TOPICS IN CATEGORY THEORY

THESIS

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Robert Patrick Miller, B. A.

Denton, Texas

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The purpose of this paper is to examine some basic topics in category theory. A category consists of a class of mathematical objects along with a morphism class having an associative composition.

The paper is divided into two chapters. Chapter I deals with intrinsic properties of categories. Various "subobjects" and properties of morphisms are defined and examples are given.

Chapter II deals with morphisms between categories called functors and the natural transformations between functors. Special types of functors are defined and examples are given.

PREFACE

The origins of category theory are in algebraic topology. The basic concepts of category, functor and natural transformation were formulated by Samuel Eilenberg and Saunders MacLane in 1945 in their paper, "General Theory of Natural Equivalences." Since then, category theory has grown into a discipline in its own right. The main strength of category theory is two-fold. First, it has applications in other branches of mathematics and it unifies many disciplines in the sense that many concepts can be expressed in functorial language. As a consequence of this unification, category theory provides a groundwork for comparing different branches of mathematics by comparing their isomorphisms.

It is the purpose of this paper to explore some basic notions in category theory. These notions include both intrinsic characteristics of a category and how categories may be compared. Category theory tries to abstract concepts from many of the different disciplines.

A category consists of two things, the mathematical objects and the morphisms between these objects. Many of the internal characteristics of a category are nothing more than abstractions of a similar concept in an already existing discipline. For most of the notions mentioned in this paper, examples are given in specific categories that suggest the origin of the concept.

Categories are compared using categorical morphisms or functors. A functor consists of two things, an assignment of an object of the domain category to an object of the codomain category, and an assignment of morphisms. Many times it is desirable to compare functors, and the tool here is a natural transformation.

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CHAPTER I

OBJECTS AND MORPHISMS

Most of the definitions and theorems come from the book Categories and Functors, by Bodo Pareigis. Also some of the examples appear in this book. In all cases, however, the proofs are original.

Let $\mathcal{C}$ consist of two things, (1) an object class, written ob $\mathcal{C}$, of mathematical objects, and (2) a family of mutually disjoint sets $\{Mor_{\mathcal{C}}(A,B)\}$ for all objects $A,B \in$ ob $\mathcal{C}$ whose elements $f,g,h \cdots \in Mor_{\mathcal{C}}(A,B)$ are called morphisms. Also, a family of maps

$\{Mor_{\mathcal{C}}(A,B) \times Mor_{\mathcal{C}}(B,C) \ni (f,g) \to gf \in Mor_{\mathcal{C}}(A,C)\}$

for all $A,B,C \in$ ob $\mathcal{C}$ called compositions exists. When we have $f \in Mor_{\mathcal{C}}(A,B)$ we will often indicate this by $A \overset{f}{\to} B$ or $f: A \to B$, where A is the domain of f and B is the codomain. Then $\mathcal{C}$ is called a category if $\mathcal{C}$ fulfills the following axioms.

(1) Associativity: For all $A,B,C,D \in$ ob $\mathcal{C}$ and all $f \in Mor_{\mathcal{C}}(A,B)$, $g \in Mor_{\mathcal{C}}(B,C)$ and $h \in Mor_{\mathcal{C}}(C,D)$, we have $h(gh) = (hg)f$.

(2) Identity: For each object A in $\mathcal{C}$ there is a morphism $1_A \in Mor_{\mathcal{C}}(A,A)$ called the identity such that for all $B \in$ ob $\mathcal{C}$, $C \in$ ob $\mathcal{C}$, $f \in Mor_{\mathcal{C}}(A,B)$ and $g \in Mor_{\mathcal{C}}(C,A)$ we have $f1_A = f$ and $1_A g = g$.

The following are some examples of categories. More examples appear in Appendix I.

(1) Set. The objects of this category are all sets. If A and B are sets, then $Mor_{Set}(A,B) = \{f \mid f$ is a function from A to B$\}$. The composition in Set is the usual composition of functions. The identity function $1_A : A \to A$ defined by $1_A(a) = a$ for every $a \in A$ satisfies axiom 2 since if $A \xrightarrow{f} B$ and $C \xrightarrow{g} A$, then $(f\,1_A)(a) = f(1_A(a)) = f(a)$, and $(1_A\,g)(a) = 1_A(g(a)) = g(a)$ for each $a \in A$. Composition of functions is associative, since if $A \xrightarrow{f} B \xrightarrow{g} C \to D$, then for every $a \in A$ $((hg)f)(a) = (hg)(f(a)) = h(g(f(a)) = h((gh)(a)) = (h(gf))(a)$. Therefore, Set is a category.

(t) Top. The object class in Top is the collection of all topological spaces. If (X,T) and (Y,S) are topological spaces (usually just writen as X and Y), then $Mor_{Top}(X,Y) = \{f \mid f$ is a continuous function from X to Y$\}$ where f is continuous means $f^{-1}(U) \in T$ for every $U \in S$. Composition is the usual composition of continuous functions. The composition is known to preserve continuity. This composition is associative since the composition of functions is associative. The identity function on X is a continuous function and satisfies axiom 2. Therefore, Top is a category.

(3) Gp. The object class in Gp is the collection of all groups. If $(A,\cdot)$ and $(B,*)$ are groups (usually written as A and B), then $Mor_{Gp}(A,B) = \{f \mid f$ is a group homomorphism

from A to B} composition of morphisms is defined to be the usual composition of homomorphisms which gives a homomorphism. This composition is associative since composition of functions is associative. The identity function is a homomorphism. Hence the collection of all groups together with their homomorphisms forms a category.

The following notations will be used. Capital Latin letters will denote objects and small Latin letters will denote morphisms between objects. When there is no ambiguity, $\mathrm{Mor}_{\mathcal{C}}(A,B)$ will be abbreviated to $\mathrm{Mor}(A,B)$. $\mathrm{Mor}\,\mathcal{C}$ will denote $\cup\,\mathrm{Mor}(A,B)$ where the union is taken over all objects A,B in $\mathcal{C}$.

We would now like to construct a new category $\mathcal{C}^{op}$ from a given category $\mathcal{C}$. The class of objects of $\mathcal{C}^{op}$ is the same as the class of objects of $\mathcal{C}$, that is, $\mathrm{ob}\,\mathcal{C}^{op} = \mathrm{ob}\,\mathcal{C}$. If $A,B \in \mathrm{ob}\,\mathcal{C}^{op}$, then $\mathrm{Mor}_{\mathcal{C}^{op}}(A,B) = \mathrm{Mor}_{\mathcal{C}}(B,A)$. Compositions are defined by the rule:

$(f,g) \in \mathrm{Mor}_{\mathcal{C}^{op}}(A,B) \times \mathrm{Mor}_{\mathcal{C}^{op}}(B,C)$, $(f,g) \to fg \in \mathrm{Mor}_{\mathcal{C}^{op}}(A,C)$ with fg formed in $\mathcal{C}$. Suppose $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathcal{C}^{op}$. Then $D \xrightarrow{h} C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathcal{C}$ and $(fg)h = f(gh)$ in $\mathcal{C}$. Therefore, $h(gf) = (hg)f$ in $\mathcal{C}^{op}$, which means the composition is associative. The identity morphism on A, 1_A, in $\mathcal{C}$ is also a morphism in $\mathcal{C}^{op}$. Let $A \xrightarrow{f} B$ and $C \xrightarrow{g} A$ in $\mathcal{C}^{op}$. Then $B \xrightarrow{f} A$ and $A \xrightarrow{g} C$. Since $1_A f = f$ and $g\,1_A = g$ in $\mathcal{C}$, we have $f\,1_A = f$ and $1_A g = g$ in $\mathcal{C}^{op}$. Therefore $\mathcal{C}^{op}$ is a category and it is called the dual category of $\mathcal{C}$.

To indicate that an object A or morphism f in a category $\mathcal{C}$ is being considered as an object or morphism in the dual category $\mathcal{C}^{op}$, we often write A^{op} or f^{op}. Also $(\mathcal{C}^{op})^{op} = \mathcal{C}$. When we have a true statement about a category $\mathcal{C}$ we can obtain a corresponding true statement about the dual category $\mathcal{C}^{op}$ by reversing the direction of all morphisms involved.

We would now like to study some of the internal properties of categories. In this study there are two things to consider: the morphisms and the objects.

Morphisms

In the study of these morphisms we will try to generalize some properties of the morphisms in some specific categories. In Set, suppose $g,h : A \to B$ and $f : B \to C$ such that $fg = fh$. Then we know that $g = h$ if f is injective (1-1), and if $fg = fh$ implies $g = h$ for every pair of functions g and h, then we can show f is 1-1. Let $A = \{(a,b) \mid a,b \in B \text{ and } f(a) = f(b)\}$. Define g and h to be the first and second projection functions respectively from A to B. Then for every $(a,b) \in A$, we have $(fg)(a,b) = f(a) = f(b) = (fh)(a,b)$. Hence, by hypothesis, $g = h$. Therefore, $a = b$ implies f is 1-1. With this in mind we make the following definition.

Let $\mathcal{C}$ be a category and $f : A \to B$ in $\mathcal{C}$. Then f is a <u>monomorphism</u> in $\mathcal{C}$ if $fg = fh$ implies $g = h$ for all $C \in \text{ob}\,\mathcal{C}$ and for all $g,h \in \text{Mor}_{\mathcal{C}}(C,A)$; that is, f is left cancellable. We sometimes shorten f is a monomorphism to f is monic.

Now, in Set, suppose $f : A \to B$ and $g,h : B \to C$ such that $gh = hf$ and f is surjective (onto). Then $B = f(A)$, and if $b \in B$, then there is an $a \in A$ such that $f(a) = B$. Now $g(b) = g(f(a)) = (gf)(a) = (hf)(a) = h(f(a)) = h(b)$. Therefore $h = g$. Conversely, if $gh = hf$ implies $g = h$ whenever $A \overset{f}{\to} B \overset{g}{\underset{h}{\rightrightarrows}} C$, then we want to show f is onto. Let $S = \{\{f^{-1}(b)\} \mid b \in f(A)\} \cup \{0\} \cup \{1\}$ where $\{0\}$ and $\{1\}$ are disjoint sets and $0 \neq 1 \neq \{f^{-1}(b)\}$ for any $b \in f(A)$. Define $h,g : B \to S$ by $h(b) = g(b) = \{f^{-1}(b)\}$ if $b \in f(a)$, $h(b) = 0$ and $g(b) = 1$ if $b \notin f(A)$. If $a \in A$ and $f(a) = b$, then $(gf)(a) = g(f(a)) = g(b) = h(b) = h(f(a)) = (hf)(a)$. Therefore $gf = hf$ and by hypothesis $g = h$, which is a contradiction if $f(A) \neq B$. Therefore $f(A) = B$, which means f is onto. Generalizing this to other categories, we have the following.

Let $\mathcal{C}$ be a category and $f : A \to B$ in $\mathcal{C}$. Then f is an <u>epimorphism</u> in $\mathcal{C}$ if $gf = hf$ implies $g = h$ for all $C \in \text{ob}\,\mathcal{C}$ and for all $g,h \in \text{Mor}_{\mathcal{C}}(B,C)$; that is, if f is right cancellable. The morphism f is also called epi. The notion of epimorphism is dual to the notion of monomorphism. The dual statement of the definition of monomorphism would read: a morphism $f^{op} : A \to B$ is a monomorphism in $\mathcal{C}^{op}$ if $f^{op} g^{op} = f^{op} h^{op}$ implies $g^{op} = h^{op}$ for every $C \in \text{ob}\,\mathcal{C}^{op}$ and for every $g^{op}, h^{op} \in \text{Mor}_{\mathcal{C}^{op}}(C,A)$. In $\mathcal{C}$ this would look like $B \overset{f}{\to} A \overset{g}{\underset{h}{\rightrightarrows}} C$, and $hf = gf$ implies $h = g$, or exactly the notion of epimorphic. Hence for every statement about monomorphisms there is a corresponding statement about epimorphisms and the converse.

Lemma 1.1. Let f and g be morphisms in a category which can be composed. Then

(1) If fg is a monomorphism, then g is a monomorphism.

(2) If f and g are monomorphisms, then fg is a monomorphism.

(3) If fg is an epimorphism, then f is an epimorphism.

(4) If f and g are epimorphisms, then fg is an epimorphism.

Proof. Let $A \xrightarrow{g} B \xrightarrow{f} C$.

(1) Let $h, h' : D \to A$ such that $gh = gh'$. Then $f(gh) = f(gh')$. Therefore $(fg)h = (fh)h'$, and since fg is a monomorphism we have $h = h'$. Hence g is left cancellable.

(2) Let $h, h' : D \to A$ such that $(fg)h = (fg)h'$. Then $f(gh) = f(gh')$ implies $gh = gh'$ since f is a monomorphism, and since g is a monomorphism $h = h'$. Therefore fg is a monomorphism.

(3) This statement is the dual of statement (1) and hence true. A direct proof would be as follows. Let $A \xrightarrow{g} B \xrightarrow{f} C \overset{h}{\underset{h'}{\rightrightarrows}} D$ such that $hf = h'f$. Then $(hf)g = (h'f)g$ or $h(fg) = h'(fg)$. Therefore $h = h'$, since fg is an epimorphism.

(4) This statement is the dual assertion of (2) and hence is also true. Directly, let $A \xrightarrow{g} B \xrightarrow{f} C \overset{h}{\underset{h'}{\rightrightarrows}} D$ such that

$h(fg) = h'(fg)$. Then $(hf)g = (h'f)g$, which implies $hf = h'f$. Therefore $h = h'$. Hence fg is an epimorphism.

Therefore Lemma 1.1 is proved.

That monomorphism is equivalent to injection and epmorphism is equivalent to surjection in Set follows from the discussion preceding the definitions. We can generalize the following lemma.

Lemma 1.2. For a category $\mathcal{C}$ whose objects can be considered as sets and whose morphisms can be considered as functions, injective implies monomorphic and surjective implies epimorphic.

Proof. Let $\mathcal{C}$ be a category satisfying the hypothesis. Let $f : A \to B$ be 1-1 and suppose $h, h' : C \to A$ in $\mathcal{C}$ such that $fh' = fh$. Then for $c \in C$, $f(h'(c)) = (fh')(c) = (fh)(c) = f(h(c))$, and since f is injective, $h'(c) = h(c)$. Therefore, $h = h'$ and f is monomorphic. Now let $f : A \to B$ be surjective and suppose $h, h' : B \to C$ in $\mathcal{C}$ such that $hf = h'f$. Then for $b \in B$ there is an $a \in A$ such that $f(a) = b$, since $f(A) = B$. Therefore $h(b) = h(f(a)) = (hf)(a) = (h'f)(a) = h'(f(a)) = h'(b)$. Therefore $h = h'$ and f is epimorphic. Thus Lemma 1.2 is proved.

The converse of this lemma is not true for some categories, as indicated by the following examples.

This example appears in the book Categories and Functors. Let $f : A \to B$ be a dense continuous map in Hd (Appendix I). A continuous map is dense if for every nonempty open set U in B there is an $a \in A$ such that $f(a) \in U$. Show f is an epimorphism. Suppose $g,h : B \to C$ in Hd such that $gf = hf$. Let $b \in B$ such that $g(b) \neq h(b)$. Then there exist disjoint open sets U and V in C such that $g(b) \in U$ and $h(b) \in V$. Then $b \in g^{-1}(U) \cap h^{-1}(V)$, which is open in B. Since f is dense, there is an $a \in A$ such that $f(a) \in g^{-1}(U) \cap h^{-1}(U)$. Therefore $(gf)(a) \in g(g^{-1}(U) \cap h^{-1}(V)) \subseteq U \cap g(h^{-1}(V))$ and $(hf)(a) \in h(g^{-1}(U) \cap h^{-1}(V)) \subseteq h(g^{-1}(U)) \cap V$. Then $(gf)(a) \neq (hf)(a)$, contrary to hypothesis. Therefore $g = h$ and a dense continuous map in Hd is an epimorphism.

Now the reals $\mathbb{R}$ with the usual topology is Hausdorff and the rationals Q with the inherited topology is Hausdorff. Then the embedding $i : Q \to R$ is a dense continuous map but it is not surjective. Therefore Hd is a category in which an epimorphism need not be surjective as a set map.

Define a category $\mathcal{C}$ by ob $\mathcal{C} = \{\{a,b\},\{b\}\}$ and let $\{a,b\} = A$ and $\{b\} = B$. Define the morphism sets as follows: $\mathrm{Mor}(A,A) = \{1_A\}$, $\mathrm{Mor}(B,B) = \{1_B\}$, $\mathrm{Mor}(A,B) = \{h \mid h(a) = b, h(b) = b\}$ and $\mathrm{Mor}(B,A) = \{k \mid k(b) = a\}$. Composition in $\mathcal{C}$ is the composition of set maps, and we know that this is associative. The identities are given. Now h is a monomorphism since there is only one morphism into A from A, and only one morphism from B into A, but h is not an injection. Also, k is an epimorphism since $\mathrm{Mor}(A,A)$ and $\mathrm{Mor}(A,B)$ consist of only one element and k is not surjective as a set map. Hence $\mathcal{C}$ is a category where the monomorphisms need not be injective nor the epimorphisms surjective as set maps.

Some categories besides Set whose objects can be considered as sets and whose morphisms can be considered as functions where we have the equivalence of monomorphisms and injective functions are Gp, its subcategory Ab (Appendix I), Top and some of its subcategories T_4 spaces, normal spaces, T_3, completely regular, regular, T_1 and T_0 spaces, along with Hd and CH, the category of compact Hausdorff spaces. The morphisms in these subcategories are the continuous functions between the objects. Some more algebraic categories where this result is true are Rm, Rg and Ri. The proofs for these are all similar.

For Gp, let $f: A \to B$ be a monomorphism. We know the product $A \times A$ is a group under coordinate-wise multiplication. Show that $C = \{(x,y) \in A \times A \mid f(x) = f(y)\}$ is a subgroup of $A \times A$. Let $(x,y),(w,z) \in C$. Show $(xw^{-1}, yz^{-1}) \in C$. Now $f(xw^{-1}) = f(x)\,f(w^{-1}) = f(y)\,f(z^{-1}) = f(yz^{-1})$ since f is a homomorphism. Therefore $(xw^{-1}, yz^{-1}) \in C$, and hence C is a subgroup of $A \times A$. Therefore $C \in Gp$. Define $p_1, p_2 : C \to A$ by $p_1(x,y) = x$ and $p_2(x,y) = y$. These are well defined and we need to know that p_1 and p_2 are homomorphisms. Let (x,y) and $(w,z) \in C$. Then $p_1((x,y)(w,z)) = p_1(xw,yz) = xw = p_1(x,y)p_1(w,z)$. Therefore p_1 is a homomorphism. Similarly p_2 is a homomorphism. Then for $(x,y) \in C$, $(fp_1)(x,y) = f(x) = f(y) = fp_2(x,y)$. Since f is a monomorphism, $p_1 = p_2$. Therefore $p_1(x,y) = p_2(x,y)$ for every $(x,y) \in C$, and hence $x = y$ whenever $f(x) = f(y)$. Hence f is injective. The same proof also shows that in Ab, monomorphism implies injection, since $A \times A$ is abelian if A is, and hence, C would be abelian.

For Top, let $f : (X,T) \to (Y,S)$ be a monomorphism. Let $C = \{(a,b) \mid f(a) = f(b),\ a,b \in X\}$. Define $p_1, p_2 : C \to X$ by $p_1(a,b) = a$ and $p_2(a,b) = b$. Let C have the weak topology determined by p_1 and p_2. Hence p_1 and p_2 are continuous. Then for every $(a,b) \in C$, $fp_1(a,b) = f(a) = f(b) = fp_2(a,b)$ and as before $p_1 = p_2$. Therefore $p_1(a,b) = p_2(a,b)$ for every $(a,b) \in C$. Therefore if $f(a) = f(b)$, then $a = b$ and f is injective. This same construction with the discrete topology on C also works for T_4, normal, completely regular, regular, Hausdorff, T_1 and T_0 spaces with their respective continuous functions, since the discrete topology has each separation property.

Let $f : A \to B$ be a nonempty monomorphism in CH. Then $A \times A$ is compact and Hausdorff. Let $C = \{(a,b) \mid f(a) = f(b)\}$. Show C is closed in $A \times A$. If $(a,b) \notin C$, then $f(a) \neq f(b)$, and hence there are disjoint open sets U and V containing $f(a)$ and $f(b)$ respectively. Now $f^{-1}(U)$ and $f^{-1}(V)$ are open sets containing a and b, and thence $f^{-1}(U) \times f^{-1}(V)$ is an open set of (a,b) disjoint from C. Therefore C is closed and hence compact. Also C is Hausdorff since Hausdorffness is hereditary ($A,B \neq$ empty set). Define $p_1, p_2 : C \to A$ by $p_1(a,b) = a$

$p_2(a,b) = b$. The functions p_1 and p_2 are continuous since they are the restrictions to C of the projection functions. Now, as before, $fp_1 = fp_2$, and hence f is an injection.

In the category Rm, let $f : A \to B$ be a monomorphism. We would like to show that the cartesian product $A \times A$ is an R-module under the operations $(a,b) + (c,d) = (a+c,\ b+d)$ and $\alpha(a,b) = (\alpha a, \alpha b)$, $\alpha \in R$. We know $A \times A$ is an abelian group under the addition. Then

(1) $(\alpha+\beta)(a,b) = ((\alpha+\beta)a, (\alpha+\beta)b) = (\alpha a + \beta a, \alpha b + \beta b) = (\alpha a, \alpha b) + (\beta a, \beta b) = \alpha(a,b) + \beta(a,b)$ since A is an R-module.

(2) $\alpha((a,b) + (c,d)) = \alpha(a+c,\ b+d) = (\alpha(a+c),\ \alpha(b+d)) = (\alpha a + \alpha c, \alpha b + \alpha d) = (\alpha a, \alpha b) + (\alpha c, \alpha d) = \alpha(a,b) + \alpha(c,d)$.

(3) $\alpha(\beta(a,b)) = \alpha(\beta a, \beta b) = (\alpha(\beta a), \alpha(\beta b)) = ((\alpha\beta)a, (\alpha\beta)b) = (\alpha\beta)(a,b)$ for every $\alpha, \beta \in R$ and $(a,b), (c,d) \in A \times A$.

Therefore $A \times A$ is an R-module.

Let $C = \{(a,b) \mid f(a) = f(b)\}$. Show C is a submodule of $A \times A$. Let $(a,b), (c,d) \in C$.

$$f(a-c) = f(a) - f(c) = f(b) - f(d) = f(b-d)$$

Therefore $(a-c, b-d) \in C$ and hence $(a,b)-(c,d) \in C$. Let $\alpha \in R$. Then $f(\alpha a) = \alpha f(a) = \alpha f(b) = f(\alpha b)$, which means $(\alpha a, \alpha b) \in C$. Therefore $\alpha(a,b) \in C$ for every $\alpha \in R$ and $(a,b) \in C$. Thus C is a submodule of $A \times A$ and hence an R-module.

Define $p_1, p_2 : C \to A$ by $p_1(a,b) = a$ and $p_2(a,b) = b$. We know p_1 and p_2 are group homomorphisms; show $p_1(\alpha(a,b)) = \alpha p_1(a,b)$.

$$p_1(\alpha(a,b)) = p_1(\alpha a, \alpha b) = \alpha a = \alpha p_1(a,b).$$

It is similar for p_2. Therefore p_1 and p_2 are morphisms in Rm. Then for every $(a,b) \in C$, $fp_1(a,b) = f(a) = f(b) = fp_2(a,b)$, which means $p_1 = p_2$ and therefore $a = b$.

Therefore f is injective. Therefore monomorphism implies injection in Rm.

Let $f : A \to B$ in Rg be a monomorphism. Then $A \times A$ is a ring under coordinate-wise addition and multiplication, since A is a ring.

Let $C = \{(a,b) \mid f(a) = f(b)\}$. Show C is a subring of $A \times A$. We know C is a group under +. Let $(a,b), (c,d) \in C$. Then $f(ac) = f(a)f(c) = f(b)f(d) = f(bd)$. This implies $(ac,bd) \in C$ and hence $(a,b)(c,d) \in C$. The associative and distributive laws are inherited from $A \times A$.

Again define $p_1,p_2 : C \to A$ as before. Now $p_1((a,b)(c,d)) = p_1(ac,bd) = ac = p_1(a,b)p_1(c,d)$, and since p_1 was shown to be a group homomorphism, p_1 is now a Rg-homomorphism. Also, $fp_1(a,b) = f(a) = f(b) = fp_2(a,b)$. Therefore $p_1 = p_2$, and as before f is an injection. Therefore in Rg monomorphism implies injective.

The example also works for the category Ri since C will have the identity (1,1) as an element where 1 is the identity in A.

The construction of the object C leads to the definition of a pullback of a morphism in a category. Let $f : A \to B$ in a category $\mathcal{C}$. The triple (C,p_1,p_2) is a <u>pullback</u> of f means

(1) $fp_1 = fp_2$ and (2) if $fg_1 = fg_2$ for any $g_1, g_2 \in Mor(D,A)$, then there is a unique $k : D \to C$ such that $p_1k = g_1$ and $p_2k = g_2$.

For Top, let $f : A \to B$ and let C, p_1 and p_2 be as before. (C has the weak topology determined by p_1 and p_2.) Then p_1 and p_2 are continuous. Show (C, p_1, p_2) is a pullback of f.

Now $fp_1(x,y) = f(x) = f(y) = fp_2(x,y)$ for every $(x,y) \in C$. Suppose there is a $D \in ob\,Top$ such that for $g_1, g_2 : D \to A$ $fg_1 = fg_2$. Then define $k : D \to C$ by $k(d) = (g_1(d), g_2(d))$, $d \in D$, which is well defined since g_1 and g_2 are, and $fg_1 = fg_2$. Let $p_i^{-1}(V)$ be a subbase element of C where V is open in A. Since g_i is continuous, $g_i^{-1}(V)$ is open in D. For every $d \in D$ $(p_ik)(d) = p_i(g_1(d), g_2(d)) = g_i(d)$. Therefore $p_ik = g_i$. Therefore $k^{-1}(p_i^{-1}(V)) = p_ik)^{-1}(V) = g_i^{-1}(V)$, which is open in D since g_i is continuous. Therefore k is continuous. To show k is unique, suppose we have $k' : D \to C$ such that $p_1k' = g_1$ and $p_2k' = g_2$. Let $d \in D$, and $k'(d) = (x,y) \in C$. Then $(p_1k')(d) = p_1(x,y) = x = g_1(d)$, and $(p_2k')(d) = p_2(x,y) = y = g_2(d)$ by hypothesis. Therefore $(g,(d), g_2(d)) = (x,y)$, which means $k(d) = k'(d)$. Therefore $k = k'$ and hence k is unique. Therefore (C, p_1, p_2) is a pullback of f. Hence every morphism has a pullback in Top.

Another way to do this construction is to give C the inherited topology from $A \times A$. Then p_1 and p_2 are just the restrictions to C of the projection maps and hence are continuous.

Then the function k is still continuous since subbase elements for C will be of the form $P_i^{-1}(V) \cap C$ where P_i is the ith projection function on $A \times A$. However, this agrees with $p_i^{-1}(V)$. Everything else remains the same. In this way we can see that if we are in a topological cateogry where the spaces have only properties that are productive and hereditary, then this will be a category where every morphism (nonempty, if necessary to avoid nonempty product) has a pullback. In particular, since T_0, T_1, T_2, regular, T_3 (regular and T_1), completely regular and Tychanoff are hereditary and productive, every morphism has a pullback.

For the category of T_4-spaces we will show that every 1-1 function has a pullback. The set C used above has the form $C = \{(x,x) \mid x \in A\}$ if $f : A \to B$ is 1-1. The restricted projections are now just the same function, say p_1. Now p_1 is a continuous bijection. Show p_1 is an open function. Let U be open in C and let $y \in p_1(U)$. Then $(y,y) \in U$. Since U is open, there is an open set $V_1 \times V_2$ such that $(y,y) \in (V_1 \times V_2) \cap C$. Therefore $y \in V_1$, $y \in V_2$. Hence $y \in V_1 \cap V_2$, which is open in A, and $V_1 \cap V_2 \subseteq p_1(U)$. Hence p_1 is an open function. Therefore p_1 is a homeomorphism and $A \cong C$ (homeomorphic). Since A is T_4, then C is T_4.

Suppose we have $fg_1 = fg_2$, where $g_1, g_2 : D \to A$. Since f is injective, $g_1 = g_2$. Define $k : D \to C$ by $k(d) = (g_1(d), g_1(d))$ for every $d \in D$. Then $p_1 k = g_1$. Show k is continuous. If U is open in C, then $p_1(U)$ is open in A. Hence $g_1^{-1}(f,(U))$ is open in D, but $(g_1^{-1}p_1)(U)$. (If $k(d) \in U$, then $p_1(g_1(d), g_1(d)) = g_1(d) \in p_1(U)$. Therefore $d \in g_1^{-1}(p_1(U))$. If $g_1(d) \in p_1(U)$, then $k(d) = (g_1(d), g_1(d)) \in U$.) Hence k is continuous. Suppose $k' : D \to C$ such that $p_1 k' = g_1$. Then $p_1 k' = g_1 = p_1 k$. Since p_1 is injective, p_1 is left cancellable and hence $k' = k$. Therefore (C, p_1, p_1) is a pullback of f when f is injective.

For the category Rm, show every morphism has a pullback. Let $f : A \to B$ in Rm and define C, p_1 and p_2 as before. Show (C, p_1, p_2) is a pullback of f.

Suppose $fg_1 = fg_2$ where $g_1, g_2 : D \to A$. Define $k : D \to C$ by $k(d) = (g_1(d), g_2(d))$. This map is well defined since $fg_1 = fg_2$ by hypothesis. Show k is a morphism in Rm. For every $d, d' \in D$ and $r \in R$ we have $k(d + d') = (g_1(d + d'), g_2(d + d')) = (g_1(d) + g_1(d'), g_2(d) + g_2(d')) = (g_1(d), g_2(d)) + (g_1(d'), g_2(d')) = k(d) + k(d')$. Therefore k is a group homomorphism and $k(rd) = (g_1(rd), g_2(rd)) = (rg_1(d), rg_2(d)) = r(g_1(d), g_2(d)) = r\,k(d)$. Therefore k is an R-homomorphism.

The uniqueness of k is the same as in Top. Therefore every morphism in Rm has a pullback. In fact, since C, p_1 and p_2 can be considered groups or abelian groups as A is, and p_1 and p_2 are group homomorphisms, (C, p_1, p_2) is a pullback for morphisms in Gp and Ab.

Now we would like to find some categories whose objects can be considered as sets and whose morphisms can be considered as functions where the epimorphisms are exactly the surjective functions. We know this is true for Set.

In Top, let $f: B \to C$ be an epimorphism (abbreviate f is epi). Suppose $f(B) \neq C$. Let $A = \{f^{-1}(c)\}_{c\in C} \cup \{\infty\} \cup \{-\infty\}$ where ∞ and $-\infty$ are two objects not equal to $f^{-1}(c)$ for any $c \in C$. Let A have the indiscrete topology and define $h,g:$ $C \to A$ by $g(c) = f^{-1}(c)$ if $c \in f(B)$ or $g(c) = \infty$ $c \notin f(B)$ and $h(c) = f^{-1}(c)$ if $c \in f(B)$ or $h(c) = -\infty$ $c \notin f(B)$. Then g and h are continuous since A has the indiscrete topology. Let $b \in B$. $f(b) = c$. Then $(gf)(b) = g(c) = f^{-1}(c) = h(c) = (hf)(b)$. Since f is epi, $h = g$, which is a contradiction. Therefore, $f(B) = C$ and an epimorphism in Top is surjective.

Let Fgp stand for the category whose objects are finite groups and whose morphisms are the group homomorphism between them. This example is outlined in the book Categories and Functors. Let $f: G' \to G$ be epi. Then $f(G') = H$ is a subgroup of G. Let G/H be the set of left cosets of H in G. Then Perm $(G/H \cup \{\infty\})$ is a finite group where ∞ is an object not in G/H.

Define $\sigma: G/H \cup \{\infty\} \to G/H \cup \{\infty\}$ by $\sigma(gH) = gH, g \notin H$, $\sigma(H) = \infty$ and $\sigma(\infty) = H$. Then σ is well defined and a bijection, and therefore $\sigma \in$ Perm $(G/H \cup \{\infty\})$. Since $\sigma(\sigma(gH)) = \sigma(gH)$, $\sigma(\sigma(H)) = \sigma(\infty) = H$ and $\sigma(\sigma(\infty)) = \sigma(H) = \infty$, we have $\sigma^2 = id$, the identity map on $G/H \cup \{\infty\}$. Define $t: G \to$ Perm $(G/H \cup \{\infty\})$ by $t(g): G/H \cup \{\infty\} \to G/H \cup \{\infty\}$ where $t(g)(g'H) =$

$gg'H$ and $t(g)(\infty) = \infty$. Suppose $g = \overline{g}$. Then $t(g)(g'H) = gg'H = \overline{g}g'H = t(\overline{g})(g'H)$ and $t(g)(\infty) = \infty = t(\overline{g})$. Therefore t is a function. Show $t(g)$ is a bijection for every $g \in G$. Suppose $t(g)(g'H) = t(g)(\overline{g}H)$. Then $gg'H = g\overline{g}H$, which means $\overline{g}g^{-1}gg' \in H$. Therefore $\overline{g}^{-1}g \in H$. Therefore $\overline{g}H = g'H$ and $t(g)$ is an injection. The map is onto, for if $g'H \in G/H \cup \{\infty\}$, then $t(g)(g^{-1}g'H) = g'H$ and $t(g)(\infty) = \infty$. Therefore $t(g)$ is a bijection and hence t is well defined.

Show t is a gp-homomorphism. Let $g, \overline{g} \in G$ and $g'H \in G/H$. Then $t(g\overline{g})(g'H) = g\overline{g}g'H = t(g)(\overline{g}g'H) = t(g)(t(\overline{g})(g'H)) = (t(g)t(\overline{g}))(g'H)$ and $t(g\overline{g})(\infty) = \infty = (t(g)t(\overline{g}))(\infty)$. Therefore t is a gp-homomorphism.

Now define $s : G \to \mathrm{Perm}(G/H \cup \{\infty\})$ by $s(g) = \sigma t(g)\sigma$. Then s is well defined and we need to show s is a gp-homomorphism. If $g, \overline{g} \in G$, then $s(g\overline{g}) = \sigma t(g\overline{g})\sigma = \sigma t(g)t(\overline{g})\sigma = \sigma t(g)\sigma\sigma t(\overline{g})\sigma = s(g)s(\overline{g})$. Let $gH \in G/H$, $h \in H$. Then $t(h)(gH) = t_h(gH) = hgH$ and $s(h)(gH) = (\sigma t_h \sigma)(gH) = \sigma t_h(gH) = \sigma(hgH) = hgH$. Also, $t(h)(\infty) = \infty$ and $s(h)(\infty) = (\sigma t_h \sigma)(\infty) = \sigma t_h(H) = \sigma(hH) = \sigma(H) = \infty$. Therefore $t(h) = s(h)$ for every $h \in H$.

Now $i : H \to G$ by $i(h) = h$ is an epimorphism and $ti = si$ implies $t = s$ for every $g \in G$. Then if $g \in G$, $gH = tg(H) = s(g)H = (\sigma tg\sigma)(H) = \sigma tg(\infty) = \sigma(\infty) = H$. Therefore $gH = H$ for every $g \in G$. Hence $H = G$ and f is a surjection. Therefore, in Fgp, epi implies surjective.

In the category of abelian groups Ab, let $f : A \to B$ be epi. Now $f(A) = H$ is a normal subgroup. Then B/H is an abelian group. Let $n : B \to B/H$ by $n(b) = bH$. Then n is a homomorphism. Let $v : B \to B/H$ by $v(b) = H$, and v is a homomorphism. Let $a \in A$ and $f(a) = b$. Then $(nf)(a) = n(b) = bH = H = v(b) = (vf)(a)$. Therefore $n = v$ since f is epi. Therefore $bH = H$ for every $b \in B$. Hence $H = B$ and f is surjective.

Let $\mathcal{C}$ be a category. We define $f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$ to be an <u>isomorphism</u> if there is a morphism $g \in \mathrm{Mor}_{\mathcal{C}}(B,A)$ such that $fg = 1_B$ and $gf = 1_A$. Two objects A and B in $\mathcal{C}$ are called isomorphic ($A \cong B$) if $\mathrm{Mor}_{\mathcal{C}}(A,B)$ contains an isomorphism. Two morphisms $f : A \to B$ and $g : A' \to B'$ are called isomorphic ($f \cong g$) if there are morphisms $h \in \mathrm{Mor}_{\mathcal{C}}(A,A')$ and $k \in \mathrm{Mor}_{\mathcal{C}}(B,B')$ such that $gh = kf$.

Since $f : A \to B$ being an isomorphism implies that there is a $g : B \to A$ such that $fg = 1_B$, and $gf = 1_A$, g is also an isomorphism and g is usually denoted by f^{-1} because it is uniquely determined by f.

We want to now show that the composition of isomorphisms is an isomorphism. Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ with f and g isomorphisms. Then $C \xrightarrow{g^{-1}} B \xrightarrow{f^{-1}} A$, and $(gf)(f^{-1}g^{-1}) = g(ff^{-1})g^{-1} = g1_Bg^{-1} = 1_C$. Also $(f^{-1}g^{-1})(gf) = f^{-1}(g^{-1}g)f = f^{-1}1_Bf = 1_A$. Then $f^{-1}g^{-1} = (gf)^{-1}$ and gf is an isomorphism.

Also 1_A is an isomorphism for every A in ob $\mathcal{C}$. Therefore the relation of objects being isomorphic is an equivalence relation. Similarly, the relation of morphisms being isomorphic is an equivalence relation.

Lemma 1.3. If f is an isomorphism, then f is a monomorphism and an epimorphism.

Proof. Suppose $f : A \to B$ is an isomorphism and there exists $h,g : C \to A$ such that $fh = fg$. Since f is an isomorphism, there exists $f^{-1} : B \to A$. Now $f^{-1}(fh) = f^{-1}(fg) = (f^{-1}f)h = (f^{-1}f)g$. Therefore $1_A h = 1_A g$ or $h = g$. Therefore f is left cancellable and hence a monomorphism.

Now suppose there exists $h,g : B \to C$ such that $gf = hf$. Therefore $(gf)f^{-1} = (hf)f^{-1}$ or $g(ff^{-1}) = h(ff^{-1})$. Hence, $g 1_B = h 1_B$ or $g = h$ and f is right cancellable. Therefore, f is an epimorphism. Thus Lemma 1.3 has been proved.

The converse of this lemma is not true. An example in Hd will be shown in the section concerning functors. A category in which the converse is true is called a balanced category. Two quick examples are Set and Ab, since in these categories monic implies 1-1 and epi implies onto, so an inverse is guaranteed.

Objects

We now want to collect monomorphisms and generalize some notions like "subset" in set theory. Let $\mathcal{C}$ be a category and $\mathcal{M}$ be the class of monomorphisms. Define two monomorphisms $f : A \to B$ and $g : C \to D$ to be equivalent $(f \sim g)$ if $B = D$ and there are two morphisms $h : A \to C$ and $k : C \to A$ such that $gh = f$ and $fk = g$. Another way of saying this is that the following diagrams commute.

Lemma 1.4. $\sim$ is an equivalence relation on $\mathfrak{M}$.

Proof.

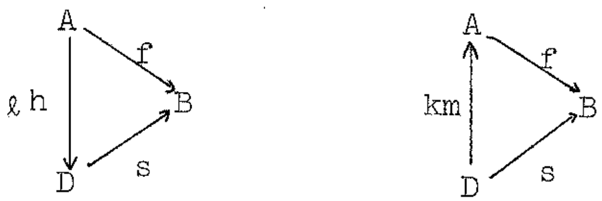
(1) Let $f : A \to B$ be monic. Then we have $f\,1_A = 1_A = f$.

(2) Suppose $f : A \to B$ and $g : C \to B$ are monic and $f \sim g$. Then the following commutative diagrams exist.

Therefore $g \sim f$.

(3) Assume the hypothesis of (2) and further, that $g \sim s : D \to B$ where s is monic. Show $f \sim s$. Then the diagrams in (2) exist along with these two commutative diagrams:

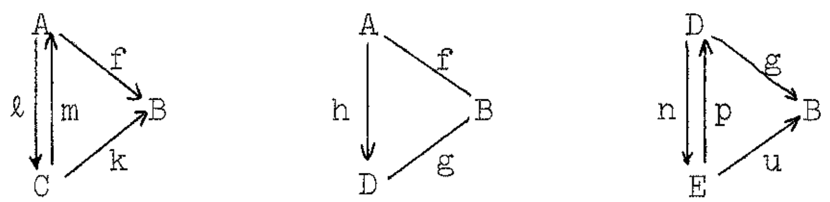
Then we would like to show the following diagrams are commutative:



Now $s(\ell h) = (s\ell)h = gh = f$ and $f(km) = (fk)m = gm = s$. Hence $f \sim s$. Therefore $\sim$ is an equivalence relation and Lemma 1.4 has been proved.

If $f \sim g$ then $f = gh$ and $fk = g$, hence $f = fkh$ and $g = ghk$. Since f and g are left cancellable, we get $1_A = kh$ and $1_C = hk$. Therefore $A \cong C$.

We are now ready to define a subobject. A <u>subobject</u> of an object B in a category $\mathcal{C}$ is the equivalence class of a monomorphism f in $\mathcal{M}$, with range B; we write "$\langle f \rangle$" for equivalence class of f. Alternatively, let $\mathcal{U}$ be a complete set of representatives for the equivalence relation. Then if $B \in \text{ob}\,\mathcal{C}$, a subobject of B is a monomorphism in $\mathcal{U}$ with range B. A subobject $\langle f \rangle$ of B is said to be smaller than a subobject $\langle g \rangle$ of B if there is a morphism h of $\mathcal{C}$ such that $f = gh$. This order relation is well defined, for suppose $\langle f \rangle \leq \langle g \rangle$ and $f \sim k$, $g \sim u$ where the following diagrams are commutative.



Now $hm : C \to D$ and $g(hm) = (gh)m \leq fm = k$. Also $nhm : C \to E$ and $u(nhm) = (un)(hm) = g(hm) = k$. Therefore $\langle k \rangle \leq \langle m \rangle$. Also, since $f = gh$ and f and g are monomorphisms, and by Lemma 1.1 (1), h is a uniquely determined monomorphism. We will often use only f to stand for a subobject of B, or further abbreviating, we will use only the domain of the subobject, assuming the monomorphism is known, and write $A' \subseteq A$ when $f : A \to B$, $g : A' \to B$ and $f \leq g$.

Lemma 1.5. The subobjects of an object $B \in \text{ob}\, \mathcal{C}$ form an ordered class.

Proof. Let $f \leq g$ and $g \leq h$ be subobjects of B where $f : A \to B$, $g : C \to B$, $h : D \to B$. Then there exist morphisms k and ℓ such that $f = gk$ and $g = h\ell$. Then $f = (h\ell)k = h(\ell k)$. Therefore $f \leq h$. Now $f \leq f$, since $f = f\, 1_A$. If $f \leq g$ and $g \leq f$, then there exists morphisms k and ℓ such that $f = gk$ and $g = f\ell$. Hence f and g are equivalent monomorphisms. Therefore the class of subobjects forms an ordered class.

Before continuing with some definitions we would like to show the subobjects in certain categories. Let $B \in \text{ob Set}$ and $f : A \to B$ be a subobject of B. Then f is 1-1. Show $i : f(A) \subseteq B \sim f$. We know i is 1-1.

Define $\overline{f} : A \to f(A)$ by $\overline{f}(a) = f(a)$ and define $h : f(A) \to A$ by $h(f(a)) = a$. Then $\overline{f}$ and h are functions. Now $(i\overline{f})(a) = i(\overline{f}(a)) = i(f(a)) = f(a)$ and $(fh)(f(a)) = f(a) = i(f(a))$. Therefore $i \sim f$. We know that f(A) is a subset of B and $i : f(A) \to B$ is a subobject of B. Obviously if C is a subset of B, then $i : C \to B$ is a subobject. There the subobjects of a set B are the subsets of B.

In Gp a subobject $f : A \to B$ of $B \in \text{ob Gp}$ is a subgroup since f is 1-1 and f(A) is a subgroup of B. Then use the same argument as in Set. It is not true in Top, however, that subobjects are always subspaces. Let R denote the reals with the usual topology and let I_D denote the interval [0,1] with the discrete topology. The inclusion $i : I_D \to R$ is continuous since I_D has the discrete topology. Hence i is a subobject of R in Top.

Now in Top, if $X \cong Y$, then there exists $h : X \to Y$ such that h is continuous, $h^{-1} : Y \to X$ is continuous, $hh^{-1} = 1_Y$ and $h^{-1}h = 1_X$. In other words, h is an isomorphism of the category Top. Therefore h is monic and epi and therefore 1-1 and onto. Hence h is a homeomorphism. Therefore X is homeomorphic to Y. Since subobjects have the property that their domains are isomorphic, the question becomes: can there be a subspace of R such that it has cardinality $c = |R|$ and has as its relative topology the discrete topology?

Assume the hypothesis for S. Since S is a subspace of a second countable space R, then S must be second countable. Therefore S must be separable. This is a contradiction to S having cardinality c. Therefore $i : I_D \to R$ is a subobject of R but is not a subspace. In Top, any continuous inclusion map is a subobject.

The ordered class of subobjects of an object $B \in \text{ob}\,\mathcal{C}$ is called the <u>power class</u> of B. If the power class of each object of a category $\mathcal{C}$ is a set, then $\mathcal{C}$ is called a <u>locally small category</u> and the power classes are referred to as <u>power sets</u>. For example, the category CH is locally small. Let $B \in \text{ob}\,\text{CH}$. Let $\mathcal{U}_B$ be the ordered class of subobjects of B.

Then $\mathcal{U}_B$ is the collection of continuous 1-1 functions into B, since monic is equivalent to 1-1 in CH. Now a 1-1 continuous function from a compact space onto (its image) a Hausdorff space is a homeomorphism. Then if $f : A \to B$ is monic, $f(A)$ is a subspace of B. Therefore $\mathcal{U}_B$ = collection of all $\langle f : A \to B \rangle$ such that f is monic is the same as $\{\langle f : A \to B\rangle \mid A \in 2^B\}$, which is a set. Therefore CH is locally small.

The category Gp is also locally small. Let $\mathcal{U}_G$ denote the power class of $G \in \text{ob}\,\text{Gp}$. We want to show $\mathcal{U}_G$ is a set. Define $F : \mathcal{U}_G \to 2^G$ by $F(\langle f : H \to G \rangle) = f(H)$. We need to show F is well defined and 1-1. To show well definition, suppose $\langle f : H \to G \rangle = \langle g : K \to G \rangle$. Then there exist monomorphisms h,k such that $gh = f$ and $fk = g$. Also h,k are 1-1. We must show $f(H) = g(k)$. Let $a \in f(H)$. Then there exists a unique $b \in H$ such that $f(b) = a$ since f is 1-1 onto $f(H)$. Then $f^{-1}(a) = b \in H$ and $h(b) \in K$. Now $g(h(b)) = (gh)(b) = f(b)$ implies $f(b) \in g(K)$ or $a \in g(K)$. Hence $f(H) \subseteq g(K)$.

Similarly, $g(K) \subseteq f(H)$. Hence F is well defined.

Now suppose $F(\langle f : H \to G \rangle) = F(\langle g : K \to G \rangle)$ or $f(H) = g(K)$. Define $h : H \to K$ by $h = g^{-1}f$. Now $g^{-1} : g(K) \to K$ is 1-1 and onto since g is 1-1, onto. Then h is well defined and $gh = f$.

Define $k : K \to H$ by $k = f^{-1}g$. Then k is well defined and $fk = g$. Therefore $\langle f \rangle = \langle g \rangle$ and F is 1-1. Hence $\mathcal{U}_G$ is a set and Gp is locally small. This also shows Set is locally small.

Let $\mathcal{C}$ be a locally small category. Let U be a subset of the power set of an object B in $\mathcal{C}$. A subobject $A \in U$ is said to be minimal (maximal) in U if $A' \in U$ and $A' \subseteq A$ $(A \subseteq A')$ always implies $A' = A$. The power set of B is called artinian (noetherian) if in each nonempty subset of the power set there is a minimal (maximal) subobject. If the power set is artinian (noetherian), then B is called an artinian (noetherian) object. If all objects in $\mathcal{C}$ are artinian or noetherian then $\mathcal{C}$ is artinian or noetherian, respectively. A subset K of the power set is called a chain if whenever $A', A \in K$, we have $A' \subseteq A$ or $A \subseteq A'$. An object $B \in \text{ob}\,\mathcal{C}$ complies with the minimum (maximum) condition for chains if each nonempty chain in the power set of B contains a minimal (maximal) element.

Lemma 1.6. An object $\mathcal{B} \in \mathcal{C}$ complies with the minimum condition (maximum condition) for chains if and only if $\mathcal{B}$ is artinian (noetherian). Instead of proving this particular lemma, we shall dualize the notions of subobjects, locally small, artinian and noetherian, and state and prove the dual assertion of this lemma.

Let $\mathcal{C}$ be a category and $\mathcal{E}$ be the class of epimorphisms of $\mathcal{C}$. Define $f : A \to B \sim g : C \to D$ if and only if $A = C$ and the following diagrams exist and are commutative.

This defines an equivalence relation on $\mathcal{E}$. (The proof is similar to the proof involving monomorphisms.) As before, if $f : A \to B$ and $g : A \to C$ are equivalent, it is easy to show that $C \cong B$. Let $\mathcal{U}^*$ be a complete set of representatives for this equivalence relation, and we have the following definitions. A <u>quotient object</u> of $A \in \text{ob}\,\mathcal{C}$ is an epimorphism in $\mathcal{U}^*$ with domain A. A quotient object f of A is said to be <u>smaller</u> than a quotient object g of A if there is a morphism h in $\mathcal{C}$ such that $f = hg$. Then h is a uniquely determined epimorphism. The dual to Lemma 5 can be stated as follows.

<u>Lemma 1.5^{op}</u>. The quotient objects of an object $\mathcal{B} \in \text{ob}\,\mathcal{C}$ form an ordered class. The proof can be done by reversing arrows in the proof of Lemma 1.5.

The <u>copower class</u> of an object $A \in \text{ob}\,\mathcal{C}$ is the ordered class of the quotient objects of A. A category is <u>locally cosmall</u> if the copower class of each object A in $\mathcal{C}$ is a set. In this case, we have <u>copower sets</u>. The dual notions of artinian and noetherian are <u>coartinian</u> and <u>conoetherian</u>.

<u>Lemma 1.6^{op}</u>. An object $\mathcal{B} \in \text{ob}\,\mathcal{C}$ complies with the minimum condition (maximum condition) for chains with respect to quotient objects if and only if $\mathcal{B}$ is coartinian (conoetherian).

<u>Proof</u>. Suppose $\mathcal{B} \in \text{ob}\,\mathcal{C}$ complies with the minimum conditions for chains with respect to quotient objects. Let U be a subset of the copower set of $\mathcal{B}$. Suppose U does not

have a minimal element. Then for each $A_i \in U$ there is an $A_{i+1} \in U$ such that $A_{i+1} \leq A_i$ and $A_{i+1} \neq A_i$. Construct a chain in this manner, choosing $A_{i+2} \leq A_{i+1}$ and $A_{i+2} \neq A_{i+1}$. Then this chain can have no minimal element contrary to hypothesis.

The other direction follows from the definition. The statement using maximum condition for chains is similar. Therefore, Lemma 1.6^{op} has been proved and its dual is also true.

We would like to now give some examples of quotient objects and some categories that are not artinian, noetherian, and some locally cosmall categories, and some categories that are not conoetherian.

The category Gp is not artinian. Let $(Z,+)$ denote the group of integers under addition. For every $n \in Z^+$, let $S_n = \{2^n j \mid j \in Z\}$. Then $(S_n,+)$ is a group. Define $i_n : S_n \to Z$ by $i_n(2^n j) = 2^n j$. Then i_n is 1-1, hence a monomorphism, and i_n is a homomorphism. We would now like to show $S_{n+1} \subseteq S_n$; that is, $i_{n+1} \leq i_n$ for every $n \in Z^+$. Define $h : S_{n+1} \to S_n$ by $h(2^{n+1}j) = 2^n \cdot 2j$. Then h is a 1-1 homomorphism and $(i_n h)(2^{n+1}j) = i_n(2^n \cdot 2j) = 2^n \cdot 2j = 2^{n+1}j = i_{n+1}(2^{n+1}j)$. Therefore $i_{n+1} \leq i_n$. Suppose there exists a k such that $i_{n+1}k = i_n$. Then $i_n \sim i_{n+1}$ and $1_{S_n} = hk$, $1_{S_{n+1}} = kh$. Therefore $k = h^{-1}$ and $hk(2^n \cdot 1) = h(2^{n+1}j) = 2^n \cdot 2j = 1_{S_n}(2^n \cdot 1) = 2^n$ for some $j \in Z$. Therefore $2^n 2j = 2^n$. Hence $j = 1/2$, contrary to $j \in Z$. Therefore $i_n \not\leq i_{n+1}$. Then $\{\langle i_n \rangle\}_{n=1}^{\infty}$ is a chain which has no minimal element. Hence Gp is not artinian. Since all the maps here are functions and all the groups are abelian, this also shows Set and Ab are not artinian.

The category Set is not noetherian. Let Z^+ denote the positive integers. Define $S_1 = \{1\}$, $S_2 = \{1,2\}$ and in general $S_n = \{1,2,3,\cdots,n\}$ for every $n \in Z^+$. Define $i_n : S_n \to Z^+$ by $i_n(m) = m$. Then $\{\langle i_n : S_n \to Z^+\rangle\}_{n=1}^{\infty}$ is a collection of subobjects of Z^+. Show $i_n \leq i_j$ whenever $n \leq j$. Define $h : S_n \to S_j$ by $h(k) = k$. This is well defined since as sets $S_n \subseteq S_j$. The map h is also 1-1 and $(i_j h)(k) = ij(k) = k = i_n(k)$. Also, $i_j \not\leq i_n$ since if there were an $\ell : S_j \to S_n$ such that $i_n \ell = i_j$. then $h\ell = 1_{S_j}$. Since $n + 1 \in S_j$ we have $(hk)(n+1) = h(k(n+1)) = n+1$. Since h is 1-1 we have $k(n+1) = n+1$ contrary to the definition of S_n. Hence $\{\langle i_n\rangle\}_{n=1}^{\infty}$ is a chain of subobjects and it has no maximal element. Therefore, Set is not noetherian. We would now like to show that Set is locally cosmall. Let $A \in$ ob Set. Show $\mathcal{U}_A^*$, the copower class of A, is a set. In Set, epimorphism implies onto and each onto function sets up an equivalence relation on the domain. For example, let $\langle f : A \to B\rangle$ be a quotient object of A. Then R defined by $aRb \equiv f(a) = f(b)$ is an equivalence relation on A, and since $R \subseteq A \times A$, R is a set. Then the collection of all equivalence relations on A is contained in the power set of $A \times A$ on $2^{A \times A}$ (Set theory-wise). Define $F : \mathcal{U}_A^* \to 2^{A \times A}$ by $F(\langle f : A \to B\rangle) = R$, where R is the equivalence relation on A induced by f.

Show F is well defined. Suppose $\langle f : A \to B\rangle = \langle g : A \to C\rangle$. Then there exists $h : B \to C$ and $k : C \to B$ such that $f = hg$ and $g = hf$. Let f determine the equivalence relation R, and g determine the equivalence relation S. Suppose aRb. Then $f(a) = f(b)$ implies $g(a) = (hf)(a) = h(f(a)) = h(f(b)) = (hf)(b) = g(b)$. Therefore, aSb. Suppose aSb. Then $f(a) = (kg)(a) = k(g(a)) = k(g(b)) = (kg)(b) = f(b)$. Therefore aRb. Therefore R = S and F is well defined. Show F is 1-1. Suppose $F(\langle f\rangle) = F(\langle g\rangle)$. Then R = S where R and S are as before. Define $h : B \to C$ by $h(b) = g(\{f^{-1}(b)\})$. Show h is well defined. If $c, d \in \{f^{-1}(b)\}$, then $f(c) = f(d) = b$ and R = S implies $g(c) = g(d)$. Then $g(\{f^{-1}(b)\}) \in C$. Therefore h is well defined. If $a \in A$, then $h(f(a)) = g(\{f^{-1}(f(a))\}) = g(a)$. Therefore $hf = g$. Similarly, define $k : C \to B$ to get $f = kg$. Hence $f \sim g$ and F is 1-1. Therefore $\mathcal{U}_A^*$ is embedded in a set and hence $\mathcal{U}_A^*$ is a set. Therefore Set is locally cosmall.

To show Set is not conoetherian, consider again Z^+. Let $S_n = \{1,2,3,\cdots,n\}$ for every $n \in Z^+$. Define $f_1 : Z^+ \to S_1$ by $f_1(z) = 1$ for every $z \in Z^+$. Define $f_2 : Z^+ \to S_2$ by $f_2(1) = 1$ and $f_2(z) = 2$ if $z \geq 2$. In general, define $f_n : Z^+ \to S_n$ by $f_n(z) = z$ if $1 \leq z < n$ and $f_n(z) = n$ if $n \leq z$. Now each f_n is an onto function; hence $\langle f_n : Z^+ \to S_n\rangle$ is a quotient object of Z^+. Show $\langle f_n\rangle \leq \langle f_{n+1}\rangle$ for every n. Define $h : S_{n+1} \to S_n$ by $h(s) = s$ if $1 \leq s \leq n$ and $h(s) = n$ if $s = n+1$.

Then h is well defined, and if $z \in Z^+$, then $(hf_{n+1})(z) = h(z)$ if $1 \leq z \leq n$. Then $h(z) = z = f_n(z)$. If $z \geq n+1$, then $(hf_{n+1})(z) = h(n+1) = n = f_n(n+1)$. Therefore $hf_{n+1} = f_n$. Now $f_{n+1} \not\leq f_n$, since there can be no onto function from S_n to S_{n+1}. Therefore $\langle f_n \rangle \leq \langle f_{n+1} \rangle$. Therefore $\{\langle f_n \rangle\}_{n=1}^{\infty}$ is a chain of quotient objects of Z^+. Since there can be no onto map for S_n to S_{n+1}, this chain has no maximal element. Hence Z^+ does not comply with the maximum condition for chains with respect to quotient objects and, by the lemma, Z^+ is not conoetherian. Therefore Set is not conoetherian.

The empty set ϕ plays a special role when we consider the functions associated with it. For any other set A, there is only one function from ϕ to A, namely the empty function. The empty topological space has a similar property. The set, $\{\phi\}$, has the property that for any other set A there is only one function from A into $\{\phi\}$, namely that function which assigns everything in A to $\phi \in \{\phi\}$. If we give $\{\phi\}$ the indiscrete topology, then $\{\phi\}$ has a similar property--there is only one continuous function into it. We can generalize this notion to an arbitrary category.

An object A in a category $\mathcal{C}$ is called an <u>initial object</u> if $\mathrm{Mor}_{\mathcal{C}}(A,B)$ consists of exactly one element for all $B \in \mathrm{ob}\,\mathcal{C}$. Dually, we define A to be a <u>final object</u> if $\mathrm{Mor}_{\mathcal{C}}(B,A)$ consists of exactly one element for all $B \in \mathrm{ob}\,\mathcal{C}$. An object is called a <u>zero object</u> if it is an initial and a final object.

<u>Lemma 1.7</u>. All initial objects are isomorphic.

<u>Proof</u>. Let A and C be initial objects in a category $\mathcal{C}$. Then 1_A and 1_C are the only elements in $\text{Mor}_{\mathcal{C}}(A,A)$ and $\text{Mor}_{\mathcal{C}}(C,C)$, respectively. Let $\text{Mor}_{\mathcal{C}}(A,C)$ consist of the one element h and $\text{Mor}_{\mathcal{C}}(C,A)$ consist of the one element k. Then $hk = 1_C$ and $kh = 1_A$. Therefore, $\text{Mor}_{\mathcal{C}}(A,C)$ contains an isomorphism. Hence A and C are isomorphic. The dual is also true.

<u>Lemma 1.8</u>. A zero object 0 of a category $\mathcal{C}$ is a subobject of each object $B \in \text{ob}\,\mathcal{C}$.

<u>Proof</u>. Let $B \in \text{ob}\,\mathcal{C}$. $\text{Mor}_{\mathcal{C}}(0,B)$ consists of one element $h : 0 \to B$. Suppose $g,k : C \to 0$ such that $hg = hk$. Since 0 is a final object, $\text{Mor}_{\mathcal{C}}(C,0)$ has at most one element. Therefore $g = k$ and h is a monomorphism. Hence 0 is a subobject of B.

A morphism $f : A \to B$ in a category $\mathcal{C}$ is called a <u>left zero morphism</u> if $fg = fh$ for all $g,h \in \text{Mor}(C,A)$ and all $C \in \text{ob}\,\mathcal{C}$. Dually, we define a <u>right zero morphism</u>. A <u>zero morphism</u> is both a right and left zero morphism.

<u>Lemma 1.9</u>. (1) If f is a right zero morphism and g is a left zero morphism, and if fg is defined, then fg is a zero morphism.

(2) Let A be an initial object. Then $f : A \to B$ is always a right zero morphism.

(3) Let 0 be a zero object. Then $f : 0 \to B$ and $g : C \to 0$ and consequently, $fg : C \to B$ are zero morphisms.

<u>Proof</u>. (1) Suppose $f : B \to A$ is a right zero morphism and $g : C \to B$ is a left zero morphism. Suppose $h, \ell : D \to C$

and $k,m : A \to D$. Then $(fg)h = f(gh) = f(g\ell) = (fg)\ell$ since g is a left zero morphism. Therefore fg is a left zero morphism. Now $k(fg) = (kf)g = (mf)g = m(fg)$ since f is a right zero morphism. Therefore fg is a zero morphism.

(2) For an initial object A and $B \in \text{ob}\,\mathcal{C}$, let $\text{Mor}_{\mathcal{C}}(A,B)$ consist of the one element $f : A \to B$. Suppose $g,h : B \to C$. Then $gf,hf : A \to C$ and $gf = hf$, since there is only one element in Mor(A,C). Hence f is a right zero morphism.

(3) Assume the hypothesis. From (2), f is a right zero morphism. Since Mor(D,0) consists of only one element for each $D \in \text{ob}\,\mathcal{C}$, f is a left zero morphism. Hence f is a zero morphism. Since Mor(0,D) consists of only one element, g is a right zero morphism. Since Mor(D,0) consists of only one element, g is a left zero morphism. From (1), fg is a zero morphism.

A category $\mathcal{C}$ is called a <u>category with zero morphisms</u> if there is a family $\{0(A,B) \in \text{Mor}(A,B)$ for all $A,B \in \mathcal{C}\}$ with $f\,0(A,B) = 0(A,C)$ and $0(B,C)g = 0(A,C)$ for all $A,B,C \in \text{ob}\,\mathcal{C}$ and all $f \in \text{Mor}(B,C)$ and $g \in \text{Mor}(A,B)$. We must show 0(A,B) is a zero morphism. Let $g,h : C \to A$. Then $0(A,B)g = 0(C,B) = 0(A,B)h$. Therefore 0(A,B) is a left zero morphism. Now $g\,0(A,B) = 0(A,C) = h\,0(A,B)$. Therefore 0(A,B) is a right zero morphism. Hence 0(A,B) is a zero morphism. This family is uniquely determined, since if $\{0'(A,B)\}$ is another family, then $0(A,B) = 0(A,B)\,0'(A,B) = 0'(A,B)$. Hence the families are the same.

<u>Lemma 1.10</u>. A category $\mathcal{C}$ with a zero object Z is a cetegory with zero morphisms.

<u>Proof</u>. Assume the hypothesis: let $A,B \in \mathcal{C}$. Then from Lemma 1.9 (3), $f : Z \to B$, and $g : A \to Z$ are zero morphisms and so is $fg : A \to B$. Show $fg = 0(A,B)$. Let $h : B \to C$ and $k : Z \to C$, where k is the only element in Mor(Z,C). Now kg is a zero morphism from A to C. Show $h(fg) = kg$. Since $hf : Z \to C$, we know $k = hf$. Then since g is a zero morphism, $h(fg) = (hf)g = kg$. Hence the first condition is satisfied. For the second condition, let $k : C \to Z$ be the unique morphism. Then $gh = k$ and $(fg)h = f(gh) = fk$. Hence the second condition is satisfied. Therefore the family $\{fg : A \to B$ where $A \overset{g}{\to} Z \overset{f}{\to} B\}$ is the family of zero morphisms.

In Top* (pointed topological spaces with pointed continuous functions), we would like to show that the one point-pointed topological spaces are zero objects. Let $(\{a\},a)$ be a one point-pointed topological space. Then if $g,f : (B,b) \to (\{a\},a)$ are pointed continuous functions, then $f(b) = g(b)$ and $f(c) = g(c) = a$ for every $c \in B$. Hence $f = g$. Therefore $(\{a\},a)$ is a final object. To show $(\{a\},a)$ is an initial object, let $g,f : (\{a\},a) \to (B,b)$ be pointed continuous functions. Then $f(a) = g(a) = b$. Therefore $f = g$. Therefore $(\{a\},a)$ is an initial object. Hence $(\{a\},a)$ is a zero object in Top*.

In Gp, show the one point group $(e,\cdot)$ is a zero object. Suppose $g,f : G \to (e,\cdot)$. Then $f(a) = e = g(n)$ and $(e,\cdot)$ is a final object. If $g,f : (e,\cdot) \to G$, then $f(e) = g(e) = e'$ where e' is the identity in G since g and f are homomorphisms. Therefore $f = g$ and $(e,\cdot)$ is an initial object, hence a zero object. The one point group is also a zero object in Ab.

We would now like to show that the family of homomorphisms that map everything to the identity is the family of zero morphisms in Ab. Denote elements in this family by $O(A,B)$ for $A,B \in \text{ob}\,\text{Ab}$. Let $f : B \to C$. Show $f\,O(A,B) = O(A,C)$. Element-wise we have for all $a \in A$ $(f\,O(A,B))(a) = f(O(A,B)(a)) = f(e_B) = e_C$, since f is a homomorphism. By definition, $O(A,C)(a) = e_C$. Hence $f\,O(A,B) = O(A,C)$. Let $g : A \to B$ and show $O(B,C)g = O(A,C)$. For every $a \in A$ $(O(B,C)g)(a) = O(B,C)(g(a)) = e_C$ and $O(A,C)(a) = e_C$. Therefore $O(B,C)g = O(A,C)$.

Let $\mathcal{C}$ be a category and $f,g : A \to B$ in $\mathcal{C}$. A morphism $i : C \to A$ is called a <u>difference kernel</u> of the pair (f,g) if $fi = gi$ and if for each $D \in \text{ob}\,\mathcal{C}$ and each morphism $h : D \to A$ such that $fh = gh$, there is exactly one morphism $h' : D \to C$ such that $h = ih'$.

<u>Lemma 1.11</u>. Each difference kernel is a monomorphism.

<u>Proof</u>. Let $f,g : A \to B$ and let $i : C \to A$ be a difference kernel of the pair (f,g). To show i is left cancellable, let $h,k : D \to C$ be such that $ih = ik$. Then $f(ih) = (fi)h = (gi)h = h(ih)$, and by definition there exists a unique $k' : D \to C$ such that $ik = ik'$. Therefore $ih = ik'$ and h' is unique

implies $k' = h'$. Also $ih = ih'$, and h' unique implies $h = h'$ and $ik = ik'$ and k' unique implies $k = k'$. Thus $k = k' = h' = h$. Therefore, i is left cancellable and hence a monomorphism.

Lemma 1.12. If $i : C \to A$ and $i' : C' \to A$ are difference kernels of the pair (f,g) $f,g : A \to B$, then there is a uniquely determined isomorphism $k : C \to C'$ such that $i = i'k$.

Proof. Since i and i' are difference kernels, they are monomorphisms by Lemma 1.11. We know $fi = gi$ and $fi' = gi'$. Then, since i' is a difference kernel and $fi = gi$, there is exactly one morphism $k : C \to C'$ such that $i = i'k$. Since i is a difference kernel and $fi' = gi'$, there is exactly one morphism $h : C' \to C$ such that $i' = ih$. Then $i = i'k = ihk$, and since i is monic, we have $1_C = hk$. Also $i' = ih = ikh$, and i' monic implies $1_{C'} = kh$. Therefore k is an isomorphism and k is unique. Therefore Lemma 1.12 has been proved.

In a category $\mathcal{C}$ with zero morphisms, let $f : A \to B$. A morphism $g : C \to A$ in $\mathcal{C}$ is called a kernel of f if $fg = 0(C,B)$, and if, to each morphism $h : D \to A$ with $fh = 0(D,B)$, there is exactly one morphism $k : D \to C$ with $h = gk$.

Lemma 1.13. Let g be a kernel of f. Then g is a difference kernel of $(f, 0(A,B))$ where $f : A \to B$.

Proof. Let $g : C \to A$. Show $fg = 0(A,B)g$. Since g is a kernel of f, we know $fg = 0(CB)$ and $0(A,B)$ has the property that $0(A,B)g = 0(C,B)$. Hence g satisfies the first condition. For the second condition, let $h' : D \to A$ such that

$fh = O(A,B)h$. Then $fh = O(D,B)$, and since g is a kernel, there is exactly one morphism $h' : D \to C$ with $h = gh'$. Thus g is a difference kernel of $(f, O(A,B)$. By Lemma 1.11, g is a monomorphism.

By dualizing, we can define a <u>difference</u> <u>cokernel</u> and a <u>cokernel</u>. Then each difference cokernel is an epimorphism dualizing Lemma 1.11. The dual to Lemma 1.13 tells us that if g is a cokernel of f, then g is a difference cokernel of $(f, O(B,A))$ where $f : B \to A$.

We would like to find some difference kernels, kernels, and cokernels. In Gp, let $f,g : G \to G'$ and let $C = \{c \in G \mid f(c) = g(c)\}$. Then $C \neq \phi$ (since $f(e) = g(e)$), and we know that C is a subgroup of $G \times G$. Let $i : C \to G$ be the inclusion homomorphism. Let $D \in \text{ob}\, Gp$, and suppose $h : D \to G$ is such that $fh = gh$. Show there is exactly one morphism $h' : D \to C$ such that $h = ih'$. Define $h' : D \to C$ by $h'(d) = h(d)$. Then $f(h(d)) = (fh)d = (gh)(d) = g(h(d))$, and therefore $h(d) \in C$. Hence h' is well defined and a homomorphism. Also $(ih')(d) = i(h'(d)) = i(h(d)) = h(d)$ (for all $d \in D$). Show h' is unique. Suppose there exists $k : D \to C$ such that $h = ik$. Let $d \in D$. Then $h'(d) = h(d) = (ik)(d) = i(k(d))$. But $(ik)(d) = k(d)$. Therefore $h'(d) = k(d)$, and h' is unique. Hence $i : C \to A$ is a difference kernel of f and g. This also shows i is a difference kernel of f and g if f and g are in Set or Ab. A category in which every pair of morphisms has a difference kernel is called a <u>category with difference kernels</u>. Hence Set, Ab and Gp are categories with difference kernels.

In Top, let $f, g : A \to B$ and let $C = \{x \in A \mid f(x) = g(x)\}$ with the relative topology from A. Again, let $i : C \to A$ be the inclusion function which is continuous. Then, similar to the above, i is a difference kernel of f and g, and hence Top has difference kernels. In Rm, C is a submodule of A and $i : C \to A$ is an R-homomorphism. Again, i is a difference kernel and Rm has difference kernels.

We would now like to give an example of a kernel and a cokernel in Ab. Let $(Z,+) = A$ and $C = (\{2z \mid z \in Z\},+) = C$. Then C is a normal subgroup and A/C is abelian. Define $f : A \to A/C = B$ by $f(a) = a + C$. Define $g : C \to A$ by $g(2z) = 2z$. Then $(fg)(2z) = f(2z) = C + 2z = C = 0(C,B)(2z)$. (We have shown what the family of zero morphisms in Ab is.) Suppose $h : D \to A$ with $fh = 0(D,B)$. Show there is exactly one morphism $k : D \to C$ with $h = gk$. Define $k(d) = h(d)$. This is possible since $fh = 0(D,B)$; that is, if $d \in D$ then $(fh)(d) = f(h(d)) = C + h(d) = C$. Hence $h(d) = 2z$ for some $z \in Z$. Therefore $h(d) \in C$. Now $(gk)(d) = g(k(d)) = k(d) = h(d)$. Show k is unique. Suppose there exists $k' : D \to C$ such that $h = gk'$. Then $k(d) = h(d) = (gk')(d) = g(k'(d)) = k'(d)$. Therefore $k = k'$ and $g : C \to A$ is a kernel of f. The group theoretic kernel of f is C. If f were an arbitrary homomorphism and C the kernel of f, then $i : C \to A$, the inclusion homomorphism, would be the kernel of f by almost the same argument.

Now in Ab the difference kernel of two homomorphisms $f, g : A \to B$ is the set $C = \{x \in A \mid f(x) = g(x)\}$, which is the same as $\{x \in A \mid f(x) - g(x) = e_B\}$, which is the kernel of $f - g$.

Now let $f : B \to A$ be a homomorphism in Ab. We know that $f(B)$ is a normal subgroup of A and that $A/f(B) = C$ is an abelian group. Define $g : A \to C$ by $g(a) = a + f(B)$. Then g is a homomorphism. Show g is a cokernel of f. Show $gf = 0(B,C)$. Now for every $b \in B$, $(gf)(b) = g(f(b)) = f(B) + f(b) = f(B) = e_C = 0(B,C)(b)$. Let $h : A \to D$ with $hf = 0(B,D)$. Show there is exactly one morphism $k : C \to D$ with $h = kg$. Define $k : C \to D$ by $k(f(B) + a) = h(a)$. If $f(B) + a = f(B) + b$, then $a - b \in f(B)$ and $h(a - b) = (hf)(c)$ for some $c \in B$. Then $(hf)(c) = 0(B,D)(C) = e_D$ implies $h(a-b) = e_D$. Therefore $h(a)-h(b) = e_D$ and $h(a) = h(b)$. Hence k is well defined. Since h is a homomorphism, so is k. If $a \in A$, then $(kg)(a) = k(g(a)) = k(f(B) + a) = h(a)$. Therefore $h = kg$. To show k is unique, suppose that $k' : C \to D$ is such that $h = k'g$. Let $f(B) + a \in C$. Then $k(f(B) + a) = h(a) = (k'g)(a) = k'(f(B) + a)$. Therefore $k = k'$. Hence $g : A \to C$ is a cokernel of f in Ab, and cokernels are characterized in Ab.

CHAPTER II

FUNCTORS AND NATURAL TRANSFORMATIONS

In Chapter I we discussed some intrinsic properties of categories. These properties dealt mainly with the morphisms of the category. In this chapter we will discuss morphisms between categories and some of their properties.

Let $\mathcal{B}$ and $\mathcal{C}$ be categories. We say $F : \mathcal{B} \to \mathcal{C}$ is a <u>covariant functor</u> if (1) $F : \text{ob}\,\mathcal{B} \to \text{ob}\,\mathcal{C}$ where $F(B) \in \text{ob}\,\mathcal{C}$ for every $B \in \text{ob}\,\mathcal{B}$, and (2) for every $f \in \text{Mor}_{\mathcal{B}}(A,B)$, $F(f) \in \text{Mor}_{\mathcal{C}}(F(A),F(B))$ and the assignment satisfies $F(1_A) = 1_{F(A)}$ for every $A \in \text{ob}\,\mathcal{B}$, and if $A \overset{g}{\to} B \overset{f}{\to} C$ in $\mathcal{B}$ then $F(fg) = F(f)F(g)$ in $\mathcal{C}$. We say $F : \mathcal{B} \to \mathcal{C}$ is a <u>contravariant functor</u> if condition (1) above is satisfied along with (2′) for every $f \in \text{Mor}_{\mathcal{B}}(A,B)$, $F(f) \in \text{Mor}_{\mathcal{C}}(F(B),F(A))$ and the assignment satisfies $F(1_A) = 1_{F(A)}$ and if $A \overset{g}{\to} B \overset{f}{\to} C$ then $F(fg) = F(g)F(f)$. If there is no ambiguity, we will write "FA" for $F(A)$, and "Ff" for $F(f)$. A functor is sometimes called a categorical morphism. We will call a covariant functor simply a functor, and we will use F,G,H,K, and L mostly for functors (unless otherwise indicated).

<u>Lemma 2.1</u>. Composition of two covariant functors or two contravariant functors is a covariant functor.

Proof. Let $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ be two covariant functors. The composition GF will be defined by the composition of the defining maps for F and G. That is, for $A \in \mathrm{ob}\mathcal{B}$, $(GF)(A) = G(F(A)) \in \mathrm{ob}\mathcal{D}$ and if $f : A \to B$ in $\mathcal{B}$ then $(GF)(f) = G(F(f)) : G(F(A)) \to G(F(B))$. Now $(GF)(1_A) = G(F(1_A)) = G(1_{FA}) = 1_{G(FA)} = 1_{(GF)A}$. If $A \xrightarrow{g} B \xrightarrow{f} C$ in $\mathcal{B}$, then $(GF)(fg) = G(F(fg)) = G(FfFg) = G(Ff)G(Fg) = (GF)f(GF)g$. Hence GF is a covariant functor.

For F and G contravariant functors, the definition is the same in that we compose the defining maps. Now $(GF)(1_A) = G(F(1_A)) = G(1_{FA}) = 1_{(GF)A}$ and if $A \xrightarrow{g} B \xrightarrow{f} C$ then $(GF)(fg) = G(FgFf) = (GF)f(GF)g$. Hence GF is a contravariant functor. Thus, Lemma 2.1 has been proved.

When composing two functors of the opposite sense, that is, one covariant and the other contravariant, the resultant functor is contravariant. Condition 1 in the definition is easily satisfied. For condition 2, suppose G is contravariant and F is covariant, and we can form the composition GF. Then $(GF)(fg) = G(FfFg) = G(Fg)G(Ff) = (GF)g(GF)f$. The other case is similar.

Suppose $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{E}$ are covariant functors. Then for $B \in \mathrm{ob}\,\mathcal{B}$ $((HG)F)(B) = (HG)(FB) = H(G(FB) = H((GF)(B)) = (H(GF))(B)$ and if $f : A \to B$ in $\mathcal{B}$, then $((HG)F)(f) = (HG)(Ff) = H(G(Ff)) = H((GF)f) = (H(GF))(f)$. Therefore the composition of functors is associative. Let $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ denote the functor that assigns each object to itself and each morphism to itself. This defines a functor, since the composition of morphisms in a category is already defined. Then for $F : \mathcal{C} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ we have $F1_{\mathcal{C}} = F$ and $1_{\mathcal{C}}G = G$.

We are almost ready to conclude that the collection of all categories with the functors between them forms a category. However, we must insure that the morphisms between two categories form a set. To do this, it is necessary to require that we collect only those categories whose object class is actually a set; that is, we allow only small categories (sometimes called diagram schemes) in the object class of this new category. Now we can collect the functors between two small categories into morphism sets and we form the category of small categories and their functors called Cat.

We would now like to give some examples of functors.

The identity functor $1_{\mathcal{C}}$ for a category $\mathcal{C}$ has already been mentioned. Some categories have objects that can be considered as sets with certain other structures imposed on the set. Likewise, the morphisms in these categories are functions with other properties. Examples are Top, Gp, Rg, and Rm, among others. We can therefore define a covariant functor $F : \mathcal{C} \to$ Set by assigning the underlying set to each object in $\mathcal{C}$ and the underlying function to each morphism in $\mathcal{C}$. Functors of this type are called forgetful functors. Not all forgetful functors have codomain Set. For example, a ring is also an abelian group under the addition, and a ring homomorphism is also a group homomorphism. Hence, we can get a forgetful functor from Rg to Ab. Similarly, functors from any subcategories of Top to Top may be defined (Appendix I).

The concept of duality may be expressed using a contravariant functor, $Op : \mathcal{C} \to \mathcal{C}^{op}$, where $Op(A) = A^{op} = A$, and if $f : A \to B$ in $\mathcal{C}$, then $Opf = f^{op} : B \to A$. Then to show Op is actually a contravariant functor we must show that $Op(1_A) = 1_A$, and if $A \xrightarrow{g} B \xrightarrow{f} C$ in $\mathcal{C}$, then $Op(fg) = Opg\,Opf$. Now $Op(1_A) = 1_A^{op} = 1_A$, and $Op(fg) = (fg)^{op} = g^{op}f^{op} = Op(g)\,Op(f)$ by definition of the dual category.

Let $\mathcal{C}$ be a category and $A \in ob\,\mathcal{C}$. Define $h^A : \mathcal{C} \to Set$ by $h^A(B) = Mor_{\mathcal{C}}(A,B)$, and if $f : B \to C$, then define $h^A(f) : h^A(B) \to h^A(C)$ by $h^A(f)(g) = fg$, where $g \in Mor_{\mathcal{C}}(A,B)$. Define $h_A : \mathcal{C} \to Set$ by $h_A(B) = Mor_{\mathcal{C}}(B,A)$, and if $f : B \to C$, then define $h_A(f) : h_A(C) \to h_A(B)$ by $h_A(f)(g) = gf$ for all $g \in Mor_{\mathcal{C}}(C,A)$.

Lemma 2.2. h^A and h_A are covariant and contravariant functors, respectively.

Proof. We will show that h_A is a contravariant functor. The proof for h^A is similar. Let $B \xrightarrow{1_B} B \xrightarrow{g} A$. Now $h_A(1_B)(g) = g\,1_B = g$. Therefore $h_A(1_B) = 1_{h_A(B)}$. Suppose $B \xrightarrow{g} C \xrightarrow{f} D$ and let $D \xrightarrow{h} A$. Then $(h_A(g)h_A(f))(h) = h_A(g)(h_A(f)(h)) = h_A(g)(hf) = (hf)(g) = h(fg) = (h_A(fg))h$. Therefore h_A defines a contravariant functor from $\mathcal{C}$ to Set.

Lemma 2.3. If $F : \mathcal{C} \to \mathcal{D}$ is a functor and f in $\mathcal{C}$ is an isomorphism with inverse isomorphism f^{-1}, then $F(f)$ is an isomorphism in $\mathcal{D}$.

Proof. Assume the hypothesis. Then $F(f)F(f^{-1}) = F(ff^{-1}) = F(1) = 1 = F(f^{-1})F(f)$. Hence we can write $F(f)^{-1} = F(f^{-1})$. Thus Lemma 2.3 has been proved.

The category Hd provides an example of a morphism which is a monomorphism and an epimorphism, but it is not an isomorphism. The embedding $i : Q \to \mathbb{R}$ was shown to be an epimorphism and since it is 1-1, it is a monomorphism. Suppose that i is an isomorphism and let $F : \text{Hd} \to \text{Set}$ be the forgetful functor. Then F(f) is an isomorphism in Set. Now in Set we have shown that monic is equivalent to 1-1 and epi is equivalent to onto. Since an isomorphism is a monomorphism and an epimorphism, F(f) must be 1-1 and onto. However, there is no 1-1 function from Q onto R.

Lemma 2.4. If $F : \mathcal{D} \to \mathcal{C}$ is a functor that is injective on the class of objects, then the image of F is a category.

Proof. Assume the hypothesis. The only thing that needs to be checked is that possible combinations of morphisms in the image are in the image. Suppose $Ff : FA \to FB$ and $Fg : FC \to FD$ where $FB = FC$. Then since F is injective on the objects, $B = C$. Therefore $f : A \to B$ and $g : B \to D$ and gf is defined in $\mathcal{D}$. Therefore $F(gf) = FgFf$. Hence FgFf is in the image of F. Clearly, each object in the image has an identity morphism in the image and the composition is associative. Therefore the image of F is a category.

We would now like to give an example of a specific functor. Let Top* denote the category of pointed topological spaces with pointed continuous functions. Then Top* is a subcategory of Topp (Appendix I). We will define a functor $\pi_1 : \text{Top}^* \to \text{Gp}$.

Define a loop based at $p \in X \in \text{Top}$ to be a continuous function $f : (I,\{0,1\}) \to (X,p)$, where $f(0) = f(1) = p$, $I = [0,1]$ and $(X,p) \in \text{Top}^*$. Two loops f and g at p are homotopic ($\sim$) means there exists a continuous function $H : I \times I \to X$ satisfying (1) $H(x,0) = f(x)$ for every $x \in I$, (2) $H(x,1) = g(x)$ for every $x \in I$, and (3) $H(0,y) = H(1,y) = p$ for every $y \in I$. Show this homotopy defines an equivalence relation. If f is a loop, then $H : I \times I \to X$ defined by $H(x,y) = f(x)$ shows $f \sim f$. If $f \sim g$ by H, then $G : I \times I \to X$ defined by $G(x,y) = H(x,1-x)$ shows $g \sim f$. If $f \sim g$ by H and $g \sim h$ by G, then F defined by $F(x,y) = H(x,2y)$ if $0 \leq y \leq \frac{1}{2}$ and $F(x,y) = G(x,2y-1)$ if $\frac{1}{2} \leq y \leq 1$ shows $f \sim h$. Therefore $\sim$ is an equivalence relation. Let $[f]$ denote the equivalence class of f.

We will now define a multiplication on the equivalence classes. Define $[f][g] = [fg]$ where $(fg)(t) = f(2t)$ if $0 \leq t \leq \frac{1}{2}$ and $(fg)(t) = g(2t-1)$ if $\frac{1}{2} \leq t \leq 1$. We must show fg defines a loop. Now $(fg)(0) = f(0) = p$ and $(fg)(1) = g(1) = p$. Also, $fg\big|_{[0,\frac{1}{2}]} = f$ and $fg\big|_{[\frac{1}{2},1]} = g$. Therefore fg is continuous. To show this multiplication is well defined, suppose $f \sim f'$ by H and $g \sim g'$ by F. Then $G : I \times I \to X$ defined by $G(x,y) = H(2x,y)$ if $0 \leq x \leq \frac{1}{2}$ and $G(x,y) = F(2x-1,y)$ if $\frac{1}{2} \leq x \leq 1$ shows that $fg \sim f'g'$. Hence the multiplication is well defined. This multiplication forms a group. We now define $\pi_1(X,p)$ to be the group defined by this multiplication of p-based loops.

If $f : (X,p) \to (Y,q)$, define $\pi_1(f) : \pi_1(X,p) \to \pi_1(Y,q)$ by $\pi_1(f)[g] = [fg]$, which is well defined since $fg : I \to Y$ is continuous and $(fg)(0) = f(g(0)) = f(p) = q = (fg)(1)$.

Show π_1 is a functor. Let $1_X : X \to X$ be the identity pointed continuous function on (X,p). Then for every $[f] \in \pi_1(X,p)$, $\pi_1(1_X)[f] = [f\,1_X] = [f] = 1_{\pi_1(X,p)}[f]$. Suppose $(X,p) \xrightarrow{g} (Y,q) \xrightarrow{f} (Z,r)$ in Top*. Then $\pi_1(fg) : \pi_1(X,p) \to \pi_1(Z,r)$. Let $[h] \in \pi_1(X,p)$. Then $(\pi_1(fg))[h] = [(fg)(h)] = [f(gh)] = \pi_1(f)[gh] = \pi_1(f)(\pi_1(g)[h]) = (\pi_1(f)\pi_1(g))[h]$. Therefore π_1 is a covariant functor.

Now we would like to define a central concept in the study of categories. Let $\mathcal{B}$ and $\mathcal{C}$ be categories and let $F,G : \mathcal{B} \to \mathcal{C}$ be covariant functors. A <u>natural transformation</u> $\varphi : F \to G$ is a family of morphisms $\{\varphi(A) : F(A) \to G(A)\}$ for all $A \in \text{ob}\,\mathcal{B}$ such that we have $\varphi(B)F(f) = G(f)\varphi(A)$ for all morphisms $f : A \to B$ in $\mathcal{B}$. The defining equation can be restated by stipulating that the following diagram is commutative.

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi(A)} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\varphi(B)]{} & G(B) \end{array}$$

If F and G are contravariant, the following diagram must be commutative.

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi(A)} & G(A) \\ {\scriptstyle F(f)}\uparrow & & \uparrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\varphi(B)]{} & G(B) \end{array}$$

When there is no ambiguity, $\varphi(A)$ is written as "φA" and φA is often called a component of φ.

Now if $\varphi: F \to G$ and $\psi: G \to H$ are natural transformations, then so is $\psi\varphi: F \to H$ defined by $(\psi\varphi)(A) = \psi A \varphi A$. Since ψ and φ are natural transformations, the small squares are commutative diagrams in the following.

$$\begin{array}{ccccc} F(A) & \xrightarrow{\varphi(A)} & G(A) & \xrightarrow{\psi(A)} & H(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle H(f)} \\ F(B) & \xrightarrow[\varphi(B)]{} & G(B) & \xrightarrow[\psi(B)]{} & H(B) \end{array}$$

Then $(\psi\varphi)(B)F(f) = \psi(B)\varphi(B)F(f) = \psi(B)G(f)\varphi(A) = H(f)\psi(A)\varphi(A) = H(f)(\psi\varphi)(A)$. Therefore the large diagram commutes and $\psi\varphi$ is a natural transformation.

This composition is also associative. Suppose $\varphi: F \to G$, $\psi: G \to H$ and $\rho: H \to K$ are natural transformations. Componentwise, we will show $(\rho\psi)\varphi = \rho(\psi\varphi)$. By definition $((\rho\psi)\varphi)(A) = (\rho\psi)(A)\varphi A = (\rho(A)\psi(A))\varphi(A)$. These separate components are morphisms in a category, and since the composition of morphisms is associative, we have $(\rho(A)\psi(A))\varphi(A) = \rho(A)(\psi(A)\varphi(A)) = \rho(A)(\psi\varphi)(A) = (\rho(\psi\varphi))(A)$. Therefore the components of these two natural transformations are the same. Hence the composition is associative.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Show that the family of morphisms $\{1_{FC} : FC \to FC\}$ determines a natural transformation 1_F where $1_F(C) = 1_{FC}$. Suppose $f : A \to B$ in $\mathcal{C}$. Then $1_{F(B)}Ff = Ff = Ff\,1_{FA}$; hence 1_F is the identity natural transformation on the functor F.

Let $\mathcal{A}$ be a small category and $\mathcal{B}$ be any category. The natural transformations between two functors F and G from $\mathcal{A}$ to $\mathcal{B}$ form a set, since they are a subset of the power set of $\bigcup_{A \in \mathrm{ob}\mathcal{A}} \mathrm{Mor}_{\mathcal{B}}(F(A),G(A))$. Therefore we can define morphism sets between functors from a small category. Define a new category whose object class is the class of all functors from a small category $\mathcal{A}$ to a category $\mathcal{B}$ and whose morphism sets are the sets of natural transformations between the two functors. This is a category since we have an identity and an associative composition. Call this category $\mathrm{Funct}(\mathcal{A},\mathcal{B})$.

A natural transformation $\tau : F \to G$ where $F,G : \mathcal{D} \to \mathcal{C}$ are covariant functors is a <u>natural isomorphism</u> if there is a natural transformation $\varphi : G \to F$ such that $\tau\varphi = 1_G$ and $\varphi\tau = 1_F$. In this case the functors are isomorphic and we write $\tau : F \cong G$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are <u>isomorphic</u> if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG = 1_{\mathcal{D}}$ and $GF = 1_{\mathcal{C}}$. Two categories are <u>equivalent</u> if the functors F and G are such that $FG \cong 1_{\mathcal{D}}$ and $GF \cong 1_{\mathcal{C}}$.

Then F and G are called <u>equivalences</u>. If F and G are contravariant, the categories are <u>dual</u> to each other.

The relation of functors being naturally isomorphic is an equivalence relation on the collection of functors between two categories. Obviously $1_F : F \cong F$, where 1_F is the identity natural transformation. Suppose $F, G : \mathcal{B} \to \mathcal{C}$, and $\tau : F \cong G$. Then there exists $\varphi : G \to F$ such that φ is a natural transformation, and $\tau\varphi = 1_G$, and $\varphi\tau = 1_F$. Hence $\varphi : G \cong F$. Further, $\varphi A \tau A = 1_{FA}$, and $\tau A \varphi A = 1_{GA}$ for every $A \in \text{ob}\,\mathcal{B}$. Hence τA is an isomorphism for every $A \in \text{ob}\,\mathcal{B}$ and $\tau A^{-1} = \varphi A$. Therefore the family of morphisms $\{\tau D^{-1} \mid D \in \text{ob}\,\mathcal{B}\}$ defines the natural transformation φ, and we say $\tau^{-1} = \varphi$.

If $\tau : F \cong G$ and $\varphi : G \cong H$, then $\varphi\tau : F \to H$ is a natural transformation. To show $\varphi\tau$ is a natural isomorphism, consider $\tau^{-1}\varphi^{-1} : H \to F$. Now $(\varphi\tau)(\tau^{-1}\varphi^{-1}) = \varphi 1_G \varphi^{-1} = \varphi\varphi^{-1} = 1_H$ and $(\tau^{-1}\varphi^{-1})(\varphi\tau) = \tau^{-1} 1_G \tau = \tau^{-1}\tau = 1_F$. Therefore $\varphi\tau : F \cong H$. Hence, functors being naturally isomorphic is an equivalence relation.

We would now like to give an example of a natural transformation and an example of a natural isomorphism.

We know $h_A : \text{Set} \to \text{Set}$ defined by $h_A(B) = \text{Mor}_{\text{Set}}(B,A)$ and $h_A(f)(g) = gf$ for all $f : B \to C$, and $g : C \to A$ is a contravariant functor for each $A \in \text{ob Set}$. Let h_A^2 denote the covariant functor $h_A h_A$. Let I denote 1_{Set}. Fix $B \in \text{ob Set}$. Show $\{\varphi A : I(A) \to h_B^2(A) \mid \varphi A$ is the function determined by evaluating each function in $\text{Mor}_{\text{Set}}(A,B) = h_B(A)$ at a fixed element a in A$\}$ defines a natural transformation $\varphi : I \to h_B^2$. Then for each $g : A \to C$ we must show that the following diagram commutes.

$$\begin{array}{ccc} I(A)=A & \xrightarrow{\varphi A} & h_B^2(A) \\ {\scriptstyle I(g)=g}\downarrow & & \downarrow{\scriptstyle h_B^2(g)} \\ I(C)=C & \xrightarrow[\varphi C]{} & h_B^2(C) \end{array}$$

Let $a \in A$. Then $[h_B^2(g)\varphi A](a) = h_B^2(g)((\varphi A)(a))$ and $[(\varphi c)g](a)$ are functions from $Mor_{Set}(C,B)$ to $\mathcal{B}$. Show that these functions are actually the same function. Doing this element-wise, let $f : C \to B$. Then $[h_B^2(g)(\varphi A)(a)](f) = [(\varphi A)(a)\, h_B(g)](f) = (\varphi A)(a)(h_B(g)(f)) = (\varphi A)(a)(fg) = (fg)(a)$. This is true, since $h_B^2(g)(k) = k\, h_B(g)$, where $k \in h_B^2(A)$ and $(\varphi A)(a) \in h_B^2(A)$. Also, $fg \in h_B(A)$, and φA evaluates functions at elements in A. Now $[((\varphi C)(g))(a)](f) = [(\varphi C)(g(a))](f) = f(g(a)) = (fg)(a)$, since φC evaluates functions at elements of C and $g(a) \in C$. Therefore the diagram commutes and φ is a natural transformation.

Let $Vect_{\mathbb{R}}$ be the category of finite dimensional vector spaces over the reals with the linear transformations between them. Define $T : Vect_{\mathbb{R}} \to Vect_{\mathbb{R}}$ by $T(L) = Hom(L,\mathbb{R})$, where $Hom(L,\mathbb{R}) = \{f \mid f$ is a linear transformation from L to $\mathbb{R}\}$, which is a finite dimensional vector space over $\mathbb{R}$, and if $f : L_1 \to L_2$ is a linear transformation, then $T(f) : T(L_2) \to T(L_1)$ is defined by $T(f)(g) = gf$ for linear transformations $g : L_2 \to \mathbb{R}$. (Note: gf is a linear transformation from L_1 to $\mathbb{R}$.) Show T is a contravariant functor. Now $T(1_L)(f) = f\,1_L = f$ for all $f : L \to \mathbb{R}$. Therefore $T(1_L) = 1_{T(L)}$. Suppose $L_1 \xrightarrow{g} L_2 \xrightarrow{f} L_3$ and let $h : L_3 \to \mathbb{R}$. Then $[T(fg)]h = h(fg)$

and $[T(g)T(f)](h) = T(g)(hf) = (hf)(g) = h(fg)$. Therefore $T(g)T(f) = T(fg)$ and T is a contravariant functor. Let T^2 denote the covariant functor TT and let I denote $1_{\mathrm{Vect}_{\mathbb{R}}}$.

Now define a natural transformation $\tau : I \to T^2$ by the family of linear transformations $\{\tau(L) : I(L) \to T^2(L) \mid \tau(L)(x) : \mathrm{Hom}(L,\mathbb{R}) \to \mathbb{R}$ by $[\tau(L)(x)](f) = f(x)$ for $f : L \to \mathbb{R}\}$. We must first show $\tau(L)(x)$ is a linear transformation from $\mathrm{Hom}(L,\mathbb{R})$ to $\mathbb{R}$. Let $f,g \in \mathrm{Hom}(L,\mathbb{R})$. Then $[\tau(L)(x)](f+g) = (f+g)(x) = f(x)+g(x) = [\tau(L)(x)](f)+[\tau(L)(x)](g)$ and $[\tau(L)(x)](\alpha f) = (\alpha f)(x) = \alpha f(x) = \alpha[\tau(L)(x)](f)$ for every $\alpha \in \mathbb{R}$.

Now we must show $\tau(L)$ is a linear transformation from $I(L) = L$ to $T^2(L)$. Element-wise we have $[\tau(L)(x+y)](f) = f(x+y) = f(x)+f(y) = [\tau(L)(x)](f)+[\tau(L)(y)](f) = [\tau(L)(x)+\tau(L)(y)](f)$ and $[\tau(L)(\alpha x)](f) = f(\alpha x) = \alpha f(x) = \alpha[\tau(L)(x)](f)$ for $f : L \to \mathbb{R}$ and $x,y \in L$. Therefore $\tau(L)$ is a morphism in $\mathrm{Vect}_{\mathbb{R}}$. We must now show that the following diagram commutes for every $f : L_1 \to L_2$.

$$\begin{array}{ccc}
I(L_1)=L_1 & \xrightarrow{\tau(L_1)} & T^2(L_1) = \mathrm{Hom}(\mathrm{Hom}(L_1,\mathbb{R}),\mathbb{R}) \\
{\scriptstyle I(f)=f}\downarrow & & \downarrow{\scriptstyle T^2(f)} \\
I(L_2)=L_2 & \xrightarrow[\tau(L_2)]{} & T^2(L_2) = \mathrm{Hom}(\mathrm{Hom}(L_2,\mathbb{R}),\mathbb{R})
\end{array}$$

Let $x \in L_1$. Then we must show the following functions are the same, $[T^2(f)\tau(L_1)](x) = \tau(L_1)(x)T(f)$ and

$[\tau(L_2)(f)](x) = \tau(L_2)(f(x))$, each of which is a linear transformation from $\mathrm{Hom}(L_2,\mathbb{R}) \to \mathbb{R}$. Again we show this elementwise. Let $g : L_2 \to \mathbb{R}$. Then $[\tau(L_1)(x)T(f)](g) = \tau(L_1)(x)(gf) = (gf)(x)$ and $[\tau(L_2)(f(x))](g) = g(f(x)) = (gf)(x)$. Therefore the diagram is commutative, and hence τ is a natural transformation. Show τ is actually a natural isomorphism. It is sufficient to show $\tau(L)$ is a vector space isomorphism for each $L \in \mathrm{Vect}_{\mathbb{R}}$. Show kernel of $\tau(L)$ is just $\{0\}$. Suppose $\tau(L)(x)$ is the zero function. Then $[\tau(L)(x)](f) = f(x) = 0$ for every $f \in \mathrm{Hom}(L,\mathbb{R})$. However, there is an $f \in \mathrm{Hom}(L,\mathbb{R})$ such that $f(x) \neq 0$. Hence, x must be 0. (For suppose $x = b_1a_1 + b_2a_2 + \cdots + b_na_n$, where $\{a_1,a_2,\cdots,a_n\}$ form a basis. Let f be such that $f(a_1) = 1$ and $f(a_i) = 0$ if $2 \le i \le n$. Then $0 = f(x) = f(b_1a_1 + b_2a_2 + \cdots + b_na_n) = b_1f(a_1) + b_2f(a_2) + \cdots + b_nf(a_n) = b_1 \cdot 1 = b_1$. Therefore $b_1 = 0$ and there is an f for each i.) Therefore the kernel of $\tau(L)$ is $\{0\}$, and hence $\tau(L)$ is 1-1. Show $\tau(L)$ is onto. Since $\dim L = \dim T^2(L)$ and $\tau(L)(L)$ is a submodule of $T^2(L)$, we have $\tau(L)(L) = T^2(L)$, or $\tau(L)$ is onto. Hence $\tau(L)^{-1}$ exists for every L and we can define a natural transformation τ^{-1}. Therefore I and T^2 are naturally isomorphic.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. The <u>product category</u> $\mathcal{A} \times \mathcal{B}$ is defined by $\mathrm{ob}(\mathcal{A} \times \mathcal{B}) = \mathrm{ob}\,\mathcal{A} \times \mathrm{ob}\,\mathcal{B}$, and $\mathrm{Mor}_{\mathcal{A}\times\mathcal{B}}((A,B),(A',B')) = \mathrm{Mor}_{\mathcal{A}}(A,A') \times \mathrm{Mor}_{\mathcal{B}}(B,B')$. Composition is that induced by $\mathcal{A}$ and $\mathcal{B}$. To show this actually defines a category, we must show that the composition is associative and the existence of an identity. Let

$$(A_4,B_4)\xrightarrow{(f_3,g_3)}(A_3,B_3)\xrightarrow{(f_2,g_2)}(A_2,B_2)\xrightarrow{(f_1,g_1)}(A_1,B_1)$$

in $\mathcal{A}\times\mathcal{B}$. Then $(f_1,g_1)((f_2,g_2)(f_3,g_3)) = (f_1,g_1)(f_2f_3,g_2g_3) = (f_1(f_2f_3),g_1(g_2g_3)) = ((f_1f_2)f_3,(g_1g_2)g_3) = (f_1f_2,g_1g_2)(f_3,g_3) = ((f_1,g_1)(f_2,g_2))(f_3,g_3)$. Let $(A,B)\in \text{ob}\,\mathcal{A}\times\mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are categories, we have 1_A and 1_B, the identities on $\mathcal{A}$ and $\mathcal{B}$, respectively.

Let $(A_1,B_1)\xrightarrow{(h,k)}(A,B)\xrightarrow{(1_A,1_B)}(A,B)\xrightarrow{(f,g)}(A_2,B_2)$. Then $(f,g)(1_A,1_B) = (f\,1_A,g\,1_B) = (f,g)$ and $(1_A,1_B)(h,k) = (1_Ah,1_Bk) = (h,k)$. Therefore $(1_A,1_B)$ is the identity on (A,B). Hence $\mathcal{A}\times\mathcal{B}$ is a category. Similarly, we can define the product of any finite number of categories.

A functor from a product category of two (n) categories into a category $\mathcal{C}$ is called a <u>bifunctor</u> (<u>multifunctor</u>). For example, let $\mathcal{A}\times\mathcal{B}$ be a product category. Then $P_{\mathcal{A}} : \mathcal{A}\times\mathcal{B}\to\mathcal{A}$ defined by $P_{\mathcal{A}}(A,B) = A$ and $P_{\mathcal{A}}(f,g) = f$ is a functor and is called the projection functor.

<u>Lemma 2.5</u>. Let $F_B : \mathcal{A}\to\mathcal{C}$ and $G_A : \mathcal{B}\to\mathcal{C}$ be functors for all $A\in\text{ob}\,\mathcal{A}$ and $B\in\text{ob}\,\mathcal{B}$. If we have $F_BA = G_AB$ and $F_{B'}(f)G_A(g) = G_{A'}(g)F_B(f)$ for all $A,A'\in\text{ob}\,\mathcal{A}$, $B,B'\in\text{ob}\,\mathcal{B}$ and all morphisms $f : A\to A'$ and $g : B\to B'$, then there is exactly one bifunctor $H : \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ with $H(A,B) = G_A(B)$ and $H(f,g) = F_{B'}(f)G_A(g)$.

<u>Proof</u>. Define $H : \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ by $H(A,B) = G_A(B)$ and $H(f,g) = F_{B'}(f)G_A(g)$. Show H defines a functor. Now $H(1_A,1_B) =$

$F_B(1_A)G_A(1_B) = 1_{F_B(A)}1_{G_A(B)} = 1_{G_A(B)} = 1_{H(A,B)}$ since $F_B(A) = G_A(B)$. Since F_B and G_A are functors for every $A \in \text{ob}\,\mathcal{A}$ and $B \in \text{ob}\,\mathcal{B}$, we have the following commutative diagram, where $A^* \xrightarrow{h} A \xrightarrow{f} A'$ in $\mathcal{A}$ and $B^* \xrightarrow{k} B \xrightarrow{g} B'$ in $\mathcal{B}$.

$$\begin{array}{ccccc}
F_{B'}(A^*) = G_{A^*}(B') & \xrightarrow{F_{B'}(h)} & F_{B'}(A) = G_A(B') & \xrightarrow{F_{B'}(f)} & F_{B'}(A') = G_{A'}(B') \\
G_{A^*}(g)\uparrow & & G_A(g)\uparrow & & G_{A'}(g)\uparrow \\
F_B(A^*) = G_{A^*}(B) & \xrightarrow{F_B(h)} & F_B(A) = G_A(B) & \xrightarrow{F_B(f)} & F_B(A') = G_{A'}(B) \\
G_{A^*}(k)\uparrow & & G_A(k)\uparrow & & G_{A'}(k)\uparrow \\
F_{B^*}(A^*) = G_{A^*}(B^*) & \xrightarrow{F_{B^*}(h)} & F_{B^*}(A) = G_A(B^*) & \xrightarrow{F_{B^*}(f)} & F_{B^*}(A') = G_{A'}(B^*)
\end{array}$$

Then $H((f,g)(h,k)) = H(fh,gk) = F_{B'}(fh)G_{A^*}(gk) = F_{B'}(f)(F_{B'}(h)G_{A^*}(g))G_{A^*}(k) = (F_{B'}(f)G_A(g))(F_B(h)G_{A^*}(k)) = H(f,g)H(h,k)$. Hence, H is a functor. It is unique because any other functor H' satisfying the conditions $H'(A,B) = G_A(B)$ and $H'(f,g) = F_{B'}(f)G_A(g)$ would be defined exactly as H. Thus, Lemma 2.5 has been proved.

<u>Lemma 2.6</u>. Let H and H' be bifunctors, $H,H' : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$. A family of morphisms $\varphi(A,B) : H(A,B) \to H'(A,B)$, $A \in \text{ob}\,\mathcal{A}$ and $B \in \text{ob}\,\mathcal{B}$ is a natural transformation if and only if it is a natural transformation in each variable; that is, $\varphi(\_\_,B)$ and $\varphi(A,\_\_)$ are natural transformations.

Note: If $H : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a fixed bifunctor, then define F_B for each $B \in \text{ob}\,\mathcal{B}$ by $F_B : \mathcal{A} \to \mathcal{C}$ where $F_B(A) = H(A,B)$ and $F_B(f) = H(f,1_B) = H(f,B)$. Then F_B is a functor, since $F_B(1_A) =$

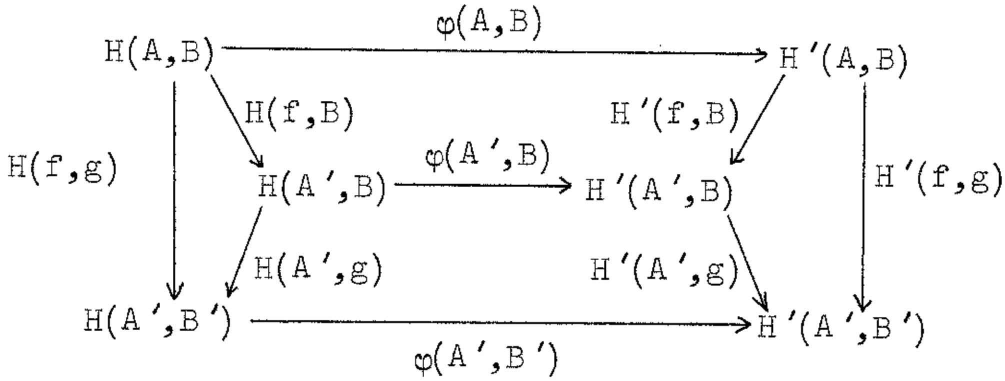
$H(1_A, 1_B) = 1_{H(A,B)} = 1_{F_B(A)}$ and $F_B(fg) = H(fg, 1_B) = H(fg, 1_B 1_B) =$ $H(f, 1_B)H(g, 1_B) = F_B(f)F_B(g)$. We will use the notation $H(\_\_, B)$ for F_B. Similarly, one can define $G_A = H(A, \_\_) : \mathcal{B} \to \mathcal{C}$. Then $H(\_\_, B)$ and $H(A, \_\_)$ satisfy the conditions of Lemma 2.5.

Proof of Lemma 2.6. Let $H(\_\_, B)$ and $H'(\_\_, B)$ be defined as above and suppose the family $\{\varphi(A,B) : H(A,B) \to H'(A,B)$ for every $(A,B) \in \text{ob}\, \mathcal{A} \times \mathcal{B}\}$ defines a natural transformation. Let $B \in \text{ob}\, \mathcal{B}$. Show $\varphi(\_\_, B) : H(\_\_, B) \to H'(\_\_, B)$ defined by $\varphi(\_\_, B)(A) = \varphi(A,B)$ defines a natural transformation. Then we must show that the following diagram commutes for each $f : A \to A'$.

$$\begin{array}{ccc} H(A,B) & \xrightarrow{\varphi(A,B)} & H'(A',B) \\ {\scriptstyle H(f,B)}\downarrow & & \downarrow{\scriptstyle H'(f,B)} \\ H(A',B) & \xrightarrow[\varphi(A',B)]{} & H'(A',B) \end{array}$$

This diagram commutes since φ is a natural transformation and $(f, 1_B) : A \times B \to A' \times B$. Hence φ is a natural transformation in the first variable. Similarly, φ is a natural transformation in the second variable.

Now suppose that $\varphi(\_\_, B)$ and $\varphi(A, \_\_)$ are natural transformations for each $A \in \text{ob}\, \mathcal{A}$ and $B \in \text{ob}\, \mathcal{B}$. Show that the family $\{\varphi(A,B) : H(A,B) \to H'(A,B)\}$ defines a natural transformation. We must show the commutativity of the outer square whenever $(f,g) : (A,B) \to (A',B')$.



Now small upper and lower quadrilaterals commute, since $\varphi(\_\_,B)$ is a natural transformation. The small outer triangles commute, since $H(\_\_,B)$, $H(A,\_\_)$, $H'(\_\_,B)$ and $H'(A,\_\_)$ satisfy conditions of Lemma 2.5. That is, $H(f,g) = H(f,B')H(A,g) = H(A',g)H(f,B)$. Similarly for $H'(f,g)$. Therefore the outer square is commutative. Thus Lemma 2.6 has been proved.

Lemma 2.7. Let $\mathcal{C}$ be a category. $Mor_{\mathcal{C}}(\_\_,\_\_) : \mathcal{C}^{op} \times \mathcal{C} \to Set$ is a bifunctor defined by $Mor_{\mathcal{C}}(\_\_,\_\_)(A,B) = Mor_{\mathcal{C}}(A,B)$, and if $(f^{op},g) : A \times B \to A' \times B'$, then $Mor(\_\_,\_\_)(f^{op},g) = Mor(f^{op},g) : Mor_{\mathcal{C}}(A,B) \to Mor_{\mathcal{C}}(A',B')$ is defined by $Mor_{\mathcal{C}}(f^{op},g)h = ghf$.

Proof. The proof is by definition of a bifunctor. Now $Mor_{\mathcal{C}}(\_\_,\_\_)1_{A\times B} = Mor_{\mathcal{C}}(1_A,1_B) : Mor_{\mathcal{C}}(A,B) \to Mor_{\mathcal{C}}(A,B)$. Then for $h \in Mor_{\mathcal{C}}(A,B)$ we have $Mor_{\mathcal{C}}(1_A,1_B)(h) = 1_B h 1_A = h$. Therefore $Mor_{\mathcal{C}}(1_A,1_B) = 1_{Mor_{\mathcal{C}}(A,B)}$.

Let $(A^*,B^*) \xrightarrow{(h^{op},k)} (A,B) \xrightarrow{(f^{op},g)} (A',B')$ in $\mathcal{C}^{op} \times \mathcal{C}$. We need to show $Mor_{\mathcal{C}}(f^{op}h^{op},gk) = Mor_{\mathcal{C}}(f^{op},g)Mor_{\mathcal{C}}(h^{op},k) : Mor_{\mathcal{C}}(A^*,B^*) \to Mor_{\mathcal{C}}(A',B')$. Let $r \in Mor_{\mathcal{C}}(A^*,B^*)$. Then

$\mathrm{Mor}_{\mathcal{C}}(f^{op}h^{op},gk)(r) = gkrhf$ and $(\mathrm{Mor}_{\mathcal{C}}(f^{op},g)\mathrm{Mor}_{\mathcal{C}}(h^{op},k))(r) = \mathrm{Mor}_{\mathcal{C}}(f^{op},g)(krh) = gkrhf$. Therefore $\mathrm{Mor}_{\mathcal{C}}(\_\_,\_\_)$ is a bifunctor. Therefore Lemma 2.7 has been proved.

We would like to define two natural transformation that will be useful later on. If $f: A \to A'$ and $g: B \to B'$ in $\mathcal{C}$ and $h^A : \mathcal{C} \to \mathrm{Set}$ is the covariant representable functor and $h_A : \mathcal{C} \to \mathrm{Set}$ is the contravariant representable functor, then define $h^f : h^{A'} \to h^A$ and $h_g : h_B \to h_{B'}$ by $h^f(C)(k) = kf$ and $h_g(D)(\ell) = g\ell$ for all $k: A' \to C$ and $\ell : D \to B$ in $\mathcal{C}$. We will show h^f is a natural transformation. Let $h: C \to D$ in $\mathcal{C}$. Then we must show the following diagram commutes.

$$\begin{array}{ccc} h^{A'}(C) & \xrightarrow{h^f(C)} & h^A(C) \\ {\scriptstyle h^{A'}(h)}\downarrow & & \downarrow{\scriptstyle h^A(h)} \\ h^{A'}(D) & \xrightarrow[h^f(D)]{} & h^A(D) \end{array}$$

Element-wise, let $k: A' \to C$. Then $[h^A(h)h^f(C)](k) = h^A(h)(kf) = h(kf) = (hk)f = h^f(D)(hk) = [h^f(D)h^{A'}(h)](k)$. Similarly, h_g is a natural transformation.

Before stating the next lemma, we would like to give some motivation. Let $A, B \in \mathrm{ob\,Ab}$. Then $A \times B \in \mathrm{ob\,Ab}$. Let $C \in \mathrm{ob\,Ab}$. A function $f: A \times B \to C$ is a <u>bilinear function</u> means $f(a_1+a_2,b) = f(a_1,b)+f(a_2,b)$ and $f(a,b_1+b_2) = f(a,b_1)+f(a,b_2)$. A triple $(A \times B, t, T)$ is a <u>tensor product</u> of $A \times B$ means $t: A \times B \to T$ is a bilinear function, and if

$K \in ob\,Ab$ such that if $g : A \times B \to K$ is bilinear, then there exists a unique homomorphism $h : T \to K$ such that $ht = g$.

Let $Hom(A,B) = \{f : A \to B \mid f$ is a group homomorphism$\}$. Fix $(A \times B, t, A \otimes B)$ a tensor product of A and B. We want to show $Hom(A \otimes B,C)$ is a group isomorphic to $Hom(A, Hom(B,C))$. Define $F : Hom(A \otimes B,C) \to Hom(A, Hom(B,C))$ by $F(f) : A \to Hom(B,C)$ where $F(f)(a) : B \to C$ and $[F(f)(a)](b) = f(t(a,b)) \in C$ for all $f : A \otimes B \to C$ and $a \in A$, $b \in B$. First we must show $F(f)(a)$ is a group homomorphism. Let $b_1, b_2 \in B$. Then $[F(f)(a)](b_1+b_2) = f(t(a,b_1+b_2)) = f(t(a,b_1)+t(a,b_2)) = ft(a,b_1)+ft(a,b_2) = [F(f)(a)](b_1)+[F(f)(a)](b_2)$. Hence $F(f)(a)$ is a group homomorphism. Now we must show $F(f)$ is a group homomorphism.

Let $a_1, a_2 \in A$. Show $F(f)(a_1+a_2) = F(f)(a_1)+F(f)(a_2)$. We will show the maps act on elements in the same way. Let $b \in B$. Then $[F(f)(a_1+a_2)](b) = f(t(a_1+a_2,b)) = f(t(a_1,b)+t(a_2,b)) = ft(a_1,b)+ft(a_2,b) = [F(f)(a_1)](b)+[F(f)(a_2)](b)$. Hence $F(f)$ is a group homomorphism. Now F is a well defined function and we must show that F is a group isomorphism. Let $f,g \in Hom(A \otimes B,C)$. Show $F(f+g) = F(f)+F(g)$ in $Hom(A, Hom(B,C))$. Let $a \in A$ and show $[F(f+g)](a) = [F(f)+F(g)](a)$ in $Hom(B,C)$. Then for every $b \in B$, $[F(f+g)(a)](b) = (f+g)(t(a,b)) = f(t(a,b))+g(t(a,b)) = [F(f)(a)](b) + [F(g)(a)](b)$. Therefore F is a group homomorphism. To show F is 1-1, suppose $F(f) = F(g)$, where $f,g \in Hom(A \otimes B,C)$. Define $h,k : A \times B \to C$ by $h(a,b) = ft(a,b)$ and $k(a,b) = gt(a,b)$ for all $(a,b) \in A \times B$. Show h and k are bilinear functions.

Let $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Then $h(a_1+a_2,b) = ft(a_1+a_2,b) = f(t(a_1,b)+t(a_2,b)) = ft(a_1,b)+ft(a_2,b) = h(a_1,b)+h(a_2,b)$ and $h(a,b_1+b_2) = ft(a,b_1+b_2) = f(t(a,b_1)+t(a,b_2)) = ft(a,b_1)+ft(a,b_2) = h(a,b_1)+h(a,b_2)$. Hence h is bilinear and similarly k is bilinear. Now $h(a,b) = f(t(a,b)) = [F(f)(a)](b) = [F(g)(a)](b) = g(t(a,b)) = k(a,b)$. Therefore $h = k$. Now since $(A \times B, t, A \otimes B)$ is a tensor product, there exists a unique homomorphism $r : A \otimes B \to C$ such that $rt = h = k = ft = gt$. Therefore $f = g = r$ and F is 1-1. Now to show that F is onto, let $h \in \mathrm{Hom}(A, \mathrm{Hom}(B,C))$. Define $k : A \times B \to C$ by $k(a,b) = [h(a)](b)$. Show k is bilinear. Then $k(a_1+a_2,b) = [h(a_1+a_2)](b) = [h(a_1)](b)+[h(a_2)](b) = k(a_1,b)+k(a_2,b)$ and $k(a,b_1+b_2) = [h(a)](b_1+b_2) = [h(a)](b_1)+[h(a)](b_2) = k(a,b_1)+k(a,b_2)$. Therefore k is bilinear. Since $(A\times B,t,A \otimes B)$ is a tensor product of $A \times B$, there exists a unique $r : A \times B \to C$ such that $rt = k$. Now $F(r) \in \mathrm{Hom}(A, \mathrm{Hom}(B,C))$ and for every $a \in A$ $[F(r)](a) : B \to C$ and $h(a) : B \to C$. Show these maps are the same. Let $b \in B$; then $[F(r)(a)](b) = rt(a,b) = k(a,b) = h(a)(b)$. Therefore $F(r)(a)$ and $h(a)$ are the same for every $a \in A$. Hence $F(r) = h$ and F is onto. Hence F is a 1-1, onto, group homomorphism, and therefore $\mathrm{Hom}(A \otimes B,C) \cong \mathrm{Hom}(A,\mathrm{Hom}(B,C))$.

Now in a categorical sense we would like to show that the product of two small categories, categories whose object class is a set, behaves like the tensor product of abelian groups. We have the following lemma.

Lemma 2.8. Let $\mathcal{A}$ and $\mathcal{B}$ be small categories and $\mathcal{C}$ be an arbitrary category. Then $\mathrm{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C}) \cong \mathrm{Funct}(\mathcal{A},\mathrm{Funct}(\mathcal{B},\mathcal{C}))$.

Proof. We will define $K : \mathrm{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C}) \to \mathrm{Funct}(\mathcal{A},\mathrm{Funct}(\mathcal{B}\ \mathcal{C}))$. Let $H \in \mathrm{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})$. Then H is a bifunctor. Define $K(H) : \mathcal{A} \to \mathrm{Funct}(\mathcal{B},\mathcal{C})$ by $K(H)(A) = H(A,\_\_)$ where $H(A,\_\_)(B) = H(A,B)$ and $H(A,\_\_)(g) = H(A,g)$ for $B \in \mathrm{ob}\,\mathcal{B}$ and $g : B \to B'$ in $\mathcal{B}$. Then we know $K(H)(A)$ is a functor from $\mathcal{B}$ to $\mathcal{C}$ for each $A \in \mathcal{A}$. We must show how $K(H)$ assigns morphisms in $\mathcal{A}$. Let $f : A \to A'$ in $\mathcal{A}$. Then $K(H)(f) : K(H)(A) \to K(H)(A')$ and $K(H)(f) : H(A,\_\_) \to H(A',\_\_)$ must be a natural transformation. Define $K(H)(f)$ by the family $\{[K(H)(f)](B) : H(A,B) \to H(A',B)\}$ where $[K(H)(f)](B) = H(f,B)$. Let $g : B \to B'$. Show the diagram commutes.

$$\begin{array}{ccc} H(A,B) & \xrightarrow{H(f,B)} & H(A',B) \\ {\scriptstyle H(A,g)}\downarrow & & \downarrow{\scriptstyle H(A',g)} \\ H(A,B') & \xrightarrow[H(f,B')]{} & H(A',B') \end{array}$$

From the definition of H and the proof of Lemma 2.6, this diagram commutes; hence $K(H)(f)$ is a natural transformation. To show $K(H)$ defines a functor, we must now show $K(H)(1_A) = 1_{K(H)(A)}$. Now $K(H)(1_A)$ is defined by the family $\{[K(H)(1_A)](B)\} = \{H(1_A,B)\} = \{H(1_A,1_B)\}$ by the notation, for every $B \in \mathrm{ob}\,\mathcal{B}$. Also, if $A^* \overset{h}{\to} A \overset{f}{\to} A'$ in $\mathcal{A}$, then $K(H)(fh)$ is defined by the family $\{H(fh,B) : H(A^*,B) \to H(A',B)\}$ for $B \in \mathrm{ob}\,\mathcal{B}$, and since H is a bifunctor, $H(fh,B) = H(f,B)H(h,B) = (K(H)(f)K(H)(h))(B)$. Therefore $K(H)$ defines a functor from $\mathcal{A}$ into $\mathrm{Funct}(\mathcal{B},\mathcal{C})$.

Now we must show how K assigns morphisms in $\mathrm{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})$. Morphisms in $\mathrm{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})$ are natural transformations between bifunctors. Let $\varphi : H \to H'$ in $\mathrm{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})$ where φ is a natural transformation. Define $K(\varphi) : K(H) \to K(H')$ by the family of morphisms $\{K(\varphi)(A) : K(H)(A) \to K(H')(A)\}$ for every $A \in \mathrm{ob}\,\mathcal{A}$, where $K(\varphi)(A) = \varphi(A,\_\_)$, as defined in Lemma 2.6. Now $\varphi(A,\_\_)$ is a natural transformation for all $B \in \mathrm{ob}\,\mathcal{B}$. Hence $K(\varphi)$ is a natural transformation from $K(H)$ to $K(H')$. It remains to be shown that K is a functor with an inverse functor. Now $K(1_H)$ is a natural transformation defined by the family $\{K(1_H)(A)\} = \{1_H(A,\_\_)\} = \{1_{H(A,\_\_}\} = \{1_{K(H)(A)}\}$. Therefore $K(1_H) = 1_{K(H)}$. Now suppose $H^* \overset{\psi}{\to} H \overset{\varphi}{\to} H'$ in $\mathrm{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})$. Show $K(\varphi\psi) = K(\varphi)K(\psi)$. Component-wise, we have $K(\varphi\psi)(A) : H^*(A,\_\_) \to H'(A,\_\_)$, where $K(\varphi\psi)(A) = \varphi\psi(A,\_\_) = \varphi(A,\_\_)\psi(A,\_\_) = [K(\varphi)K(\psi)](A)$. Therefore K is a functor.

We would now like to define $L : \mathrm{Funct}(\mathcal{A}, \mathrm{Funct}(\mathcal{B}, \mathcal{C})) \to \mathrm{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})$. Let $F \in \mathrm{Funct}(\mathcal{A},\mathrm{Funct}(\mathcal{B},\mathcal{C}))$. Now $L(F): \mathcal{A}\times\mathcal{B} \to \mathcal{C}$ must be a functor. Define $L(F)(A,B) = F(A)(B) \in \mathrm{ob}\,\mathcal{C}$, and if $(f,g) : A\times B \to A'\times B'$, let $L(F)(f,g): F(A((B) \to F(A')(B')$ be defined by $L(F)(f,g) = (F(f)(B'))(F(A)(g))$. This can be seen in the following commutative diagram since $F(f) : F(A) \to F(A')$ is a natural transformation.

$$\begin{array}{ccc} F(A)(B) & \xrightarrow{F(f)(B)} & F(A')(B) \\ {\scriptstyle F(A)(g)}\downarrow & & \downarrow{\scriptstyle F(A\ )(g)} \\ F(A)(B') & \xrightarrow[F(f)(B')]{} & F(A')(B') \end{array}$$

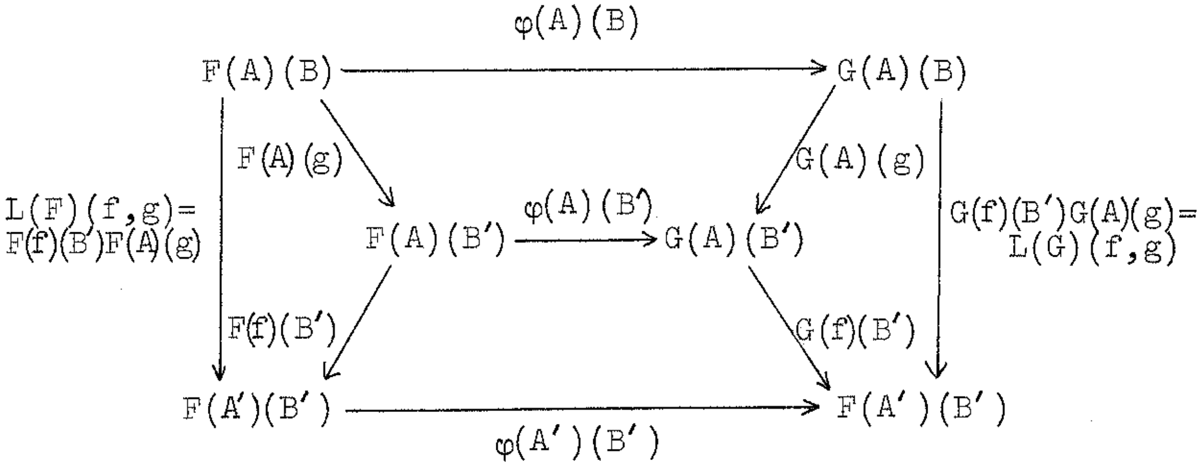
Show $L(F)$ is a functor. Now $L(F)(1_{(A,B)}) = L(F)(1_A,1_B)$ $= (F(1_A)(B))(F(A)(1_B)) = (1_{F(A)}(B))(1_{F(A)(B)}) =$ $1_{F(A)(B)}1_{F(A)(B)} = 1_{F(A)(B)} = 1_{L(F)(A,B)}$. Suppose $(A^*,B^*)\xrightarrow{(h,k)}(A,B)\xrightarrow{(f,g)}(A',B')$ in $\mathcal{A}\times\mathcal{B}$. Then $L(F)(f,g)L(F)(h,k) = (F(f)(B'))(F(A)(g))(F(h)(B))F(A^*)(k) =$ $(F(f)(B'))(F(h)(B'))(F(A^*)(g))(F(A^*)(k)) = (F(fh)(B'))(F(A^*)(gk))$ $= L(F)(fh,gk)$. Since $F(h)$ is a natural transformation and $F(A^*)$ and $F(A)$ are functors, the following commutative diagram exists and explains the substitution.

$$\begin{array}{ccc} F(A^*)(B) & \xrightarrow{F(h)(B)} & F(A)(B) \\ {\scriptstyle F(A^*)(g)}\downarrow & & \downarrow{\scriptstyle F(A)(g)} \\ F(A^*)(B') & \xrightarrow[F(h)(B')]{} & F(A)(B') \end{array}$$

Therefore $L(F)$ is a functor.

We must now show how L acts on morphisms in $\text{Funct}(\mathcal{A}, \text{Funct}(\mathcal{B},\mathcal{C}))$. Let $\varphi: F \to G$ be in $\text{Funct}(\mathcal{A},\text{Funct}(\mathcal{B},\mathcal{C}))$; that is, φ is a natural transformation. Then $L(\varphi)$: $L(F) \to L(G)$ must be a natural transformation. Now $L(F), L(G) : \mathcal{A}\times\mathcal{B} \to \mathcal{C}$ are bifunctors. Define $L(\varphi)$ by the family of morphisms $\{L(\varphi)(A,B) : L(F)(A,B) \to L(G)(A,B)\}$ where $L(\varphi)(A,B) = \varphi(A)(B)$. This is well defined, since $\varphi(A)$:

$F(A) \to G(A)$ is a natural transformation for all $A \in \text{ob } \mathcal{A}$ and $\varphi(A)(B) : F(A)(B) \to G(A)(B)$ is defined for all $B \in \text{ob } \mathcal{B}$. To show $L(\varphi)$ is a natural transformation between the bifunctors $L(F)$ and $L(G)$, let $(f,g) : (A,B) \to (A',B')$ in $\mathcal{A} \times \mathcal{B}$. Since $\varphi(A)$ is a natural transformation for all $A \in \text{ob } \mathcal{A}$, and by the definition of $L(F)$, we have the following commutative diagram.



Therefore $L(\varphi)$ is a natural transformation.

Show L is a functor. Let $1_F : F \to F$ be the identity natural transformation on $F : \mathcal{A} \to \text{Funct}(\mathcal{B},\mathcal{C})$. Then $L(1_F)$ is defined by the family of morphisms $\{L(1_F)(A\ B)\} = \{1_F(A)(B)\} = \{1_{F(A)}(B)\} = \{1_{F(A)(B)}\}$, which defines $1_{L(F)}$, the identity natural transformation on $L(F)$. Suppose $H \overset{\psi}{\to} F \overset{\varphi}{\to} G$ in $\text{Funct}(\mathcal{A},\text{Funct}(\mathcal{B},\mathcal{C}))$. Then $F,G,H : \mathcal{A} \to \text{Funct}(\mathcal{B},\mathcal{C})$ are functors and φ, ψ are natural transformations. Show $L(\varphi\psi) = L(\varphi)L(\psi) : L(H) \to L(G)$. Now $L(\varphi\psi)(A,B) = (\varphi\psi)(A)(B) = [\varphi(A)\psi(A)](B) = \varphi(A)(B)\psi(A)(B) = L(\varphi)(A,B)L(\psi)(A,B) = (L(\varphi)L(\psi))(A\ B)$, since $\varphi(A)$, $\psi(A)$, $L(\varphi)$ and $L(\psi)$ are all natural transformations. Hence L is a functor.

Finally we must show $KL = 1_{\mathrm{Funct}(\mathcal{A},\mathrm{Funct}(\mathcal{B},\mathcal{C}))}$ and $LK = 1_{\mathrm{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})}$. Let $F \in \mathrm{Funct}(\mathcal{A},\mathrm{Funct}(\mathcal{B},\mathcal{C}))$. $(KL)(F)$ is a functor from $\mathcal{A}$ to $\mathrm{Funct}(\mathcal{B},\mathcal{C})$. Show this functor is the same as F. Let $A \in \mathrm{ob}\,\mathcal{A}$. Then $[(KL)(F)](A) : \mathcal{B} \to \mathcal{C}$ is a functor, and so is $F(A) : \mathcal{B} \to \mathcal{C}$. To show these functors are the same, let $B \in \mathrm{ob}\,\mathcal{B}$. Then $([KL)(F)](A))(B) = ([K(L(F))](A))(B) = (L(F)(A,\_\_))(B) = L(F)(A,B) = F(A)(B)$. Now let $g : B \to B'$ in $\mathcal{B}$. $([(KL)(F)](A))(g) = (L(F)(A,\_\_))(g) = L(F)(A,g) = F(A)(g)$. Therefore $[(KL)(F)](A) = F(A)$. We must now show that $(KL)(F)$ and F act on morphisms in A the same. Let $f : A \to A'$ in $\mathcal{A}$. $[(KL)(F)](f) = [K(L(F))](f) = L(F)(f,\_\_) = F(f)$ by definition. (See the diagram where $L(F)$ is defined.) Therefore $(KL)F = F$ and KL and the identity functor agree on objects in $\mathrm{Funct}(\mathcal{A},\mathrm{Funct}(\mathcal{B},\mathcal{C}))$. We will now show that they agree on the morphisms. Let $\varphi : F \to G$ be a natural transformation where $F,G : \mathcal{A} \to \mathrm{Funct}(\mathcal{B},\mathcal{C})$ are functors. We must show $(KL)(\varphi) : K(L(\varphi)) : K(L(F) = F \to K(L(G)) = G$ is the same natural transformation as φ. We will show that the defining family of maps are the same. Let $A \in \mathrm{ob}\,\mathcal{A}$. Then $\varphi(A), K(L(\varphi))(A) : F(A) \to G(A)$ are natural transformations. Again, show their components are the same. Let $B \in \mathrm{ob}\,\mathcal{B}$. Then $(K(L(\varphi))(A))(B) = L(\varphi)(A,\_\_)(B) = L(\varphi)(A,B) = \varphi(A)(B)$. Since this is true for all $B \in \mathrm{ob}\,\mathcal{B}$, the components of $K(L(\varphi))(A)$ and $\varphi(A)$ are the same, and since this is true for all $A \in \mathrm{ob}\,\mathcal{A}$, the components of $K(L(\varphi))$ and φ are the same.

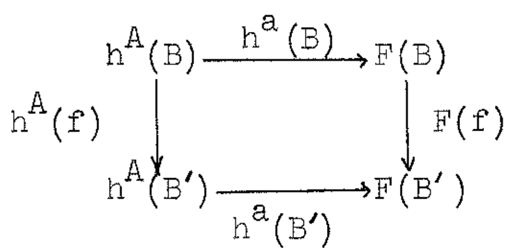
Hence $(KL)\varphi$ and φ are the same natural transformations. Therefore $KL = 1_{\text{Funct}(\mathcal{A},\text{Funct}(\mathcal{B},\mathcal{C}))}$.

We are now ready to show $KL = 1_{\text{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})}$. Let $H : \mathcal{A}\times\mathcal{B} \to \mathcal{C}$. Then $(LK)(H) = L(K(H)) : \mathcal{A}\times\mathcal{B} \to \mathcal{C}$. Let $(A,B) \in \text{ob}\,\mathcal{A}\times\mathcal{B}$. $L(K(H))(A,B) = (K(H)(A))(B) = H(A,B)$. Let $(f,g) : A\times B \to A'\times B'$. Then $(L(K(H))(f,g) = K(H)(f)(B')K(H)(A)(g) = H(f,B)H(A,g) = H(f,g)$. Therefore $(LK)H = H$ and LK and $1_{\text{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})}$ act on objects the same. For morphisms, let $\varphi : H \to H'$ be in $\text{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})$. Show $(LK)\varphi = \varphi$. Now φ is defined by the family $\{\varphi(A,B) : H(A,B) \to H'(A,B)\}$ and $(LK)(\varphi) = L(K(\varphi))$ is defined by the family $\{L(K(\varphi))(A,B) = (K(\varphi)(A))(B) : L(H)(A,B) = H(A,B) \to L(H')(A,B) = H'(A,B)\}$. Then $(K(\varphi)(A))(B) = \varphi(A,\_\_)(B) = \varphi(A,B)$; hence these two families are the same. Therefore $LK = 1_{\text{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C})}$. Hence $\text{Funct}(\mathcal{A}\times\mathcal{B},\mathcal{C}) \cong \text{Funct}(\mathcal{A},\text{Funct}(\mathcal{B},\mathcal{C}))$. Therefore Lemma 2.8 has been proved.

In the following sequence of proofs, we have to generalize some concepts and notations. Let $F,G : \mathcal{C} \to \mathcal{D}$ be functors and let $\text{Mor}_f(F,G)$ be the collection of all natural transformations between them. Now $\text{Mor}_f(F,G)$ in this general context will not be taken to be a set or a class, but if $\mathcal{C}$ is a small category, $\text{Mor}_f(F,G)$ is a set, as we saw in Chapter I. Whenever $\varphi : F \to G$ is a natural transformation, we write "$\varphi \in \text{Mor}_f(F,G)$," and if $\mathcal{C}$ is a small category, this will be taken to mean "φ is tn the set $\text{Mor}_f(F,G)$." Let X be a set or a class. Then an <u>application</u> $\tau : \mathrm{Mor}_f(F,G) \to X$ will assign to each natural transformation an element of X. The instruction will be denoted by τ.

<u>Theorem 2.1</u> (Yoneda Lemma). Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to \mathrm{Set}$ be a covariant functor and $A \in \mathrm{ob}\,\mathcal{C}$. Then the application $\tau : \mathrm{Mor}_f(h^A,F) \to F(A)$ where $\tau(\varphi) = \varphi(A)1_A \in F(A)$ is unique and invertible. The inverse of this application is $\tau^{-1} : F(A) \to \mathrm{Mor}_f(h^A,F)$ where $\tau(a) = h^a \in \mathrm{Mor}_f(h^A,F)$ and $h^a(B)(f) = F(f)(a)$, for all $f : A \to B$ in $\mathcal{C}$ and $a \in F(A)$.

<u>Proof</u>. Now τ is uniquely defined since $\varphi(A) : h^A(A) \to F(A)$. We must show τ^{-1} is well defined. That is, we must show h^a is a natural transformation from h^A to F. Let $f : B \to B'$ in $\mathcal{C}$. Show the following diagram is commutative.



Let $g : A \to B$. Then $[F(f)h^a(B)](g) = F(f)F(g)(a)$ and $[h^a(B')h^A(f)](g) = h^a(B')(fg) = F(fg)(a) = F(f)F(g)(a)$. Hence τ^{-1} is well defined.

Show $\tau\tau^{-1} = 1_{F(A)}$ and $\tau^{-1}\tau = 1_{\mathrm{Mor}_f(h^A,F)}$. Let $a \in F(A)$. Then $(\tau\tau^{-1})(a) = \tau(h^a) = h^a(A)(1_A) = F(1_A)(a) = 1_{F(A)}(a) = a$. Hence $\tau\tau^{-1} = 1_{F(A)}$.

Let $\varphi \in \mathrm{Mor}_f(h^A,F)$. Then $(\tau^{-1}\tau)(\varphi) = \tau^{-1}(\varphi(A)1_A) = h^{\varphi(A)1_A}$. Show $h^{\varphi(A)1_A} = \varphi$. Let $B \in \mathrm{ob}\,\mathcal{C}$. Then show

$h^{\varphi(A)1_A}(B) = \varphi(B) : h^A(B) \to F(B)$. Let $f : A \to B$ in $\mathcal{C}$. Then $[h^{\varphi(A)1_A}(B)](f) = F(f)(\varphi(A)1_A) = (F(f)\varphi(A))(1_A)$. Now since φ is a natural transformation, we have the following commutative diagram.

$$\begin{array}{ccc} h^A(A) & \xrightarrow{\varphi(A)} & F(A) \\ {\scriptstyle h^A(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ h^A(B) & \xrightarrow[\varphi(B)]{} & F(B) \end{array}$$

Therefore $F(f)\varphi(A) = \varphi(B)h^A(f)$. Therefore $(F(f)\varphi(A))(1_A) = (\varphi(B)h^A(f))(1_A) = \varphi(B)(f)$. Since this is true for all $B \in \operatorname{ob}\mathcal{C}$, $h^{\varphi(A)1_A} = \varphi$. Hence $\tau^{-1}\tau = 1_{\mathrm{Mor}_f(h^A,F)}$. Therefore Theorem 2.1 has been proved.

<u>Lemma 2.9</u>. Let F and G be functors from $\mathcal{C}$ into Set and let $\varphi : F \to G$ be a natural transformation. Let $f : A \to B$ be a morphism in $\mathcal{C}$. Then the following diagrams are commutative.

$$\begin{array}{ccc} \mathrm{Mor}_f(h^A,F) & \xrightarrow{\tau} & F(A) \\ {\scriptstyle \mathrm{Mor}_f(h^A,\varphi)}\downarrow & & \downarrow{\scriptstyle \varphi(A)} \\ \mathrm{Mor}_f(h^A,G) & \xrightarrow[\tau]{} & G(A) \end{array} \qquad \begin{array}{ccc} \mathrm{Mor}_f(h^A,F) & \xrightarrow{\tau} & F(A) \\ {\scriptstyle \mathrm{Mor}_f(h^f,F)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathrm{Mor}_f(h^B,F) & \xrightarrow[\tau]{} & F(B) \end{array}$$

where $\mathrm{Mor}_f(h^A,\varphi)(\psi) = \varphi\psi$ and $\mathrm{Mor}_f(h^f,F)(\psi) = \psi h^f$.

<u>Proof</u>. Let $\psi : h^A \to F$ be a natural transformation. Then $[\varphi(A)\tau](\psi) = \varphi(A)(\psi(A)(1_A)) = (\varphi(A)\psi(A))(1_A) = (\varphi\psi)(A)1_A = \tau(\varphi\psi) = \tau(\mathrm{Mor}_f(h^A,\varphi)(\psi)) = [\tau\mathrm{Mor}_f(h^A,\varphi)](\psi)$. Hence the first diagram commutes.

Since ψ is a natural transformation, the following diagram is commutative.

$$\begin{array}{ccc} h^A(A) & \xrightarrow{\psi(A)} & F(A) \\ {\scriptstyle h^A(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ h^A(B) & \xrightarrow[\psi(B)]{} & F(B) \end{array}$$

Then $[F(f)\tau](\psi) = F(f)(\psi(A)1_A) = (F(f)\psi(A))(1_A) = [\psi(B)h^A(f)](1_A) = \psi(B)(f\,1_A) = \psi(B)(f)$ and $[\tau\,Mor_f(h^f,F)](\psi) = \tau(\psi h^f) = [(\psi h^f)(B)](1_B) = \psi(B)(h^f(B)(1_B)) = \psi(B)(1_B f) = \psi(B)(f)$. Therefore the second diagram commutes. Therefore, Lemma 2.9 has been proved.

We would now like to define what is called the contra-variant representation functor. Let $\mathcal{C}$ be a small category. Define $h^- : \mathcal{C} \to Funct(\mathcal{C},Set)$ by $h^-(A) = h^A$ and if $f : A \to B$ in $\mathcal{C}$ then $h^-(f) = h^f : h^B \to h^A$. Let $1_A : A \to A$. Then $h^-(1_A) = h^{1_A}$. If $B \in ob\,\mathcal{C}$ and $f : A \to B$, then $h^{1_A}(B)(f) = f\,1_A = f$. Therefore $h^{1_A}(B) = 1_{h^A(B)}$. Hence $h^-(1_A) = 1_{h^-(A)}$. Let $C \overset{g}{\to} A \overset{f}{\to} B$. Show $h^{fg} = h^g h^f$. Let $D \in ob\,\mathcal{C}$ and $h : B \to D$ in $\mathcal{C}$. Then $h^{fg}(D)(h) = hfg$ and $(h^g h^f)(D)(h) = (h^g(D)h^f(D))(h) = h^g(D)hf = hfg$. Therefore h is a contravariant functor. Similarly, we define the covariant representation functor $h_- : \mathcal{C} \to Funct(\mathcal{C}^{op}, Set)$.

Lemma 2.10. Let $\mathcal{C}$ be a small category. Then $Mor_f(h^-,\_) : \mathcal{C} \times Funct(\mathcal{C},Set) \to Set$ and $\Phi : \mathcal{C} \times Funct(\mathcal{C},Set) \to Set$ are bi-functors. $Mor_f(h^-,\_)$ is defined by $Mor_f(h^-,\_)(A,F) =$

$\mathrm{Mor}_f(h^A,F)$ and $\mathrm{Mor}_f(h^-,\_)(f,\varphi) = \mathrm{Mor}_f(h^f,\varphi)$. Φ is defined by $\Phi(A,F) = F(A)$ and $\Phi(f,\varphi) = \varphi(B)F(f) = G(f)\varphi(A)$. In both cases, $\varphi : F \to G$ is a natural transformation of the functors $F,G : \mathcal{C} \to \mathrm{Set}$. Further, the application τ is a natural isomorphism of the bifunctors. Φ is called the evaluation functor.

Proof. Let $(A,F) \in \mathrm{ob}(\mathcal{C} \times \mathrm{Funct}(\mathcal{C},\mathrm{Set}))$. Then $(1_A,1_F)$ is the identity for (A,F). Show $\mathrm{Mor}_f(h^-,\_)(1_A,1_F) = 1_{\mathrm{Mor}_f(h^-,\_)(A,F)}$. That is, show $\mathrm{Mor}_f(h^{1_A},1_F) = 1_{\mathrm{Mor}_f(h^A,F)}$. Let $\varphi : h^A \to F$ be a natural transformation. Then $\mathrm{Mor}_f(h^{1_A},1_F)(\varphi) = \varphi h^{1_A}$, and if $D \in \mathrm{ob}\,\mathcal{C}$ and $g : A \to D$, then $[(\varphi h^{1_A})(D)](g) = [\varphi(D)h^{1_A}(D)](g) = \varphi(D)(g\,1_A) = \varphi(D)(g)$. Hence $\mathrm{Mor}_f(h^{1_A},1_F)(\varphi) = \varphi$. Hence $\mathrm{Mor}_f(h^{1_A},1_F) = 1_{\mathrm{Mor}_f(h^A,F)}$. Now suppose $(C,H) \xrightarrow{(g,\psi)} (A,F) \xrightarrow{(f,\varphi)} (B,G)$ in $\mathcal{C} \times \mathrm{Funct}(\mathcal{C},\mathrm{Set})$. Show $\mathrm{Mor}_f(h^{fg},\varphi\psi) = \mathrm{Mor}_f(h^f,\varphi)\mathrm{Mor}(h^g,\psi) : \mathrm{Mor}_f(h^C,H) \to \mathrm{Mor}_f(h^B,G)$. Then for $\rho : h^C \to H$ a natural transformation, we have $\mathrm{Mor}_f(h^{fg},\varphi\psi)(\rho) = (\varphi\psi)(\rho)(h^{fg}) = \varphi\psi\rho h^g h^f = \varphi(\psi\rho h^g)h^f$ and $[\mathrm{Mor}_f(h^f,\varphi)\mathrm{Mor}_f(h^g,\psi)](\rho) = \mathrm{Mor}_f(h^f,\varphi)(\psi\rho h^g) = \varphi(\psi\rho h^g)(h^f)$. Hence $\mathrm{Mor}_f(h^-,\_)$ is a bifunctor.

Show Φ is a bifunctor. Let $(A,F) \in \mathcal{C} \times \mathrm{Funct}(\mathcal{C},\mathrm{Set})$. Then $\Phi(1_A,1_F) = 1_F(A)F(1_A) = 1_{F(A)}1_{F(A)} = 1_{F(A)} = \Phi(A,F)$. Suppose $(C,H) \xrightarrow{(g,\psi)} (A,F) \xrightarrow{(f,\varphi)} (B,G)$. Then $(fg,\varphi\psi) = \varphi\psi(B)H(fg) = \varphi(B)\psi(B)H(f)H(g)$. Since ψ is a natural transformation, we have the following commutative diagram.

$$\begin{array}{ccc} H(A) & \xrightarrow{\psi(A)} & F(A) \\ {\scriptstyle H(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ H(B) & \xrightarrow[\psi(B)]{} & F(B) \end{array}$$

Then $\Phi(f,\varphi)\,\Phi(g,\psi) = \varphi(B)(F(f)\,\psi(A))H(g) = \varphi(B)\,\psi(B)H(f)H(g)$. Hence Φ is a bifunctor.

Now $\tau(\_,F)$ and $\tau(A,\_)$ are natural transformations follows from Lemma 2.9. Hence τ is a natural transformation from Lemma 2.6.

Now $\tau^{-1} : \Phi \to Mor_f(h^{-},\_)$ where $\tau^{-1}(A,F) : F(A) \to Mor_f(h^A,F)$ is defined by $\tau^{-1}(A,F)(a) = h^a$ and $h^a(B)(f) = F(f)(a)$. Let $(A,F) \xrightarrow{(f,\varphi)} (B,G)$. Then we must show the diagram commutes.

$$\begin{array}{ccc} F(A) & \xrightarrow{\tau^{-1}(A,F)} & Mor_f(h^A,F) \\ {\scriptstyle G(f)\varphi(A)= \varphi(B)F(f)= \Phi(f,\varphi)}\downarrow & & \downarrow{\scriptstyle Mor_f(h^f,\varphi)} \\ G(B) & \xrightarrow[\tau^{-1}(B,G)]{} & Mor_f(h_B,G) \end{array}$$

Let $a \in F(A)$. Then $(Mor_f(h^f,\varphi)\,\tau^{-1}(A,F))(a) = Mor_f(h^f,\varphi)(h^a)$ $= \varphi h^a h^f$, and $(\tau^{-1}(B,G)\,\varphi(B)F(f))(a) = \tau^{-1}(B,G)\,\varphi(B)\,h^a(B)(f)$ $= h^{(\varphi(B)h^a(B))(f)}$, each of which is a natural transformation from h^B to G. Let $C \in$ ob Set and $g : B \to C$. Then $((\varphi h^a h^f)(C))(g) =$ $(\varphi(C)h^a(C)h^f(C))(g) = \varphi(C)h^a(C)(h^f(C)(g)) = \varphi(C)H^a(C)(gf) =$ $\varphi(C)F(gf)(a)$. Since $\varphi : F \to G$ is a natural transformation, we have $\varphi(C)F(g) = G(g)\varphi(B)$. Now $(h^{(\varphi(B)h^a(B))(f)}(C))(g) =$ $G(g)\varphi(B)h^a(B)(f) = G(g)\varphi(B)F(f)(a) = \varphi(C)F(g)F(f)(a) =$ $\varphi(C)F(gf)(a)$. Therefore τ^{-1} is a natural transformation.

Show $\tau\tau^{-1} = 1_{\Phi}$. Let $(A,F) \in \mathcal{C} \times \text{Funct}(\mathcal{C},\text{Set})$ and $a \in F(A)$. Then $((\tau\tau^{-1})(A,F))(a) = (\tau(A,F)\tau^{-1}(A,F))(a) = \tau(A,F)h^{a} = h^{a}(A)1_A = F(1_A)(a) = 1_{F(A)}(a) = a$. Therefore $\tau\tau^{-1} = 1_{\Phi}$.

Show $\tau^{-1}\tau = 1_{\text{Mor}_f(h^{-},\_)}$. Show $\tau^{-1}\tau(A,F) = 1_{\text{Mor}_f(h^{A},F)}$. Let $\psi : h^{A} \to F$ be a natural transformation and $B \in \text{ob}\,\mathcal{C}$ and $f : A \to B$ in $\mathcal{C}$. Then $[\tau^{-1}\tau(A,F))(\psi)(B)]\,(f) = [(\tau^{-1}(A,F)\tau(A,F))(\psi)(B)](f) = (\tau^{-1}(A,F)\psi(A)(1_A))(B)(f) = h^{\psi(A)(1_A)}(B)(f) = F(f)(\psi(A)(1_A)) = \psi(B)h^{A}(f)1_A = \psi(B)(f1_A) = \psi(B)(f)$ since $\psi(B)h^{A}(f) = F(f)\psi(A)$. Hence $\tau^{-1}\tau = 1_{\text{Mor}_f(h^{-},\_)}$. Therefore τ is a natural isomorphism between the bifunctors. Hence Lemma 2.10 has been proved.

Lemma 2.11. Let $A,B \in \text{ob}\,\mathcal{C}$. Then

(1) for $f \in \text{Mor}_{\mathcal{C}}(A,B)$, let $\lambda(f) = h^{f} \in \text{Mor}_f(h^{B},h^{A})$. Then λ is a bijection;

(2) the bijection of (1) induces a bijection between the isomorphism in $\text{Mor}_{\mathcal{C}}(A,B)$ and the natural isomorphisms in $\text{Mor}_f(h^{B},h^{A})$;

(3) for contravariant functors $F : \mathcal{C} \to \text{Set}$, we have a unique invertible application between $\text{Mor}_f(h_A,F)$ and $F(A)$;

(4) for $f \in \text{Mor}_{\mathcal{C}}(A,B)$, let $\sigma(f) = h_f \in \text{Mor}_f(h_A,h_B)$. Then σ is a bijection inducing a bijection between the isomorphisms in $\text{Mor}_{\mathcal{C}}(A,B)$ and the natural isomorphisms in $\text{Mor}_f(h_A,h_B)$.

Proof. (1) In the Yoneda Lemma, let $F = h^{A} : \mathcal{C} \to \text{Set}$. Then $\tau^{-1} : h^{A}(B) \to \text{Mor}_f(h^{B},h^{A})$ is λ in the hypothesis. Then τ^{-1} is unique and invertible.

(2) Let f be an isomorphism in $\mathrm{Mor}_{\mathcal{C}}(A,B)$. Show h^f is a natural isomorphism in $\mathrm{Mor}_f(h^B,h^A)$. Since f is an isomorphism, we know there exists g such that $fg = 1_B$ and $gf = 1_A$. Then $h^g h^f = h^{fg} = h^{1_B} = 1_{h^B}$ and $h^f h^g = h^{gf} = h^{1_A} = 1_{h^A}$. Therefore h^f is a natural isomorphism.

Suppose h^f is a natural isomorphism in $\mathrm{Mor}_f(h^A,h^B)$. We can choose h^f since the application is a bijection. We must show that f is an isomorphism. Since h^f is a natural isomorphism, there exists $h^g \in \mathrm{Mor}_f(h^A,h^B)$ such that $h^f h^g = 1_{h^A}$ and $h^g h^f = 1_{h^B}$. Then $1_{h^A} = h^f h^g = h^{gf}$ implies $gf = 1_A$ and $1_{h^B} = h^g h^f = h^{fg}$ implies $fg = 1_B$. Hence f is an isomorphism.

(3) This is the dual assertion of the Yoneda lemma.

(4) This is the dual assertion of (1) and (2). Therefore Lemma 2.11 has been proved.

We would like to give some other properties that some functors have, before proceeding. A <u>full functor</u> is a functor which induces surjective maps on the morphism sets. A <u>faithful functor</u> is a functor which induces injective maps on the morphism sets. Then by Lemma 2.11 (1), h^- is a full and faithful functor, and by Lemma 2.11 (4), h_- is full and faithful.

An interesting property of full and faithful functors is that it is always the case that the image of a full and faithful functor is a category. As in the beginning of this chapter, we must check whether or not FgFf is in the image of F when

$f : A \to B$, $g : C \to D$ and $F(B) = F(C)$. Now $1_{FB} \in \mathrm{Mor}_{\mathcal{D}}(FB, FB) = \mathrm{Mor}_{\mathcal{D}}(FB, FC)$. Since F is a bijection on the morphism sets, there is a morphism $h \in \mathrm{Mor}_{\mathcal{C}}(B, C)$ such that $Fh = 1_{FB}$. Then $F(ghf) = FgFhFf = Fg\,1_{FB}\,Ff = FgFf$ and ghf is the desired morphism. Therefore the image of the functor will form a category.

Lemma 2.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor. Let $\mathcal{A}$ and $\mathcal{B}$ be small categories and $G : \mathcal{A} \to \mathcal{C}$, $G' : \mathcal{B} \to \mathcal{D}$ be functors. ($\mathcal{A}$ and $\mathcal{B}$ are sometimes called diagram schemes and G and G′ are called diagrams when the domains are diagram schemes.) Further, let $E : \mathcal{A} \to \mathcal{B}$ be a functor which is bijective on the objects such that the diagram is commutative.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{E} & \mathcal{B} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G'} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

Then there is exactly one diagram $H : \mathcal{B} \to \mathcal{C}$ such that $FH = G'$ and $HE = G$.

Proof. We will define H in the following way: since E is bijective on the objects, for each $B \in \mathrm{ob}\,\mathcal{B}$ there is a unique $A \in \mathrm{ob}\,\mathcal{A}$ such that $E(A) = B$. Define $H(B) = G(A) \in \mathrm{ob}\,\mathcal{C}$. Now let $f : B \to C$ in $\mathcal{B}$. Let $A, A' \in \mathrm{ob}\,\mathcal{A}$ be such that $E(A) = B$ and $E(A') = C$. Since F is full and faithful, F is bijective on the morphism sets. Hence $\mathrm{Mor}_{\mathcal{C}}(GA, GA') \cong \mathrm{Mor}_{\mathcal{D}}((FG)A, (FG)A') \cong \mathrm{Mor}_{\mathcal{D}}((G'E)A, (G'E)A')$ since the diagram is commutative. Now by definition of H and choice of A,A′ we have $\text{Mor}_{\mathcal{C}}(HB,HC) \cong \text{Mor}_{\mathcal{B}}(G'B,G'C)$. Since $G'f \in \text{Mor}_{\mathcal{B}}(G'B,G'C)$, there is a unique $h \in \text{Mor}_{\mathcal{C}}(HB,HC)$ such that $Fh = G'f$. Define $Hf = h$. Now H is well defined.

Show H is a functor. Let $1_B : B \to B$ in $\mathcal{B}$. Then $H1_B = h$ where $Fh = G'1_B = 1_{G'B}$, but $F1_{HB} = 1_{(FH)B}$. We need to show that $FHB = G'B$. Now $(FH)B = (FG)A = (G'E)A = G'B$, where $EA = B$. Since F is full and faithful, then $1_{HB} = h$. Let $B \xrightarrow{g} C \xrightarrow{f} D$ in $\mathcal{B}$. Show $H(fg) = HfHg$. Let $H(fg) = h$ where $Fh = G'(fg) = F'fG'g$. Let k,ℓ be such that $Fk = G'f$ and $F\ell = G'g$. Then $G'fG'g = FkF\ell = F(k\ell)$, which implies $h = k\ell$, since F is full and faithful. Hence $(FH)(fg) = Fh = F(k\ell) = F(HfHg)$, and since F is faithful $H(fg) = HfHg$. Hence H is a functor.

Show $HE = G$ and $G' = FH$. Let $A \in \text{ob}\,\mathcal{A}$. Now $(HE)A = H(EA) = GA$. If $f : A \to B$ in $\mathcal{A}$, then $(HE)f = H(Ef) = h$ where $Fh = G'(Ef) = (FG)f$. Then $h = Gf$, since F is faithful. Hence $HE = G$. Let $B \in \text{ob}\,\mathcal{B}$. Then $F(H(B)) = F(GA)$ where $EA = B$. Then $(FG)A = (G'E)A = G'B$. Let $f : B \to C$ in $\mathcal{B}$. Then $(FH)f = Fh$ where $Fh = G'f$. Hence $F(Hf) = G'f$. Therefore Lemma 2.12 has been proved.

APPENDIX

The following are examples of categories. Most of these examples are given in the book <u>Categories and Functors</u>.

(1) <u>The empty category</u>. The object class is the empty set and Mor $\mathcal{C}$ is also the empty set. This is a category vacuously.

(2) <u>The category of ordered sets--Ord</u>. The object class is the collection of all sets with an order relation $\leq$ satisfying (1) $a \leq a$, (2) if $a \leq b$ and $b \leq a$, then $a = b$, and (3) if $a \leq b$, $b \leq c$, then $a \leq c$. The morphism set for $A, B \in$ ob Ord is $\text{Mor}(A,B) = \{f \mid f : A \to B$ and f is an order-preserving function$\}$. The composition of morphisms is defined to be the composition of functions. We must show that this composition is order-preserving. If $A \xrightarrow{f} B \xrightarrow{g} C$ in Ord and $a \leq b$ in A, then $f(a) \leq f(b)$ in B. Therefore $g(f(a)) \leq g(f(b))$ in C or $gf(a) \leq gf(b)$. Hence this composition is well defined. This composition is associative, since the composition of functions is associative. The identity function 1_A on A is an order-preserving function and serves as the identity morphism for A in Ord. Hence Ord is a category.

(3) <u>An ordered set as category</u>. For any ordered set $(S, \leq)$, usually just written as "S," let ob $\mathcal{C}$ = S and $a, b \in S$

define $\mathrm{Mor}(a,b) = \{(a,b)\}$ if $a \leq b$ and the empty set ϕ otherwise. Composition is defined to be the unique element in $\mathrm{Mor}(a,c)$ or the empty set whenever we compose (a,b) and (b,c). Since $\leq$ is transitive, this gives a unique composition. Since the morphism sets consist of at most one element, the composition is associative and $a \leq a$ guarantees an identity. Hence an ordered set forms a category.

(4) <u>The category of pointed sets--Set*</u>. A pointed set is a pair (A,a) where A is a set and $a \in A$. The collection of all pointed sets forms the object class of Set*. A pointed map between two pointed sets (A,a) and (B,b) is a function $f : A \to B$ such that $f(a) = b$. The collection of all pointed maps between (A,a) and (B,b) forms $\mathrm{Mor}((A,a),(B,b))$. Composition is the composition of functions. We must show that the composition of pointed maps is a pointed map. Suppose $(A,a) \xrightarrow{f} (B,b) \xrightarrow{g} (C,c)$. Then $(gf)(a) = g(f(a)) = g(b) = c$. Therefore the composition is well defined. Again the composition is associative since the morphisms are functions. The identity map 1_A on A is a pointed map. Hence Set* is a category.

(5) <u>An equivalence relation as a category</u>. Let M be a set and R be an equivalence relation on M. Let $\mathrm{ob}\,\mathcal{C} = M$. Then let $\mathrm{Mor}_{\mathcal{C}}(a,b) = \{(a,b)\}$ if aRb and the empty set otherwise. As in example (3), this defines a category.

A category $\mathcal{B}$ is called a <u>subcategory</u> of a category $\mathcal{C}$ if $\mathrm{ob}\,\mathcal{B} \subseteq \mathrm{ob}\,\mathcal{C}$ and $\mathrm{Mor}_{\mathcal{A}}(A,B) \subseteq \mathrm{Mor}_{\mathcal{C}}(A,B)$ for all $A,B \in \mathrm{ob}\,\mathcal{B}$. Also the composition of morphisms in $\mathcal{B}$ must coincide with the composition of the same morphisms in $\mathcal{C}$ and the identity in $\mathcal{B}$ of an object A must be the same as the identity of A taken in $\mathcal{C}$.

(6) The category of abelian groups--Ab. The object class of Ab is the collection of all abelian groups and the morphisms are all group homomorphisms between abelian groups. Now these homomorphisms are also homomorphisms in Gp. The composition and the identities are the same as in Gp also. Then by the same proof as for Gp, Ab is a category. By the above definition, Ab is a subcategory of Gp.

(7) A group as a category. Let G be a group. Let ob $\mathcal{C}$ = B where B is any object. Define $\mathrm{Mor}_{\mathcal{C}}(B,B) = G$ such that the composition is the multiplication of elements of G. Since the multiplication is associative, the composition is associative. The identity of the group is the identity morphism. Thus $\mathcal{C}$ forms a category.

(8) The category of rings--Rg (not necessarily with multiplicative identity). The collection of all rings form the object class and $\mathrm{Mor}_{Rg}(A,B)$ is the set of all ring homomorphisms between A and B. Composition is the usual composition, and this is associative. For each $A \in \mathrm{ob}\, Rg$ there is an identity ring homomorphism. Hence Rg is a category.

(9) The category of rings with identity--Ri. Let ob Ri be the collection of all rings with a multiplicative identity.

The morphisms are the usual ring homomorphisms with the usual composition. Then Ri forms a subcategory of Rg.

(10) The category of all modules over a ring R--Rm. Let ob Rm be the collection of all modules over R and let $Mor_{Rm}(A,B)$ be the collection of all R-homomorphisms. Composition is the usual composition, which is associative. The identity R-homomorphism serves as the identity. Thus Rm is a category.

(11) The category of vector spaces over a field F--$Vect_F$. Let ob $Vect_F$ be the collection of all vector spaces over F, and the morphism sets are the collections of linear transformations between vector spaces. $Vect_F$ forms a subcategory of Rm.

(12) The category of topological pairs--Topp. A topological pair is an ordered pair (X,A) where X is a topological space and $A \subseteq X$. A morphism from (X,A) to (Y,B) is a continuous function $f : X \to Y$ such that $f(A) \subseteq B$. The composition is the composition of functions, and if $(X,A) \xrightarrow{f} (Y,B) \xrightarrow{g} (Z,C)$, then $f(A) \subseteq B$, and hence $(gf)A \subseteq C$. Therefore the composition is well defined. The composition is associative since the composition of functions is associative and the identity function is a morphism in Topp. Hence Topp is a category.

(13) The category of Hausdorff topological spaces--Hd. Let ob Hd be the collection of all Hausdorff topological spaces and the collection of morphism is the collection of

continuous functions between them. Then Hd is a subcategory of Top.

Similarly. we get other subcategories of Top by collecting all those topological spaces with a certain property and using the continuous functions between them. Some of these are T_4, T_3, T_1 and T_0 spaces, compact Hausdorff spaces CH, and locally compact Hausdorff spaces LCH.

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