CHEBYSHEV SUBSETS IN SMOOTH NORMED LINEAR SPACES

THESIS

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MASTER OF ARTS

By

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This paper is a study of the relation between smoothness of the norm on a normed linear space and the property that every Chebyshev subset is convex. Every normed linear space of finite dimension, having a smooth norm, has the property that every Chebyshev subset is convex. In the second chapter two properties of the norm, uniform Gateaux differentiability and uniform Fréchet differentiability where the latter implies the former, are given and are shown to be equivalent to smoothness of the norm in spaces of finite dimension. In the third chapter it is shown that every reflexive normed linear space having a uniformly Gateaux differentiable norm has the property that every weakly closed Chebyshev subset, with non-empty weak interior that is norm-wise dense in the subset, is convex.
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CHAPTER I

INTRODUCTION

First some notations and terms that are used in stating the problem are introduced below.

Notation: If $E$ be a normed linear space: the norm of $E$ will be denoted by $|,|$ - that is, if $x$ is in $E$, then $|x|$ is the norm of $x$; $B(E)$ and $S(E)$ are the unit ball and unit sphere of $E$ respectively; and, $\rho$ is the metric on $E$ induced by the norm - i.e., if $x$ and $y$ are in $E$, then $\rho(x,y) = |x-y|$, and if $A$ and $B$ are subsets of $E$, then $\rho(A,B) = \inf\{|x-y|: x \in A \text{ and } y \in B\}$. Usually letters of the English alphabet will be used to denote members of a vector space and Greek letters to denote members of the scalar field.

Definition: If $E$ is a normed linear space and $C$ is a subset of $E$, then $C$ has unique nearest point property (or $C$ is Chebyshev) if and only if for every $x$ in $E$ there is a unique $y$ in $C$ such that $\rho(x,y) \leq \rho(x,z)$ for every $z$ in $C$.

Definition: If $E$ is a normed linear space, then the norm on $E$ is strictly convex if and only if for every $x$ and $y$ in $E$ such that $|x| = |y| = 1$ and $x \neq y$ and for every real number $0 < \alpha < 1$, $|\alpha x + (1-\alpha)y| < 1$.

Definition: If $E$ is a normed linear space, then the norm on $E$ is smooth if and only if for every $x$ in $S(E)$ there is a unique closed hyperplane of support of $B(E)$ that
contains \( x \). Note that a condition equivalent to smoothness of the norm, but sometimes more useful, is that for every \( x \) in \( S(E) \) there is a unique \( f \) in \( E^* \) (\( E^* \) denotes the space of all bounded linear functionals on \( E \)) such that \( |f| = 1 \) and \( f(x) = 1 \).

In his book *Convex Sets*, Valentine gives a result that was originally stated by Motzkin and then extended by Busemann and others - in a normed linear space of finite algebraic dimension that has a smooth and strictly convex norm, a subset is Chebyshev if and only if it is closed and convex.\(^1\) It was further shown that in a normed linear space of finite algebraic dimension, for all of the Chebyshev subsets to be closed and convex it is sufficient that the norm be smooth.\(^2\)

It does not seem to be known whether this last result of Busemann extends to normed linear spaces of infinite dimension. The following chapter gives conditions on the norm that are similar to and imply smoothness, but are more restrictive. The last chapter presents a theorem that attempts to clarify the relation between smoothness of the norm and the property that every Chebyshev subset of a normed linear space be convex.

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\(^2\)Ibid., p.96.
CHAPTER II

SMOOTH NORMED LINEAR SPACES

Smoothness of the norm on a normed linear space can be defined differently with the equivalent condition that the norm be "Gateaux differentiable." ¹

Definition: Let $E$ be a normed linear space. The norm on $E$ is Gateaux differentiable if and only if for every $x$ in $E$ such that $|x|=1$,

$$G(x:y) = \lim_{\lambda \to 0} \frac{|x+\lambda y|-|x|}{\lambda}$$

exists for every $y$ in $E$.

By adding uniformity conditions to the limit $G(x:y)$ above we obtain two conditions on the norm of $E$, that are similar to smoothness of the norm - in fact, identical in the case of normed linear spaces of finite algebraic dimension - but characterize fewer spaces, making it easier to extend to some degree the result that smoothness of the norm in normed linear spaces of finite dimension implies that every Chebyshev subset is closed and convex.

Definition: Let $E$ be a normed linear space the norm of which is smooth. The norm is uniformly Gateaux differentiable

if and only if for every $y$ in $E$, $(|x+\lambda y|-|x|)/\lambda$ converges to $G(x:y)$ uniformly for all $x$ of norm 1 in $E$ - i.e., for every positive number $\epsilon$ there is a positive number $\delta(\epsilon)$ such that if $x \in E$ such that $|x|=1$ and $\lambda$ is a real number such that $0<|\lambda|<\delta$ then

$$\left|\frac{|x+\lambda y|-|x|}{\lambda} - G(x:y)\right| < \epsilon.$$ 

**Definition:** Let $E$ be a normed linear space the norm of which is smooth. The norm on $E$ is uniformly Fréchet if and only if $(|x+\lambda y|-|x|)/\lambda$ converge to $G(x:y)$ uniformly for all $x$ and $y$ in $E$ such that $|x|=|y|=1$ - i.e., for every positive number $\epsilon$ there is a positive number $\delta(\epsilon)$ such that if $x$ and $y$ are members of $E$ such that $|x|=|y|=1$ and $\lambda$ is a real number such that $0<|\lambda|<\delta(\epsilon)$ then

$$\left|\frac{|x+\lambda y|-|x|}{\lambda} - G(x:y)\right| < \epsilon.$$ 

The following theorem gives a condition on the norm of a real normed linear space that is equivalent to the norm being uniformly Gateaux differentiable, and is easier to work with than the definition given above.

**Theorem 2.1** If $E$ is a real normed linear space, then the following condition is equivalent to the statement that the norm on $E$ is uniformly Gateaux differentiable:

if $y$ is in $E$ and $|y|>1$, then

$$\{x||x|+1<1/2(|x+y|+|x-y|)\}$$

is bounded.
**Proof:** First the sufficiency of the condition will be shown. Assume that the condition holds in $E$. Let $y$ be a non-zero member of $E$ and suppose $\varepsilon$ is a positive number less than 1. There is a positive number $\eta$ dependent only on the choice of $y$ such that $\eta |y|=1$. Then $|\eta y|=1$. Then there is a non-negative number $\delta$ dependent on $y$ and $\varepsilon$ such that, if $zeE$ and

$$|z|+1 \leq |z+(8\eta/\varepsilon)y|+|z-(8\eta/\varepsilon)y|)/2,$$

then $|z|\leq \delta$. Suppose $x$ is in $E$ and of norm 1. There are two cases to consider - first where $\delta$ is strictly positive and secondly where $\delta$ is zero.

**Case 1.** Assume $\delta$ is strictly positive. If $\lambda$ is a real number and $0<|\lambda|<8\eta/\varepsilon\delta$, then

$$|(8\eta/\varepsilon\lambda)x|+1>(|(8\eta/\varepsilon\lambda)x+(8\eta/\varepsilon)y|+|(8\eta/\varepsilon\lambda)x-(8\eta/\varepsilon)y|)/2,$$

$$2|x/\lambda|+2\varepsilon/8\eta>|x/\lambda+y|+|x/\lambda-y|,$$

and

$$\varepsilon/4\eta |x/\lambda+y|+|x/\lambda-y|-|x/\lambda|.$$

Suppose $\lambda_1$ and $\lambda_2$ are real numbers such that

$$0<|\lambda_1|<(8\eta/\varepsilon\delta)$$

and

$$0<|\lambda_2|<8\eta/\varepsilon\delta.$$ Then

$$\varepsilon/4\eta |x/|\lambda_1|+y|+|x/|\lambda_1|-y|-|x/|\lambda_1|, $$

and

$$\varepsilon/4\eta |x/|\lambda_2|+y|+|x/|\lambda_2|-y|-|x/|\lambda_2|.$$ Then

$$\varepsilon/2\eta (|x/|\lambda_1|+y|-|x/|\lambda_1|)+|x/|\lambda_2|-y|-|x/|\lambda_2|)+$$

$$(|x/|\lambda_2|+y|-|x/|\lambda_2|)+|x/|\lambda_1|-y|-|x/|\lambda_1|).$$
Note that if \( \alpha \) and \( \beta \) are positive numbers, then
\[
|\alpha x| + |\beta x| = |(\alpha + \beta)x| \\
= |\alpha x + \beta x - y| \\
\leq |\alpha x + y| + |\beta x - y|.
\]

Then
\[
\frac{\varepsilon}{4\eta} > \frac{|x/|\lambda_1| + y - |x/|\lambda_1| + |x/|\lambda_1| - y - |x/|\lambda_1|}{|\lambda_1|} \quad (1)
\]
\[
= \left| \frac{x + |\lambda_1|y - |x| + x - |\lambda_1|y - |x|}{|\lambda_1|} \right|,
\]
\[
\frac{\varepsilon}{4\eta} > \frac{|x/|\lambda_2| + y - |x/|\lambda_2| + |x/|\lambda_2| - y - |x/|\lambda_2|}{|\lambda_2|} \quad (2)
\]
\[
= \left| \frac{x + |\lambda_2|y - |x|}{|\lambda_2|} + \frac{x - |\lambda_2|y - |x|}{|\lambda_2|} \right|,
\]
\[
\frac{\varepsilon}{2\eta} > \frac{|x/|\lambda_1| + y - |x/|\lambda_1| + |x/|\lambda_2| - y - |x/|\lambda_2|}{|\lambda_1|} \quad (3)
\]
\[
= \left| \frac{x + |\lambda_1|y - |x|}{|\lambda_1|} + \frac{x - |\lambda_2|y - |x|}{|\lambda_1|} \right| + \\
\left| \frac{x + |\lambda_2|y - |x|}{|\lambda_2|} + \frac{x - |\lambda_2|y - |x|}{|\lambda_1|} \right|.
\]

There are three possibilities to consider - 1) both \( \lambda_1 \) and \( \lambda_2 \) are negative; 2) both \( \lambda_1 \) and \( \lambda_2 \) are positive; and 3) one of \( \lambda_1 \) and \( \lambda_2 \) is negative and the other is positive.
Subcase 2.1. Assume both \( \lambda_1 \) and \( \lambda_2 \) are negative. Then, from (1) above

\[
\frac{\epsilon}{4\eta} > \frac{|x - |\lambda_1|y| - x|}{-|\lambda_1|} - \frac{|x + |\lambda_1|y| - x|}{|\lambda_1|}
\]

and from (3)

\[
\frac{\epsilon}{2\eta} > \frac{|x + |\lambda_1|y| - x|}{|\lambda_1|} - \frac{|x - |\lambda_2|y| - x|}{-|\lambda_2|}
\]

Then

\[
\frac{3\epsilon}{4\eta} > \frac{|x + \lambda_2 y| - |x|}{\lambda_1} - \frac{|x + \lambda_2 y| - |x|}{\lambda_2}
\]

Subcase 2.2. Assume both \( \lambda_1 \) and \( \lambda_2 \) are positive. Then by manipulation of (2) and (3) above, just as was done in subcase 2.1,

\[
\frac{3\epsilon}{4\eta} > \frac{|x + \lambda_1 y| - |x|}{\lambda_1} - \frac{|x + \lambda_2 y| - |x|}{\lambda_2}
\]

Subcase 2.3. Assume one of \( \lambda_1 \) and \( \lambda_2 \) is positive and the other is negative. Without any loss of generality, say \( \lambda_1 \) is positive and \( \lambda_2 \) is negative. Then, from (3) above

\[
\frac{\epsilon}{2\eta} > \frac{|x + |\lambda_1|y| - x|}{|\lambda_1|} - \frac{|x - |\lambda_2|y| - x|}{-|\lambda_2|}
\]

\[
= \frac{|x + \lambda_1 y| - |x|}{\lambda_1} - \frac{|x + \lambda_2 y| - |x|}{\lambda_2}
\]
Therefore, from subcase 2.2 through 2.3 above, for every pair of real numbers \( \lambda_1 \) and \( \lambda_2 \) such that \( 0 < |\lambda_1| < 8\eta/\varepsilon\delta \) and \( 0 < |\lambda_2| < 8\eta/\varepsilon \),

\[
\varepsilon/\eta > \left( |x+\lambda_1 y|-|x|\right)/\lambda_1 - \left( |x+\lambda_2 y|-|x|\right)/\lambda_2 .
\]

**Case 2.** Assume \( \delta \) is zero. Then, for any non-zero number \( \lambda \)

\[
|(8\eta/\varepsilon\lambda)x|=8\eta/\varepsilon\lambda>0=\delta
\]

which implies, as in Case 1 above, that

\[
|(8\eta/\varepsilon\lambda)x|+1>(|8\eta/\varepsilon\lambda)x+y|+(8\eta/\varepsilon\lambda)x-y)/2.
\]

Then proceeding in exactly the same argument as in Case 1 above, it follows that for any non-zero numbers \( \lambda_1 \) and \( \lambda_2 \),

\[
\varepsilon/\eta > \left( |x+\lambda_1 y|-|x|\right)/\lambda_1 - \left( |x+\lambda_2 y|-|x|\right)/\lambda_2 .
\]

Since \( x \) was chosen arbitrarily from \( S(E) \), and \( \delta \) depended only on \( y \) and \( \varepsilon \), and \( \varepsilon \) was chosen independent of \( y \), it follows that, for any positive number \( \varepsilon \) there is a positive number \( \delta(\varepsilon,y) \) dependent on \( \varepsilon \) and \( y \) such that, for every \( x \) in \( S(E) \):

1) in the case that \( \delta(\varepsilon,y)>0 \), if \( \lambda_1 \) and \( \lambda_2 \) are non-zero real numbers such that \( 0 < |\lambda_1| < 8\eta/\varepsilon\delta(\varepsilon,y) \) and \( 0 < |\lambda_2| < 8\eta/\varepsilon\delta(\varepsilon,y) \)

Then

\[
\varepsilon/\eta > \left( |x+\lambda_1 y|-|x|\right)/\lambda_1 - \left( |x+\lambda_2 y|-|x|\right)/\lambda_2 ;
\]

and
2) in the case that \( \delta(\varepsilon,y) = 0 \), for every pair of non-zero real numbers \( \lambda_1 \) and \( \lambda_2 \),

\[
\varepsilon/n > \frac{|x + \lambda_1 y - |x|/\lambda_1 - (|x + \lambda_2 y - |x|)/\lambda_2|}{\lambda_1 - \lambda_2}.
\]

First consider \( \{(|x + \lambda_1 y - |x|)/\lambda)|\lambda \in \mathbb{R}, \lambda \neq 0 \} \) is a Cauchy net in \( \mathbb{R} \), and so it has a limit, \( \xi \), in \( \mathbb{R} \). Then, for every positive number \( \varepsilon \) there is a positive number \( \delta(\eta \varepsilon, y) \) such that if \( \lambda \) is a non-zero real number such that \( |\lambda| < \delta(\eta \varepsilon, y) \) then

\[
((|x + \lambda y - |x|)/\lambda - \xi)|<\varepsilon \eta/\varepsilon = \varepsilon.
\]

Then \( \xi \) is just the limit

\[
G(x:y) = \lim_{\lambda \to 0} (|x + \lambda y - |x|)/\lambda
\]

of the definition of Gateaux differentiability of the norm.

Suppose \( \varepsilon \) is a positive number. Then there is a non-negative number \( \delta(\varepsilon \eta/2, y) \) dependent on \( \varepsilon \) and \( y \), such that, letting

\[
\delta'(\varepsilon \eta/2, y) = \begin{cases} 
\delta(\varepsilon \eta/2, y) & \text{if } \delta(\varepsilon \eta/2, y) > 0 \\
1 & \text{if } \delta(\varepsilon \eta/2, y) = 0.
\end{cases}
\]
if \( x \) is in \( S(E) \) and \( \lambda_1 \) and \( \lambda_2 \) are real numbers such that
\[
0 < |\lambda_1|, |\lambda_2| < 8\eta/\varepsilon \delta(\epsilon\eta/2, y)
\]
then
\[
\left( \frac{|(x + \lambda_1 y) - |x|)}{\lambda_1} - \frac{|(x + \lambda_2 y) - |x|)}{\lambda_2} \right) < \epsilon \eta/2n = \epsilon/2.
\]

Since \( G(x:y) \) exists, there is a positive number \( \gamma \) such that
if \( \lambda \) is a real number and \( 0 < |\lambda| < \gamma \)
then
\[
\left( \frac{|(x + \gamma y) - |x|)}{y} - G(x:y) \right) < \epsilon/2.
\]
Then it follows easily that, for every real number \( \lambda \) such
that \( 0 < |\lambda| < 8\eta/\varepsilon \delta'(\epsilon\eta/2, y) \),
\[
\left( \frac{|(x + \lambda y) - |x|)}{\lambda} - G(x:y) \right) < \epsilon.
\]
Then, for \( y \) which was chosen arbitrarily as a non-zero
member of \( E \), for every positive number \( \varepsilon \) there is a positive
number \( \delta(\epsilon\eta, y) \) such that, for every \( x \) of norm one in \( E \) and
real number \( \lambda \) such that \( 0 < |\lambda| < \delta(\epsilon\eta, y) \),
\[
\left( \frac{|(x + \lambda y) - |x|)}{\lambda} - G(x:y) \right) < \epsilon.
\]
Then the norm on \( E \) is uniformly Gateaux differentiable.

Secondly, necessity of the condition will be shown.
Assume that \( E \) is uniformly Gateaux differentiable. Suppose
\( y \) is in \( E \) and \( |y| \geq 1 \). Then there is a positive number \( \delta \) such
that if \( x \) is a member of \( E \) of norm one and \( \lambda \) is a real number
such that \( 0 < |\lambda| < \delta \),
then

\[ |(|x+\lambda y| - |x|)/\lambda - G(x:y)| < 1 \]

and

\[ |(|x-\lambda y| - |x|)/-\lambda - G(x:y)| < 1. \]

If \( x \in E \) and \( |x| > 1/\delta \) then \( \delta > 1/|x| \)

and

\[
|\lambda y| - |x| + |x-y| - |x| = (|x/|x| + (1/|x|) y - |x/|x| |)/ (1/|x|) + \\
|G(x:y) - G(x:y)| - (1/|x|) y - |x/|x| |)/ (1/|x|) + \\
G(x:y) - G(x:y)
\]

\[
\leq |(|x/|x| + (1/|x|) y - |x/|x| |)/ (1/|x|) - G(x:y)| + \\
|G(x:y) - G(x:y)| - (1/|x|) y - |x/|x| |)/ (1/|x|) - G(x:y)| < 2,
\]

implying that

\[ |x+y| + |x-y| < 2 + 2|x|. \]

Therefore, if \( x \) is in \( E \) and \( |x| > 1/\delta \)

then \( (|x+y| + |x-y|)/2 < |x| + 1. \)

Then \( \{x \in E | x| + 1 \leq (|x+y| + |x-y|)/2\} \) is bounded.

**Definition:** If \( E \) is a normed linear space with a uniformly Gateaux differentiable norm, then for every \( y \) in \( E \) of norm greater than or equal to one

\[ \Delta(y) = \sup\{|x| | x \in E \text{ and } |x| + 1 \leq (|x+y| + |x-y|)/2\}. \]
Theorem 2.2  If $E$ is a normed linear space with a uniformly Gateaux differentiable norm, then the norm on $E$ is uniformly Frechet differentiable if and only if, for every positive number $\delta \geq 1$, \{\Delta(y) | y \in E \text{ and } |y| = \delta\} is bounded.

Proof: First assume that the norm on $E$ is uniformly Frechet differentiable. Let $\delta$ be a positive number and $\delta \geq 1$. There is a positive number $\eta$ such that if $x$ is a member of $E$ of norm $1$ and $y$ is a member of $E$ of norm $1$ and $\lambda$ is a real number such that $0 < |\lambda| \leq 1$ then

$$\left| \left( \frac{|x + \lambda y| - |x|}{\lambda} \right) - G(x:y) \right| < \frac{1}{\delta}$$

Let $y$ be a member of $E$ such that $|y| = \delta$. Then $|y/\delta| = 1$. Suppose $x$ is a member of $E$ such that $|x| > \delta/\eta$. Then $1/|x/\delta| < \eta$.

Then

$$|x/\delta + y/\delta| - |x/\delta| + |x/\delta - y/\delta| - |x/\delta|$$

$$= \left( |x/\delta| x/\delta | + (1/|x/\delta|) (y/\delta) | - |x/\delta| x/\delta | \right) / (1/|x/\delta|) +$$

$$\left( |x/\delta| x/\delta | - (1/|x/\delta|) (y/\delta) | - |x/\delta| x/\delta | \right) / (1/|x/\delta|) +$$

$$G(x:y) - G(x:y)$$

$$\leq \left( |x/|x| + (1/|x/\delta|) (y/\delta) | - |x/|x| | \right) / (1/|x/\delta|) - G(x:y) +$$

$$\left( |x/|x| - (1/|x/\delta|) (y/\delta) | - |x/|x| | \right) / (1/|x/\delta|) - G(x:y)$$

$$\leq \frac{2}{\delta}.$$
Then

\[ |x+y| + |x-y| - 2|x| < 2 \]

Therefore, if \( x \) is a member of \( E \) such that \( |x| > \delta/\eta \), then

\[ \frac{(|x+y| + |x-y|)}{2} < |x| + 1. \]

Then

\[
\Delta(y) = \sup \{|x| \mid x \in E \text{ and } |x| + 1 \leq \frac{(|x+y| + |x-y|)}{2}\} \leq \delta/\eta.
\]

Since \( y \) was chosen arbitrarily so that \( |y| = \delta \), \{\( \Delta(y) \mid y \in E \text{ and } |y| = \delta \} \) is bounded by \( \delta/\eta \).

Secondly, assume that, for every positive number \( \delta \geq 1 \), \{\( \Delta(y) \mid y \in E \text{ and } |y| = \delta \} \) is bounded.

Let \( \varepsilon \) be a positive number. Then there is a positive number \( \eta \) such that, for every \( y \) in \( E \) such that \( |y| = 16/\varepsilon \), \( \Delta(y) \leq \eta \). Suppose each of \( x \) and \( y \) are members of \( E \) of norm one. Suppose \( \lambda \) is a real number such that \( 0 < |\lambda| < 16/\varepsilon \eta \).

Then

\[ |16x/\varepsilon \lambda| = 16/\varepsilon \lambda > \eta \geq \Delta(16y/\varepsilon). \]

Then

\[ |16x/\varepsilon \lambda| + 1 > (|16x/\varepsilon \lambda + 16y/\varepsilon| + |16x/\varepsilon \lambda - 16y/\varepsilon|)/2 \]

Then

\[ \varepsilon/8 > |x/\lambda + y| + |x/\lambda - y| - 2|x/\lambda|, \]
Then, for every real number \( \lambda \) such that \( 0 < |\lambda| < 16/\epsilon_n \),

\[
\epsilon/8 > |x/\lambda + y| - |x/\lambda| + |x/\lambda - y| - |x/\lambda|
\]

Suppose \( \lambda_1 \) and \( \lambda_2 \) are real numbers such that

\( 0 < |\lambda_1| < 16/\epsilon_n \) and \( 0 < |\lambda_2| < 16/\epsilon_n \).

Then

\[
\epsilon/8 > |x/|\lambda_1| + y| - |x/|\lambda_1| | + |x/|\lambda_1| - y| - |x/|\lambda_1| |
\]

\[
\epsilon/8 > |x/|\lambda_2| + y| - |x/|\lambda_2| | + |x/|\lambda_2| - y| - |x/|\lambda_2| |
\]

and

\[
\epsilon/4 > (|x/|\lambda_1| + y| - |x/|\lambda_1| | + |x/|\lambda_1| - y| - |x/|\lambda_1| |) +
\]

\[
(|x/|\lambda_2| + y| - |x/|\lambda_2| | + |x/|\lambda_2| - y| - |x/|\lambda_2| |).
\]

As in the proof of sufficiency of the condition given in theorem 2.1 above, it follows that

\[
\epsilon/2 > \left|\frac{|x+\lambda_1y| - |x|}{\lambda_1} - \frac{|x+\lambda_2y| - |x|}{\lambda_2}\right|
\]

Then, for every pair of real numbers \( \lambda_1 \) and \( \lambda_2 \) such that

\( 0 < |\lambda_2| < 16/\epsilon_n \),

\[
\epsilon/2 > \left|\frac{|x+\lambda_1y| - |x|}{\lambda_1} - \frac{|x+\lambda_2y| - |x|}{\lambda_2}\right|
\]

Suppose \( \lambda \) is a real number and \( 0 < |\lambda| < 16/\epsilon_n \). Since \( E \) has uniformly Gateaux differentiable norm, there is a positive number \( \xi \) such that if \( y \) is a real number and \( 0 < |y| < \xi \), then

\[
\left|\frac{|x+y| - |x|}{y} - G(x:y)\right| < \epsilon/2.
\]

Choose \( 0 < y < \min\{\xi, 16/\epsilon_n\} \).
Then
\[ \varepsilon/2 > \left( |x + \lambda y| - |x| \right)/\lambda - \left( |x + y| - |x| \right)/\gamma \]
and
\[ \varepsilon/2 > \left( |x + y| - |x| \right)/\gamma - G(x:y) \]
Then
\[ \varepsilon > \left( |x + \lambda y| - |x| \right)/\lambda - G(x:y). \]
Therefore, for any \( x \) and \( y \) of norm one and real number \( \lambda \) such that \( 0 < |\lambda| < 16/\varepsilon n \),
\[ |(|x + \lambda y| - |x|)/\lambda - G(x:y)| < \varepsilon. \]
Therefore, \( E \) is uniformly Fréchet differentiable.

**Theorem 2.3** If \( E \) is a normed linear space of finite algebraic dimension, then the following conditions are equivalent:

1) the norm on \( E \) is smooth;
2) the norm on \( E \) is uniformly Gateaux differentiable;
3) and the norm on \( E \) is uniformly Fréchet differentiable.

**Proof:** First assume that the norm on \( E \) is smooth.

Suppose, for some \( y \) in \( E \) of norm greater than or equal to one, for every positive integer \( n \) there is an \( x_n \) in \( E \) such that \( |x_n| \geq n \) and
\[ |x_n| + 1 \leq (|x_n + y| + |x_n - y|)/2. \]
Then
\[ |x_n/|y||+1/|y| \leq (|x_n/|y|+y/|y|+|x_n/|y|-y/|y|))/2 \]
For each positive integer n let \( z_n = x_n/|x_n| \). Then \( \{z_n\}_{n=1}^{\infty} \)
is a sequence in \( S(E) \). Since \( E \) is of finite algebraic
dimension, \( B(E) \) is compact in the norm topology. Then \( S(E) \)
is compact and first countable. Then there is a subsequence
\( \{z_n(k)\}_{k=1}^{\infty} \) of \( \{z_n\}_{n=1}^{\infty} \) such that, for some \( z \) in \( S(E) \),
\[ z = \lim_{k \to \infty} z_n(k). \]
Note that, for every \( n \), \( |z_n| = 1 \). Then \( |z| = 1 \),
since the norm is continuous.

Suppose \( \delta \) is a positive number. Then, there is a
positive integer \( n_0 \) such that if \( n \geq n_0 \) and \( n \) is a positive
integer, then \( \lambda > |y|/n \geq |y|/|x_n| \). Suppose \( n \geq n_0 \) and recall
that \( |x_n| \geq n \) and that \( |z_n| = 1 \).

Then
\[ |x_n/|y||+1/|y| \leq (|x_n/|y|+y/|y|+|x_n/|y|-y/|y|))/2 \]
\[ |x_n/|z_n||y||+1/|y| \leq (|x_n/|z_n||y|+y/|y|+|x_n/|z_n||y|-y/|y|))/2 \]
\[ \leq (|x_n/|z_n||y|+y/|y|+|x_n/|z_n||y|-y/|y|))/2 \]
\[ (n_0/|y|)|z_n|+(|x_n/|z_n||y|)/|y|)|z_n|+1/|y| \]
\[ \leq (|x_n/|z_n||y|+y/|y|+|x_n/|z_n||y|-y/|y|))/2 \]
\[ |n_0z_n/y| + 1/y \leq (\|x_n/z_n/y + y/y| - (|x_n| - n_0)/y| + \\
\|x_n/z_n/y - y/y| - (|x_n| - n_0)/y|)/2 \]

\[ |n_0z_n/y| + 1/y \leq (|n_0z_n/y + y/y| + |n_0z_n/y - y/y|)/2. \]

Since \( n \) was chosen arbitrarily as a positive integer greater than or equal to \( n_0 \), the last inequality holds for all such positive integers.

Since \( z = \lim_{k \to \infty} z_n(k) \),

\[ n_0z/y = \lim_{k \to \infty} (n_0z_n(k)/|y|), \]

\[ n_0z/y + y = \lim_{k \to \infty} (n_0z_n(k)/|y| + y), \]

and

\[ n_0z/y - y = \lim_{k \to \infty} (nz_n(k)/|y| - y). \]

Then, since the norm is continuous,

\[ |n_0z/y| = \lim_{k \to \infty} (nz_n(k)/|y|), \]

\[ |n_0z/y + y/y| = \lim_{k \to \infty} (nz_n(k)/|y| + y/y|) \]

and

\[ |n_0z/y - y/y| = \lim_{k \to \infty} (nz_n(k)/|y| - y/y|). \]

Suppose \( \varepsilon \) is a positive number. Then there is a positive number \( k_0 \) such that if \( k \geq k_0 \),
then
\[ \| (n_0 z/y) \| - (n_0 z/(n(k)/y)) \| < \frac{\varepsilon}{2}, \]
\[ \| (n_0 z/y) + (y/y) \| - (n_0 z/(n(k)/y)) + (y/y) \| < \frac{\varepsilon}{2}, \]
and
\[ \| (n_0 z/y) -(y/y) \| - (n_0 z/(n(k)/y)) -(y/y) \| < \frac{\varepsilon}{2} \]

For some $k_1 > k_0$, $n(k_1) > n_0$.

If $k > k_1$, then
\[ \| (n_0 z/y) \| + 1/|y| \leq \| (n_0 z/(n(k)/y)) \| + \varepsilon/2 + 1/|y| \]
\[ \leq \| (n_0 z/(n(k)/y)) \| + (y/y) + \| (n_0 z/(n(k)/y)) -(y/y) \| /2 + \varepsilon/2 \]
\[ \leq \| (n_0 z/y) \| + (y/y) + \| (n_0 z/y) -(y/y) \| + 2\varepsilon)/2 + \varepsilon/2 \]
\[ = \| (n_0 z/y) \| + (y/y) + \| (n_0 z/y) -(y/y) \| /2 + \varepsilon. \]

Since $\varepsilon$ was arbitrarily a positive number,
\[ \| (n_0 z/y) \| + 1/|y| \leq \| (n_0 z/y) \| + (y/y) \| + \| (n_0 z/y) -(y/y) \| /2. \]

Then
\[ 2/|y| < \| (n_0 z/y) \| + (y/y) \| - \| (n_0 z/y) \| + (n_0 z/y) -(y/y) \| - \]
\[ \| (n_0 z/y) \| + \Theta(z/y) -(y/y) - \Theta(z/y)/|y|. \]
\[ \leq \| (z + (y/n_0)(y/y)) \| - |z|/|y/n_0) - \Theta(z/y)/|y| \| + \]
\[ \| (z - (y/n_0)(y/y)) \| - |z|/(|y/n_0) - \Theta(z/y)/|y| \|. \]
Since this holds for arbitrary choice of a positive number \( \delta \), for every positive \( \delta \) there is a real number \( \lambda \) such that

\[ |\lambda| < \delta \]

and

\[ \frac{1}{|y|} \left( |z + \lambda(y/|y|)| - |z| \right) / \lambda - G(z:y/|y|) \]

which contradicts the fact that, since \( E \) is smooth and each of \( z \) and \( y/|y| \) is of norm one,

\[ G(z:y/|y|) = \lim_{\lambda \to 0} \left( \frac{|z + \lambda(y/|y|)| - |z|}{\lambda} \right) / \lambda, \]

exists.

Then there is no sequence of \( x_n \)'s for \( y \) such that, for every positive integer \( n \), \( |x_n| > n \) and

\[ |x_n| + \frac{1}{2} < \left( |x_n + y| + |x_n - y| \right) / 2. \]

Then, since \( y \) was arbitrarily a member of \( E \) of norm greater than or equal to one, for every \( y \) in \( E \) such that \( |y| > 1 \),

\[ \{ x \in E | x + 1 \leq (|x + y| + |x - y|) / 2 \} \]

is bounded. Then \( E \) is uniformly Gateaux differentiable.

Secondly assume that the norm of \( E \) is uniformly Gateaux differentiable. Suppose \( \{ \Delta(y) | y \in E \ and |y| = y \} \) where \( y > 1 \), is not bounded. Then, for every positive integer \( n \), there is a \( y_n \) in \( E \) such that \( |y_n| = y \) and \( \Delta(y) \geq n + 1. \)

Suppose \( n \) is a positive integer. Since \( \Delta(y) \geq n + 1, \) there is an \( x_n \) in \( E \) such that

\[ |x_n| + 1 \leq \left( |x_n + y_n| + |x_n - y_n| \right) / 2 \]

and \( \Delta(y) \geq |x_n| > n. \)
Then $2|x_n| > n$ and $1 < |x_n|$. Then

$$|2x_n| + 2 \leq (|2x_n + 2y_n| + |2x_n - 2y_n|)/2$$

For every positive integer $n$, let $z_n = 2x_n$.

Since $\{y_n\}^\infty_{n=1}$ is a sequence in $\gamma S(E)$ and $\gamma S(E)$ is compact and first countable, there is a subsequence $\{y_n(k)\}^\infty_{k=1}$ and a $y$ in $\gamma S(E)$ such that $y = \lim_{k \to \infty} y_n(k)$. Then

$$2y = \lim_{k \to \infty} 2y_n(k).$$

Then there is a $k_0$ such that $k > k_0$ implies that

$$|2y - 2y_n(k)| < 1.$$ 

Suppose $k$ is a positive integer. Then $|z_n(k)| \geq n(k)$ and

$$|z_n(k)| + 2 \leq (|z_n(k) + 2y_n(k)| + |z_n(k) - 2y_n(k)|)/2$$

Then

$$|z_n(k)| \leq (|z_n(k) + 2y_n(k)| + |z_n(k) - 2y_n(k)|)/2$$

$$\leq (|z_n(k) + 2y_n(k)| - |2y_n(k) - 2y| + |z_n(k) - 2y_n(k)| - |2y - 2y_n(k)|)/2$$

$$\leq (|z_n(k) + 2y| + |z_n(k) - 2y|)/2.$$ 

Then

$$\{x \in E | |x| + 1 \leq (|x + 2y| + |x - 2y|)/2\}$$

is not bounded - a contradiction of the assumption that $E$ is uniformly Gateaux differentiable.
Therefore, for every positive number \( \gamma \geq 1 \), 
\( \{ A(y) | y \in E \text{ and } |y| = \gamma \} \) is bounded. Then \( E \) is uniformly Fréchet differentiable.

Note that if \( E \) is uniformly Fréchet differentiable, then, given an \( x \) of norm one in \( E \), the limit \( G(x:y) \) exists for every \( y \) in \( E \), which by definition implies that \( E \) is Gateaux differentiable (or equivalently, smooth).

**Notation:** Let \( E \) be a normed linear space with a smooth norm. Then, for every \( x \) of norm one in \( E \), let \( T_x \) be the unique bounded linear functional such that \( |T_x| = 1 \) and \( T_x(x) = 1 \).

Note that if \( E \) has a smooth norm, then for every \( x \) in \( E \), \( T_x \) is such that \( T_x(y) = G(x:y) \) for every \( y \) in \( E \). \(^2\)

**Theorem 2.4** If \( E \) is a normed linear space with a uniformly Gateaux differentiable norm and \( \{ x_i \}_{i \in \mathcal{D}} \) is a net in \( S(E) \) converging weakly to some \( x \) in \( S(E) \), then

\[
T_x = \text{weak}^* \lim_{i \in \mathcal{D}} T_{x_i}.
\]

**Proof:** Let \( \{ x_i \}_{i \in \mathcal{D}} \) be a net in \( S(E) \) that converges to some \( x \) in \( E \).

Suppose there is a positive number \( n \) and a \( y \) in \( E \) such that, for every \( i \) in \( D \), there is a \( j > i \) such that

\[
|T_x(y) - T_{x_j}(y)| > n.
\]

Let \( \xi = n/|y| \) and \( z = y/|y| \). Then, for every \( i \) in \( \mathcal{D} \), there is

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a \ j > 1 \ such \ that

\[ |T(x(z) - T(x_j(z))| \geq \xi. \]

There is a positive integer \( n_0 \) such that, for every \( n \geq n_0 \),

\[ T(z) + \xi/2 < n\xi/2. \]

Suppose \( n \) is a positive integer and \( n > n_0 \). Then there is an \( i_0 \) in \( D \) such that if \( i > i_0 \), then

\[ |T(x) - T(x_i)| < 1/n, \]

since \( x = \text{weak}\lim_{i \in D} x_i \). There is an \( i > i_0 \) such that

\[ |T(z) - T(z)| > \xi. \]

Then

\[ 1 - 1/n < T(x_i) \]

\[ n\xi/2 - \xi/2 < T(n\xi x_i/2) \]

\[ n\xi/2 - \xi/2 + T(z) < T(n\xi x_i/2 + z). \]

Since \( |T(p)| \leq |T(x)| |p| = |p| \) for every \( p \) in \( E \),

\[ n\xi/2 - \xi/2 + T(z) < |n\xi x_i/2 + z|. \]

Then

\[ 0 < |n\xi x_i/2 + z| - n\xi/2 + \xi/2 - T(z). \] (1)

Similarly,

\[ 0 < |n\xi x_i/2 - z| - n\xi/2 + \xi/2 + T(z). \] (2)
Either $T_x(z) \geq T_{x_1}(z)$ or $T_{x_1}(z) > T_x(z)$.

**Case 1** Suppose $T_x(z) > T_{x_1}(z)$. Then

$$\frac{\xi}{2} \leq T_x(z) - T_{x_1}(z)$$

$$\frac{\xi}{2} \leq T_x(z) - T_{x_1}(z) - \frac{\xi}{2}$$

Then, combining with (1) above,

$$\frac{\xi}{2} \leq \left| n\delta x_1 / 2 + z \right| - \frac{\xi}{2} / 2 + T_x(z) + T_z(z) - T_{x_1}(z) - \frac{\xi}{2}$$

$$= \left| n\delta x_1 / 2 + z \right| - \frac{\xi}{2} / 2 - T_{x_1}(z)$$

$$= \left| x_1 + 2(n\delta/z) - x_1 \right| \left| - \frac{2/n\delta}{2/n\delta} - T_{x_1}(z) \right|$$

**Case 2** Suppose $T_{x_1}(z) > T_x(z)$. Then

$$\frac{\xi}{2} \leq T_{x_1}(z) - T_x(z)$$

$$\frac{\xi}{2} \leq T_{x_1}(z) - T_x(z) - \frac{\xi}{2}$$

Then, combining with (2),

$$\frac{\xi}{2} \leq \left| n\delta x_1 / 2 - z \right| - \frac{\xi}{2} / 2 + T_{x_1}(z) + T_{x_1}(z) - T_x(z) - \frac{\xi}{2}$$

$$= \left| n\delta x_1 / 2 - z \right| - \frac{\xi}{2} / 2 + T_{x_1}(z)$$

$$= \left| x_1 - (2/n\delta)z \right| - x_1 \left| \frac{2/n\delta}{2/n\delta} + T_{x_1}(z) \right|$$

$$= \left| x_1 - (2/n\delta)z \right| - x_1 \left| \frac{-2/n\delta}{-2/n\delta} - T_{x_1}(z) \right|.$$
Then in either case there is a real number $\lambda$ such that $|\lambda|=2/n\xi$ and
\[
\xi/2 \leq \left( \frac{|x_i + \lambda z|}{|x_i|} - |x_i| \right)/\lambda - T_{x_1}(z).
\]
Note that $T_{x_1}(z)=G(x_1,z)$ for every $i$ in $D$. Then, for every positive integer $n$, there is a real number $\lambda$ such that $\lambda=2/n\xi$ and
\[
\xi/2 \leq \left( \frac{|x_1 + \lambda z|}{|x_1|} - |x_1| \right)/\lambda - G(x_1).z).
\]
Then there does not exist a positive number $\delta$ such that, for every real number $\lambda$ such that $0<|\lambda|<\delta$, and every $u$ in $E$ of norm one,
\[
\xi/2 > \left( \frac{|u + \lambda z|}{|u|} - |u| \right)/\lambda - G(u;z).
\]
This contradicts the assumption that the norm on $E$ is uniformly Gateaux differentiable.

The result of theorem 2.4 can be strengthened considerably in a normed linear space having a uniformly Frechet differentiable norm, as is done in theorem 2.5 below.

**Definition:** Let $E$ be a normed linear space. The norm on $E$ is **uniformly convex** if and only if for every positive number $\varepsilon$ such that $0<\varepsilon<2$, there is a positive number $\delta(\varepsilon)$ such that if $x$ and $y$ are members of $E$ of norm one and $|x-y|>\varepsilon$, then
\[
|\xi/2 x + \xi/2 y| < 1 - \delta(\varepsilon).
\]
The norm on $E$ is locally uniformly convex if and only if, for every $x$ in $E$, if $\varepsilon$ is a positive number then there is a positive number $\delta(x,\varepsilon)$ such that if $y \in E$ and $|x-y| \geq \varepsilon$ then $|\frac{1}{2}x + \frac{1}{2} y| < 1 - \delta(x,\varepsilon)$.

In *Normed Linear Spaces*, Day notes that the norm of a normed linear space $E$ is uniformly Frechet differentiable if and only if the norm of $E^*$ (the dual of $E$) is uniformly convex.  

**Theorem 2.5** If $E$ is a normed linear space and $E^*$ is locally uniformly convex and $\{x_i\}_{i \in D}$ is a net in $S(E)$ that converges weakly to some $x$ in $S(E)$, then $T_x = \lim_{i \in D} T_{x_i}$.

**Proof:** Let $\{x_i\}_{i \in D}$ be a net in $S(E)$ that converges to some $x$ in $S(E)$. Suppose $\varepsilon$ is a positive number. Then there is a positive number $\delta(T_x,\varepsilon)$ such that if $f$ is of norm one in $E^*$ and $|T_x - f| \geq \varepsilon$, then $|\frac{1}{2}T_x + \frac{1}{2} f| < 1 - \delta(T_x,\varepsilon)$. Since $x = \text{weak lim}_{i \in D} x_i$, there is an $i_0$ in $D$ such that if $i \geq i_0$, then $|T_x(x_i) - 1| < \delta(T_x,\varepsilon)$. Suppose $i > i_0$, then

$|T_x(x_i) - 1| < \delta(T_x,\varepsilon)$.

$1 - \delta(T_x,\varepsilon) < T_x(x_i) \leq 1$

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Then

\[ \frac{1}{2} |2T_x + 1| = \frac{1}{2} (|T_x + T_{x_1}|)(|x_1|) \]

\[ \geq \frac{1}{2} |T_x(x_1) + T_{x_1}(x_1)| \]

\[ > (1 - \delta(T_x, \epsilon))/2 + 1/2 \]

\[ = 1 - \delta(T_x, \epsilon)/2 \]

\[ > 1 - \delta(T_x, \epsilon). \]

Then \[ |T_x - T_{x_1}| < \epsilon. \]
CHAPTER III

CONVEXITY OF CERTAIN WEAKLY CLOSED SUBSETS OF NORMED LINEAR SPACES THAT HAVE UNIFORMLY GATEAUX DIFFERENTIABLE NORMS.

If \((E, |, |)\) is a reflexive normed linear space and the norm on \(E\) is uniformly Gateaux differentiable, then every weakly closed Chebyshev subset of \(E\) having a non-empty weak interior that is norm-wise dense in \(E\) is convex. The proof of this statement will be facilitated by constructing a certain topology on every uniformly bounded family of closed balls in \(E\) (a family of subsets \(C\) of \(E\) is uniformly bounded if and only if there is a positive number \(\delta\) such that every member of \(C\) is a subset of \(\delta B(E)\)).

Notation: If \(A\) and \(B\) are two sets then the complement of \(A\) in \(B\) will be denoted by \(B\setminus A\).

Definition: Let \(X\) be a space with a topology \(\mathcal{T}\) and let \(\{A_\lambda\}_{\lambda \in \Lambda}\) be a net of subsets of \(X\). Then

\[
\lim \inf_{\lambda \in \Lambda} A_\lambda = \{x \in X | \text{for every neighborhood } U \text{ of } x \}\{\lambda \in \Lambda | A_\lambda \cap U \neq \emptyset\} \text{ is residual in } \Lambda
\]

and

\[
\lim \sup_{\lambda \in \Lambda} A_\lambda = \{x \in X | \text{for every neighborhood } U \text{ of } x \}\{\lambda \in \Lambda | A_\lambda \cap U \neq \emptyset\} \text{ is cofinal in } \Lambda
\]
The net \( \{A_\lambda\}_{\lambda \in \Delta} \) is said to \( \Psi \)-inf sup converge to \( A \) (and \( \Psi \)-lim \( A_\lambda = A \)) if and only if \( \limsup_{\lambda \in \Delta} A_\lambda = A = \liminf_{\lambda \in \Delta} A_\lambda \).

It immediately follows from the above definition that, for any net \( \{A_\lambda\}_{\lambda \in \Delta} \) of subsets of a topological space \((X, \Psi)\),

\[
\liminf_{\lambda \in \Delta} A_\lambda \subseteq \limsup_{\lambda \in \Delta} A_\lambda.
\]

**Lemma 3.1** Let \((X, \Psi)\) be a topological space and \( \{A_\lambda\}_{\lambda \in \Delta} \) be a net of subsets of \( X \). Then the following are true:

1) if \( A_\lambda = A \) for every \( \lambda \in \Delta \) and \( A \) is a closed subset of \( X \), then \( A = \Psi \)-lim \( A_\lambda \);

2) if \( A \) is a subset of \( X \) and \( A = \Psi \)-lim \( A_\lambda \), then every subnet of \( \{A_\lambda\}_{\lambda \in \Delta} \) \( \Psi \)-inf sup converges to \( A \);

3) and, if there is a subset \( A \) of \( X \) such that every subnet of \( \{A_\lambda\}_{\lambda \in \Delta} \) has a subnet that \( \Psi \)-inf sup converges to \( A \), then \( \Psi \)-lim \( A_\lambda = A \).

**Proof:** 1) Assume that \( A \) is a closed subset of \( X \) and that \( A_\lambda = A \) for every \( \lambda \in \Delta \). Since \( A \) is closed, \( E_\lambda A \) is an open subset of \( X \). Then, for every \( x \) in \( E_\lambda A \), \( E_\lambda A \) is a neighborhood of \( x \) such that \( \{\lambda \in \Delta | A_\lambda \cap (E_\lambda A) = \emptyset \} \) is not cofinal in \( \Delta \). Then \( E_\lambda A \) is a subset of \( \text{Evlim sup } A_\lambda \). Then \( \text{lim sup } A_\lambda \) is contained in \( A \). If \( x \) is in \( A \), then, for any neighborhood \( U \) of \( x \), \( x \in \bigcup \Delta A = \bigcup A_\lambda \) for every \( \lambda \) in \( \Delta \) which implies that
\{\lambda \in \Lambda | A_{\lambda} \neq \emptyset\} = A is certainly residual in A. Then A is a subset of \(\liminf_{\lambda \in \Lambda} A_{\lambda}\). Then

\[
\limsup_{\lambda \in \Lambda} A_{\lambda} \subseteq A \subseteq \liminf_{\lambda \in \Lambda} A_{\lambda}
\]

Then \(\limsup_{\lambda \in \Lambda} A_{\lambda} = A = \liminf_{\lambda \in \Lambda} A_{\lambda}\).

Then \(A = \Psi\lim A_{\lambda}\).

2) Assume \(\{A_{\tau(d)}\}_{d \in D}\) is a subnet of \(\{A_{\lambda}\}_{\lambda \in \Lambda}\) and A is a subset of X such that \(A = \Psi\lim A_{\lambda}\).

Suppose \(x \in \liminf_{\lambda \in \Lambda} A_{\lambda}\) and suppose \(U\) is a neighborhood of \(x\). Then \(\{\lambda \in \Lambda | A_{\lambda} \cap U \neq \emptyset\}\) is residual in \(\Lambda\). Then, by definition of residual, there is a \(\lambda_0\) in \(\Lambda\) such that if \(\lambda > \lambda_0\) then \(A_{\lambda} \cap U \neq \emptyset\). Since \(\{A_{\tau(d)}\}_{d \in D}\) is a subnet, there is \(d_0\) in \(D\) such that \(\lambda_0 \leq \tau(d_0)\). If \(d > d_0\), then \(\tau(d) > \tau(d_0) \geq \lambda_0\) which implies that \(A_{\tau(d)} \cap U \neq \emptyset\). Then \(\{d \in D | A_{\tau(d)} \cap U \neq \emptyset\}\) is residual in \(D\). Then \(x\) is in \(\liminf_{d \in D} A_{\tau(d)}\). Then

\[
\liminf_{\lambda \in \Lambda} A_{\lambda} \subseteq \liminf_{d \in D} A_{\tau(d)}.
\]

Let \(x\) be in \(\limsup_{d \in D} A_{\tau(d)}\) and suppose \(U\) is a neighborhood of \(x\). Suppose \(\lambda\) is in \(A\). Then there is a \(d\) in \(D\) such that \(\lambda \leq \tau(d)\). Since \(x\) is in \(\limsup_{d \in D} A_{\tau(d)}\), \(\{d \in D | A_{\tau(d)} \cap U \neq \emptyset\}\) is cofinal in \(D\). Then there is a \(d'\) in \(D\) such that \(d' \geq d\) and \(A_{\tau(d')} \cap U \neq \emptyset\). Then \(\tau(d') \geq \tau(d) \geq \lambda\) and \(A_{\tau(d')} \cap U \neq \emptyset\).
Then \( \{ \lambda \in A | A_\lambda \cap U \neq \emptyset \} \) is cofinal in \( A \). Then \( x \) is in \( \lim \sup_\lambda A_\lambda \).

Then

\[
\lim_{d \in D} \sup_\lambda A_\tau(d) \subseteq \lim_{\lambda \in \Lambda} \sup_\lambda A_\lambda.
\]

Therefore, since \( A = \Psi - \lim_{\lambda \in \Lambda} A_\lambda \),

\[
\lim_{d \in D} \sup_\lambda A_\tau(d) \subseteq \lim_{\lambda \in \Lambda} \sup_\lambda A_\lambda = \lim_{\lambda \in \Lambda} \inf_\lambda A_\lambda \subseteq \lim_{d \in D} \inf_{\lambda \in \Lambda} A_\tau(d).
\]

Then

\[
\lim_{d \in D} \sup_\lambda A_\tau(d) = A = \lim_{d \in D} \inf_{\lambda \in \Lambda} A_\tau(d).
\]

Then \( A = \Psi - \lim_{d \in D} A_\tau(d) \). Since \( \{ A_\tau(d) \}_{d \in D} \) was arbitrarily chosen a subnet of \( \{ A_\lambda \}_{\lambda \in \Lambda} \), \( A = \Psi - \lim_{d \in D} \Psi(d) \) holds for every subnet \( \{ A_\psi(d) \}_{d \in D} \), of \( \{ A_\lambda \}_{\lambda \in \Lambda} \).

3) Assume that \( A \) is a subset of \( X \) and every subnet of \( \{ A_\lambda \}_{\lambda \in \Lambda} \) has a subnet that \( \Psi \)-inf sup converges to \( A \).

First that \( \lim_{\lambda \in \Lambda} \sup_\lambda A_\lambda \) is contained in \( A \) will be shown.

Suppose \( x \) is in \( \lim_{\lambda \in \Lambda} \sup_\lambda A_\lambda \). Let \( U_x \) be the neighborhood system of \( x \) and \( D = \{ (U, \lambda) | U \text{ is in } U_x \text{ and } U \cap A_\lambda \neq \emptyset \} \).

Let \( (U, \lambda) \leq (P, \delta) \) if and only if \( P \subseteq U \) and \( \lambda \leq \delta \). Since "\( \leq \)" with respect to elements of \( A \) and "\( " \) are each reflexive and transitive clearly "\( \leq \)" with respect to members of \( D \) is reflexive and transitive. To show that \( D \) is a directed set, all that remains to be shown is that, for every \( (U, \lambda) \) and \( (P, \delta) \) in \( D \), there is an \( (M, \gamma) \) in \( D \) such that \( (U, \lambda) \leq (M, \gamma) \) and \( (P, \delta) \leq (M, \gamma) \).
Suppose \((U,\lambda)\) and \((P,\delta)\) are in \(D\). Then, since \(U \in U_x\) and \(P \in U_x\), \(P \cap U\) is in \(U_x\); and, since \(\lambda \in \Lambda\) and \(\delta \in \Lambda\), there is a \(\gamma\) in \(\Lambda\) such that \(\lambda \leq \gamma\) and \(\delta \leq \gamma\). Since \(x\) is in \(\limsup_{\lambda \in \Lambda} A_{\lambda}\), there is an \(\eta \in \Lambda\) such that \(\gamma \leq \eta\) and \(A_{\eta} \cap (P \cap U) \neq \emptyset\). Then 
\[(U, \lambda) \leq (P \cap U, \eta)\) and \((P, \delta) \leq (P \cap U, \eta)\) and \((P \cap U, \eta) \in D\). Therefore \(D\) is a directed set. Let \(\tau : D \to \Lambda\) be defined as follows: 
\(\tau(U, \lambda) = \lambda\) for every \((U, \lambda)\) in \(D\). Clearly \(\tau\) is well-defined.

If \((U, \lambda)\) and \((P, \delta)\) are in \(D\) and \((U, \lambda) \leq (P, \delta)\), then, by definition, \(\lambda \leq \delta\) which implies that \(\tau(U, \lambda) \leq \tau(P, \delta)\). Suppose \(\lambda \in \Lambda\). Then \(X\) is a neighborhood of \(x\), and certainly \(A_{\lambda} \cap X \neq \emptyset\).

Then \((X, \lambda)\) is in \(D\) and \(\lambda \leq \tau(X, \lambda)\). Therefore \(\{A_{\tau(U, \lambda)}\}(U, \lambda) \in D\) is a subnet of \(\{A_{\lambda}\}_{\lambda \in \Lambda}\). Then there is a subnet \(\{A_{\tau(\psi(m))}\}_{m \in M}\) of \(\{A_{\tau(U, \lambda)}\}(U, \lambda) \in D\) such that \(A = \limsup_{m \in M} A_{\tau(\psi(m))}\).

Let \(U\) be a neighborhood of \(x\) and \(m_0\) be a member of \(M\). Since \(\{\lambda \in \Lambda \mid A_{\lambda} \cap U \neq \emptyset\}\) is cofinal in \(\Lambda\), there is a \(\lambda \in \Lambda\) such that \(A_{\lambda} \cap U \neq \emptyset\). Then \((U, \lambda)\) is in \(D\). Then there is a \(m\) in \(M\) such that \((U, \lambda) \leq \psi(m)\). Then there is a \(m'\) in \(M\) such that \(m_0 \leq m'\) and \(m \leq m'\). Then \((U, \lambda) \leq \psi(m) \leq \psi(m')\). Since \(\psi(m')\) is in \(D\), 
\(\psi(m') = (V, \delta)\) where \(V \in U_x\) and \(\delta \in \Lambda\) and \(V \cap A_{\delta} \neq \emptyset\). Then \((U, \lambda) \leq (V, \delta)\) implies that \(V \cap U\). Then, since \(\delta = \tau(V, \delta) = \tau(\psi(m'))\), 
\(V \cap A_{\tau(\psi(m'))} \neq \emptyset\) and \(m_0 \leq m'\). Then \(\{m \in M \mid A_{\tau(\psi(m))} \cap U \neq \emptyset\}\) is cofinal
in \(M\). Then \(x\) is in \(\limsup_{m \in M} A_{\tau(\psi(m))}\), and \(A = \limsup_{m \in M} A_{\tau(\psi(m))}\).
since $A = \psi\lim_{\lambda \in \Lambda} A_\lambda(\psi(m))$. Then $x \in A$. Therefore,
\[
\limsup_{\lambda \in \Lambda} A_\lambda \subseteq A.
\]

Secondly that $A \subseteq \liminf_{\lambda \in \Lambda} A_\lambda$ will be shown. Suppose $x \in A$ and $U$ is a neighborhood of $x$. Let $N = \{\lambda \in \Lambda | A_\lambda \cap U = \phi\}$. Suppose $N$ is cofinal in $\Lambda$. Then $\{A_\lambda\}_{\lambda \in N}$ is a subnet of $\{A_\lambda\}_{\lambda \in \Lambda}$. Then some subnet $\{A_\tau(m)\}_{m \in M}$ $\psi$-inf $\sup$ converges to $A$. But, for every $m$ in $M$, $\tau(m)$ is in $N$ and $A_\tau(m) \cap U = \phi$.

Then $\{m \in M | A_\tau(m) \cap U \neq \phi\}$ is not cofinal in $U$. Then $x$ is not in
\[
\limsup_{m \in M} A_\tau(m).
\]
Then, since $x \in A$, $\limsup_{m \in M} A_\tau(m) \neq A$. Then $A \neq \psi\lim_{\lambda \in \Lambda} A_\lambda$ - a contradiction. Therefore, $N$ is not
cofinal in $\Lambda$. Then $\{\lambda \in \Lambda | A_\lambda \cap U \neq \phi\}$ is residual in $\Lambda$. Since
this holds for an arbitrary neighborhood $U$ of $x$, $x$ is in
\[
\liminf_{\lambda \in \Lambda} A_\lambda.
\]
Then $A$ is contained in $\liminf_{\lambda \in \Lambda} A_\lambda$.

Therefore
\[
\limsup_{\lambda \in \Lambda} A_\lambda \subseteq A \subseteq \liminf_{\lambda \in \Lambda} A_\lambda.
\]

Then
\[
\limsup_{\lambda \in \Lambda} A_\lambda = A = \liminf_{\lambda \in \Lambda} A_\lambda.
\]

Then $A = \psi\lim_{\lambda \in \Lambda} A_\lambda$. 

Notation: Let $E$ be a normed linear space. Then a net
\[
\{A_i\}_{i \in D}
\]
of subsets of $E$ converges to a subset $A$ of $X$ in the
sense of the "inf sup convergence", defined above, with
respect to the weak topology on $E$ will be denoted:
\[
\{A_i\}_{i \in D} \text{ w-inf sup converges to } A,
\]
or $A = \text{w-lim}_{i \in D} A_i$. If $X$ is a
subset of $E$, then $\text{int } X$ denotes the interior of $X$ with
respect to the norm (or metric) topology on $E$ and $\text{weak int } X$
denotes the interior of $X$ with respect to the weak topology
on $E$.

Lemma 3.2 If \{x_i + \alpha_i B(E)\}_{i \in D} is a net of closed balls
in a normed linear space $E$, $x$ is in $E$, $\alpha$ is a positive number,
$\alpha = \text{lim}_{i \in D} \alpha_i$ and $x = \text{weak lim}_{i \in D} x_i$, then
\[
x + \alpha B(E) = \text{w-lim}_{i \in D} (x_i + \alpha_i B(E)).
\]

Proof: First it will be shown that $x + \alpha B(E)$ is
contained in $\text{lim inf}_{i \in D} (x_i + \alpha_i B(E))$. Since $x = \text{weak lim}_{i \in D} x_i$,
for every weak neighborhood $U$ of $x$, there is an $i_0$ in $D$
such that if $i \geq i_0$, then $x_i \in U$. Then $x \in \text{lim inf}_{i \in D} (x_i + \alpha_i B(E))$.
Let $y$ be in $x + \alpha B(E)$ and $y \neq x$. Suppose $W$ is a weak neighbor-
hood of $y$. Then there is a positive number $\delta$ such that
$y + \alpha [\text{int } B(E)]$ is a subset of $W$ since the weak topology is
contained in the norm topology. Then there is a $z = \lambda x + (1-\lambda)y$
for some positive number $\lambda$ less than one such that
\[ z \in y + \delta[\text{int} B(E)]. \] Then \( z \in W \) and
\[
|x-z| = |x-\lambda x + (1-\lambda)y|
= (1-\lambda)|x-y|
< |x-y| < \alpha
\]

Since \( W+x-z \) is a weak neighborhood and contains \( x \) and since \( x = \text{weak lim } x_i \), there is an \( i_0 \) in \( D \) such that if \( i > i_0 \) then \( x_i \in W+x-z \) which implies that \( x_i - x + \zeta \in W \). Also there is an \( i_1 \) in \( D \) such that if \( i_1 > i \) then
\[
|\alpha - \alpha_i| < \alpha - |x-z|
\]
\[
\alpha - \alpha_i < \alpha - |x-z|
\]
\[
- \alpha_i < |x-z|
\]
\[
\alpha_i > |x-z|
\]

There is an \( i_2 \) in \( D \) such that \( i_0 < i_2 \) and \( i_1 < i_2 \). If \( i > i_2 \), then \( x_i - x + z \) is in \( W \) and
\[
|x_i - (x_i - x + z)| = |x-z|
< \alpha_i
\]

Then, if \( i > i_2 \) then \( x_i - x + \zeta \in W \cap [x_i + \alpha_i B(E)] \). Then
\[
\gamma \text{lim inf}_{i \in D} [x_i + \alpha_i B(E)]. \text{ Therefore } x + \alpha U \text{ is a subset of}
\]
\[
\text{lim inf}_{i \in D} [x_i + \alpha_i B(E)].
\]
Secondly, it needs to be shown that \( \lim \sup_{i \in D} [x_i + \alpha_i B(E)] \) is a subset of \( x + \alpha B(E) \). Suppose \( y \) is not in \( x + \alpha B(E) \). Then there is a positive number \( \delta \) such that \( [y + \delta B(E)] \cap [x + \alpha B(E)] = \emptyset \). Then there is an \( f \) in \( E^* \) and a positive number \( \lambda \) such that

\[
f(y + \delta [\text{int}B(E)]) < \lambda < f(y + \alpha B(E))
\]

since each of \( y + \delta [\text{int}B(E)] \) and \( x + \alpha B(E) \) are convex and have non-empty interiors.

Let \( \epsilon_1, \epsilon_2, \) and \( \gamma \) be positive numbers such that

\[
\epsilon_1 < \lambda - f(y),
\]
\[
\epsilon_2 < \min\{(\lambda - \epsilon_1 - f(y))/2, \epsilon_1\}, \text{ and}
\]
\[
\gamma = (\epsilon_1 \alpha/(f(x) - \lambda)).
\]

Then, since \( \alpha = \lim_{i \in D} x_i \) and \( x = \text{weak lim } x_i \), there is an \( i_0 \) in \( D \) such that if \( i > i_0 \), then \( |\alpha - \alpha_i| < \delta \) and

\[
f(x) - \epsilon_2 < f(x_i) < f(x) + \epsilon_2
\]

Suppose \( i > i_0 \). Then,

\[
|\alpha - \alpha_i| < \delta
\]
\[
\alpha_i - \alpha < \delta
\]
\[
\alpha_i < \alpha + \delta;
\]

and, since \( f(x) > \lambda \) and \( \epsilon_1 > 0 \),

\[
\lambda - f(x) - \epsilon_1 < 0.
\]

Then

\[
\frac{\alpha(\lambda - f(x) - \epsilon_1)/\alpha_i}{\alpha_i} < \frac{\alpha(\lambda - f(x) - \epsilon_1)/\alpha + \delta}{\alpha + \delta}.
\]
Suppose \( z \) is in \( x_1 + \alpha_1 B(E) \) and \( f(z) < f(y) + \varepsilon_2 \). Then

\[
f(x + \alpha(z - x_1)/\alpha_1) \leq f(x) + \alpha((f(y) + \varepsilon_2 - f(x))/\alpha_1)
\]

\[
= f(x) + \alpha((f(y) + 2\varepsilon_2 - f(x))/\alpha_1)
\]

\[
\leq f(x) + \alpha((f(y) + \lambda - \varepsilon_1 - f(y) - f(x))/\alpha_1)
\]

\[
= f(x) + \alpha(\lambda - f(x) - \varepsilon_1)/\alpha_1
\]

\[
< f(x) + \alpha(\lambda - f(x) - \varepsilon_1)/(\alpha + \delta)
\]

\[
= f(x) + (\lambda - f(x) - \varepsilon_1) - \delta(\lambda - f(x) - \varepsilon_1)/(\alpha + \delta)
\]

\[
= \lambda - \varepsilon_1 - (\varepsilon_1 \alpha/(f(x) - \lambda)) (\lambda - f(x) - \varepsilon_1)/(\alpha + \varepsilon_1 \alpha/(f(x) - \lambda))
\]

\[
= \lambda - \varepsilon_1 - \varepsilon_1 \alpha/(f(x) - \lambda) (\lambda - f(x) - \varepsilon_1)/(\alpha f(x) - \alpha \lambda + \varepsilon_1 \alpha)
\]

\[
= \lambda - \varepsilon_1 - \varepsilon_1 \alpha/(\lambda - f(x) - \varepsilon_1)/(\alpha f(x) + \varepsilon_1 - \lambda)
\]

\[
= \lambda - \varepsilon_1 + \varepsilon_1
\]

\[
= \lambda
\]

Then

\[
f(x + \alpha(z - x_1)/\alpha_1) < \lambda.
\]

Then, since

\[
\lambda < f(x + \alpha B(E)),
\]

\( x + \alpha(z - x_1)/\alpha_1 \) is not in \( x + \alpha B(E) \). But

\[
|(x + \alpha(z - x_1)/\alpha_1) - x| = (\alpha/\alpha_1) |z - x_1|
\]

\[
\leq (\alpha/\alpha_1) \alpha_1
\]

\[
= \alpha,
\]
since $z$ is in $x_1 + \alpha_1 B(E)$. Then $z \in x_1 + \alpha_1 B(E)$ implies $f(y) + \varepsilon_2 \leq f(z)$. If $i > 0$ then $f(y) + \varepsilon_2 \leq f(x_1 + \alpha_1 B(E))$. Then $y$ is contained in the weak neighborhood $W = \{ z \in E | f(z) < f(y) + \varepsilon_2 \}$ and $\{ i \in D \mid [x_1 + \alpha_1 B(E)] \cap W \neq \emptyset \}$ is not cofinal in $D$. Then

$y \not\in \limsup_{i \in D} [x_1 + \alpha_1 B(E)]$.

Therefore $y \not\in \limsup_{i \in D} [x_1 + \alpha_1 B(E)]$ implies that $y \in x + \alpha B(E)$.

Then

$$\limsup_{i \in D} [x_1 + \alpha_1 B(E)] \subseteq x + \alpha B(E) \subseteq \liminf_{i \in D} [x_1 + \alpha_1 B(E)].$$

Then

$$x + \alpha B(E) = \text{w-lim}_{i \in D} [x_1 + \alpha_1 B(E)].$$

**Lemma 3.3** If $C$ is a uniformly bounded family of closed balls in a reflexive normed linear space $E$ and $\{ x_1 + \alpha_1 B(E) \}_{i \in D}$ is a net in $C$ such that, for some $x + \alpha B(E)$ in $C$,

$$x + \alpha B(E) = \text{w-lim}_{i \in D} [x_1 + \alpha_1 B(E)],$$

then $x = \text{weak lim}_{i \in D} x_1$ and $\alpha = \text{lim}_{i \in D} \alpha_1$.

**Proof**: Since $C$ is uniformly bounded, there is a positive number $\lambda$ such that every member of $C$ is a subset of $\lambda B(E)$. Then, for every $i$ in $D$, $x_i \in \lambda B(E)$ and $\alpha_1$ is in the interval $[0, \lambda]$. 
Suppose \{x_\mu(m)\}_{m \in M} is an arbitrary subnet of \{x_i\}_{i \in D}.
Then \{(x_\mu(m), \alpha_\mu(m))\}_{m \in M} is a net in \lambda \beta(B(E) \times [0, \lambda]). Since E is reflexive, \lambda \beta(B(E)) is compact in the weak topology on E.
Then \lambda \beta(B(E)) \times [0, \lambda] is compact in the Tychonoff topology given by the topology on \lambda \beta(B(E)) relative to the weak topology on [0, \lambda].
Then, some subnet \{(x_\mu(\psi(t)), \alpha_\mu(\psi(t)))\}_{t \in T} converges in that Tychonoff topology to some \( (x', \alpha') \) in \lambda \beta(B(E)) \times [0, \lambda].
Then \( x' = \text{weak lim}_{t \in T} x_\mu(\psi(t)) \) and \( \alpha' = \lim_{t \in T} \alpha_\mu(\psi(t)) \). Then, by lemma 3.2,
\[ x' + \alpha' \beta(E) = \text{w-lim}_{t \in T} [x_\mu(\psi(t)) + \alpha_\mu(\psi(t))] \beta(E). \]
But \( \{x_\mu(\psi(t)) + \alpha_\mu(\psi(t)) \beta(E)\}_{t \in T} \) is a subnet of \( \{x_i + \alpha_i \beta(E)\}_{i \in D} \).
Then, by lemma 3.1,
\[ x + \alpha \beta(E) = \text{w-lim}_{t \in T} [x_\mu(\psi(t)) + \alpha_\mu(\psi(t))] \beta(E). \]
Then, from the definition of inf sup convergence
\( x + \alpha \beta(E) = x' + \alpha' \beta(E) \). Then clearly \( x = x' \) and \( \alpha = \alpha' \). Then
\( x = \text{weak lim}_{t \in T} x_\mu(\psi(t)) \). Then, since \( \{x_\mu(m)\}_{m \in M} \) was an arbitrary subnet of \( \{x_i\}_{i \in D} \), and \( \{x_\mu(\psi(t))\}_{t \in T} \) is a subnet of
\( \{x_\mu(m)\}_{m \in M} \), \( x = \text{weak lim}_{i \in D} x_i \).

By the same argument, only choosing an arbitrary subnet
\( \{\alpha_\mu(m)\}_{m \in M} \) of \( \{\alpha_i\}_{i \in D} \), first and then looking at the net
\( \{(x_\mu(m), \alpha_\mu(m))\}_{m \in M} \), it follows that \( \alpha = \lim_{i \in D} \alpha_i \).
Lemma 3.4  If E is a reflexive normed linear space and E is a uniformly bounded family of closed balls in E, and \( \{x_i + \alpha_i B(E)\}_{i \in D} \) is a net in C such that, for some \( x + \alpha B(E) \), 
\( x + \alpha B(E) = \omega-lim [x_i + \alpha_i B(E)] \) and for every \( i \) in D there is a net 
\( \{x_i^n + \alpha_i^n B(E)\}_{n \in \mathbb{N}} \) such that 
\( x_i + \alpha_i B(E) = \omega-lim [x_i^n + \alpha_i^n B(E)] \); then some subnet of 
\( \{x_i^n + \alpha_i^n B(E)\}_{i \in D, n \in \mathbb{N}} \) (\( \{x_i^n + \alpha_i^n B(E)\}_{i \in D, n \in \mathbb{N}} \) is ordered lexicographically) \( \omega-inf \ sup \) converges to \( x + \alpha B(E) \).

Proof: By lemma 3.3:

\[
x = \lim_{i \in D} x_i
\]

and \( \alpha = \lim_{i \in D} x_i \)

and, for every \( i \) in D,

\[
x_i = \lim_{n \in \mathbb{N}} x_i^n
\]

and \( \alpha_i = \lim_{n \in \mathbb{N}} \alpha_i^n. \)

Consider \( X = E \times \mathbb{R} \) (where \( \mathbb{R} \) is the real numbers) with the Tychonoff topology given by the weak topology on E and the usual topology on \( \mathbb{R} \). Then \( \{(x_i, \alpha_i)\}_{i \in D} \) converges to \( (x, \alpha) \) and, for every \( i \) in D, \( \{(x_i^n, \alpha_i^n)\}_{n \in \mathbb{N}} \) converges to \( (x_i, \alpha_i) \) in \( E \times \mathbb{R} \). Then some subnet \( \{(x_{\mu(t)}, \alpha_{\mu(t)})\}_{t \in T} \) of the net 
\( \{(x_i^n, \alpha_i^n)\}_{i \in D, n \in \mathbb{N}} \) converges to \( (x, \alpha) \). Then \( \{x_{\mu(t)} + \alpha_{\mu(t)} B(E)\}_{t \in T} \) is a subnet of 
\( \{x_i^n + \alpha_i^n B(E)\}_{i \in D, n \in \mathbb{N}} \) and \( x = \lim_{t \in T} x_{\mu(t)} \) and

\[
x = \lim_{i \in D} x_i
\]
\[ x + \alpha B(E) = \text{w-lim}_{t \in T} \left[ x_\mu(t) + \alpha \mu(t) B(E) \right]. \]

**Theorem 3.5** Let \( E \) be a reflexive normed linear space and \( C \) a uniformly bounded family of closed balls in \( E \). Then if "closure" of a subset \( M \) of \( C \) is defined by
\[ \overline{M} = \{ A \in C | A = \text{w-lim}_{i} A_i \text{ for some net } \{ A_i \}_{i \in D} \in C \}, \]
then a Hausdorff topology \( C_w \) on \( C \) is given by that closure operation and net convergence in \( C_w \) corresponds to w-inf sup convergence.

**Proof:** By lemmas 3.1 through 3.4, w-inf sup convergence on \( C \) satisfies all of the requirements for a definition of "net convergence" on a space \( X \) to determine a topology on \( X \), by the closure operation
\[ \overline{M} = \{ x | x = \text{lim}_{i} x_i \text{ for some net } \{ x_i \}_{i \in D} \in M \}, \]
in which net convergence is just the "net convergence" previously defined - namely that:

1) if \( x_i = x \), for every \( i \) in \( D \), then the net \( x_i \) converges to \( x \);

2) if a net \( \{ x_i \}_{i \in D} \) converges to \( x \), then every subnet converges to \( x \);

3) if every subnet of a net \( \{ x_i \}_{i \in D} \) has a subnet that converges to \( x \), then \( \{ x_i \}_{i \in D} \) converges to \( x \); and
4) the diagonal principle (which was proven in the case of \( C \) and w-inf sup convergence in lemma 3.4).  

**Corollary 3.6** Let \( E \) be a reflexive normed linear space and \( C \) a uniformly bounded family of closed balls in \( E \) with the topology \( C_w \) given in theorem 3.5, and let \( d:C \to \mathbb{R} \) (where \( \mathbb{R} \) is the real numbers) be defined by \( d(x+\alpha B(E))=\alpha \) for every \( x+\alpha B(E) \) in \( C \). Then \( d \) is continuous.

**Proof:** Suppose \( \{x_1+\alpha_1 B(E)\}_{i \in D} \) is a net in \( C \) and \( x+\alpha B(E) \) is in \( C \) and \( x+\alpha B(E)=w-lim_{i \in D}[x_1+\alpha_1 B(E)] \). Then by lemma 3.3, \( \alpha=\lim_{i \in D} \alpha_1 \).

**Corollary 3.7** Let \( E \) be a reflexive normed linear space. If \( C \) is a uniformly bounded family of closed balls in \( E \) and \( \{x_1+\alpha_1 B(E)\}_{i \in D} \) is a net in \( C \) then some subnet of \( \{x_1+\alpha_1 B(E)\}_{i \in D} \) w-inf sup converges to a closed ball in \( E \).

**Proof:** Since \( C \) is uniformly bounded, there is a positive number \( \delta \) such that every member of \( C \) is a subset of \( \delta B(E) \).
Suppose \( \{x_1+\alpha_1 B(E)\}_{i \in D} \) is a net in \( C \). Then, for every \( i \) in \( D \), \( x_1 \in \delta B(E) \) and \( 0 \leq \alpha \leq \delta \). Then \( \{(x_1, \alpha_1)\}_{i \in D} \) is a net in \( \delta B(E) \times [0,\delta] \). Since \( \delta B(E) \times [0,\delta] \) with the Tychonoff topology given by the topology on \( \delta B(E) \) relative to the weak topology on \( E \) and by the usual topology on \( [0,\delta] \) is compact, there is

---

a subnet \( \{ (x_{\mu(m)}, \alpha_{\mu(m)}) \}_{m \in M} \) of \( \{ (x_i, \alpha_i) \}_{i \in D} \) that converges to an \( (x, \alpha) \) in \( \delta B(E) [0, \delta] \). Then \( x = \text{weak lim} \ x_i \) and \( \alpha = \text{lim} \ \alpha_i \).

Then, by lemma 3.2,

\[
x+\alpha B(E) = \text{w-lim} \left[ x_{\mu(m)} + \alpha_{\mu(m)} B(E) \right]
\]

Therefore every net in \( C \) has a subnet that is \( w-\inf \sup \) convergent to some closed ball in \( E \).

The following lemmas will also be needed.

**Lemma 3.8** Let \( E \) be a normed linear space with a uniformly Gateaux differentiable norm. If \( z_0 = \sum_{i=1}^{n} \alpha_i x_i \) such that \( x_i \in E \) and \( 0 < \alpha_i < 1 \) for \( i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} \alpha_i = 1 \) and \( \varepsilon \) is a positive number, then there is a positive number \( \eta \) such that if \( \delta > \eta \) then, for every \( y \) in \( E \) such that \( z_0 \in y + \delta B(E) \), \( [y + \delta B(E)] \cap [x_i + \varepsilon \text{int} B(E)] \neq \emptyset \) for some \( 1 \leq i \leq n \).

**Proof:** If \( n \) is a positive integer and \( z_0 \) is a member of \( E \), then \( P(n, z_0) \) if and only if wherever \( z_0 = \sum_{i=1}^{n} \alpha_i x_i \), for \( x_i \in E \) and \( \alpha_i \) a real number such that \( 0 < \alpha_i < 1 \) for every \( 1 \leq i \leq n \), and \( \varepsilon \) is a positive number, there is a positive number \( \eta \) such that if \( \delta > \eta \) then, for every \( y \) in \( E \) such that \( z_0 \in y + \delta B(E) \),

\[
(y + \delta B(E)) \cap (x_i + \varepsilon \text{int} B(E)) \neq \emptyset
\]

for some \( 1 \leq i \leq n \).

First it will be shown that \( P(2, \theta) \) where \( \theta \) is the zero in \( E \). Suppose \( \theta = \alpha z_1 + (1-\alpha)z_2 \) for a real number \( 0 < \alpha < 1 \) and
members $z_1$ and $z_2$ of $E$. Assume without loss of generality that $\alpha \leq (1-\alpha)$. Note that $z_1 = -(1-\alpha)z_2/\alpha$. Let $Y$ be the set of all $y$ in $E$ such that

$$[y+|y|B(E)] \cap [z_1 + \epsilon \text{int}B(E)] = \emptyset$$

and

$$[y+|y|B(E)] \cap [z_2 + \epsilon \text{int}B(E)] = \emptyset.$$

Suppose $y \in Y$. Then each of $(|y|z_1/|y-z_1|) + (1 - |y|/|y-z_1|)y$ and $(|y|z_2/|y-z_2|) + (1 - |y|/|y-z_1|)y$ are in $y+|y|B(E)$, since

$$
(|y|z_1/|y-z_1|) + (1 - |y|/|y-z_1|)y - y = \frac{|y|}{|y-z_1|}z_1 - y = |y|
$$

and by the same argument

$$
(|y|z_2/|y-z_2|) + (1 - |y|/|y-z_2|)y - y = |y|.
$$

Then

$$(1 - (|y/z_2/|y-z_2|))|y-z_1| = (|y|z_1/|y-z_1|) + (1 - |y|/|y-z_1|)y - y \geq \epsilon,$$

since $[y+|y|B(E)] \cap [z_1 + \epsilon \text{int}B(E)] = \emptyset$, and by the same argument

$$(1 - (|y|/|y-z_2|))|y-z_1| \geq \epsilon.$$

Note that neither $z_1$ nor $z_2$ are in $y+|y|B(E)$. Then $|y-z_1| > |y|$ and $|y-z_2| > |y|$ which implies that $1 - |y|/|y-z_1|$ and $1 - |y|/|y-z_2|$ are both positive.
Then

\[|y-z_1|-|y| \geq \varepsilon;\]

and

\[|y-z_2|-|y| \geq \varepsilon.\]

If \(|y-(1-\alpha)z_2/\alpha| < |y-z_2|\), then \((1-\alpha)z_2/\alpha\) is in \(y+|y-z_2|\text{int}B(E)\) and, since \(|y|<|y-z_1|\), the \(\theta\) of \(E\) is in \(y+|y-z_2|\text{int}B(E)\) which implies that

\[z_2 = (\alpha/(1-\alpha))(1-\alpha)z_2/\alpha + (1-\alpha/(1-\alpha))\theta\]

is in \(y+|y-z_2|\text{int}B(E)\) which is convex. But if \(z_2 \in y+|y-z_2|\text{int}B(E)\) then \(|y-z_2| < |y-z_2|\). Therefore

\[|y+z_1| = |y-(1-\alpha)z_2/\alpha| \geq |y-z_2|.

Then

\[|y+z_1|-|y| > |y-z_2|-|y| \geq \varepsilon\]

Therefore, for every \(y\) in \(Y\),

\[|y-z_1|-|y| \geq \varepsilon\]

and

\[|y+z_1|-|y| \geq \varepsilon.\]

Then, for every \(y\) in \(Y\), \(|(y/|y|)|=1\) and

\[(|(y/|y|)+(1/|y|)z_1|-|(y/|y|)|)/(1/|y|) = |y+z_1|-|y| \geq \varepsilon,\]

and

\[(|(y/|y|)-(1/|y|)z_1|-|(y/|y|)|)/(1/|y|) = |y-z_1|-|y| \geq \varepsilon.\]
Then
\[ 2\epsilon \leq \left| \left( \frac{y}{|y|} \right) + \left( \frac{1}{|y|} \right) z_1 \right| - \left| \left( \frac{y}{|y|} \right) \right| \left( \frac{1}{|y|} \right) - G\left( \frac{y}{|y|} : z_1 \right) + \left| G\left( \frac{y}{|y|} : z_1 \right) - \left( \frac{y}{|y|} \right) \left( \frac{1}{|y|} \right) z_1 - \left( \frac{y}{|y|} \right) \right| \right| - \left( \frac{1}{|y|} \right) - G\left( \frac{y}{|y|} , z_2 \right) \right| .
\]

Since \( E \) is uniformly Gateaux differentiable, there is a positive number \( \xi \) such that if \( \lambda \) is a real number and \( 0 < |\lambda| < \xi \) then, for every \( x \) of norm \( 1 \) in \( E \),
\[ \left| \left( \left| x + \lambda z_1 \right| - |x| \right) / \lambda - G(x : z_1) \right| < \epsilon. \]

If \( y \in Y \) and \( |y| > 1 / \xi \) then \( 1 / |y| < \xi \) and
\[ 2\epsilon \left| \left( \left( \frac{y}{|y|} \right) + (z_2 / |y|) \right) - \left( \frac{y}{|y|} \right) \right| \left( \frac{1}{|y|} \right) - G\left( \frac{y}{|y|} , z_2 \right) + \left| \left( \left( \frac{y}{|y|} \right) - (z_2 / |y|) \right) - \left( \frac{y}{|y|} \right) \right| \left( \frac{1}{|y|} \right) - G\left( \frac{y}{|y|} , z_2 \right) \right| .
\]

Therefore if \( y \in Y \), then \( |y| \leq 1 / \xi \).

Let \( \delta = \max\{|z_1|, |z_2|\} + 1 / \xi \).

Suppose \( \lambda \) is a positive number and \( \lambda > \delta \), and suppose \( y \) is a member of \( E \) such that \( \theta \in y + \lambda B(E) \). First consider the case where \( |y| > 1 / \xi \). Then \( y \) is not in \( Y \) which implies that
\( (y + |y| B(E)) \cap (x + \epsilon \text{int} B(E)) \neq \emptyset \) where \( x \) is one of \( z_1 \) or \( z_2 \).

Since \( \theta \in y + \lambda B(E) \), however, \( |y| \leq \lambda \). Then \( y + |y| B(E) \) is a subset of \( y + \lambda B(E) \). Then \( (y + \lambda B(E)) \cap (x + \epsilon \text{int} B(E)) \neq \emptyset \). If \( |y| \leq 1 / \xi \) then, letting \( x \) be either \( z_1 \) or \( z_2 \),
\[ |y - x| \leq |y| + |x| \leq 1 / \xi + \max\{|z_1|, |z_2|\} \]
\[ = \delta \]
\[ < \lambda , \]
which implies that

\[ x \in (y + \lambda B(E)) \cap (x + \varepsilon [\text{int}B(E)]) \]

Therefore, if \( \theta = \alpha z_1 + (1-\alpha)z_2 \) for some members \( z_1 \) and \( z_2 \) of E and positive number \( 0 < \varepsilon \) and \( \varepsilon \) is a positive number, then there is a positive number \( \delta \) such that if \( \lambda > \delta \) then, for every \( y \) in E such that \( \theta \in y + \lambda B(E) \), \((y + \lambda B(E)) \cap (x + \varepsilon [\text{int}B(E)]) \neq \emptyset \)

where \( x \) is one of \( z_1 \) or \( z_2 \). Then \( P(2, \theta) \) where \( \theta \) is the zero of E.

To extend the fact that \( P(2, \theta) \) to \( P(2, z_0) \) for every \( z_0 \) in E, is a simple matter. Suppose \( z_0 \neq \emptyset \) and \( z_0 = \alpha z_1 + (1-\alpha)z_2 \) for some \( z_1 \) and \( z_2 \) in E and some positive number \( 0 < \varepsilon < 1 \). Then \( \theta = \alpha (z_1 - z_0) + (1-\alpha)(z_2 - z_0) \). Since \( P(2, \theta) \), there is a positive number \( \delta \) such that if \( \lambda > \delta \) then, for every \( y \) in E such that \( \theta \in y + \lambda B(E) \),

\[ (y + \lambda B(E)) \cap (x + \varepsilon [\text{int}B(E)]) \neq \emptyset \]

where \( x \) is one of \( z_1 - z_0 \) or \( z_2 - z_0 \).

Suppose \( \lambda > \delta \).

Suppose \( y \in E \) such that \( z_0 \in y + \lambda B(E) \). Then \( \theta \in y - z_0 + \lambda B(E) \).

Assume without loss of generality that

\[ (y - z_0 + \lambda B(E)) \cap (z_1 - z_0 + \varepsilon [\text{int}B(E)]) \neq \emptyset . \]

Then there exists some \( p \) such that

\[ p \in (y - z_0 + \lambda B(E)) \cap (z_1 - z_0 + \varepsilon [\text{int}B(E)]) . \]
Then
\[ p + z_0 \varepsilon (y + \lambda B(E)) \cap (z_1 + \varepsilon [\text{int} B(E)]) \].

Since \( \lambda \) was chosen arbitrarily greater than \( \delta \), and \( y \) was chosen with the only stipulation that \( z_0 \varepsilon y + \lambda B(E) \), it follows that \( P(2, z_0) \). Therefore \( P(2, z) \) for every \( z \) in \( E \).

Suppose \( m \) is a positive integer, \( m \geq 2 \) and for every positive integer \( j < m \), \( P(j, z) \) for every \( z \) in \( E \). Suppose \( z_0 \in E \), and suppose \( z_0 = \sum_{i=1}^{m} \alpha_i x_i \) where \( x_i \in E \) and \( 0 < \alpha_i < 1 \) for \( i = 1, 2, \ldots, m \) such that \( \sum_{i=1}^{m} \alpha_i = 1 \). Then
\[ z_0 = \alpha x + (1 - \alpha) q \]
where
\[ q = \sum_{i=1}^{m-1} \alpha_i x_i / (1 - \alpha) \].

Note that
\[ \sum_{i=1}^{m-1} \frac{\alpha_i + \alpha = 1}{m} \]
\[ \sum_{i=1}^{m-1} \frac{\alpha_i = 1 - \alpha}{m} \]
\[ \frac{1}{(1 - \alpha)} \sum_{i=1}^{m-1} \alpha_i = 1. \]

Then, some \( m-1 < m \), \( P(m-1, q) \). Then there is a positive number \( \delta_1 \) such that if \( \lambda > \delta_1 \) then for every \( y \) in \( E \) such that \( q \varepsilon y + \lambda B(E) \),
\[ (y + \lambda B(E)) \cap (x_1 + (\varepsilon/2) [\text{int} B(E)]) \neq \emptyset \]
for some \( 1 \leq i \leq m - 1 \).
Since $P(2,z)$ for every $z$ in $E$, there is also a positive number $\delta_2$ such that if $\lambda > \delta_2$ then, for every $y$ in $E$ such that $z_0 \in y + \lambda B(E)$,

$$ (y + \lambda B(E)) \cap (x + (\varepsilon/2) [\text{int}(B(E))] \neq \emptyset $$

where $x$ is one of $x_m$ or $q$.

Let $\delta = \max\{\delta_1, \delta_2\}$.

Suppose $\lambda > \delta$. Let $y$ be a member of $E$ such that $z_0 \in y + \lambda B(E)$.

Since $\lambda > \delta > \delta_2$,

$$ (y + \lambda B(E)) \cap (x + (\varepsilon/2) [\text{int}(B(E))] \neq \emptyset $$

where $x$ is one of $x_m$ or $q$. Suppose $x = q$. Then $q \in y + (\lambda + \varepsilon/2) B(E)$. Then, since $\lambda > \delta > \delta_1$, $\lambda + \varepsilon/2 > \delta_1$ and

$$ (y + (\lambda + \varepsilon/2) B(E)) \cap (x_i + (\varepsilon/2) [\text{int}(B(E))] \neq \emptyset $$

for some $1 \leq i \leq m-1$ - that is there exists a $p$ in $E$ such that $|y - p| \leq \lambda + \varepsilon/2$ and $|p - x_i| < \varepsilon/2$ for some $1 \leq i \leq m-1$.

Then $|y - x_i| < \lambda + \varepsilon$.

Then

$$ \lambda = |(\lambda / |y - x_i|)(x_i - y) + y - y| = |(\lambda / |y - x_i|)(x_i - y)| = |\lambda - |y - x_i|| $$

and

$$ \lambda = |(\lambda / |y - x_i|)(x_i - y) + y - x_i| = |(\lambda / |y - x_i| - 1)(x_i - y)| = |\lambda - |y - x_i|| $$
If $|y-x_1| > \lambda$ then

$$|(\lambda/|y-x_1|)(x_1-y)+y-x_1|=|y-x_1|-\lambda < \varepsilon,$$

which implies that

$$(\lambda/|y-x_1|)(x_1-y)+y\varepsilon(y+\lambda B(E))\cap(x_1+\varepsilon[\text{int}B(E)]).$$

If $|y-x_1| \leq \lambda$ then

$$x_1\varepsilon(y+\lambda B(E))\cap(x_1+\varepsilon[\text{int}B(E)])$$

Therefore, if $x=q$, then

$$(y+\lambda B(E))\cap(x_1+\varepsilon[\text{int}B(E)])\neq\emptyset,$$

and if $x=x_m$ then

$$(y+\lambda B(E))\cap(x_m+\varepsilon[\text{int}B(E)])\neq\emptyset.$$ Therefore, for arbitrary $z_0$ in $E$, if $z_0=\sum_{i=1}^m x_i$ where $x_i \in E$ and $0<\alpha_i<1$, for every $1 \leq i \leq m$, and $\sum_{i=1}^m \alpha_i = 1$ then there is a positive number $\delta$ such that if $\lambda > \delta$ then, for every $y$ in $E$ such that $z_0 \varepsilon y+\lambda B(E)$,

$$(y+\lambda B(E))\cap(x_1+\varepsilon[\text{int}B(E)])\neq\emptyset$$ for some $1 \leq i \leq n$. Then $P(m,z)$ for every $z$ in $E$.

Then, by induction, for every positive integer $n$ and $z$ a member of $E$, $P(n,z)$.

**Corollary 3.8** If $E$ is a normed linear space with a uniformly Gateaux differentiable norm and $Z$ is a subset of
$E, \ z_0 \in \text{conv}(\text{int}Z) \cap Z$ and
$Y = \{y \in E | |y-z_0| \leq p(y,Z)\}$,
then there is a positive number $\xi$ such that $p(y,Z) \leq \xi$
for every $y$ in $Y$.

Proof: If $z_0 \in \text{conv}(\text{int}Z) \cap Z$ then, for some positive
integer $n$, $z_0 = \sum_{i=1}^{n} \alpha_i x_i$ where $x_i \in \text{int}Z$ and $0 < \alpha_i < 1$ for every
$1 \leq i \leq n$ and $\sum_{i=1}^{n} \alpha_i = 1$. Since, for every $1 \leq i \leq n$, $x_i \in \text{int}Z$, there
is a positive number $\varepsilon$ such that $x_i + \varepsilon [\text{int}B(E)]$ is contained
in $Z$ for each $1 \leq i \leq n$. Then there is a positive number $\delta$
such that if $\lambda > \delta$ then for every $y$ in $E$ such that $z_0 \in y+\lambda B(E)$,
there is an $1 \leq i \leq n$ such that
$$(y+\lambda B(E)) \cap (x_i + \varepsilon [\text{int}B(E)]) \neq \emptyset.$$
Suppose $|y-z_0| > \delta$ then
$$(y+|y-z_0| B(E)) \cap (x_i + \varepsilon [\text{int}B(E)])$$
contains some member $q$ of $E$. Then $|y-q| \leq |y-z_0|$ and
$|q-x_i| < \varepsilon$.

Then $|y-x_i| < |y-z_0| + \varepsilon$.

Let $\eta = |y-z_0| + \varepsilon - |y-x_i|$.

Then
$$|((\varepsilon-\eta/2)/|y-x_i|)(y-x_i)+x_i-x_0| = |\varepsilon-\eta/2|$$
and

\[
\left| ((\varepsilon-n/2)/|y_{-1}|)(y_{-1}+x_{1}-y) \right|
= \left| (|x_{1}-y|+n/2-\varepsilon/2) \right|
= \left| (2|x_{1}-y|+|y-z_0|+\varepsilon-|y_{-1}|+2\varepsilon)/2 \right|
= \left| (|x_{1}-y|+|y-z_0|-\varepsilon)/2 \right|.
\]

Suppose either \(|x_{1}-y|<\varepsilon\) or \(\varepsilon<n/2\). If \(|x_{1}-y|<\varepsilon\),
then \(y\in\mathbb{Z}\) implying that \(p(y,z)=0<\delta<|y-z_0|\); but, if \(\varepsilon<n/2<\delta\) then \(|y_{-1}|<|y-z_0|\), since \(n=|y-z_0|+\varepsilon-|y_{-1}|\),
which implies that \(p(y,z)<|y_{-1}|<|y-z_0|\).

Suppose neither \(|x_{1}-y|<\varepsilon\) nor \(\varepsilon<n/2\). Then

\[
\frac{((\varepsilon-n/2)/|y_{-1}|)(y_{-1}+x_{1}-y)}{=\varepsilon-n/2} <\varepsilon
\]

and

\[
\frac{((\varepsilon-n/2)/|y_{-1}|)(y_{-1}+x_{1}-y)}{=(|y_{-1}|-\varepsilon+|y-z_0|)/2} \leq (|y-z_0|+|y-z_0|)/2
\]

\[
= |y-z_0|.
\]

Then \( ((\varepsilon-n/2)/|y_{-1}|)(y_{-1}+x_{1}) \in\mathbb{Z} \)
and

\[
p(y,z)\leq |((\varepsilon-n/2)/|y_{-1}|)(y_{-1}+x_{1}-y)| < |y-z_0|.
\]

Then, in all possible cases, \(p(y,z)<|y-z_0|\).
Therefore, if \(|y-z_0| \leq p(y,Z)\) then \(|y-z_0| \leq \delta\).

If \(x \in Z\) then

\[
p(y,Z) \leq |y-x| \leq |y-z_0+z_0-x| \leq \delta + p(z_0,Z).
\]

Let \(\xi = \delta + p(z_0,Z)\). Then, for every \(y\) in \(Y\), \(p(y,Z) \leq \xi\).

**Lemma 3.9** Suppose \(E\) is a reflexive normed linear space with a uniformly Gateaux differentiable norm, \(Z\) a weakly closed subset of \(E\) having a non-empty weak interior that is norm-wise dense in \(Z\), and \(z_0 \in \text{conv} (\text{int} Z)^* \cap Z\). Let

\[
F = \{x+\alpha B(E) | x \in E, \, \alpha > 0 \text{ and } z_0 \in x + \alpha B(E) \text{ and } [x+\alpha \text{int} B(E)] \cap Z = \emptyset\}.
\]

Then there is a positive number \(\delta\) such that

\[
\delta = \max \{\alpha | x+\alpha B(E) \in F \text{ for some } x \in E\}.
\]

**Proof:** By lemma 3.8 there is a positive number \(\eta\) such that if \(y \in E\) and \(|y-z_0| \leq p(y,Z)\), then \(p(y,Z) \leq \eta\).

If \(x+\alpha B(E) \in F\), since \(z_0 \in x+\alpha B(E)\) and

\[
(x+\alpha \text{int} B(E)) \cap Z = \emptyset, \, |x-z_0| \leq \alpha p(x,Z).
\]

If \(q \in x+\alpha B(E)\) for some \(x+\alpha B(E)\) in \(F\), then \(|x-q| \leq \alpha \leq \eta\).

Then

\[
|q-z_0| \leq |x-q| + |x-z_0| \leq 2\eta.
\]
Then $F$ is a uniformly bounded family of closed balls in $E$. If it can be shown that, a net in $F$ w-inf sup converging to a closed ball $B$ in $E$ implies $B\in F$, then from corollary 3.7, $F$ is compact in its $F_w$ topology. Since, by corollary 3.6, $d:F\to \mathbb{R}$ defined by $d(x+\alpha B(E))=\alpha$, for every $x+\alpha B(E)$ in $F$, is continuous with respect to the $F_w$ topology; $F$ being compact would imply that there is a non-negative number

$$\delta = \max \{d(x+\alpha B(E)) | x+\alpha B(E) \in F\},$$

and $z_0 \notin Z$ would imply that $0<\delta$.

It suffices, therefore, to show that if $B=w-lim A_{i\in D}$ for some net $\{A_i\}_{i\in D}$ in $F$, then $B\in F$. Let $\{x_i+\alpha_i B(E)\}_{i\in D}$ be a net in $F$ and $x+\alpha B(E)$ be a closed ball in $E$ such that $x+\alpha B(E)=w-lim \{x_i+\alpha_i B(E)\}$.

Since, for every $i\in D$, $z_0 \notin x_i+\alpha_i B(E)$; for every weak neighborhood $W$ of $z_0$, $\{i\in D | x_i+\alpha_i B(E) \cap W \neq \emptyset\}=D$ and so is residual in $D$. Then $z_0 \notin \lim \inf_{i\in D} [x_i+\alpha_i B(E)]=x+\alpha B(E)$.

Suppose there is a $q$ in the weak interior of $Z$ that is also in $x+\alpha \text{int}B(E)$. Then $|x-q|<\alpha$. Let $\gamma$ be a positive number such that $|x-q|<\gamma<\alpha$. Since $F$ is uniformly bounded, by lemma 3.3, $x=\lim \inf_{i\in D} x_i$ and $\alpha=\lim_{i\in D} \alpha_i$. Then there is an
Let $D^+ = \{ i \in D \mid i > i_0 \}$. Then $x = \text{weak lim}_{i \in D^+} x_i$ and $\gamma = \text{lim}_{i \in D^+} [\alpha_i - (\alpha - \gamma)]$.

Then, by lemma 3.2,

$$x + \gamma B(E) = \text{w-lim}_{i \in D^+} [x_i + (\alpha_i - (\alpha - \gamma))B(E)].$$

Note that, since $x_i + (\alpha_i - (\alpha - \gamma))B(E)$ is contained in $x_i + \alpha_i \text{int} B(E)$ for each $i$ in $D^+$,

$$[x_i + (\alpha_i - (\alpha - \gamma))B(E) \cap Z = \emptyset$$

for every $i$ in $D^+$. Then $\{ i \in D^+ \mid [x_i + (\alpha_i - (\alpha - \gamma))B(E) \cap Z \neq \emptyset \}$ is not cofinal in $D^+$ and $Z$ is a weak neighborhood of $q$.

Then $q$ is not in $\text{lim sup}_{i \in D^+} [x_i + (\alpha_i - (\alpha - \gamma))B(E)]$ which is $x + \gamma B(E)$ — a contradiction of the assumption that $|x - q| < \gamma < \alpha$.

If $q$ is in the weak interior of $Z$, then $q$ is not in $x + \alpha \text{int} B(E)$. If there is a $y$ in $Z$ that is also in $x + \alpha \text{int} B(E)$, then, since the weak interior of $Z$ is non-empty and norm-wise dense in $Z$, there is a $q$ in the weak interior of $Z$ that is also in $x + \alpha \text{int} B(E)$. Therefore

$$[x + \alpha \text{int} B(E) \cap Z = \emptyset.$$ 

Then $x + \alpha B(E) \in F$. 

Lemma 3.10. Let $E$ be a reflexive normed linear space with a uniformly Gateaux differentiable norm, $x \in S(E)$ and $f \in E^*$ such that $f(x) = |f| = 1$, and $z \in B(E)$ such that $f(z) < 1$. Let $H = \{ y \in E | |f(y)| < 1 \}$. Then there are positive numbers $\alpha$ and $\eta$ such that for every real number $0 < \lambda < \eta$

$$[H + \alpha(z-x)] \cap B(E) + \lambda(z-x) \subseteq B(E)$$

Proof: Suppose for every positive number $\varepsilon$ there is a $y_\varepsilon$ in $E$ such that

1. $f(y_\varepsilon) = 1$,
2. $y_\varepsilon + \varepsilon(z-x) \notin B(E)$, and
3. for some positive number $0 < \lambda < \varepsilon$,

$$y_\varepsilon + \lambda(z-x) \in B(E).$$

Suppose $\varepsilon$ is a positive number. Then exists $\tau_\varepsilon = \sup\{ \lambda \in \mathbb{R} | \lambda < \varepsilon \text{ and } y_\varepsilon + \lambda(z-x) \in B(E) \} < \varepsilon$

Then $m_\varepsilon = y_\varepsilon + \tau_\varepsilon(z-x)$ is in $S(E)$. Then there is a $g_\varepsilon$ in $E^*$ such that $g_\varepsilon(m_\varepsilon) = |g| = 1$. Then

$$|g_\varepsilon| < g_\varepsilon(y_\varepsilon + \varepsilon(z-x)) = g_\varepsilon(y_\varepsilon + \tau_\varepsilon(z-x)) + (\varepsilon - \tau_\varepsilon)g_\varepsilon(z-x)
= g_\varepsilon(m_\varepsilon) + (\varepsilon - \tau_\varepsilon)g_\varepsilon(z-x)$$

Then $g_\varepsilon(z-x) > 0$. Since $f(x) = |f| = 1$ and $f(z) < 1$,

$$|f(x-z) - g_\varepsilon(x-z)| = |f(x-z) + g_\varepsilon(z-x) - f(x-z)| > f(x-z).$$

Then, for every positive number $\varepsilon$, there is an $m_\varepsilon \in S(E)$ such that $m_\varepsilon = y_\varepsilon + \tau_\varepsilon(z-x)$ where $f(y_\varepsilon) = 1$ and $\tau_\varepsilon < \varepsilon$ and,
for $g_\varepsilon$, the unique member of $E^*$ such that $g_\varepsilon(m_\varepsilon) = |g_\varepsilon| = 1$,

$$|g_\varepsilon(x-z) - f(x-z)| \geq f(x-z).$$

Ordering the positive numbers as follows $-\alpha < 0 \beta$ if and only if $\beta < \alpha$, then \{m_\varepsilon\} $\varepsilon \in \mathbb{R}$ (where $\mathbb{R}$ is all real numbers) is a net in $B(E)$. Then some subnet \{m_\psi(i)\} $i \in D$ converges weakly to an $m$ in $B(E)$ since $B(E)$ is weakly compact. Then

$$f(m) = \lim_{i \in D} f(m_\psi(i))$$

$$= \lim_{i \in D} [f(y_\psi(i) + \psi(i)f(z-x))]$$

$$= \lim_{i \in D} f(y_\psi(i))$$

$$= 1$$

Then, since $m \in B(E)$, $|m| = 1$. Let $g$ denote the unique member of $E^*$ such that $g(m) = |g| = 1$. Then, by theorem 2.4,

$$g = \text{weak}^* \lim_{i \in D} g_\psi(i).$$

Since, for each $i \in D$,

$$|g_\psi(i)(x-z) - f(x-z)| \geq f(x-z)$$

and $f(x-z)$ is positive, $g \neq f$. But $g(m) = |g| = 1 = f(m) = |f|$ which contradicts the assumption of smoothness of the norm in the definition of uniform Gateaux differentiability of the norm.

Therefore, there is a positive number $\varepsilon$ such that for every $y$ in $E$ such that $f(y) = 1$ either $y + \varepsilon(z-x) \in B(E)$ or, for
every positive number $\lambda$ less than $\varepsilon$, $y_\varepsilon + \lambda(z-x) \not\in B(E)$.

Suppose $y \in [H+(\varepsilon/2)(z-x)]B(E)$. Then $y = y' + (\varepsilon/2)(z-x) \in B(E)$
for $y' \in E$ such that $f(y') > 1$.

Note that $f(y' + (\varepsilon/2)(z-x)) = f(y) < 1$.

Then

$$0 < (f(y') - 1)/f(z-x) < \varepsilon/2.$$  

Then

$$y = [y' + (f(y') - 1)(z-x)/f(z-x)] + [\varepsilon/2 - (f(y') - 1)/f(z-x)](z-x),$$  

and

$$f(y' + (f(y') - 1)(z-x)/f(z-x)) = 1.$$  

Let $y'' = y' + (f(y') - 1)(z-x)/f(z-x)$. Then $y'' \in (z-x) \in B(E)$.

Then, since $B(E)$ is convex $y + \lambda(z-x) \in B(E)$ for every

$\lambda \in [0, \varepsilon/2]$.

Then

$$[H+(\varepsilon/2)(z-x)]B(E) + \lambda(z-x) \in B(E)$$

for every $\lambda \in [0, \varepsilon/2]$.

The following lemma was suggested by a similar lemma given by Victor Klee to a proof that every weakly closed Chebyshev subset of a normed linear space, having a norm that is both uniformly Gateaux differentiable and uniformly convex, is convex.  

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Lemma 3.11 Let $E$ be a reflexive normed linear space with a uniformly Gateaux differentiable norm. Suppose $Z$ is a weakly closed subset of $E$ having a non-empty weak interior that is norm-wise dense in $Z$ and $z_0 \in \text{conv}(\text{int}Z) \cap Z$. Let $Y = \{y \in E | p(y,Z) \geq |y-z_0|\}$. Then there is a $y$ in $Y$ such that there are two distinct members $z_1$ and $z_2$ of $Z$ such that $|y-z_1| = |y-z_2| = p(y,Z)$.

Proof: First a certain uniformly bounded family of weakly closed subsets of $Z$ having finite intersection property is constructed.

Let $F = \{x+\alpha B(E) | x \in E, \alpha > 0, z_0 \in x+\alpha B(E), \text{ and } (x+\alpha \text{int}B(E)) \cap Z = \phi\}$. Then $y \in Y$ if and only if $y + p(y,Z) B(E) \in F$. By lemma 3.9, there is a positive number

$$\delta = \max \{\alpha | x + \alpha B(E) \in F \text{ for some } x \in E\}.$$

Then $F$ is uniformly bounded since every member of $F$ is a subset of $(|z_0| + \delta) B(E)$, and there is an $x$ in $E$ such that $x + \delta B(E) \in F$.

Since $(x + \delta \text{int}B(E)) \cap Z = \phi$, $\delta < p(x,Z)$. Since $z_0 \in x + \delta B(E)$, $z_0 \in x + p(x,Z) B(E)$ implying that $x + p(x,Z) B(E)$ is in $F$. Then $p(x,Z) < \delta$. Then, for every positive number $\epsilon \leq 1$,

$$(x + (\delta + \epsilon) B(E)) \cap Z \neq \phi.$$ Then \{(x + (\delta + \epsilon) B(E)) \cap Z | 0 < \epsilon < 1\} is a family of weakly closed subsets of $x + (\delta + 1) B(E)$ having the
finite intersection property. Since $E$ is reflexive, $x+(\delta+1)B(E)$ is compact in the weak topology on $E$. Then

$$
\bigcap_{0<\epsilon<1} ([x+(\delta+\epsilon)B(E)]\cap Z) \neq \emptyset.
$$

Then there is a $q$ in $Z$ that is also in $x+\delta B(E)$. Then $q \in x+\delta S(E)$.

Let $H=\{\lambda \in \mathbb{R} \mid \lambda > 0 \text{ and, for every positive number } \gamma \leq \lambda, [x+\gamma (z_0-q)+\delta B(E)] \in F \text{ and } q \in [x+\gamma (z_0-q)-\delta B(E)]\}$, where $\mathbb{R}$ is the real numbers.

Suppose $\lambda \in H$. Then $x+\lambda (z_0-q)+\delta B(E) \in F$. Then $z_0=\lambda (z_0-q)+y$ for some $y$ in $x+\delta B(E)$. Then

$$
\lambda |z_0-q| = |\lambda (z_0-q)+y-y| = |z_0-y| \leq |z_0-x| + |x-y| \leq 2\delta.
$$

implying that $\lambda \leq 2\delta/|z_0-q|$. Then $H$ is bounded above.

Let $\zeta=\sup H$. Suppose $p \in x+\zeta (z_0-q)+\delta [\text{int}B(E)]$. Then $p=\zeta (z_0-q)+y$ where $y \in x+\delta [\text{int}B(E)]$.

If, for every positive number $\lambda$, there is a positive number $\gamma \leq \lambda$ such that

$$
|y+\gamma (z_0-q)-x| \geq \delta,
$$

that is

$$
|y-x|+\gamma |z_0-q| \geq \delta,
$$
then $|y-x| \geq \delta$ - contradicting the assumption that $y \in x+\delta[\text{int} B(E)]$.

Then there is a positive number $\lambda_0$ such that if $\lambda$ is a positive number and $\lambda \leq \lambda_0$ then

$$|y+\lambda(z_0-q)-x| < \delta,$$

that is $[y+\lambda(z_0-q)-x] \in x+\delta[\text{int} B(E)]$.

Let $\lambda = \min\{\zeta/2, \lambda_0\}$. Then $p = (\zeta-\lambda)(z_0-q) + y + \lambda(z_0-q)$.

Since $\zeta = \sup H$, there is a $\gamma$ in $H$ such that $\zeta - \lambda < \gamma < \zeta$.

Then $[x+ (\zeta-\lambda)(z_0-q) + \delta B(E)] \in F$. Then

$$(x+(\zeta-\lambda)(z_0-q)+\delta[\text{int} B(E)]) \cap Z = \phi.$$

Then $p \notin Z$. Since $p$ was arbitrarily in $x+\zeta(z_0-q) + \delta[\text{int} B(E)]$,

$$(x+\zeta(z_0-q)+\delta[\text{int} B(E)]) \cap Z = \phi.$$

If $\lambda$ is a positive number less than $\zeta$ then there is a $\gamma$ in $H$ such that $\lambda \gamma < \zeta$ implying that $[x+\lambda(z_0-q)+\delta B(E)] \in F$ and $q \in [x+\lambda(z_0-q)+\delta B(E)]$.

Suppose $\lambda$ is a positive number and $\lambda < \zeta$. Then $z_0 \in [x+\lambda(z_0-q)+\delta B(E)]$ and

$$|x+\zeta(z_0-q)-z_0| \leq |x+\zeta(z_0-q)-(z+\lambda(z_0-q))| + |x+\lambda(z_0-q)-z_0|$$

$$\leq (|\zeta-\lambda|)|z_0-q|+\delta.$$

Similarly

$$|x+\zeta(z_0-q)-q| \leq (|\zeta-\lambda|)|z_0-q|+\delta.$$
Then, since \( \lambda \) was arbitrarily a positive number less than \( \zeta \),
\[
|x + \zeta(z_0 - q) - z_0| \leq \epsilon + \delta
\]
and
\[
|x + \zeta(z_0 - q) - q| \leq \epsilon + \delta,
\]
for every positive number \( \epsilon < \zeta \). Then each of \( z_0 \) and \( q \) is in \( x + \zeta(z_0 - q) + \delta B(E) \). \( [x + \zeta(z_0 - q) + \delta B(E)] \epsilon P \). Let \( m = x + \zeta(z_0 - q) \).

Since \( q \in Z \), \( |m - q| = \delta \). Then exist unique \( f \in E^* \) such that \( f((q - m)/\delta) = |f| = 1 \). Then \( f(q - m) = \delta \).

Before the right family of weakly closed subsets of \( Z \) can be constructed a weak open neighborhood \( W \) of \( q \) must be given such that for some positive number \( \psi \),
\[
(m + \lambda(z_0 - q) + \delta B(E)) \cap W \subseteq m + \delta B(E)
\]
for every positive number \( \lambda < \psi \). There are two cases to consider - first where \( f(z_0 - m) \neq \delta \) and secondly where \( f(z_0 - m) = \delta \).

**Case 1**  Assume \( f(z_0 - m) \neq \delta \). Then \( f(z_0 - m) < \delta \).

By lemma 3.10, there are positive numbers \( \beta \) and \( \eta \) such that
\[
[H + \beta(z_0 - q)/\delta] \cap B(E) + \lambda(z_0 - q)/\delta \subseteq B(E)
\]
for every \( \lambda \in [0, \eta] \).

Then
\[
[\delta H + m + \beta(z_0 - q)] \cap (m + \delta B(E)) + \lambda(z_0 - q) \subseteq m + \delta B(E).
\]
Let $W=\delta H_{f+m+\beta(z_0-q)}$. Note that
\[
f(q-m-\beta(z_0-q))=f(q-m)+\beta f(q-m)-\beta f(z_0-m)
=\delta+\beta(\delta-f(z_0-m))
>\delta,
\]
Then $q\in\delta H_{f+m+\beta(z_0-q)}$.

Suppose $y\in(m+\lambda(z_0-q)+\delta B(E))\cap W$ for some $\lambda\in[0,\eta]$.

Then $y\in\delta H_{f+m+\beta(z_0-q)}$ and $y=y'+\lambda(z_0-q)$ for some $y'\in m+\delta B(E)$.

Then
\[
f(y-m-\beta(z_0-q))>\delta
f(y'-m+(\beta-\lambda)(z_0-q))>\delta.
\]
Then $y'\in\delta H_{f+m+(\beta-\lambda)(z_0-q)}\subseteq\delta H_{f+m+\beta(z_0-q)}$.

Then $y\in[\delta H_{f+m+\beta(z_0-q)}]\cap[m+\delta B(E)]+\lambda(z_0-q)\subseteq m+\delta B(E)$.

Then $(m+\lambda(z_0-q)+\delta B(E))\cap W\subseteq m+\delta B(E)$ for every $\lambda\in[0,\eta]$.

**Case 2** Assume $f(z_0-m)=\delta$.

Note that
\[
\lambda_0=\sup\{\lambda\in\mathbb{R}|\lambda>0 \text{ and } [(\lambda(q-z_0)+z_0)\in[m+\delta B(E)]\}
\geq 1.
\]

Since $[\text{span}\{q,z_0\}]\cap[m+\delta B(E)]$ is a closed subset of $E$, $y=\lambda_0(q-z_0)+z_0$ is in $m+\delta B(E)$. For every $\lambda>\lambda_0$, let
\[
y_\lambda=\lambda(q-z_0)+z_0
=q+(1-\lambda)(z_0-q)
\]
Note that
\[
f((v-m)/\delta) = f((q-m)/\delta) + (1-\lambda)[f((z_0-m)/\delta) - f((q-m)/\delta)]
\]
\[= 1.
\]
Note also that
\[
(v-m)/\delta = \lim_{(\lambda-\lambda_0) \to 0} (v_{\lambda}-m)/|v_{\lambda}-m|,
\]
for if \(\lambda > \lambda_0\) then
\[
| (v-m)/\delta - (v_{\lambda}-m)/\delta | = (1/\delta) |\lambda_0(q-z_0)+z_0-\lambda(q-z_0) - z_0 |
\]
\[= (\lambda-\lambda_0) |q-z_0|(1/\delta)
\]
and
\[
| (v_{\lambda}-m)/\delta - ((v_{\lambda}-m)/|v_{\lambda}-m|) | = 1/\delta - (1/|v_{\lambda}-m|)(|v_{\lambda}-m|)
\]
\[= |v_{\lambda}-m|/\delta - 1
\]
\[= |v_{\lambda}-m|/|v-m| - 1.
\]

For every \(\lambda > \lambda_0\) let \(g_{\lambda}\) be the unique member of \(E^*\) such that \(g_{\lambda}((v_{\lambda}-m)/|v_{\lambda}-m|) = |g_{\lambda}| = 1\).

By theorem 2.4, \(f = \text{weak}^* \lim_{(\lambda-\lambda_0) \to 0} g_{\lambda}\).

By lemma 3.10, there are two positive numbers \(\alpha\) and \(\eta\) such that if \(\lambda \in [0,\eta]\) then
\[
[H_f - \alpha(q-m)/\delta] \cap B(E) = \lambda(q-m)/\delta \in B(E).
\]

Then there is a positive number \(\lambda > \lambda_0\) such that
\[
\delta \alpha \geq |g_{\lambda}(\alpha(q-m)+z_0-m) - f(\alpha(q-m)+z_0-m)|.
\]
\[
\delta + \delta = f(\alpha(q-m) + z_0 - m)
\]
\[
< g_\lambda(\alpha(q-m) + z_0 - m) + \delta \alpha
\]
\[
\delta < g_\lambda(\alpha(q-m) + z_0 - m).
\]

Let \( H_\lambda = \{ y \in E | 1 < g_\lambda(y) \} \).

Note that \( g_\lambda((v_\lambda - m)/|v_\lambda - m|) = 1 \). Then \( g_\lambda((v_\lambda - m)/\delta) > 1 \).

Then \( Q = \{ \gamma | (z_0 - m)/\delta + \gamma (q - z_0)/\delta \notin H_\lambda \} \) is bounded above by \( \lambda \), since \( (v_\lambda - m)/\delta = \lambda(q - z_0)/\delta + (z_0 - m)/\delta \).

Let \( \xi = \inf Q \). Then \( (z_0 - m)/\delta + (\xi + 1)(q - z_0)/\delta \notin H_\lambda \) and
\[
(q-m)/\delta = (z_0 - m)/\delta + (\xi + 1)(q-m+m-z_0)/\delta + \xi(q_0 - q)/\delta.
\]

Then
\[
(q-m)/\delta \in H_\lambda + \xi(z_0 - q)/\delta.
\]

Suppose \( y \in H_\lambda \).

Then
\[
g_\lambda(y + \xi(z_0 - q)/\delta + \alpha(q-m)/\delta)
\]
\[
= g_\lambda(y + \xi(z_0 - q)/\delta - (z_0 - m)/\delta + \alpha(q-m)/\delta + (z_0 - m)/\delta)
\]
\[
= g_\lambda(y) - g_\lambda((z_0 - m)/\delta + \xi(q - z_0)) + g_\lambda(\alpha(q-m)/\delta + (z_0 - m)/\delta)
\]
\[
> 1.
\]

Then \( y + \xi(z_0 - q)/\delta \notin H_\lambda + \alpha(m-q)/\delta \).

Therefore \( H_\lambda + \xi(z_0 - q)/\delta \notin H_\lambda + \alpha(m-q)/\delta \).
Then, for every γ ∈ [0, n],

\[ [H_\lambda + \xi (z_0 - q)/\delta] \cap [H_f + \alpha (m - q)/\delta] \cap B(E) + \lambda (z_0 - q)/\delta \subseteq [H_f + \alpha (m - q)/\delta] \cap B(E) + \lambda (m - q)/\delta \subseteq B(E). \]

Then

\[ [\delta H_\lambda + m + \xi (z_0 - q)] \cap [\delta H_f + m + \alpha (m - q)] \cap [m + \delta B(E)] + \gamma (z_0 - q)/\delta \subseteq m + \delta B(E) \]

for every γ ∈ [0, n]. Let

\[ W = [\delta H_\lambda + m + \xi (z_0 - q)] \cap [\delta H_f + m + \alpha (m - q)] \]

Then by arguments similar to those in case 1 above,

\[ [m + \gamma (z_0 - q)] \cap [m + \delta B(E)] \subseteq m + \delta B(E) \]

for every γ ∈ [0, n].

Now the certain family of subsets of Z is constructed.

For every positive number λ, let

\[ C_\lambda = \bigcap_{0 < \beta < \lambda} [m + \beta (z_0 - q) + \delta B(E)]. \]

and \( C'_\lambda = C \cap (E \cap W). \)

First it will be shown that \( C'_\lambda \) is weakly closed for every positive number λ. Suppose λ is a positive number and \( y \in E \cap C'_\lambda \). Suppose \( y = y' + \beta (z_0 - q) \) for some \( y' \in m + \delta B(E) \).

If \( \beta \) is positive then there is an \( f \in E^* \) and positive number \( \alpha \) such that

\[ f(y) \geq \sup_{y' \in m + \delta B(E)} f[y' + \beta (z_0 - q)], \]

which implies that \( f(y) \geq f(C_\lambda) \).
Similarly, if \( \beta \) is negative there is a \( f \) in \( E^* \) and positive number \( \alpha \) such that

\[
\sup f[C_\lambda] \leq \sup f(m+\delta B(E)) \\
\leq \alpha \\
< f(y).
\]

Otherwise, suppose the line \( y+R(z_0-q) \), where \( R \) is the real numbers, is a subset of \( E^\circ C_\lambda \). Then there is a closed hyperplane \( H \), where \( H=\{y\in E|f(y)=\alpha\} \) for some \( f\in E^* \) and positive number \( \alpha \), that contains \( y+R(z_0-q) \) and does not intersect \( m+\delta B(E) \). Then, since \( f \) is weakly continuous and \( E \) is reflexive, there is a positive number \( \varepsilon < \alpha \) such that

\[
\max\{f(y)|y\in m+\delta B(E)\}=\alpha-\varepsilon.
\]

Then

\[
\sup f[C_\lambda] \leq \alpha-\varepsilon \\
\leq \alpha-\varepsilon/2 \\
< f(y).
\]

Therefore, in all the possible cases, if \( y\in E^\circ C_\lambda \) then \( y \) can be strictly separated from \( C_\lambda \) by a closed hyperplane. Then \( C_\lambda \) is weakly closed. Then \( C_\lambda = C_\lambda \cap [E^\circ W] \) is weakly closed, for every positive number \( \lambda \).
Suppose $\lambda$ is a positive number and $\lambda \leq \min \{1, \xi\}$, where $\xi$ is the positive number such that
\[
[m+\lambda(z_0-q)+\delta B(E)] \cap \Omega \subseteq m+\delta B(E) \text{ for every } \lambda \in [0, \xi].
\]
Then
\[
m+\lambda(z_0-q)+\delta B(E)=x+(\xi+\lambda)(z_0-q)+\delta B(E)
\]
then either $m+\lambda(z_0-q)+\delta B(E) \notin F$ or $q \notin m+\lambda(z_0-q)+\delta B(E)$.

**Case 1** Assume $m+\lambda(z_0-q)+\delta B(E) \notin F$. Since $\lambda \leq 1$,
\[
z_0 \in m+\lambda(z_0-q)+\delta B(E).
\]
Then there is $p \in (m+\lambda(z_0-q)+\delta \text{int} B(E)) \cap Z$.

If $p \in W$, then $p \in \delta \text{int} B(E)$ - a contradiction of the fact that $m+\delta B(E) \in F$. Then $C \cap Z \neq \phi$.

**Case 2** Assume $q \notin m+\lambda(z_0-q)+\delta B(E)$. Since
\[
m+\lambda(z_0-q)+p(m+\lambda(z_0-q), Z) \delta B(E) \in F, \quad p(m+\lambda(z_0-q), Z) \leq \delta.
\]
Then there is a $p \in (m+\lambda(z_0-q)+\delta B(E)) \cap Z$. If $p \in W$, then $p \neq q$ and
\[
p \in m+\delta B(E).
\]
Therefore in each possible case, there is a $p_\lambda$ in $C \cap Z$ such that either $p_\lambda \in W$ or $p_\lambda \in \delta B(E)$ and $p_\lambda \neq q$.

Suppose, for every positive number $\lambda \leq \min \{1, \eta\}$,
\[
p_\lambda \in C \cap Z. \text{ Then } \{C \cap Z | 0 < \lambda \leq \min \{1, \eta\}\} = D \text{ is a family of closed subsets in } E \text{ having finite intersection property and being such that every member of } D \text{ is a subset of } m+(\delta+2)B(E).
Then, since $E$ is reflexive, there is a $p$ in
\[ \bigcap_{0 \leq \lambda \leq \min\{1, n\}} C!nZ = [m + \delta B(E)] \cap [E \setminus W] \cap Z. \]

Then $p \in m + \delta B(E)$ and $p \neq q$.

In either case there is a member of $Y$, namely $m$, such that there are two distinct members of $Z$, $p$ and $q$ such that $|m - p| = |m - q| = p(m,Z)$.

**Theorem 3.12** If $E$ is a reflexive normed linear space with a uniformly Gateaux differentiable norm and $C$ is a weakly closed Chebyshev subset of $E$ having a non-empty weak interior that is norm-wise dense in $C$, then $C$ is convex.

**Proof:** Suppose $y \in \text{conv} C$ and $y \notin C$. Then $p(y,C) > 0$, and $y = \sum_{i=1}^{n} \alpha_i x_i$ where $0 < \alpha_i < 1$ and $x_i \in C$, for every $1 \leq i \leq n$, and $\sum_{i=1}^{n} \alpha_i = 1$. For each $1 \leq i \leq n$, since $\text{int} C$ is dense in $C$, there is a $z_i \in \text{int} C$ such that $|x_i - z_i| < p(y,C)/2n$. Then
\[ |y - x_{i=1}^{n} \alpha_i z_i| = |\sum_{i=1}^{n} \alpha_i (x_i - z_i)| \]
\[ \leq \sum_{i=1}^{n} \alpha_i p(y,C)/2n \]
\[ = p(y,C)/2. \]

But $\sum_{i=1}^{n} \alpha_i z_i \in \text{conv}(\text{int} C) \subseteq C$. Therefore, if $y \in \text{conv} C$, then $y \in C$. Then, since $C$ is clearly contained in $\text{conv} C$, $C = \text{conv} C$. Then $C$ is convex.
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