2 - \left[\frac{\delta_t}{\delta} \right]^2 x_2 \right]$$

or

$$\frac{\partial p^{(1)}}{\partial x_2} = \rho^{(1)} g_2^{(0)} - \rho \left[\frac{\delta_{tt}}{\delta} \right] x_2$$

Now using the approximation for v_2 from (39.) in all the momentum component equations and using the relations from (39c.), that is:

$$v_2 = \left[\frac{s_t}{s} \right] x_2, \quad \delta(x_1, x_3, t) = \Delta(x_1, x_3) s(t)$$

$$\frac{\delta_t}{\delta} = \frac{s_t(t)}{s(t)}; \quad \frac{\delta_{tt}}{\delta} = \frac{s_{tt}(t)}{s(t)}$$

then we have

$$\frac{\partial p^{(1)}}{\partial x_2} = \rho^{(1)} g_2^{(0)} - \rho \left[\frac{s_{tt}}{s} \right] x_2 \quad (41.)$$

$$\frac{\partial p^{(1)}}{\partial x_i} = \rho^{(1)} g_i^{(0)} - \rho \left[\left[\frac{s_t}{s} \right] \frac{\partial v_i}{\partial x_2} x_2 + \frac{\partial v_i}{\partial t} \right] + \eta \frac{\partial^2 v_i}{\partial x_2^2} \quad (42.)$$

with $i = 1, 3$.

In addition to these equations, conservation of mass requires;

$$\frac{dp}{dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (43.)$$

where

$$\nabla \cdot \mathbf{v} \equiv \frac{\partial v_i}{\partial x_i} \equiv \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

Using the order relations in (2.), that is

$$\left| \frac{\partial v_i}{\partial x_2} \right| \gg \left[\left| \frac{\partial v_i}{\partial x_1} \right|, \left| \frac{\partial v_i}{\partial x_3} \right| \right] \quad i = 1, 2, 3$$

then, consistent with previous approximations:

$$\nabla \cdot \mathbf{v} \approx \frac{\partial v_2}{\partial x_2}$$

With this approximation and the result $v_2 = (s_t/s) x_2$, then $\nabla \cdot \mathbf{v} \approx (s_t/s)$ and (43.) becomes

$$\frac{\partial \rho}{\partial t} + \rho \left[\frac{s_t}{s} \right] = 0$$

Integrating from 0 to t gives

$$\frac{\rho(t)}{\rho(0)} = \frac{s(0)}{s(t)}$$

Since $\rho(0)$ is the density at $t = 0$, it is equivalent to the equilibrium density $\rho^{(0)}$. Therefore:

$$\rho = \rho^{(0)} \left[\frac{s(0)}{s(t)} \right] \quad (44a.)$$

By definition $\rho = \rho^{(0)} + \rho^{(1)}$ and therefore

$$\rho^{(1)} = \rho^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} \quad (44b.)$$

Using the results (44.) in the momentum equations of (41) - (42) gives

$$\left. \begin{aligned} \frac{1}{\rho} \frac{\partial p^{(1)}}{\partial x_2} &= g_2^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} - \left[\frac{s_{tt}}{s} \right] x_2 \\ \frac{1}{\rho} \frac{\partial p^{(1)}}{\partial x_1} &= g_1^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} + \frac{\eta}{\rho} \frac{\partial^2 v_i}{\partial x_2^2} - \left[\frac{\partial v_i}{\partial t} + \left[\frac{s_t}{s} \right] \frac{\partial v_i}{\partial x_2} x_2 \right] \end{aligned} \right\} \quad (45.)$$

Since the interval in x_2 of the open fracture is small, then it is reasonable to expand $p^{(1)}$ in a power series in x_2 over the interval $-\delta \leq x_2 \leq \delta$, and use this series to approximate $p^{(1)}$. Thus, to second

order, with $P_\alpha^{(1)} \equiv \left[\frac{\partial^\alpha p^{(1)}}{\partial x_2^\alpha} \right]_{x_2=0}$

$$p^{(1)}(x, t) = P_0^{(1)}(x_1, x_3, t) + P_1^{(1)}(x_1, x_3, t)x_2 + \frac{1}{2}P_2^{(1)}(x_1, x_3, t)x_2^2 \quad (46a.)$$

Using this expansion in (45.) now gives for the first equation (the x_2 component equation),

$$P_1^{(1)} + P_2^{(1)}x_2 = \rho^{(0)}g_2^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} - \rho^{(0)} \left[\frac{s(0)}{s(t)} \right] \left[\frac{s_{tt}}{s} \right] x_2$$

Furthermore, we can expand $\rho^{(0)}(x)$ and $g_2^{(0)}(x)$ in similar series, so for example

$$\rho^{(0)}(x) = D_0^{(0)}(x_1, x_3) + D_1^{(0)}(x_1, x_3) x_2 + \frac{1}{2}D_2^{(0)}(x_1, x_3) x_2^2; \quad D_\alpha^{(0)} \equiv \left[\frac{\partial^\alpha \rho^{(0)}}{\partial x_2^\alpha} \right]_{x_2=0}$$

Since $-\delta \leq x_2 \leq \delta$ is a small interval, a good approximation is obtained simply by using the first term of the series for both $\rho^{(0)}$ and $g_2^{(0)}$ in the momentum equations. Therefore, taking $g_2^{(0)}$ constant and $\rho^{(0)}$ independent of x_2 , so:

$$\rho^{(0)}(x) \approx D_0^{(0)}(x_1, x_3) \quad (46b.)$$

then on has

$$P_1^{(1)}(x_1, x_3, t) + P_2^{(1)}(x_1, x_3, t)x_2 = D_0^{(0)}(x_1, x_3) g_2^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} - D_0^{(0)}(x_1, x_3) \left[\frac{s(0)}{s(t)} \right] \left[\frac{s_{tt}}{s} \right] x_2$$

Now the coefficients multiplying the powers of x_2 in this expression are independent of x_2 and therefore this expression can only be satisfied if all the coefficients of powers of x_2 vanish independently. Therefore, to second order we must have:

$$\begin{aligned} P_1^{(0)} - D_0^{(0)} g_2^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} &= 0 \\ P_2^{(1)} + D_0^{(0)} \left[\frac{s(0)}{s(t)} \right] \left[\frac{s_{tt}}{s} \right] &= 0 \end{aligned}$$

Thus, re-introducing $\rho^{(0)}$ for $D_0^{(0)}$ with the understanding that this is the value of ρ^0 at $x_2 = 0$:

$$\begin{aligned} P_1^{(0)}(x_1, x_3, t) &= \rho^{(0)} g_2^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} \\ P_2^{(1)}(x_1, x_3, t) &= -\rho^{(0)} \left[\frac{s(0)}{s(t)} \right] \left[\frac{s_{tt}}{s} \right] \end{aligned} \quad (47.)$$

Therefore; from (46a.) and (47.):

$$p^{(1)}(x, t) = P_0^{(1)}(x_1, x_3, t) + \rho^{(0)} g_2^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} x_2 - \rho^{(0)} \left[\frac{s(0)}{s(t)} \right] \left[\frac{s_{tt}}{s} \right] \frac{x_2^2}{2} \quad (48.)$$

The second set of component equations in (45.) become, using (48.) and $\rho^{(0)} \approx D_0^{(0)}$ and $g_i^{(0)}$ constant:

$$\begin{aligned} \frac{1}{\rho} \left[\frac{\partial P_0^{(1)}}{\partial x_i} + \frac{\partial \rho^{(0)}}{\partial x_i} g_2^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} x_2 - \frac{\partial \rho^{(0)}}{\partial x_i} \left[\frac{s(0)}{s(t)} \right] \left[\frac{s_{tt}}{s} \right] \frac{x_2^2}{2} \right] = \\ g_i^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} + \frac{\eta}{\rho} \frac{\partial^2 v_i^{(1)}}{\partial x_2^2} - \left[\frac{s_t}{s} \right] \frac{\partial v_i^{(1)}}{\partial x_2} x_2 - \frac{\partial v_i^{(1)}}{\partial t} \end{aligned} \quad (49.)$$

where the index i takes on the values 1 and 3 for components in the x_1 and x_3 directions. Now, as with $p^{(1)}(x, t)$, the particle velocity components $v_i(x, t)$ can be expanded in x_2 , to second order, as:

$$v_i^{(1)}(x, t) = {}_0V_i^{(1)}(x_1, x_3, t) + {}_1V_i^{(1)}(x_1, x_3, t)x_2 + \frac{1}{2} {}_2V_i^{(1)}(x_1, x_3, t)x_2^2 \quad (50.)$$

where

$${}_aV_i^{(1)}(x_1, x_3, t) \equiv \left[\frac{\partial^{(a)} v_i}{\partial x_2^{(a)}} \right]_{x_2=0}$$

Using the approximation of (50.) in (49.) gives

$$\begin{aligned} & \frac{\partial P_0^{(1)}}{\partial x_i} + \frac{\partial p^{(0)}}{\partial x_i} g_2^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} x_2 - \frac{1}{2} \frac{\partial p^{(0)}}{\partial x_i} \left[\frac{s(0)}{s(t)} \right] \left[\frac{s_{tt}}{s} \right] x_2^2 = \\ & \rho^{(0)} g_i^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} + \eta {}_2V_i^{(1)} - \rho \left[\left[\frac{s_t}{s} \right] {}_1V_i^{(1)} x_2 + \left[\frac{s_t}{s} \right] {}_2V_i^{(1)} x_2^2 + \frac{\partial {}_0V_i^{(1)}}{\partial t} + \frac{\partial {}_1V_i^{(1)}}{\partial t} x_2 + \frac{1}{2} \frac{\partial {}_2V_i^{(1)}}{\partial t} x_2^2 \right] \end{aligned}$$

Here again the coefficient multipliers of the powers of x_2 are independent of x_2 and so these coefficients must vanish independently if the equation is to be satisfied for all x_2 . Therefore we must have:

$$\frac{\partial P_0^{(1)}}{\partial x_i} = \eta {}_2V_i^{(1)} + \rho^{(0)} g_i^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} - \rho^{(0)} \left[\frac{s(0)}{s(t)} \right] \frac{\partial {}_0V_i^{(1)}}{\partial t} \quad (51a.)$$

$$\frac{\partial p^{(0)}}{\partial x_i} g_2^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} = - \rho^{(0)} \left[\left[\frac{s_t}{s} \right] {}_1V_i^{(1)} + \frac{\partial {}_1V_i^{(1)}}{\partial t} \right] \quad (51b.)$$

$$\frac{\partial p^{(0)}}{\partial x_i} \left[\frac{s_{tt}}{s} \right] = \rho^{(0)} \left[2 \left[\frac{s_t}{s} \right] {}_2V_i^{(1)} + \frac{\partial {}_2V_i^{(1)}}{\partial t} \right] \quad (51c.)$$

where the result $\rho = \rho^{(0)}(s(0) / s(t))$, from the equation of continuity, has been used repeatedly.

Now, as in the steady state case, take:

$${}_aV_i^{(1)}(x_1, x_3, t) \equiv \beta_p^a \frac{\partial P_0^{(1)}}{\partial x_i} + k_i^a(x_1, x_3, t) \quad (52a.)$$

where

$$v_i^{(1)} = {}_0V_i^{(1)} + {}_1V_i^{(1)}x_2 + \frac{1}{2} {}_2V_i^{(1)}x_2^2$$

and with the functions taken as separable in time, so that:

$$\begin{aligned} P_0^{(1)} &= P_0^{(1)}(x_1, x_3) f_p(t) \\ k_i^\alpha &= K_i^\alpha(x_1, x_3) h_\alpha^{(i)}(t) \end{aligned} \quad (52b.)$$

(Here, there is no sum on repeated greek letter indices. Further, Latin indices enclosed in parentheses are excluded from the summation convention.) Using this representation of the velocity field, the component equations of motion in (51.) become:

$$\begin{aligned} \frac{\partial P_0^{(1)}}{\partial x_i} \left[f_p - \eta \beta_p^{(2)} f_p + \rho^{(0)} \left[\frac{s(0)}{s(t)} \right] \beta_p^{(0)} \frac{df_p}{dt} \right] + \rho^{(0)} \left[\frac{s(0)}{s(t)} \right] K_i^{(0)} \frac{dh_0^{(i)}(t)}{dt} - \rho^{(0)} g_i^{(0)} \left[\frac{s(0)}{s(t)} - 1 \right] \\ - \eta K_i^{(2)} h_2^{(i)}(t) = 0 \end{aligned} \quad (53a.)$$

$$\beta_p^{(1)} \frac{\partial P_0^{(1)}}{\partial x_i} \left[\left[\frac{s_t}{s} \right] f_p + \frac{df_p}{dt} \right] + K_i^{(1)} \left[\left[\frac{s_t}{s} \right] h_1^{(i)} + \frac{dh_1^{(i)}}{dt} \right] + \left[\frac{1}{\rho^{(0)}} \right] \frac{\partial \rho^{(0)}}{\partial x_i} g_2^{(0)} \left[1 - \frac{s(t)}{s(0)} \right] = 0 \quad (53b.)$$

$$\beta_p^{(2)} \frac{\partial P_0^{(1)}}{\partial x_i} \left[2 \left[\frac{s_t}{s} \right] f_p + \frac{df_p}{dt} \right] + K_i^{(2)} \left[2 \left[\frac{s_t}{s} \right] h_2^{(i)} + \frac{dh_2^{(i)}}{dt} \right] - \left[\frac{1}{\rho^{(0)}} \right] \frac{\partial \rho^{(0)}}{\partial x_i} \left[\frac{s_{tt}}{s} \right] = 0 \quad (52c.)$$

There are many possibilities for solutions of these equations since all the functions f_p , $h_\alpha^{(i)}$, $K_i^{(\alpha)}$ and constants $\beta_p^{(\alpha)}$, for $\alpha = 0, 1, 2$ and $i = 1, 3$ can be freely chosen to satisfy the relations. These multiple possibilities reflect the many flows that are possible. Only with additional constraining equations (e.g. involving boundary conditions) can a subset of solutions be isolated and, even then, many solutions may be possible reflecting different flow patterns in different ranges of physical parameters or for different initial conditions.

Type I Transient Laminar Flow

Inspection of the equations in (53.) suggests a class of solution having characteristics of non-steady near laminar flow in a open fracture and this class of elemental solution will be developed as a type of interest for use in constructing global solutions for flow in complex fracture systems.

To develop such solutions we can chose the constants $\beta_p^{(\alpha)}$ to be:

$$\beta_p^{(0)} = 0 \quad , \quad \beta_p^{(1)} = 0 \quad , \quad \beta_p^{(2)} = \frac{1}{\eta}$$

In this case the equations in (53.) reduce to:

$$\rho^{(0)} \left[\frac{s(0)}{s(t)} \right] \left[K_i^{(0)} \frac{\partial h_0^{(i)}}{\partial t} - g_i^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} - \left[\frac{\eta}{\rho^{(0)}} \right] K_i^{(2)} \left[\frac{s(t)}{s(0)} \right] h_2^{(i)} \right] = 0 \quad (54a.)$$

$$K_i^{(1)} \left[\left[\frac{s_t}{s} \right] h_1^{(i)} + \frac{dh_1^{(i)}}{dt} \right] + \frac{1}{\rho^{(0)}} \left[\frac{\partial \rho^{(0)}}{\partial x_i} \right] g_2^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} = 0 \quad (54b.)$$

$$\frac{1}{\eta} \frac{\partial P_0^{(1)}}{\partial x_i} \left[2 \left[\frac{s_t}{s} \right] f_p + \frac{df_p}{dt} \right] + K_i^{(2)} \left[2 \left[\frac{s_t}{s} \right] h_2^{(i)} + \frac{dh_2^{(i)}}{dt} \right] - \frac{1}{\rho^{(0)}} \left[\frac{\partial \rho^{(0)}}{\partial x_i} \right] \left[\frac{s_{tt}}{s} \right] = 0 \quad (54c.)$$

These equations can now be satisfied in the following manner:

(i.) In (54c.) note that the only term involving a pressure gradient term can be eliminated if

$$\frac{df_p}{dt} + 2 \left[\frac{s_t}{s} \right] f_p = 0$$

Therefore, upon integration over the range (0,t), we get:

$$f_p(t) = f_p(0) \left[\frac{s(0)}{s(t)} \right]^2 \quad (55a.)$$

with f_p normalized to unity at $t = 0$, then

$$f_p(t) = \left[\frac{s(0)}{s(t)} \right]^2 \quad (55b.)$$

(ii.) Inspection of the equations in (54.) shows that if:

$$K_i^{(0)} = K_i^{(1)} = K_i^{(2)} = \frac{1}{\rho^{(0)}} \left[\frac{\partial \rho^{(0)}}{\partial x_i} \right] \quad (56.)$$

Then, also using (55.), the equations in (54) reduce to:

$$\left[\frac{\partial \rho^{(0)}}{\partial x_i} \right] \frac{\partial h_0^{(i)}}{\partial t} - \rho^{(0)} g_i^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} - \left[\frac{\partial \rho^{(0)}}{\partial x_i} \right] \left[\frac{\eta}{\rho^{(0)}} \right] \left[\frac{s(t)}{s(0)} \right] h_2^{(i)} = 0 \quad (57a.)$$

$$\frac{dh_1^{(i)}}{dt} + \left[\frac{s_t}{s} \right] h_1^{(i)} + g_2^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} = 0 \quad (57b.)$$

$$\frac{dh_2^{(i)}}{dt} + 2 \left[\frac{s_t}{s} \right] h_2^{(i)} - \left[\frac{s_{tt}}{s} \right] = 0 \quad (57c.)$$

(iii.) Noting that (e.g. Richards, Manual of Mathematical Physics, Pergamon, 1959; p 341):

$$\frac{dg(t)}{dt} + f_1(t)g(t) + f_2(t) = 0 \quad (58a.)$$

$$g(t) = \left[c_0 - \int_{\beta}^t f_2(t') \left[\exp \int_{\alpha}^{t'} f_1(t'') dt'' \right] dt' \right] \exp \int_{\alpha}^t f_1(t') dt' \quad (58b.)$$

where α and β are arbitrary constants, which upon changes in their values results, at most, in changes in the constant c_0 . Applying this result to (57.) allows each of the first order differential equations to be solved in closed form. In particular, from (57c.) we get:

$$h_2^{(i)}(t) = \left\{ \gamma_2^{(i)} + \int_0^t \frac{s_t t'}{s} \left[\exp \int_0^{t'} \frac{2s_{t''}}{s} dt'' \right] dt' \right\} \exp \left[-2 \int_0^t \frac{s_t'}{s} dt' \right]$$

so that

$$h_2^{(i)}(t) = \left[\frac{s(0)}{s(t)} \right]^2 \left\{ \gamma_2^{(i)} + \left[\frac{1}{s(0)} \right]^2 \int_0^t s_t t' s dt' \right\} \quad (59a.)$$

or, after integrating by parts;

$$h_2^{(i)}(t) = \gamma_2^{(i)} \left[\frac{s(0)}{s(t)} \right]^2 + \left[\frac{1}{s} \right]^2 \left\{ s(t)s_t(t) - s(0)s_t(0) - \int_0^t s_t^2 dt' \right\} \quad (59b.)$$

where $\gamma_2^{(i)}$ are constants, which may be different for $i = 1$ and 3 .

Similarly, equation (57b.) is

$$\frac{dh_1^{(i)}}{dt} + \left[\frac{s_t}{s} \right] h_1^{(i)} + g_2^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} = 0$$

and has the general solution

$$h_1^{(i)}(t) = \left\{ \gamma_1^{(i)} - \int_0^t g_2^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} \exp \left[\int_0^{t'} \frac{s_{t''}}{s} dt'' \right] dt' \right\} \exp \left[- \int_0^t \frac{s_t'}{s} dt' \right]$$

which integrates to:

$$h_1^{(i)} = \left[\frac{s(0)}{s(t)} \right] \left[\gamma_1^{(i)} + g_2^{(0)} \int_0^t \frac{s(t')}{s(0)} \left\{ 1 - \frac{s(t')}{s(0)} \right\} dt' \right] \quad (60.)$$

Here $\gamma_1^{(i)}$ represents constants for $i = 1$ and 3 .

Finally, (57a.) now becomes, using (59.) in the equation:

$$\left[\frac{\partial p^{(0)}}{\partial x_i} \right] \frac{dh_0^{(i)}}{dt} = \left[\frac{\partial p^{(0)}}{\partial x_i} \right] \left[\frac{\eta}{\rho^{(0)}} \right] \left[\frac{s(0)}{s(t)} \right] \left\{ \gamma_2^{(i)} + \left[\frac{1}{s(0)} \right]^2 \int_0^t s_{t't'} s dt' \right\} + \rho^{(0)} g_i^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\}$$

Integration over the interval (0,t) gives:

$$\begin{aligned} \left[\frac{\partial p^{(0)}}{\partial x_i} \right] [h_0^{(i)}(t) - h_0^{(i)}(0)] &= \left[\frac{\partial p^{(0)}}{\partial x_i} \right] \left[\frac{\eta}{\rho^{(0)}} \right] \left\{ \gamma_2^{(i)} \int_0^t \frac{s(0)}{s(t')} dt' + \left[\frac{1}{s(0)} \right] \int_0^t \frac{dt'}{s(t')} \int_0^{t'} s_{t''t'''} s(t'') dt'' \right\} \\ &+ \rho^{(0)} g_i^{(0)} \int_0^t \left\{ 1 - \frac{s(t')}{s(0)} \right\} dt' \end{aligned} \quad (61a.)$$

Now, collecting results we have:

$$\begin{aligned} v_i^{(1)}(x, t) &= {}_0V_i^{(1)}(x_1, x_3, t) + {}_1V_i^{(1)}(x_1, x_3, t) x_2 + \frac{1}{2} {}_2V_i^{(1)}(x_1, x_3, t) x_2^2 \\ {}_\alpha V_i(x_1, x_3, t) &= \beta_p^{(\alpha)} \frac{\partial P_0^{(1)}}{\partial x_i} + K_i^{(\alpha)} = \beta_p^{(\alpha)} \frac{\partial P_0^{(1)}}{\partial x_i} f_p(t) + K_i^{(\alpha)}(x_1, x_3) h_\alpha^{(i)}(t) \end{aligned}$$

Therefore, with $\beta_p^{(0)} = \beta_p^{(1)} = 0$ and $\beta_p^{(2)} = 1/\eta$:

$$v_i^{(1)}(x, t) = K_i^{(0)} h_0^{(i)} + K_i^{(1)} h_1^{(i)} x_2 + K_i^{(2)} h_2^{(i)} \frac{x_2^2}{2} + \beta_p^{(2)} \left[\frac{\partial P_0^{(1)}}{\partial x_i} \right] f_p(t) \frac{x_2^2}{2}$$

where $-\delta \leq x_2 \leq \delta$ and $\delta(x_1, x_3, t) = \Delta(x_1, x_3)s(t)$ defines the fracture surfaces at $x_2 = \pm \delta$. Since

$$K_i^{(0)} = K_i^{(1)} = K_i^{(2)} = \left[\frac{1}{\rho^{(0)}} \right] \frac{\partial p^{(0)}}{\partial x_i}$$

then, with $\beta_p^{(2)} = 1/\eta$, $f_p(t) = \left[\frac{s(0)}{s(t)} \right]^2$ and taking $\gamma_2^{(i)} = (\eta/\rho^{(0)})^{-1}$ for $i = 1, 3$:

$$v_i^{(1)}(x, t) = \left[\frac{1}{\rho^{(0)}} \right] \frac{\partial \rho^{(0)}}{\partial x_i} \left[h_0^{(i)}(0) + \int_0^t \frac{s(0)}{s(t')} dt' + \left[\frac{\eta}{\rho^{(0)}} \right] \int_0^t \frac{dt'}{s(t')} \int_0^{t'} s_{t''} \left[\frac{s(t'')}{s(0)} \right] dt'' + h_1^{(i)} x_2 + h_2^{(i)} \frac{x_2^2}{2} \right] \\ + g_i^{(0)} \int_0^t \left\{ 1 - \frac{s(t')}{s(0)} \right\} dt' + \frac{1}{2\eta} \left[\frac{\partial P_0^{(1)}}{\partial x_i} \right] \left[\frac{s(0)}{s(t)} \right]^2 x_2^2 \quad (62.)$$

where $i = 1, 3$ and where $h_1^{(i)}$ and $h_2^{(i)}$ are given by equations (59.) and (60.). Here also

$$v_2^{(1)}(x, t) = \left[\frac{s_t(t)}{s(t)} \right] x_2 \quad (63.)$$

$$\rho(x, t) = \rho^{(0)} + \rho^{(1)} = \rho^{(0)} \left[\frac{s(0)}{s(t)} \right] \quad (64.)$$

$$p^{(1)}(x, t) = P_0^{(1)}(x_1, x_3, t) + \rho^{(0)} g_2^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} x_2 - \frac{\rho^{(0)}}{2} \left[\frac{s(0)}{s(t)} \right] \left[\frac{s_{tt}}{s} \right] x_2^2 \quad (65.)$$

In all these expressions $\rho^{(0)}(x)$ is evaluated at $x_2 = 0$ in the fracture zone, defined by $-\delta \leq x_2 \leq \delta$, as an approximation. Therefore $\rho^{(0)}$ is, at most, a function of x_1 and x_3 only.

The velocity components $v_i^{(1)}$ given in (62.) can also be written in a different form by noting that from (64.):

$$\frac{\partial \rho^{(1)}}{\partial x_i} = \frac{\partial \rho^{(0)}}{\partial x_i} \left[\frac{s(0)}{s(t)} - 1 \right] = \frac{\rho^{(1)}}{\rho^{(0)}} \left[\frac{\partial \rho^{(0)}}{\partial x_i} \right]$$

Therefore

$$\frac{1}{\rho^{(0)}} \frac{\partial \rho^{(0)}}{\partial x_i} = \frac{1}{\rho^{(1)}} \frac{\partial \rho^{(1)}}{\partial x_i} \quad (66.)$$

and so the equilibrium density gradient can be replaced by the perturbed density gradient in (62.).

Since an equation of state for the fluid relates the density to the pressure and temperature, that is:

$$\rho = \rho(p, T) \quad (67.)$$

then for small changes in the fluid state variables, denoted by δp , δp , δT , we have:

$$\delta \rho = \left[\frac{\partial \rho}{\partial p} \right]_T \delta p + \left[\frac{\partial \rho}{\partial T} \right]_p \delta T \quad (68a.)$$

Identifying these variations in the state variables with $\rho^{(1)}$, $p^{(1)}$ and $T^{(1)}$, the variations from equilibrium associated with flow of the fluid, gives

$$\rho^{(1)} = \left[\frac{\partial \rho}{\partial p} \right]_T^{(0)} p^{(1)} + \left[\frac{\partial \rho}{\partial T} \right]_p^{(0)} T^{(1)} \quad (68a.)$$

$$\frac{\partial \rho^{(1)}}{\partial x_i} = \left[\frac{\partial \rho}{\partial p} \right]_T^{(0)} \frac{\partial p^{(1)}}{\partial x_i} + \left[\frac{\partial \rho}{\partial T} \right]_p^{(0)} \frac{\partial T^{(1)}}{\partial x_i}$$

where the superscript (0) on the partial derivatives of the density denotes evaluation of the derivatives at the equilibrium state; that is at $\rho^{(0)}$, $p^{(0)}$ and $T^{(0)}$. (In the second relation in (68b.), involving gradients, the spatial variations of the coefficients $(\partial \rho / \partial p)_T$ and $(\partial \rho / \partial T)_p$ are assumed small enough that their gradients can be neglected.) Therefore, denoting the fluid moduli represented by the derivatives of density with respect to pressure and temperature by

$$\left. \begin{aligned} \alpha_p &= - \frac{1}{\rho^{(0)}} \left[\frac{\partial \rho}{\partial p} \right]_T^{(0)} \\ \alpha_T &= - \frac{1}{\rho^{(0)}} \left[\frac{\partial \rho}{\partial T} \right]_p^{(0)} \end{aligned} \right\} \quad (69.)$$

then

$$\left[\frac{1}{\rho^{(0)}} \right] \frac{\partial \rho^{(0)}}{\partial x_i} = - \left[\alpha_p \frac{\partial p^{(1)}}{\partial x_i} + \alpha_T \frac{\partial T^{(1)}}{\partial x_i} \right] \quad (70.)$$

Using the relations (66.) and (70.) in the equation (62.) for $v_i^{(1)}(x, t)$ allows the result to be expressed in alternate forms. These results have forms similar to the steady state solutions obtained earlier.

Type II Transient Laminar Flow

If the equilibrium density $\rho^{(0)}$ is taken to be approximately constant over the spatial domain of the elemental fracture, then a second type of solution to the basic "coefficient equations" in (52.) can be easily found. In particular, if we take:

$$\beta_p^{(0)} = \frac{1}{\rho^{(0)}}, \quad \beta_p^{(1)} = 0, \quad \beta_p^{(2)} = 0 \quad (71.)$$

then the equations in (52.) become

$$\frac{\partial P_0^{(1)}}{\partial x_i} \left\{ f_0 + \left[\frac{s(0)}{s(t)} \right] \frac{df_p}{dt} \right\} + \rho^{(0)} \left[\frac{s(0)}{s(t)} \right] K_i^{(0)} \frac{dh_0^{(i)}}{dt} - \rho^{(0)} g_i^{(0)} \left\{ \frac{s(0)}{s(t)} - 1 \right\} - \eta K_i^{(2)} h_2^{(i)} = 0 \quad (72a.)$$

$$K_i^{(1)} \left[\left[\frac{s_t}{s} \right] h_i^{(i)} + \frac{dh_i^{(i)}}{dt} \right] + \frac{1}{\rho^{(0)}} \frac{\partial \rho^{(0)}}{\partial x_i} g_2^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} = 0 \quad (72b.)$$

$$K_i^{(2)} \left[2 \left[\frac{s_t}{s} \right] h_2^{(i)} + \frac{dh_2^{(i)}}{dt} \right] - \frac{1}{\rho^{(0)}} \frac{\partial \rho^{(0)}}{\partial x_i} \left[\frac{s_{tt}}{s} \right] = 0 \quad (72c.)$$

These equations can be satisfied in the same way as were those for Type I flow, that is:

- (i.) The dependence on the terms involving the pressure derivatives are eliminated by requiring that f_p satisfy:

$$\frac{df_p}{dt} + \left[\frac{s(t)}{s(0)} \right] f_p = 0$$

so that

$$f_p(t) = f_p(0) \exp \left[\frac{-1}{s(0)} \int_0^t s(t') dt' \right] ; f_p(0) \equiv 1$$

- (ii.) Take the functions $K_i^{(\alpha)}(x_1, x_3)$, where $\alpha = 0, 1, 2$, to have the form

$$K_i^{(0)} = K_i^{(1)} = K_i^{(2)} = \frac{1}{\rho^{(0)}} \frac{\partial \rho^{(0)}}{\partial x_i} = \frac{1}{\rho^{(1)}} \frac{\partial \rho^{(1)}}{\partial x_i}$$

for $i = 1$ and 3 .

In this case the equations in (72.) reduce to:

$$\frac{\partial \rho^{(0)}}{\partial x_i} \left[\frac{s(0)}{s(t)} \right] \frac{dh_0^{(i)}}{dt} - \rho^{(0)} g_i^{(0)} \left[\frac{s(0)}{s(t)} - 1 \right] - \left[\frac{\eta}{\rho^{(0)}} \right] \frac{\partial \rho^{(0)}}{\partial x_i} h_2^{(i)} = 0 \quad (73a.)$$

$$\frac{dh_1^{(i)}}{dt} + \left[\frac{s_t}{s} \right] h_1^{(i)} + g_2^{(0)} \left\{ 1 - \frac{s(t)}{s(0)} \right\} = 0 \quad (73b.)$$

$$\frac{dh_2^{(i)}}{dt} + 2 \left[\frac{s_t}{s} \right] h_2^{(i)} - \left[\frac{s_{tt}}{s} \right] = 0 \quad (73c.)$$

These equations are identical to those in (57.), which apply to the Type I case, so that the solutions for the time dependent functions $h_\alpha^i(t)$ for this second case are the same as those obtained earlier. That is, with $i = 1, 3$:

$$h_2^{(i)}(t) = \left[\frac{s(0)}{s(t)} \right]^2 \left\{ \gamma_2^{(i)} + \left[\frac{1}{s(0)} \right]^2 \int_0^t s_{tt'} s dt' \right\} \quad (74a.)$$

$$h_2^{(i)}(t) = \left[\frac{s(0)}{s(t)} \right] \left\{ \gamma_2^{(i)} + g_2^{(0)} \int_0^t \frac{s(t')}{s(0)} \left\{ 1 - \frac{s(t')}{s(0)} \right\} dt' \right\} \quad (74b.)$$

$$\begin{aligned} \left[\frac{\partial p^{(0)}}{\partial x_i} \right] [h_0^{(i)}(t) - h_0^{(i)}(0)] &= \left[\frac{\partial p^{(0)}}{\partial x_i} \right] \left[\eta / \rho^{(0)} \right] \left\{ \gamma_2^{(i)} \int_0^t \frac{s(0)}{s(t')} dt' + \left[\frac{1}{s(0)} \right] \int_0^t \frac{dt'}{s(t')} \int_0^{t'} s_{t''} s(t'') dt'' \right\} \\ &+ \rho_0 g_i^{(0)} \int_0^t \left\{ 1 - \frac{s(t')}{s(0)} \right\} dt' \end{aligned} \quad (74c.)$$

where we can take $\gamma_2^{(i)} = (\eta / \rho^{(0)})^{-1}$, as in the type I case with $\rho^{(0)} \approx$ constant. The solution for the velocity components is now given by:

$$v_i^{(1)}(x, t) = K_i^{(0)} h_0^{(i)} + K_i^{(1)} h_1^{(i)} x_2 + K_i h_2^{(i)} \frac{x_2^2}{2} + \frac{1}{\rho^{(0)}} \left[\frac{\partial P_0^{(1)}}{\partial x_i} \right] f_p(t)$$

or, using the results in (i) and (ii) above:

$$\begin{aligned} v_i^{(1)}(x, t) &= \left[\frac{1}{\rho^{(1)}} \right] \frac{\partial p^{(1)}}{\partial x_i} \left[\left\{ h_0^{(i)}(0) + H_0^{(i)}(t) \right\} + h_1^{(i)} x_2 + h_2^{(i)} \frac{x_2^2}{2} \right] + g_i^{(0)} \int_0^t \left\{ 1 - \frac{s(t')}{s(0)} \right\} dt' \\ &+ \frac{1}{\rho^{(0)}} \left[\frac{\partial P_0^{(1)}}{\partial x_i} \right] \exp \left[\frac{-1}{s(0)} \int_0^t s(t') dt' \right] \end{aligned} \quad (75.)$$

where

$$H_0^{(i)} = \int_0^t \frac{s(0)}{s(t')} dt' + \frac{\eta}{\rho^{(0)}} \int_0^t \frac{dt'}{s(t')} \int_0^{t'} s_{t''} \left[\frac{s(t'')}{s(0)} \right] dt''$$

has been introduced and $\gamma_2^{(i)} = (\eta / \rho^{(0)})^{-1}$ has been used, as in the Type I solution. This solution has the same form as does the Type I solution, except the dependence in the pressure gradient term is different. Also we note that, as with the previous solution type:

$$\left[\frac{1}{\rho^{(0)}} \right] \frac{\partial p^{(0)}}{\partial x_i} = \left[\frac{1}{\rho^{(1)}} \right] \frac{\partial p^{(1)}}{\partial x_i}$$

and, from the existence of an equation of state,

$$\frac{\partial p^{(1)}}{\partial x_i} = - \left[\rho^{(0)} \alpha_p \frac{\partial p^{(1)}}{\partial x_i} + \rho^{(0)} \alpha_T \frac{\partial T^{(1)}}{\partial x_i} \right]$$

so that these relations can be used in (75.) to express the result in alternate forms.

The expressions for $v_2^{(1)}(x, t)$, $\rho(x, t)$ and $p^{(1)}(x, t)$ for this solution are the same as those given in equations (63.)-(65.) for the Type I solution.

Summary of Results

The solutions obtained apply only to cases in which fluid flows within a fracture is quasi-laminar, in that the spatial variations in particle velocities normal to the fracture boundaries are much more rapid than those in directions tangential to the boundaries. Both steady state solutions, where the fracture boundaries do not move and the particle velocity field is independent of time, and transient, time dependent solutions in which the fracture boundaries are allowed to move in an arbitrary, but well behaved manner, have been obtained. The latter class of solutions incorporate movement of the fracture boundaries in response to fluid pressure changes in the fracture as well as in response to stress changes in the solid medium, within which the fractures exist. In particular, they satisfy the condition that the normal component of the fluid particle velocity is equal to the particle velocity of the solid material at the fracture boundary, an expression of conservation of mass at the boundary.

All the solutions contain constant parameters that are fixed by boundary and initial conditions. The commonly applied boundary condition that the fluid particle velocity components tangent to the fracture boundary should vanish would imply, in the case of moving boundaries driven by relaxing stress fields or waves in the solid, an ideal fluid response at the boundary where no tangential tractions are transmitted to the fluid; that is perfect slip at the boundary. On the other hand, perfect slip of the fluid along the boundary of the fracture would also allow solutions in which the tangential components of the fluid particle velocity are unconstrained. Thus, no good physical argument justifying direct constraints on the tangential fluid velocities at the boundary is available in a moving boundary case. However, as noted earlier in the introduction, the condition that the tractions be continuous at the boundary, which expresses conservation of momentum at the boundary, can be used to specify the boundary

movement with time in terms of the dynamic pressure $p^{(1)}$ in the fluid. In addition, a viscous fluid, as treated here, would transmit tangential tractions from the solid into the fluid and this would produce a shear flow near the boundary. In this case fluid particle velocities tangent to the boundary would not necessarily be equal to tangential solid particle velocities at the boundary, but there would be a tendency for the fluid to be dragged, tangentially, along with the solid movement. The solutions obtained here, or variants of them, should have the generality and flexibility to incorporate fluid flows of this latter type and we shall consider such cases in subsequent studies through the application of shear traction continuity conditions at the fluid-solid fracture boundary using fundamental solutions of the type obtained in this study.

In addition to conditions along the fracture surface, it is necessary to consider end conditions other than those free-flow conditions implicitly used here. That is, elemental solutions satisfying conservation of mass and momentum in the fluid within a fracture with unconstrained end conditions have been obtained, but these solutions must be constrained by end conditions applied at the spatial intersections with other fractures. In this case the solutions obtained must be "joined" to other solutions of the same form by requiring conservation of mass and momentum between the solutions across the intersection zone. Thus, for a network of intersecting fractures, solutions of the form obtained here must be joined throughout by applying additional conservation relations so that each fracture flow solution will be constrained to match these conditions. This matching will have to be accomplished by proper choice of the remaining free parameters in the solutions given here. This aspect of the problem will be considered in Part II of this study. The results given here are therefore the fundamental solutions required to complete the formulation for fluid flow in a intersecting network of fractures within a deformable and dynamically changing solid, as described in the introduction.

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