DEVELOPMENT OF A FORCE SPECIFICATION FOR A
FORCE-LIMITED RANDOM VIBRATION TEST

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ABSTRACT

Vibration testing techniques have been developed and employed that reduce the overtesting caused by the essentially infinite mechanical impedance of the shaker in conventional vibration tests. With these “force-limiting” techniques, two vibration test specifications are used: the conventional acceleration specification, and an interface force specification. The vibration level of the shake table is controlled such that neither the table acceleration nor the force transmitted to the test item exceeds its specification, hence the name “dual control” vibration test. The effect of limiting the shake table vibration to the force specification is to reduce (“notch”) the shaker acceleration near some of the test item’s resonance frequencies.

Several methods of deriving the force specification have been described in the literature [1, 2]. A new method is proposed in this paper that is based on a modal method of coupling two dynamic systems, in this case the “source” or launch vehicle, and the “load” or payload. The only information that is required is an experimentally-measurable frequency-response function (FRF) called the dynamic mass for both the source and the load. The method, referred to as the coupled system, modal approach (CSMA) method, is summarized and compared to an existing method of determining the force specification for force-limited vibration testing.

KEYWORDS

Force limited vibration test, random vibration test.

INTRODUCTION

A conventional random vibration test can result in significant overtest in some frequency bands of the test - specifically, near certain natural frequencies of the test item in its tested (fixed-base) boundary condition. Near these frequencies, the test item behaves like a vibration absorber when it is mounted on the launch vehicle: the acceleration at some locations of the test item reaches a peak (which may be damaging to the structure or components mounted at these locations), but the acceleration at the test item / launch vehicle interface is sharply reduced. This effect results in a deep valley in the interface acceleration that would be measured in flight, but the acceleration specification (envelope) that is based on the flight data “smooths over” such valleys. This is one cause of overtest in conventional vibration tests.

A force-limited vibration test attempts to reduce this source of overtest by limiting the excitation based on both the interface acceleration specification and an interface force specification. At those frequencies where the payload would act like a vibration absorber and reduce the interface acceleration of the coupled launch vehicle and payload (the “source” and the “load”), the interface force will govern the vibration level of the test. In all other frequency ranges, where there are no vibration absorber effects, the conventional base acceleration will govern the vibration level of the test.

The derivation of the specification (envelope) for the interface force spectrum is the subject of this paper. A scheme that is based on a modal method of coupling two independent dynamic systems is described and the results of two numerical simulations are discussed. The “coupled system, modal approach” (CSMA) method is described and illustrated with data from a force-limited vibration test of a small satellite. The method is also illustrated in two numerical simulations. The CSMA method is compared to an existing method of determining the force specification for force-limited vibration testing (the TDFS method of Scharpton [1]), and the assumptions and limitations of the technique are discussed.

DERIVATION OF THE CSMA FORCE SPECIFICATION

1) Required Information

The CSMA method of determining the force specification for a force-limited vibration test is based on information about the source and load that is contained in their
modeled as linear systems. If nonlinearities exist in either based on a linear model is valid.

They may also be measured in a modal “tap test,” using a hammer to excite the test item at its interface. The dynamic mass may also be estimated analytically (for example, using a finite-element model).

In addition to the source and load dynamic mass FRF’s, the interface acceleration specification (envelope) is required to calculate the interface force specification.

2) Modal Representation of the Source and Load

When the dynamic mass FRF’s of the source and load have been measured, three modal parameters for each natural mode of vibration that has a significant participation in this interface-excited response can be found. The parameters to be extracted include the natural frequency, the damping ratio, and the modal effective mass for each mode. The modal effective mass is the amount of mass that is represented by each natural mode of a structure (the sum of the modal effective masses over all the natural modes is equal to the total mass of the structure). When the dynamic system is represented as a system of one-degree-of-freedom oscillators in parallel (sometimes called an asparagus patch model), the modal effective mass is equal to the mass of each oscillator. Further discussion of the concept of modal effective mass is found in reference [3].

The process of finding the parameters that define the model that is the best fit to the experimentally-measured data is a well-posed mathematical problem, and there is a statistical measure of the “goodness of fit” for the resulting parameters. (Here, the “model” refers to a linear, multi-degree-of-freedom oscillator system.) This kind of model-fitting procedure is the basis of most experimental modal analysis software packages, and many methods are available. One method that has been found to be reliable and accurate is a non-linear least-squares technique based on the Levenberg-Marquardt algorithm [4].

Naturally, the whole modal parameter estimation process is invalid if the actual source and load cannot reasonably be modeled as linear systems. If nonlinearities exist in either structure that cause significant deviation from the response of a linear structure, then neither the CSMA method nor any other method of deriving a force specification that is based on a linear model is valid.

3) The Coupled System

The actual vibration environment (as opposed to the test environment) consists of the load attached to source, with free boundary conditions. When the modal parameters that describe the (independent) source and load have been determined, the structural response of the coupled system can be calculated, based on a modal approach. Specifically, an eigenvalue problem can be formulated that has identical eigenvalues to the actual coupled system. The formulation of this coupled system eigenvalue problem is described in Appendix 3. The number of natural modes of the coupled system model will be the sum of the number of modes extracted from the source FRF plus the number of modes extracted from the load FRF; of these, one will be a rigid body mode (assuming this is a one-dimensional problem).

The structural response of the coupled system can be computed based on the coupled-system model if a system of applied forces can be chosen. The method by which the applied forces are determined in the CSMA method is discussed below, but first an assumption about how these applied forces may vary with frequency is considered.

4) Assumptions about the Frequency Dependence of the Applied Forces

The interface acceleration specification used in a conventional random vibration test is generally specified as a power spectral density envelope. It is assumed that the envelope does not precisely follow every peak and valley in the actual flight data, but rather that it varies smoothly with frequency by “smoothing over” the valleys.

The CSMA method of determining the force specification is based on the assumption that the applied forces on the system also vary smoothly with frequency. Specifically, the forces (specified as force PSD’s) applied to the coupled system are assumed to vary piecewise-linearly (as plotted on a log-log plot), in such a way that the frequency-rate-of-change (i.e., the steepness at any frequency of the force spectrum) is limited by a user-choosable parameter. The parameter that defines the maximum rate of change of the applied forces with frequency is called “max_rate” (its units are dB/octave).

Fig. 1 illustrates this frequency rate-of-change measure. If the max_rate parameter is chosen sufficiently large (say, greater than 48 dB/octave), then the force spectrum could include very sharp spikes (narrow-band components). A very small value of max_rate (say, 0.1 dB/octave) implies a nearly white (flat) spectrum.
The assumption that the applied forces vary smoothly with frequency is reasonable in a random vibration test. If the actual forces applied to the real source and load are narrow band or have significant narrow-band components, then it is dubious that a PSD-specified vibration test, conventional or force-limited, is valid. The reasoning behind this is best illustrated with a simple example. A simple two-degree-of-freedom system representing a source and a load (each modeled as a one-degree-of-freedom system), connected to each other at their interface, is shown in Fig. 2. Figure 3 consists of three pairs of plots, each pair consisting of the applied force spectrum and the acceleration response of the system. The acceleration is measured at the interface of the two systems. In Fig. 3a, the system is excited by a flat force spectrum. Fig. 3b gives the response of the system, and also shows an acceleration envelope that could be drawn that bounds the actual acceleration spectrum. Fig. 3c shows the response of the same system to a different applied force spectrum. In this case, the force spectrum was chosen such as to cause the acceleration to exactly equal the acceleration envelope in Fig. 3b. Notice that this force spectrum is highly “notchy”: the force spectrum must vary extremely rapidly, especially in the vicinity of the system’s antiresonance frequency, in order to produce the desired acceleration spectrum. This is exactly the force spectrum that would result in a conventional random vibration test using the envelope of Fig. 3b as the acceleration specification.

The situation illustrated in Figs. 3c and 3d is unrealistic because the particular force spectrum shown in Fig. 3c is very specific to the modal dynamic properties of the particular system for which the response was calculated. In other words, if the natural frequencies of the two-degree of freedom system were slightly changed (even by only a few percent), the identical force spectrum (Fig. 3e, which is identical to Fig. 3c) would produce a very different acceleration response (Fig. 3f). But the acceleration specification used in a conventional random vibration test is usually intended to be used for multiple payloads, having at least slightly different modal properties. Therefore, it is not reasonable to use the force spectrum shown in Fig. 3c in a conventional vibration test: that interface force spectrum is “too dependent” on the particular dynamic properties of the system under test. Instead, a “more generic” force spectrum should be used. The derivation of this force spectrum will now be directly addressed.

5) Upper Bound for the Interface Force Spectra

The interface force in the coupled system model obviously depends on the particular excitation that is applied. However, the loading that is applied to the actual source structure is unknown: neither the magnitude, the frequency dependence, nor the distribution of forces can be determined in the vibration testing environment.

However, an upper bound for the applied forces can be determined by considering the response of the coupled system model when each of the source degrees of freedom is excited individually. Specifically, the force PSD that represents the largest force that can be independently applied to each of the source degrees of freedom without exceeding

![](image1.png)

Fig. 1. This force spectrum varies with frequency in a piecewise-linear fashion (plotted on log-log scales), with the frequency rate-of-change expressed in dB/octave.

![](image2.png)

Fig. 2. Two connected one-degree-of-freedom systems excited by a force applied at one mass. The acceleration is measured at the “interface” of the two sub-systems.

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the interface acceleration envelope can determined as follows:

1) At every frequency of the analysis, determine the largest force PSD that can be applied to each source degree of freedom such that the resulting interface acceleration PSD is equal to the specified interface acceleration envelope. The only restriction placed upon this process is that the frequency dependence of the applied force may not vary arbitrarily fast, but rather has a maximum rate-of-change (given by the parameter max_rate).

2) Repeat step #1 for every source degree of freedom, then form a “composite” interface force spectrum that is simply the greatest interface force PSD level (at each frequency of the analysis) of the individual interface force spectra.

The composite force spectrum then represents an upper bound: no system of forces applied to the source degrees of freedom (and that obey the maximum frequency rate-of-change criteria) could produce a greater interface force without exceeding the interface acceleration specification. With the upper-bound interface force PSD determined, a smoothly-varying envelope can be drawn around it and used as the force specification for a force-limited vibration test.

**DISCUSSION OF THE CSMA METHOD**

The consequences of the proviso concerning the maximum rate at which the applied forces are allowed to vary with frequency can be best understood by considering the two limiting cases. If the frequency dependence of the forcing functions was allowed to vary arbitrarily fast (by using a large “max_rate” parameter), then the set of forces applied to the source degrees of freedom would cause the interface acceleration PSD to equal the interface acceleration envelope at all frequencies, as in Fig. 3d. In this case, a conventional vibration test would have no overtest because no coupled-system antiresonances (which is where the vibration absorber effect of the payload is felt) would have been “smoothed over.” The CSMA method will yield an interface force that is exactly equal to the force spectrum that would result in a conventional test - i.e., no force limiting would be predicted by the CSMA method.

On the other hand, using a very small “max_rate” would allow no frequency dependence of the forcing functions, i.e., the spectra of the forcing functions would be flat, or white, over the entire frequency range of the analysis, as in Fig. 3a. The set of forces applied to the source degrees of
freedom would then cause the interface acceleration to equal the interface acceleration envelope at just one frequency for each source degree of freedom.

The CSMA method will always "try to find" the kind of force spectrum shown in Fig. 3c, but the steepness of both the peaks and valleys of the "allowable" force spectra are limited by the parameter "max_rate." The value of "max_rate" to be used in a force limited vibration test should be based on a physical understanding of the actual forces applied to launch vehicle that will carry the payload being tested. Many sources of excitation on launch vehicles are "inherently broad-band", for example acoustic loadings, pressure distributions from turbulent flows, and the variations in thrust from some rocket motors. Other types of loading are certainly not broad band, such as the forces produced by an unbalanced rotating mass. However, a random vibration test would not be used to simulate the effect of narrow-band loadings. So a physically realistic maximum rate of change of the loading with frequency should be based on the types of loading that the random vibration test is intended to simulate.

EXAMPLES OF CSMA METHOD, AND COMPARISON TO THE TDFS METHOD

The CSMA method is illustrated in the appendices with numerical simulations of two simple dynamic systems. A two-degree-of-freedom system, which consists of a one-degree-of-freedom source coupled to a one-degree-of-freedom load, is used as an example in Appendix 1. A six-degree-of-freedom system, which consists of a three-degree-of-freedom source coupled to a three-degree-of-freedom load, is illustrated in Appendix 2. In these appendices each step of the CSMA method is described and all relevant data is presented. The CSMA method is also compared to another method of determining a force specification, the "TDFS" method of Scharton [1], in each of these examples.

The CSMA method has also been used with data from an actual force limited random vibration test. This test was performed on a small (425 lbm) satellite called "FORTE ." The force specification for the FORTE test was based on the "TDFS" method because, at the time of the test, the CSMA method of determining the force spectrum did not exist. However, for the purpose of comparing these two methods, the process of determining the force specification for the FORTE test was repeated using the CSMA method.

The dynamic mass FRF of the FORTE payload was determined in a random vibration test on a shaker, with the structure mounted vertically and supported by eight force transducers. The dynamic mass FRF was measured at the proto-qualification vibration level (0.008 g^2/Hz, constant from 20 to 800 Hz), and again at a level reduced by -12 dB (0.0005 g^2/Hz, constant from 20 to 800 Hz). The level at which the measurements were made did not significantly affect the dynamic mass FRF's. The dynamic mass FRF for FORTE is shown in Fig. 4.

The dynamic mass FRF of the launch vehicle, a Pegasus-XL manufactured by Orbital Sciences Corporation, was determined in a modal "tap test". In this test, only the avionics section and the third stage of the rocket were included. The dynamic mass FRF was determined by averaging the acceleration measurements at each of the tap locations around the payload interface.

The force spectra calculated by the TDFS and the CSMA methods are shown in Fig. 5. The TDFS force spectrum, which was calculated at 1/3 octave band frequencies, was smooth enough to use without any envelope. The spectrum calculated by the CSMA method tends to vary more rapidly with frequency, so a conservative envelope of the CSMA force spectrum could be used as the force spectrum. The value of "max_rate" used for the CSMA method was 24 dB/octave. The estimated interface force PSD for a conventional test is also shown in the figure. At all frequencies where this interface force is greater than the force specification, the vibration test would be force-controlled.

To give an indication of the magnitude and frequency range of the effect of the force limiting in the dual-controlled
vibration test, the actual (as-tested) interface acceleration can be compared to the interface acceleration specification. For a conventional vibration test with no force limiting, the base acceleration would be identical to the specification (a constant 0.008 g^2/Hz spectrum from 20 to 800 Hz). In the force-limited vibration test, the shaker control system reduced the base acceleration whenever the measured interface force reached the limit force specification. The resulting acceleration spectrum is shown in Fig. 6. The force limiting had the largest effect at the resonant frequencies of the structure, in this case mainly at 106 Hz. The maximum reduction ("notch") in the base acceleration PSD was about 10 dB at 106 Hz.

The CSMA method was illustrated with data from an actual force limited vibration test of the FORTE' satellite and with two numerical simulations of simple spring-mass systems. The method was compared to the TDFS method of Scharton for each of these systems. The CSMA method yields a force specification that is similar but not identical to the TDFS method.

The CSMA method of determining the force specification for a force-limited vibration test has been implemented in a short program. The procedure to extract the modal parameters has been implemented in a second program, but this procedure can be performed by many commercial modal analysis packages.

REFERENCES
APPENDIX 1

Two-Degree-of-Freedom System Example

The simplest approach that one could use to model any launch vehicle and payload system would be to model each item as a single degree-of-freedom system. Such a system is illustrated in Fig. 7. The individual components (the source and the load) each have one natural mode of vibration; when coupled together at the payload/launch vehicle interface, the system has two natural modes, one of which is a rigid-body mode.

In an actual vibration test, neither the loading on the launch vehicle, nor the “flight data” of the actual interface acceleration is assumed to be known; only the interface acceleration specification, which is assumed to properly envelope the flight data, is given. For the purposes of this example only, an “example” loading on the source is used so that the true interface acceleration and force (information that is not available in a real test) may be compared to the interface acceleration envelope and the force specification that will be determined by the CSMA method (information that is available in a real test). The loading function and the interface acceleration envelope were chosen for simplicity in this example: the applied force PSD had a constant PSD level of 30,000 lbf/s/Hz from 10 to 500 Hz, and the envelope of the interface acceleration had a constant value of 86,000 (in/s^2)^2/Hz (0.577 g^2/Hz) from 10 to 500 Hz.

The dynamic mass FRF’s for the source and load are shown in Fig. 8. The first step of the CSMA method is to find the modal parameters that define the source and load modal models. This process was performed, with the resulting modal data listed in Table 1.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural Frequency (Hz)</th>
<th>Damping Ratio (%)</th>
<th>Modal Effective Mass (lbm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source:</td>
<td>28.5</td>
<td>4.0</td>
<td>1200.0</td>
</tr>
<tr>
<td>Load:</td>
<td>69.9</td>
<td>4.0</td>
<td>400.0</td>
</tr>
</tbody>
</table>

Next, the modal properties of the coupled system were computed. This data is given in Table 2. The CSMA method is particularly simple for this problem because there is only one source and one load degree of freedom, so there is no difference between the physical system and the parallel oscillator model, and the modal effective masses are equal to the physical masses.

The next step of the CSMA procedure was performed with several different values of the “max_rate” parameter to illustrate its effect. The interface acceleration produced in each case is plotted in Fig. 9.

![Fig. 7. Two-degree-of-freedom system, consisting of the coupled source and load oscillators.](image)

![Fig. 8. Dynamic mass FRF’s for the source and load.](image)
The interface acceleration produced with the largest value of "max_rate" (96 dB/octave) allows the coupled-system interface acceleration to equal to the interface acceleration envelope at every frequency of the analysis.

The interface acceleration produced with the intermediate values of "max_rate" are equal to the envelope at all frequencies except near the antiresonance frequencies. This is a consequence of the finite rate of change of the applied force: the applied force required to make the interface acceleration equal to the envelope at the antiresonance frequency (70 Hz) is so much larger than the applied force required to make the interface acceleration equal to the envelope at any nearby frequency that the maximum rate of change criteria comes into play. Because the applied forces cannot "grow fast enough" with frequency, the interface acceleration will be smaller than the envelope at the those frequencies where the interface acceleration is varying the fastest (primarily, near the coupled-system antiresonance frequency).

The interface acceleration produced with a small max_rate is a special version of the cases above: because the applied force now has a white (flat) spectrum, the greatest interface acceleration will always be at the coupled-system natural frequency. Thus, the amplitude of the applied force is controlled by the interface acceleration at the coupled-system natural frequency only.

After the force applied to the source degree of freedom is defined (i.e., its magnitude is defined at every frequency of the analysis), the resulting coupled-system interface force PSD can be computed. This is shown in Fig. 10. A reasonable envelope can be drawn around this interface force and used as the force specification for a force-limited vibration test. The magnitude of the force limiting for various values of max_rate is shown in Table 3.

<table>
<thead>
<tr>
<th>max_rate (dB/octave)</th>
<th>Maximum Reduction in Interface Acceleration (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>-37</td>
</tr>
<tr>
<td>12</td>
<td>-30</td>
</tr>
<tr>
<td>24</td>
<td>-23</td>
</tr>
<tr>
<td>48</td>
<td>-9.8</td>
</tr>
<tr>
<td>96</td>
<td>0</td>
</tr>
</tbody>
</table>

When the computed interface force PSD is compared to the true interface force (which, again, would not be available in a real test), it can be seen that the CSMA method has produced a force specification that is conservative, i.e., is greater than the true interface force at all frequencies. The interface force PSD that would be generated in a conventional vibration test (no force limiting) is also plotted in Fig. 10. At all frequencies where the computed force specification is less than this force PSD, a force-limited vibration test would be governed by the force specification; at all other frequencies the test would be governed by the acceleration specification.

Fig. 9. Interface acceleration PSD calculated for four different values of max_rate. Also shown is the interface acceleration envelope and the true (actual) interface acceleration.

Fig. 10. Coupled-system interface force, computed using max_rate = 24 dB/octave. Also shown is the interface force that would result with no force-limiting and the true (actual) interface force.

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Fig. 11. Maximum reduction of interface force by the CSMA method, as a function of the ratio of the load mass to the source mass.

One additional set of tests was performed with this two-degree-of-freedom model. The CSMA procedure was repeated while varying the mass of the load, in order to observe the effect that this had on the amount of predicted force limiting. Figure 11 shows the maximum depth of the force limiting, expressed in dB, as a function of the ratio of load mass to source mass. As expected, the CSMA method predicts a diminishing force limiting effect as this ratio decreases.

APPENDIX 2

Six-Degree-of-Freedom System

A somewhat more complex structural system is shown in Fig. 12, where the source and load are each modeled as three-degree-of-freedom systems. The coupled system is also shown in this figure. The dynamic mass FRF's for the source and load are shown in Figs. 13 and 14.

The CSMA procedure was followed as described in Appendix 1. Again, for the purposes of this example only, an "example" loading on the source was used, so that the interface acceleration and interface force predicted by the CSMA method could be compared to the true (actual) values. In a real vibration test this information would not be available. The force spectrum that was used is given in Table 1. The envelope that was used as the interface acceleration specification is given in Table 2. The modal properties extracted from the dynamic mass FRF's are given in Table 3. The coupled system modal properties were determined as described in the main text; this data is given in Table 4.

The largest possible system of applied forces that resulted in a valid interface acceleration was computed as described in the text. The value of "max_rate" that was used was 24 dB/octave. Fig. 15 shows the interface acceleration com-
Fig. 14. Dynamic mass FRF for the load system.

Fig. 15. Interface acceleration PSD, based on max_rate = 24 dB/octave. The interface acceleration envelope and the true interface acceleration are also shown.

The interface force computed using the system of applied forces determined via the CSMA method, and the true (actual) interface acceleration computed using the actual forcing function (defined in Table 1). Note that the interface acceleration computed using the system of applied forces determined via the CSMA method is conservative, i.e., it is greater than the true interface acceleration at all frequencies.

The interface force computed using the system of applied forces determined via the CSMA method is shown in Fig. 16. Also shown in this figure is the interface force that would result if no force-limiting were present (i.e., if the load were driven by the interface acceleration envelope at all frequencies). The true interface force is also plotted; again, the interface force computed by the CSMA method

Table 4: Applied Force Spectrum* Force applied to Source Mass #1 (888 lbf)

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Applied Force PSD (lbf^2/Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.0</td>
<td>3,000</td>
</tr>
<tr>
<td>20.0</td>
<td>300,000</td>
</tr>
<tr>
<td>30.0</td>
<td>300,000</td>
</tr>
<tr>
<td>40.0</td>
<td>30,000</td>
</tr>
<tr>
<td>200.0</td>
<td>30,000</td>
</tr>
<tr>
<td>2000.0</td>
<td>3</td>
</tr>
</tbody>
</table>

*Values at intermediate frequencies are interpolated such that the spectrum is piecewise linear when plotted on log-log plot.

Table 5: Interface Acceleration Spectrum*

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Acceleration PSD (in/sec^2)^2 / Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.0</td>
<td>120</td>
</tr>
<tr>
<td>25.0</td>
<td>35,500</td>
</tr>
<tr>
<td>70.0</td>
<td>35,500</td>
</tr>
<tr>
<td>120.0</td>
<td>1,400</td>
</tr>
<tr>
<td>200.0</td>
<td>1,400</td>
</tr>
<tr>
<td>2000.0</td>
<td>0.14</td>
</tr>
</tbody>
</table>

*Values at intermediate frequencies are interpolated such that the spectrum is piecewise linear when plotted on log-log plot.
Table 6: Modal Data for the Source and Load

<table>
<thead>
<tr>
<th>Natural Frequency (Hz)</th>
<th>Damping Ratio (%)</th>
<th>Modal Effective Mass (lbm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18.3</td>
<td>4.0</td>
<td>1597.8</td>
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<tr>
<td>42.6</td>
<td>4.0</td>
<td>162.8</td>
</tr>
<tr>
<td>56.8</td>
<td>4.0</td>
<td>15.0</td>
</tr>
<tr>
<td>Load</td>
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<td></td>
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<tr>
<td>34.9</td>
<td>4.0</td>
<td>301.9</td>
</tr>
<tr>
<td>76.4</td>
<td>4.0</td>
<td>70.4</td>
</tr>
<tr>
<td>104.2</td>
<td>4.0</td>
<td>13.7</td>
</tr>
</tbody>
</table>

Table 7: Modal Data for the Coupled System

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural Frequency (Hz)</th>
<th>Damping Ratio (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>27.8</td>
<td>4.0</td>
</tr>
<tr>
<td>3</td>
<td>39.8</td>
<td>4.0</td>
</tr>
<tr>
<td>4</td>
<td>55.8</td>
<td>4.0</td>
</tr>
<tr>
<td>5</td>
<td>68.2</td>
<td>4.0</td>
</tr>
<tr>
<td>6</td>
<td>101.0</td>
<td>4.0</td>
</tr>
</tbody>
</table>

is conservative in that it is greater than the true interface force at all frequencies.

The CSMA force specification is compared to the specification found using the TDFS method of Scharton [1] in Fig. 17. The TDFS force specification was calculated at 1/3-octave band frequencies. The two methods yield similar results in this example.

APPENDIX 3

Formulation of the Coupled-System Eigenvalue Problem

Any system of spring-mass oscillators can be represented as a “parallel-oscillator” system (sometimes called an “asparagus patch” system), consisting of a system of spring-mass oscillators connected in parallel. Fig. 18 illustrates the parallel oscillator system for one particular three-degree-of-freedom system. The masses $m^*_i$ in the parallel oscillator representation are called the modal effective masses of the system. The spring rates $k^*_i$ are equal to $m^*_i \omega_i^2$, where $\omega_i$ is the natural frequency for mode $i$ of the physical system. The two representations in Fig. 19 have identical eigenvalues, and their eigenvectors are associated in a simple way. For a given base acceleration, the total base reaction force for the two representations is identical.

An eigenvalue problem representing the coupled system composed of two (or more) connected sub-systems can be set up based on the modal parameters for the sub-systems. The modal parameters that are required are the natural frequencies, damping ratios, and modal effective masses for each mode of each sub-system. The two sub-systems are coupled by superimposing their bases, as illustrated in Fig. 18. The interface itself may be assumed to have a mass $m_I$. The equations of motion for this coupled system are given.
The eigenvalues of this system of equations are identical to the eigenvalues for the coupled system shown in Fig. 18b. The eigenvectors of the parallel oscillator system are not identical to the eigenvectors of the physical system. However, the two representations both have the same total base reaction force for any given base acceleration.

Fig. 18. (a): Two uncoupled sub-systems. (b): Coupled system, in the parallel-oscillator representation.

Fig. 19. (a): The physical three-degree-of-freedom system. (b): The parallel-oscillator representation of the same system.