INVERSE LIMIT SPACES

THESIS

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By

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Inverse systems, inverse limit spaces, and bonding maps are defined. An investigation of the properties that an inverse limit space inherits, depending on the conditions placed on the factor spaces and bonding maps is made. Conditions necessary to ensure that the inverse limit space is compact, connected, locally connected, and semi-locally connected are examined.

A mapping from one inverse system to another is defined and the nature of the function between the respective inverse limits, induced by this mapping, is investigated. Certain restrictions guarantee that the induced function is continuous, onto, monotone, periodic, or open. It is also shown that any compact metric space is the continuous image of the cantor set.

Finally, any compact Hausdorff space is characterized as the inverse limit of an inverse system of polyhedra.
The purpose of this thesis is to examine the inverse limit spaces of inverse systems and to use inverse systems to characterize some well known topological spaces.

In Chapter I, inverse systems, inverse limit spaces and bonding maps are defined. An investigation of some of the properties that the inverse limit of an inverse system can be forced to inherit from the factor spaces of the system is made.

A mapping from one inverse system to another is defined in Chapter II and the relationship between the inverse limits of these systems is examined. The second chapter will culminate with the use of inverse limit systems to prove that any compact metric space is the continuous image of the cantor set.

In Chapter III, simplices, complexes, and polyhedra are defined. Some preliminary theorems are proven, then it is shown that any compact Hausdorff space is homeomorphic to the inverse limit of an inverse system of polyhedra.

Many definitions and theorems of general topology such as those found in Willard's *General Topology* are used without reference. Most of the theorems in this paper can be found in Capel's "Inverse Limit Spaces"
Gentry's "Some Properties of the Induced Map" and Nagata's Modern General Topology. However, all proofs are original.
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CHAPTER I

INTRODUCTION

The intention of this chapter is to examine the properties inherited by \( \lim(X_{\alpha}, f_{\beta}, D) \) from the factor spaces of \( \Pi X_{\alpha} \). To accomplish this, some preliminary definitions and theorems are needed.

Definition 1.1: Let \( \{X_{\alpha}\}_{\alpha \in D} \) be a collection of topological spaces, indexed by the directed set \( D \). For \( \alpha \leq \beta \) in \( D \), let \( f_{\beta \alpha}: X_{\beta} \to X_{\alpha} \) be a continuous function such that \( f_{\alpha \alpha}: X_{\alpha} \to X_{\alpha} \) is the identity function, and for \( \alpha \leq \gamma \leq \beta \) in \( D \), \( f_{\beta \alpha} = f_{\gamma \alpha} \circ f_{\beta \gamma} \). Then \( (X_{\alpha}, f_{\beta \alpha}, D) \) is an inverse system and \( f_{\beta \alpha} \) is called a bonding map.

Let \( \Pi X_{\alpha} \) be the topological product of the \( X_{\alpha} \)'s, hereafter denoted by \( \Pi X_{\alpha} \) when no confusion may result. Let \( P_{\alpha}: \Pi X_{\alpha} \times X_{\alpha} \) be the \( \alpha \)-th projection function. For \( x \in \Pi X_{\alpha} \), let \( P_{\alpha}(x) = x_{\alpha} \).

Definition 1.2: Let \( X_\infty = \{x \in \Pi X_{\alpha} \mid \text{for } \alpha \leq \beta \text{ in } D, \ f_{\beta \alpha}(x_{\beta}) = x_{\alpha} \} \) and give \( X_\infty \) the relative product topology. Then \( X_\infty \) is called the inverse limit space of the inverse system \( (X_{\alpha}, f_{\beta \alpha}, D) \). For \( \alpha \in D \), let \( \pi_{\alpha} = P_{\alpha} : X_\infty \to X_{\alpha} \). Then \( \pi_{\alpha} \) is continuous. For \( \beta \in D \), let \( S_\beta = \{x \in \Pi X_{\alpha} \mid \text{if } \alpha \leq \beta \text{ in } D, \ f_{\beta \alpha}(x_{\beta}) = x_{\alpha} \} \). Then \( X_\infty = \bigcap_{\beta \in D} S_\beta \).
For notational purposes, let $X_\infty = \lim_{\rightarrow_+}(X_\alpha, f_{\beta_\alpha}, D)$.

Whenever $D$ is the positive integers, hereafter denoted by $\mathbb{Z}_+$, then $(X_\alpha, f_{\beta_\alpha}, \mathbb{Z}_+)$ will be called an inverse sequence.

Unless otherwise specified, the hypotheses of the theorems in Chapter I will assume that $(X_\alpha, f_{\beta_\alpha}, D)$ is an inverse system and $X_\infty = \lim_{\rightarrow_+}(X_\alpha, f_{\beta_\alpha}, D)$.

**Theorem 1.1:** If $\alpha \leq \beta$ in $D$, then $f_{\beta_\alpha} \circ \pi_\beta = \pi_\alpha$.

**Proof:** Let $x \in X_\alpha$. Then $f_{\beta_\alpha} \circ \pi_\beta(x) = f_{\beta_\alpha}(\pi_\beta(x))$.

Since $\pi_\beta(x) = x_{\beta}, f_{\beta_\alpha}(\pi_\beta(x)) = f_{\beta_\alpha}(x_{\beta})$. But $x \in X_\infty$. Therefore, $f_{\beta_\alpha}(x_{\beta}) = x_\alpha$. Since $\pi_\alpha(x) = x_\alpha, f_{\beta_\alpha} \circ \pi_\beta(x) = x_\alpha$.

Hence, $f_{\beta_\alpha} \circ \pi_\beta = \pi_\alpha$.

**Theorem 1.2:** A base for $X_\infty$ is

$$\{\pi^{-1}_\alpha(U_\alpha) | \alpha \in D \text{ and } U_\alpha \text{ is open in } X_\alpha\}.$$  

**Proof:** Let $x \in X_\infty$. Then, for $\alpha \in D, x_\alpha \in X_\alpha$. There exists an open set $U_\alpha$ of $X_\alpha$ such that $x_\alpha \in U_\alpha$. Therefore, $x \in \pi^{-1}_\alpha(U_\alpha)$. Hence, $X_\infty = \bigcup_{\alpha \in D} \pi^{-1}_\alpha(U_\alpha)$. Let

$$x \in \pi^{-1}_\alpha(U_\alpha) \cap \pi^{-1}_\beta(U_\beta)$$

where $U_\alpha$ and $U_\beta$ are open in $X_\alpha$ and $X_\beta$ respectively. Since $\alpha$ and $\beta$ are in $D$, there is $\lambda \in D$ such that $\alpha \leq \lambda$ and $\beta \leq \lambda$.

Let $U_\lambda = f^{-1}_{\beta_\alpha}(U_\alpha) \cap f^{-1}_{\lambda_\beta}(U_\beta)$. Then $U_\lambda$ is open in $X_\lambda$ and $x_\lambda \in U_\lambda$. Hence, $x \in \pi^{-1}_\lambda(U_\lambda)$. If $y \in \pi^{-1}_\lambda(U_\lambda), y_\lambda \in U_\lambda$. But $f_{\lambda_\alpha}(y_\lambda) = y_\alpha$. Therefore, $y_\alpha \in U_\alpha$. Hence, $y \in \pi^{-1}_\alpha(U_\alpha)$.

Similarly, $y \in \pi^{-1}_\beta(U_\beta)$. Consequently,

$$\pi^{-1}_\lambda(U_\lambda) \subseteq \pi^{-1}_\alpha(U_\alpha) \cap \pi^{-1}_\beta(U_\beta).$$
Therefore, \( \{ \pi^{-1}_\alpha(U_\alpha) \mid \alpha \in D \text{ and } U_\alpha \text{ is open in } X_\alpha \} \) is a base for some topology on \( X_\alpha \). Let \( V \) be a basic open set of \( \Pi X_\alpha \). Then, \( V = \bigcap_{\alpha \in F} \pi^{-1}_\alpha(U_\alpha) \) where \( F \) is a finite subset of \( D \) and \( U_\alpha \) is open in \( X_\alpha \). Let \( H \) be a basic open set of \( X_\alpha \). Then, \( H = X_\alpha \cap W \) where \( W \) is a basic open set of \( \Pi X_\alpha \). Then
\[
W = \bigcap_{\alpha \in E} \pi^{-1}_\alpha(U_\alpha).
\]
There exists \( \beta \in D \) such that \( \alpha \leq \beta \) for all \( \alpha \in E \). Let \( U_\beta = \bigcap_{\alpha \in E} f_{\beta \alpha}^{-1}(U_\alpha) \). Then \( U_\beta \) is open in \( X_\beta \). Let \( x \in H \). Then \( x \in X_\alpha \cap W \). Since \( x \in W \), \( x_\alpha \in U_\alpha \) for all \( \alpha \in E \). Since \( x \in X_\alpha \), \( x_\beta \in f_{\beta \alpha}^{-1}(U_\alpha) \) for all \( \alpha \in E \). Therefore, \( x_\beta \in U_\beta \). Hence, \( x \in \pi^{-1}_\beta(U_\beta) \). Therefore, \( H \subseteq \pi^{-1}_\beta(U_\beta) \).

Let \( z \in \pi^{-1}_\beta(U_\beta) \). Then \( z \in X_\alpha \) and \( z_\beta \in U_\beta \). Hence, \( z_\beta \in \bigcap_{\alpha \in E} f_{\beta \alpha}^{-1}(U_\alpha) \). Therefore, \( f_{\beta \alpha}(z_\beta) \in U_\alpha \) for all \( \alpha \in E \).

Since \( z \in X_\alpha \), \( f_{\beta \alpha}(z_\beta) = z_\alpha \). Therefore, \( z_\alpha \in U_\alpha \) for all \( \alpha \in E \). Hence, \( z \in \bigcap_{\alpha \in E} f^{-1}_\alpha(U_\alpha) \). Consequently, \( z \in W \cap X_\alpha \).

Therefore, \( \pi^{-1}_\beta(U_\beta) = H \). Now,
\[
\{ \pi^{-1}_\alpha(U_\alpha) \mid \alpha \in D \text{ and } U_\alpha \text{ is open in } X_\alpha \}
\]
is a base for \( X_\alpha \).

**Theorem 1.3:** The collection \( \{ S_\alpha \}_{\alpha \in D} \) is descending in the sense that if \( \alpha \) and \( \beta \) are in \( D \), then there exists \( \lambda \) in \( D \) such that \( S_\lambda \subseteq S_\alpha \cap S_\beta \).

**Proof:** If \( \alpha \) and \( \beta \) are in \( D \), there exists \( \lambda \in D \) such that \( \alpha \leq \lambda \) and \( \beta \leq \lambda \). Let \( x \in S_\lambda \). If \( \sigma \in D \) and \( \sigma \leq \lambda \), then \( f_{\lambda \sigma}(x_\lambda) = x_\sigma \). Let \( \delta \leq \beta \) in \( D \). Then \( \delta \leq \lambda \). Therefore,
\[ x_\delta = f_{\lambda \delta}(x_\lambda) = f_{\beta \delta} \circ f_{\lambda \beta}(x_\lambda) = f_{\beta \delta}(x_\beta). \]

Hence, \( x \in S_\beta \). Similarly, \( x \in S_\alpha \). Therefore, \( S_\lambda \subseteq S_\alpha \cap S_\beta \).

**Theorem 1.4:** If each \( X_\alpha \) is Hausdorff, then each \( S_\alpha \) is closed in \( \Pi X_\alpha \). Hence, \( X_\infty \) is closed in \( \Pi X_\alpha \).

**Proof:** Let \( x \in \Pi X_\alpha \setminus S_\alpha \). Then there exists \( \lambda \in D \) such that \( \lambda < \alpha \) and \( f_{\alpha \lambda}(x_\lambda) \neq x_\lambda \). Since \( X_\lambda \) is Hausdorff, there exists open disjoint sets \( U \) and \( V \) in \( X_\lambda \) such that \( x_\lambda \in U \) and \( f_{\alpha \lambda}(x_\lambda) \in V \). Now, \( f_{\alpha \lambda}^{-1}(U) \cap f_{\alpha \lambda}^{-1}(V) = \emptyset \) and each is open in \( X_\alpha \). Since \( f_{\alpha \lambda}(x_\lambda) \in V, x_\alpha \in f_{\alpha \lambda}^{-1}(V) \). Hence, \( x \in f_{\alpha \lambda}^{-1}(f_{\alpha \lambda}^{-1}(V)) \). But \( x_\lambda \in U \). Therefore, \( x \in f_{\alpha \lambda}^{-1}(V) \).

Let \( H = f_{\alpha \lambda}^{-1}(U) \cap f_{\alpha \lambda}^{-1}(f_{\alpha \lambda}^{-1}(V)) \). Then \( x \in H \) and \( H \) is open in \( \Pi X_\alpha \). If \( y \in S_\alpha \cap H \), then \( f_{\alpha \lambda}(y_\lambda) \in f_{\alpha \lambda}^{-1}(V) \). Hence, \( y_\lambda = f_{\alpha \lambda}(y_\lambda) \in V \). But \( y \in f_{\alpha \lambda}^{-1}(U) \). Therefore, \( y_\lambda \in U \).

Consequently, \( y_\lambda \in U \cap V \). This is a contradiction to \( U \cap V = \emptyset \). Hence, \( S_\alpha \cap H = \emptyset \). Therefore, \( \Pi X_\alpha \setminus S_\alpha \) is open.

Thus, \( S_\alpha \) is closed. Since \( X_\infty = \bigcap_{\alpha \in D} S_\alpha \), \( X_\infty \) is closed in \( \Pi X_\alpha \).

**Lemma 1.1:** If \( U \) is a collection of compact, nonempty, connected sets in a Hausdorff space, \( X \), such that for each \( A \) and \( B \) in \( U \) there is some \( C \) in \( U \) such that \( C \subseteq A \cap B \), then \( \bigcap U \) is compact, nonempty, and connected.

**Proof:** Since each \( C \in U \) is compact and \( X \) is Hausdorff, each \( C \in U \) is closed. Hence \( \bigcap U \) is closed. Let \( C \in U \).

Then \( \bigcap U \subseteq C \). Hence, \( \bigcap U \) is a closed subset of a compact set. Therefore, \( \bigcap U \) is compact.

Suppose that \( \bigcap U \) is empty. Let \( V \in U \). Then, for \( x \in V \), there exists \( U_x \in U \) such that \( x \notin U_x \). Since \( \{x\} \) and
If $U_x$ are compact and disjoint, there exists open disjoint sets $H_x$ and $K_x$ in $X$ such that $x \in H_x$ and $U_x \subseteq K_x$. Let
$$H = \{H_x \mid x \in V\}.$$ 
Then $H$ covers $V$. Since $V$ is compact, there exists $M$, a finite subset of $H$, covering $V$. Let $M = \{H_1, H_2, \ldots, H_m\}$.
For each $H_i \in M$, consider the corresponding $K_i$ and the
$$U_i \subseteq K_i.$$ 
Then, $\bigcap_{i=1}^{m} U_i \subseteq \bigcap_{i=1}^{m} K_i$. But $(\bigcap_{i=1}^{m} K_i) \cap (\bigcup_{i=1}^{m} H_i) = \emptyset$.

Hence, since $V \subseteq \bigcup_{i=1}^{m} H_i$, $V \cap (\bigcap_{i=1}^{m} K_i) = \emptyset$. Therefore,
$$V \cap (\bigcap_{i=1}^{m} U_i) = \emptyset.$$ 
This violates the hypothesis that the finite intersection of elements of $U$ contains an element of $U$. Hence, $\bigcap U \neq \emptyset$.

Suppose that $\bigcap U$ is not connected. Then there exists $A$ and $B$, each closed in $\bigcap U$, such that $A \cap B = \emptyset$, $A \neq \emptyset \neq B$, and $\bigcap U = A \cup B$. Let $C \in U$. Then $\bigcap U \subseteq C$. Therefore, $A \subseteq C$ and $B \subseteq C$. Since $A$ and $B$ are closed in $\bigcap U$ and $\bigcap U$ is compact, $A$ and $B$ are compact. Now $A \cap B = \emptyset$ and $X$ is Hausdorff. Therefore, there exist open disjoint sets $W$ and $V$, of $C$, such that $A \subseteq W$ and $B \subseteq V$. If $x \in C$ and $x \notin \bigcap U$, then there exists $C_x \in U$ such that $C_x \subseteq C$ and $x \notin C_x$. Hence, $x \in C \setminus C_x$ which is open in $C$. Therefore,
$$M = \{C \setminus C_x \mid x \in C \setminus \bigcap U \} \cup \{(W \cup V)\}$$ 
covers $C$.
Since $C$ is compact, there exists a finite subcollection $N$ of $M$ covering $C$. Now, $(W \cup V)$ must be an element of $N$. 

Let the other elements of \( N \) be denoted by \( C \setminus C_1, C \setminus C_2, \ldots, C \setminus C_m \).

Then there exists \( C' \in U \) such that \( C' \subseteq \bigcap_{i=1}^{m} C_i \). If \( x \in C' \), \( x \notin C \setminus C_i \) for any \( i \). Hence, \( x \notin W \cup V \). Therefore, \( C' \subseteq W \cup V \).

If \( C' \subseteq W \), then \( \bigcap U \subseteq W \). But \( \bigcap U = A \cup B \). Hence, \( A \cup B \subseteq W \).

However, \( B \subseteq V \) and \( W \cap V = \emptyset \). Therefore, \( B = \emptyset \). This is a contradiction to \( B \neq \emptyset \). Hence, \( C' \notin V \). Similarly, \( C' \notin V \).

Now \( C' \cap W \neq \emptyset \neq C' \cap V \). Therefore, since \( (C' \cap W) \cup (C' \cap V) = C', (C' \cap W) \cap (C' \cap V) = \emptyset \) and each of \( C' \cap W \) and \( C' \cap V \) are open in \( C' \), \( C' \) is not connected. This contradicts the hypothesis that \( C' \) is connected. Therefore, \( \bigcap U \) is connected.

The following example illustrates the necessity of compactness.

**Example 1.1:** For each \( n \in \mathbb{Z}_+ \), let \( X_n \) be the box in \( \mathbb{R}^2 \) with corners at \((0,0), (1,0), (0, \frac{1}{n}), (1, \frac{1}{n})\) such that \( \{(x,0)\mid 0 < x < 1 \} \) and \( \{(x, \frac{1}{n})\mid 0 < x < 1 \} \) have been deleted. Direct \( \{X_n\}_{n=1}^{\infty} \) by set inclusion. Then \( \{X_n\}_{n=1}^{\infty} = C \) satisfies the hypothesis of the lemma, except for compactness. However, \( \bigcap C = \{(0,0), (1,0)\} \). Hence, \( \bigcap C \) is not connected.

The preceding theorems now bear fruit by assisting in the proof of Theorem 1.5.

**Theorem 1.5:** If each \( X_\alpha \) is compact and Hausdorff, then \( X_\infty \) is compact. If, in addition, each \( X_\alpha \neq \emptyset \), then \( X_\infty \neq \emptyset \).

In addition, if each \( X_\alpha \) is connected, then \( X_\infty \) is connected.

**Proof:** If \( X_\alpha \) is compact and Hausdorff for each \( \alpha \in D \), then \( \Pi X_\alpha \) is compact and Hausdorff. Hence, by Theorem 1.4,
$X_\infty$ is closed in $\Pi X_\alpha$. Consequently, since a closed subset of a compact set is compact, $X_\infty$ is compact.

Suppose that for each $\alpha \in D$, $X_\alpha \neq \emptyset$. Let $\alpha \in D$. Then there is $x_\alpha \in X_\alpha$. For each $\beta \in D$ such that $\beta \leq \alpha$, $f_{\alpha \beta}(x_\alpha) \in X_\beta$. Let $x \in \Pi X_\alpha$ such that for each $\beta \leq \alpha$ in $D$, $x_\beta = f_{\alpha \beta}(x_\alpha)$ and, for $\gamma \in D$ such that $\gamma \neq \alpha$, $x_\gamma$ is some element of $X_\gamma$. Then for $\beta \leq \alpha$ in $D$, $f_{\alpha \beta}(x_\alpha) = x_\beta$. Therefore $x \in S_\alpha$. Hence, $S_\alpha \neq \emptyset$ for each $\alpha$. Thus, by Lemma 1.1, $X_\infty \neq \emptyset$.

Suppose that $X_\alpha$ is connected for each $\alpha \in D$. Let $\alpha \in D$. Suppose that $S_\alpha$ is not connected. Then $S_\alpha = A \cup B$ where $A$ and $B$ are closed in $S_\alpha$. Further, $A \cap B = \emptyset$ and $A \neq \emptyset \neq B$. Since $A$ and $B$ are closed in $S_\alpha$, they are closed in $\Pi X_\alpha$. If $x \in A$ and $y \in B$ then, $x \in \Pi X_\alpha \setminus B$, which is open. There exists a basic neighborhood $V$, of $x$, such that $V$ is a subset of $\Pi X_\alpha \setminus B$. Let

$$V = \bigcap_{\beta \in F} P_{\beta}^{-1}(V_\beta),$$

where $V_\beta$ is open in $X_\beta$ and $F$ is a finite subset of $D$. If $P_{\delta}(y) \in V_\delta$ for all $\delta \in F$, then $y \in P_{\delta}^{-1}(V_\delta)$ for all $\delta \in F$. Hence, $y \in \bigcap_{\beta \in F} P_{\beta}^{-1}(V_\beta)$. This contradicts the supposition that $y \in B$. Therefore, there is some $\delta \in F$ such that $P_{\delta}(x) \neq P_{\delta}(y)$. Since $X_\gamma$ is Hausdorff, there exist disjoint open sets $H_\gamma$ and $K_\gamma$ of $X_\gamma$ such that $P_{\gamma}(x) \in H_\gamma$ and $P_{\gamma}(y) \in K_\gamma$. 

Let \( \lambda \geq \alpha \) for all \( \alpha \in F \). Then \( f_{\lambda \gamma}^{-1}(H_{\gamma}) \cap f_{\lambda \gamma}^{-1}(K_{\gamma}) = \phi \), and \( P_{\lambda}(x) \neq P_{\lambda}(y) \). Hence \( P_{\lambda}(A) \cap P_{\lambda}(B) = \phi \). Let \( s \in X_\alpha \).

Let \( t \) be an element in \( \Pi X_\alpha \) such that \( P_\beta(t) = f_{\alpha \beta}(s) \) for each \( \beta \leq \alpha \) in \( D \). Then \( t \in S_\alpha \). Hence, \( P_\alpha(S_\alpha) \) is onto \( X_\alpha \). If \( \beta \geq \alpha \) in \( D \), then \( S_\beta \subseteq S_\alpha \). Since \( \lambda \geq \alpha \), \( S_\lambda \subseteq S_\alpha \) for each \( \alpha \in F \).

Therefore, \( S_\lambda \subseteq A \cup B \). Hence,

\[
X_\lambda = P_\lambda(S_\lambda) \subseteq P_\lambda(A \cup B) \subseteq P_\lambda(A) \cup P_\lambda(B).
\]

Since \( P_{\lambda} \) is continuous and \( A \) and \( B \) are compact, \( P_{\lambda}(A) \) and \( P_{\lambda}(B) \) are compact. Hence, \( P_{\lambda}(A) \) and \( P_{\lambda}(B) \) are closed in \( X_\lambda \). Therefore, \( X_\lambda \) is not connected. This is a contradiction to the supposition. Hence, \( S_\alpha \) is connected. Therefore, by Lemma 1.1, \( X_\infty \) is connected.

It is noteworthy to point out that as a result of Theorem 1.5, if each factor space is a continuum, then \( X_\infty \) is a continuum.

Theorems 1.6 through 1.11 will be necessary to establish the background for proving Theorems 1.12 through 1.16.

**Theorem 1.6:** If \( X_\alpha \) is compact and Hausdorff for each \( \alpha \) in \( D \), then

\[
\pi_\alpha(X_\infty) = \cap \{f_{\beta \alpha}(X_\beta) | \alpha \leq \beta \text{ in } D\}.
\]

Hence, if each \( f_{\beta \alpha} \) is onto, then \( \pi_\alpha \) is onto.

**Proof:** Let \( z_\alpha \in \pi_\alpha(X_\infty) \). Then, there is \( z \in X_\infty \) such that \( \pi_\alpha(z) = z_\alpha \). Since \( z \in X_\infty \), if \( \alpha \leq \beta \) in \( D \), \( f_{\beta \alpha}^{-1}(z_\alpha) \subseteq X_\beta \).

Hence, \( z_\alpha \in f_{\beta \alpha}(X_\beta) \). Therefore, \( z_\alpha \in \cap \{f_{\beta \alpha}(X_\beta) | \alpha \leq \beta \text{ in } D\} \).

Let \( x_\alpha \in \cap \{f_{\beta \alpha}(X_\beta) | \alpha \leq \beta \text{ in } D\} \). Suppose that \( x_\alpha \notin \pi_\alpha(X_\infty) \). Then \( P^{-1}_\alpha(x_\alpha) \cap X_\infty = \phi \). Therefore, if \( x \in P^{-1}_\alpha(x_\alpha) \), there
exists $\lambda_x \in D$ such that $x \notin S_{\lambda_x}$. Hence, $x \in \Pi \lambda x \setminus S_{\lambda_x}$. Let

$$K = \{\lambda_x | x \in P^{-1}_\alpha(x)\}.$$ 

Then,

$$\mathcal{U} = \{\Pi \lambda x \setminus S_{\lambda_x} | \lambda_x \in K\}$$

covers $P^{-1}_{\alpha}(x)$. Since $P^{-1}_{\alpha}(x)$ is closed in $\Pi X_{\alpha}$, it is compact. Therefore, there exists a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ covering $P^{-1}_{\alpha}(x)$. Let

$$\mathcal{V} = \{\Pi \lambda x \setminus S_{\lambda} | \lambda \in M\},$$

where $M$ is a finite subset of $D$. There exists $\sigma \in D$ such that $\lambda \leq \sigma$ for each $\lambda \in M$ and $\alpha \leq \sigma$. Since $\alpha \leq \sigma$, $x_{\alpha} = f_{\sigma\alpha}(x_{\sigma})$. There exists $x_{\sigma} \in X_{\sigma}$ such that $x_{\alpha} = f_{\sigma\alpha}(x_{\sigma})$.

Let $y \in \Pi X_{\alpha}$ such that $y_{\lambda}$ is $f_{\sigma\lambda}(x_{\sigma})$ if $\lambda \leq \sigma$ and some point in $X_{\lambda}$ if $\lambda \notin \sigma$. Then $y \in S_{\sigma}$ and $y \in P^{-1}_{\alpha}(x)$. Since $y \in P^{-1}_{\alpha}(x_{\alpha})$, $y \in \Pi X_{\alpha} \setminus S_{\lambda} \in \mathcal{V}$. But $\lambda \leq \sigma$. Hence, $S_{\sigma} \subseteq S_{\lambda}$.

Therefore, since $y \in S_{\sigma}$, $y \in S_{\lambda}$. This is a contradiction to $y \in \Pi X_{\alpha} \setminus S_{\lambda}$. Therefore, $P^{-1}_{\alpha}(x_{\alpha}) \cap X_{\infty} \neq \emptyset$. Thus,

$$x_{\alpha} \in \pi_{\alpha}(X_{\infty}).$$

Hence, $\cap \{f_{\beta\alpha}(X_{\beta}) | \alpha \leq \beta \text{ in } D\} \subseteq \pi_{\alpha}(X_{\infty})$. Therefore,

$$\pi_{\alpha}(X_{\infty}) = \cap \{f_{\beta\alpha}(X_{\beta}) | \alpha \leq \beta \text{ in } D\}.$$

Since each $f_{\beta\alpha}$ is onto, $f_{\beta\alpha}(X_{\beta}) = X_{\alpha}$ for each $\alpha \leq \beta$ in $D$. Therefore, $\pi_{\alpha}(X_{\infty}) = X_{\alpha}$.

To settle an obvious question, it will now be shown that if each $\pi_{\alpha}$ is onto, then $f_{\beta\alpha}(X_{\beta})$ must be onto $X_{\alpha}$ for each $\alpha \leq \beta$. If not, there exists $\beta \in D$ such that $f_{\beta\alpha}(X_{\beta})$ is
not onto $X_\alpha$. Hence, there exists $x \in X_\alpha$ such that
\[ f_{\beta\alpha}(X_\beta) \cap \{x\} = \emptyset. \]
But each $\pi_\alpha$ is onto. Therefore, there exists $y \in X_\omega$ such that $\pi_\alpha(y) = x$. However, by Theorem 1.1,
\[ f_{\beta\alpha} \circ \pi_\beta(y) = \pi_\alpha(y). \]
Hence,
\[ f_{\beta\alpha}(\pi_\beta(y)) = x. \]
This contradicts the supposition that
\[ f_{\beta\alpha}(X_\beta) \cap \{x\} = \emptyset. \]
Therefore, $f_{\beta\alpha}(X_\beta)$ is onto $X_\alpha$.

Theorem 1.6 is important because, as the following example shows, there are nonempty inverse limit spaces such that the $\pi$ function does not map onto each factor space.

**Example 1.2:** For each $n \in \mathbb{Z}_+$, let $X_n$ be the closed unit interval. Define $f_{n+1} : X_{n+1} \to X_n$ by $f_{n+1}(x) = \frac{x}{2}$ and $f_{nn} : X_n \to X_n$ to be the identity map. For $m > n$, let $f_{mn} : X_m \to X_n$ be $f_{n+1} \circ f_{n+2} \circ \ldots \circ f_{m-1}$. Since each $f_{mn}$ is continuous, $(X_n, f_{mn}, \mathbb{Z}_+)$ is an inverse system. For each $n$, $f_{n+1}(0) = 0$. Hence, $v = (0,0,\ldots) \in X_\omega$. Therefore, $X_\omega \neq \emptyset$. If $n \geq 1$ and $x \in [0,1]$, $f_{n+1}^{-1}(x) = 2^{n-1}(x)$. Let $y = (p,2p,\ldots,2^{n-1}(p),\ldots)$ be an element of $X_\omega$ such that $y \neq v$. Then $p \neq 0$. Hence $p > 0$. There exists $m \in \mathbb{Z}_+$ such that $2^{m-1}(p) > 1$. Therefore,
\[ \pi_m(y) = 2^{m-1}(x) > 1. \]
This is impossible since $\pi_m(y) \in [0,1]$. Therefore, $y = v$. 
Hence, $X_\infty$ has only one element, namely $v$. Now, since $v = (0, 0, \ldots)$, $\pi_n(X_\infty) = \{0\}$ for each $n$. Hence, $\pi_n(X_\infty)$ is not onto for any $n$.

**Lemma 1.2:** Let $C$ be a collection of compact subsets of $X$, a Hausdorff space, indexed by the directed set $D$, such that for $\alpha$ and $\beta$ in $D$, there exists $\lambda \in D$ such that $C_\lambda \subseteq C_\alpha \cap C_\beta$. Then, if $U$ is open in $X$ and $\bigcap C \subseteq U$, there is $C \in C$ such that $C \subseteq U$.

**Proof:** Suppose that for each $C \in C$, $C \notin U$. Then for each $\alpha \in D$, there exists $x_\alpha \in C_\alpha$ such that $x_\alpha \notin U$. Hence, for each $\alpha$, $X \setminus U \cap C_\alpha \neq \emptyset$. Since $U$ is open, $X \setminus U$ is closed. Therefore $(X \setminus U) \cap C_\alpha$ is closed. But since $C_\alpha$ is compact $(X \setminus U) \cap C_\alpha$ is compact. For $\alpha \in D$, let $H_\alpha = (X \setminus U) \cap C_\alpha$. Let $H = \{H_\alpha \mid \alpha \in D\}$.

If $H_\alpha$ and $H_\beta$ are elements of $H$, there exists $\delta \in D$ such that $C_\delta \subseteq C_\alpha \cap C_\beta$. Let $x \in H_\delta$. Then, $x \in (X \setminus U) \cap C_\delta$. Therefore, $x \in C_\delta$ and $x \notin U$. But since $x \in C_\delta$, $x \in C_\alpha \cap C_\beta$.

Since $x \notin U$, $x \in (X \setminus U) \cap C_\alpha$ and $x \in (X \setminus U) \cap C_\beta$. Therefore, $x \in H_\alpha \cap H_\beta$. Hence, $H_\delta \subseteq H_\alpha \cap H_\beta$. By Lemma 1.1, $\cap H \neq \emptyset$.

Let $y \in \cap H$. Then $y \notin U$. But if $y \in \cap H$, $y \in C_\alpha$ for each $\alpha$. Therefore, $y \in \cap C$. Now $\cap C \subseteq U$. Hence $y \in U$. This contradicts $y \notin U$. Therefore, there is $C \in C$ such that $C \subseteq U$.

The following theorem is now made easy by Lemma 1.2

**Theorem 1.7:** If $X_\alpha$ is compact and Hausdorff for all $\alpha \in D$ and $K$ is open in $\prod X_\alpha$ such that $X_\infty \subseteq K$, then there
exists $a \in D$ such that

$$X_\infty \subseteq S_a \subseteq K.$$ 

Proof: Since each $X_a$ is compact and Hausdorff, $\Pi X_a$ is compact and Hausdorff. Since each $S_a$ is closed in $\Pi X_a$, each $S_a$ is compact. By Lemma 1.2, the theorem is proven.

For the remaining theorems of Chapter 1 it will be assumed that if $(X_\alpha, f_{\alpha}, D)$ is an inverse system, then each $X_\alpha$ is Hausdorff.

Theorem 1.8: Let $A$ be a compact subset of $X_\infty$,

$$A_\alpha = \pi_\alpha (A), B = \Pi A_\alpha,$$

and $f_{\beta \alpha} = f_{\beta} | A_\alpha$. Then the following are true:

(a) $(A_\alpha, g_{\beta \alpha}, D)$ is an inverse system and each $g_{\beta \alpha}$ is onto.

(b) If $A_\infty = \lim (A_\alpha, g_{\beta \alpha}, D)$, then $A_\infty = B \cap X_\infty = A$.

(c) $X_\infty \setminus A = \bigcup \{ \pi_\alpha^{-1}(X_\infty \setminus A_\alpha) | a \in D \}$.

Proof:

(a) Since $f_{\beta \alpha}$ is continuous, $f_{\beta \alpha} | A_\beta$ is continuous. Therefore, each $g_{\beta \alpha}$ is continuous. Let $x \in A_\alpha$. Then there exists $y \in A$ such that $y_\alpha = x$. Therefore, if $\beta \preceq \alpha$ in $D$, then

$$g_{\alpha \beta} (x) = f_{\alpha \beta} (y_\alpha) = y_\beta.$$ 

Since $y \in A$, $y_\beta \in A_\beta$. Hence $g_{\alpha \beta} : A_\alpha \rightarrow A_\beta$. If $x \in A_\alpha$,

$$g_{\alpha \alpha} (x) = f_{\alpha \alpha} (x) = x.$$ 

Therefore $g_{\alpha \alpha}$ is the identity map on $A_\alpha$. Suppose that $\alpha \leq \lambda \leq \beta$ in $D$. Let $x_\beta \in A_\beta$. Then there exists $x \in A$ such
that $\pi_{\beta}(x) = x_{\beta}$. Since $x \in A$, $g_{\beta\alpha}(x_{\beta}) = f_{\beta\alpha}(x_{\beta})$. But $x \in X_{\omega}$. Therefore,

$$f_{\beta\alpha}(x_{\beta}) = f_{\lambda\alpha} \circ f_{\beta\lambda}(x_{\beta}) = f_{\lambda\alpha}(x_{\lambda}) = x_{\lambda}.$$ 

Since $x_{\beta} \in A_{\beta}$,

$$f_{\beta\lambda}(x_{\beta}) = g_{\beta\lambda}(x_{\beta}) = x_{\lambda}.$$ 

Hence,

$$g_{\beta\alpha}(x_{\beta}) = f_{\lambda\alpha}(g_{\beta\lambda}(x_{\beta})).$$

Since $g_{\beta\lambda}(x_{\beta}) = x_{\lambda}$ and $x_{\lambda} \in A_{\lambda}$,

$$f_{\lambda\alpha}(g_{\beta\lambda}(x_{\beta})) = g_{\lambda\alpha}(g_{\beta\lambda}(x_{\beta})).$$

Therefore,

$$g_{\beta\alpha}(x_{\beta}) = g_{\lambda\alpha} \circ g_{\beta\lambda}(x_{\beta}).$$

Hence, $(A_{\alpha}, g_{\beta\alpha}, D)$ is an inverse system.

Let $\alpha < \beta$ in $D$ and $x_{\alpha} \in A_{\alpha}$. Then there exists $x$ in $A$ such that $x_{\alpha} = \pi_{\alpha}(x)$. Then,

$$\pi_{\beta}(x) = x_{\beta} \in A_{\beta}.$$ 

Since $x \in X_{\omega}$, $f_{\beta\alpha}(x_{\beta}) = x_{\alpha}$. But

$$g_{\beta\alpha}(x_{\beta}) = f_{\beta\alpha}(x_{\beta}) = x_{\alpha}.$$ 

Hence, $f_{\beta\alpha}$ is onto.

(b) Let $\overline{\pi}_{\alpha}$ be the $\alpha$-th projection function of $\Pi A_{\alpha}$ restricted to $A_{\alpha}$. Let $x \in A_{\omega}$. Then for each $\alpha$ and $\beta$ in $D$ such that $\alpha \leq \beta$, $g_{\beta\alpha}(x_{\beta}) = x_{\alpha}$. Hence, $f_{\beta\alpha}(x_{\beta}) = x_{\alpha}$. Therefore $x \in X_{\omega}$. Since $x \in A_{\omega}, x \in \Pi A_{\alpha}$. Hence, $x \in X_{\omega} \cap B$. Therefore, $A_{\omega} \subseteq X_{\omega} \cap B$.

Let $x \in B \cap X_{\omega}$. Suppose that $x \notin A$. Then for each $y \in A$, there exists $\lambda_{y} \in D$ such that $x_{\lambda_{y}} \neq y_{\lambda_{y}}$. Therefore,
there exist disjoint sets $U_x$ and $V_y$, open in $X_y$, such
that $x_x \in U_x$ and $y_y \in V_y$. Now, each of $\pi_{y_y}^{-1}(U_x)$
and $\pi_{y_y}^{-1}(V_y)$ is open in $A_{\infty}$ and their intersection is
empty. Let
$$\mathcal{V} = \{\pi_{y_y}^{-1}(V_y) | y \in A\}.$$ 
Then $\mathcal{V}$ covers $A$. There exists a finite subset $\mathcal{V}$ of $\mathcal{V}$
covering $A$. Let
$$\mathcal{V} = \{\pi_{y_y}^{-1}(V_y) | \lambda \in M\}$$
where $M$ is a finite subset of $D$. Therefore $\sigma \in D$ such
that $x_{\alpha} \in \sigma$ for all $\alpha \in M$. For each $\lambda \in M$,
$$g_{\lambda}^{-1}(U_y) \cap g_{\sigma}^{-1}(V_{\lambda}) = \phi.$$ 
Recall that $U_x$ is the open set of $X_y$ that is disjoint from
$V_{\lambda}$ and contains $x_{\lambda}$. If $z_{\sigma} \in A_{\sigma}$, there is $z \in A$
such that $\pi_{\sigma}(z) = z_{\sigma}$. Since $z \in A$, $z \in \pi_{\sigma}^{-1}(V_{\lambda}) \in \mathcal{V}$. Hence, $z_{\lambda} \in V_{\lambda}$.
Since $z \in X_{\infty}$, $z_{\sigma} \in g_{\sigma}^{-1}(V_{\lambda})$. Therefore \{g_{\lambda}^{-1}(V_{\lambda}) | \lambda \in M\}
covers $A_{\sigma}$. But $x_{\sigma} \notin g_{\sigma}^{-1}(V_{\lambda})$ for any $\lambda \in M$. Therefore,
$x_{\sigma} \notin A_{\sigma}$. But if $x_{\sigma} \notin A_{\sigma}$ then $x \notin \Pi_{\alpha}$. This is a contradi-
tion to $x \in B$. Hence $x \in A$. Therefore $A_{\infty} \subseteq X_{\infty} \cap B \subseteq A$.

Let $x \in A$. Then $x \in X_{\infty}$ and for each $\alpha \in D$, $x_{\alpha} \in A_{\alpha}$.
Therefore, $x \in X_{\infty} \cap B$. Hence $A \subseteq X_{\infty} \cap B$.

Let $x \in B \cap X_{\infty}$. Since $x \in B$, $x_{\alpha} \in A_{\alpha}$ for each $\alpha$.
Since $x \in X_{\infty}$, for $\alpha \leq \beta$, $f_{\beta}(x_{\beta}) = x_{\alpha}$. But $x_{\beta} \in A_{\beta}$. Hence,
$$g_{\beta}(x_{\beta}) = f_{\beta}(x_{\beta}) = x_{\alpha}.$$
Therefore, \( x \in A \). Thus \( A \subseteq X_\omega \cap B \subseteq A_\omega \). Hence,
\[
A_\omega = X_\omega \cap B = A.
\]

(c) Let \( x \in X_\omega \setminus A \). Suppose that for each \( \alpha \in D \),
\( x_\alpha \in A_\alpha \). Then \( x \in \Pi A_\alpha \cap X_\omega \). By part b, \( x \in A \). This contradicts \( x \in X_\omega \setminus A \). Therefore, there exists \( \lambda \in D \) such that \( x_\lambda \notin A_\lambda \). Hence, \( x_\lambda \notin X_\lambda \setminus A_\lambda \). Therefore, \( x \notin \pi_\lambda^{-1}(X_\lambda \setminus A_\lambda) \).

Therefore, \( x \in \bigcup \{ \pi_\alpha^{-1}(X_\alpha \setminus A_\alpha) \mid \alpha \in D \} \). Hence,
\[
X_\omega \setminus A \subseteq \bigcup \{ \pi_\alpha^{-1}(X_\alpha \setminus A_\alpha) \mid \alpha \in D \}.
\]

Let \( x \in \bigcup \{ \pi_\alpha^{-1}(X_\alpha \setminus A_\alpha) \mid \alpha \in D \} \). Then there exists \( \lambda \in D \) such that \( x \notin \pi_\lambda^{-1}(X_\lambda \setminus A_\lambda) \). Hence, \( x_\lambda \notin A_\lambda \). Therefore, \( \pi_\lambda^{-1}(X_\lambda \setminus A_\lambda) \setminus A = \emptyset \). Since \( x \notin \pi_\lambda^{-1}(X_\lambda) \), \( x \notin A \). Hence, \( x \notin X_\omega \setminus A \). Thus,
\[
X_\omega \setminus A = \bigcup \{ \pi_\alpha^{-1}(X_\alpha \setminus A_\alpha) \mid \alpha \in D \}.
\]

Theorem 1.9: Let \( A \) and \( B \) be compact subsets of \( X_\omega \),
\( C = A \cap B \), \( C_\alpha = \pi_\alpha(C) \), and \( h_{\beta \alpha} = f_{\beta \alpha} \mid C_\beta \). Then
\[
C = \lim_{\alpha} (C_\alpha, h_{\beta \alpha}, D) = C_\omega.
\]

Proof: By Theorem 1.8, part a, \( (c_\alpha, h_{\beta \alpha}, D) \) is an inverse system. Let \( \pi_\alpha(A) = A_\alpha \) and \( \pi_\alpha(B) = B_\alpha \) for each \( \alpha \in D \). For \( \alpha \leq \beta \) in \( D \), let \( g_{\beta \alpha} = f_{\beta \alpha} \mid A_\beta \) and \( j_{\beta \alpha} = f_{\beta \alpha} \mid B_\beta \).

Then, by Theorem 1.8, part b, \( A_\omega = \lim_{\alpha} (A_\alpha, g_{\beta \alpha}, D) = A \) and \( B_\omega = \lim_{\alpha} (B_\alpha, j_{\beta \alpha}, D) = B \). Let \( x \in C_\omega \). Then for each \( \alpha \),
\( x_\alpha \in C_\alpha \). Therefore, \( x_\alpha \in A_\alpha \cap B_\alpha \) and for \( \alpha \leq \beta \) in \( D \),
\[
h_{\beta \alpha}(x_\alpha) = f_{\beta \alpha} \mid C_\beta(x_\alpha) = f_{\beta \alpha} \mid A_\beta \cap B_\beta(x_\beta) = x_\beta.
\]

Therefore, \( x \in A_\omega \) and \( x \in B_\omega \). Consequently \( C_\omega \subseteq A_\omega \cap B_\omega \). Let
Then for $\alpha \leq \beta$ in $D$,

$$g_{\beta \alpha}(x_{\beta}) = f_{\beta \alpha}(x_{\beta}) = x_{\alpha}.$$  

But

$$g_{\beta \alpha}(x_{\beta}) = f_{\beta \alpha}|_{A_{\beta}}(x_{\beta}) = x_{\alpha}$$

and

$$f_{\beta \alpha}(x_{\beta}) = f_{\beta \alpha}|_{B_{\beta}}(x_{\beta}) = x_{\alpha}.$$  

Hence,$$
 x_{\alpha} = f_{\beta \alpha}|_{A_{\beta} \cap B_{\beta}}(x_{\beta}) = f_{\beta \alpha}|_{C_{\beta}}(x_{\beta}).$$

Therefore, $x \in C_{\infty}$. Now, $A_{\infty} \cap B_{\infty} = C_{\infty}$. Since $A_{\infty} = A$ and $B_{\infty} = B$, $C_{\infty} = A \cap B = C$. Hence,

$$C = \lim_{\leftarrow}(C_{\alpha}, h_{\alpha \beta}, D) = C_{\infty}.$$  

**Theorem 1.10:** If $E$ is cofinal in $D$, then $(X_{\alpha}, f_{\beta \alpha}, E)$ is an inverse system and $X_{\infty} = \lim_{\leftarrow}(X_{\alpha}, f_{\beta \alpha}, D)$ is homeomorphic to $Y_{\infty} = \lim_{\leftarrow}(X_{\alpha}, f_{\beta \alpha}, E)$.  

**Proof:** For each $\alpha \leq \beta$ in $E$, $f_{\beta \alpha}$ is continuous and $f_{\beta \beta}$ is the identity map on $X_{\beta}$. If $\alpha \leq \delta \leq \beta$ in $E$, then $\alpha \leq \delta \leq \beta$ in $D$. Hence, $f_{\beta \alpha} = f_{\delta \alpha} \circ f_{\beta \delta}$. Therefore, $(X_{\alpha}, f_{\beta \alpha}, E)$ is an inverse system. Let $\pi_{\alpha}$ be the $\alpha$-th projection function of $\Pi X_{\alpha}$ restricted to $X_{\infty}$ and $\overline{\pi}_{\alpha}$ be the $\alpha$-th projection function of $\Pi X_{\alpha}$ restricted to $Y_{\infty}$. Define a relation $g: X_{\infty} \to Y_{\infty}$ by $g(x) = t$ where for all $\lambda \in E$, $\overline{\pi}_{\lambda}(t) = \pi_{\lambda}(x)$. It is apparent that $g: X_{\infty} \to Y_{\infty}$, for if $x \in X_{\infty}$ and $\alpha \leq \beta$ in $E$, then $f_{\beta \alpha}(t_{\beta}) = f_{\beta \alpha}(x_{\beta}) = x_{\alpha} = t_{\alpha}$. Hence, $t \in Y_{\infty}$. If $g(x) = w$ and $g(x) = t$, then for all $\alpha \in E$, $\overline{\pi}_{\alpha}(w) = \pi_{\alpha}(x) = \overline{\pi}_{\alpha}(t)$. Therefore, $w = z$. Hence, $g$ is a function.
Suppose that \( x \neq y \) in \( X_\infty \). Then there exists \( \lambda \in D \) such that \( x_\lambda \neq y_\lambda \). Since \( E \) is cofinal in \( D \), there exists \( \delta \in E \) such that \( \lambda \leq \delta \). Hence \( x_\lambda \neq y_\lambda \). Therefore, 

\[
\overline{\pi}_\delta(g(x)) \neq \overline{\pi}_\delta(g(y)).
\]

Hence, \( g(x) \neq g(y) \). Therefore \( g \) is one to one.

Let \( y \in Y_\infty \). If \( \alpha \in D \), there exists \( \alpha' \in E \) such that \( \alpha \leq \alpha' \). Let \( x \in \Pi X_\alpha \) such that for each \( \alpha \in D \),

\[
\pi_\alpha(x) = f_{\alpha}^\prime(\overline{\pi}_\alpha(y)).
\]

If \( \alpha' \) and \( \alpha'' \) are elements of \( E \) such that \( \alpha \leq \alpha' \) and \( \alpha \leq \alpha'' \), then there exists \( \alpha''' \) in \( E \) such that \( \alpha'' \leq \alpha''' \) and \( \alpha \leq \alpha''' \). Then,

\[
f_{\alpha'''\alpha}(\overline{\pi}_{\alpha'''\alpha}(y)) = f_{\alpha''\alpha}^\prime(\overline{\pi}_{\alpha''\alpha}(y)) = f_{\alpha\alpha'}^\prime(\overline{\pi}_{\alpha\alpha'}(y)).
\]

But

\[
f_{\alpha''\alpha}^\prime \circ f_{\alpha''\alpha}(\overline{\pi}_{\alpha''\alpha}(y)) = f_{\alpha''\alpha}(\overline{\pi}_{\alpha''\alpha}(y))
\]

and

\[
f_{\alpha\alpha'}^\prime \circ f_{\alpha\alpha'}(\overline{\pi}_{\alpha\alpha'}(y)) = f_{\alpha\alpha'}^\prime(\overline{\pi}_{\alpha\alpha'}(y)).
\]

Therefore,

\[
f_{\alpha''\alpha}(\overline{\pi}_{\alpha''\alpha}(y)) = f_{\alpha\alpha'}^\prime(\overline{\pi}_{\alpha\alpha'}(y)).
\]

Let \( \alpha \leq \beta \) in \( D \). Then,

\[
\pi_\beta(x) = f_{\beta\beta}^\prime(\overline{\pi}_{\beta\beta}(y)).
\]

Hence,

\[
f_{\beta\alpha}(\pi_\beta(x)) = f_{\beta\alpha}(f_{\beta\alpha}^\prime(\overline{\pi}_{\beta\alpha}(y))
\]

\[
= f_{\beta\alpha}(f_{\beta\alpha}^\prime(\overline{\pi}_{\beta\alpha}(y))) = f_{\beta\alpha}(f_{\beta\alpha}^\prime(\overline{\pi}_{\beta\alpha}(y)) = \pi_\alpha(x).
\]
Hence, $x \in X_{\infty}$ and

$$g(x) = y.$$ 

Therefore, $g$ is onto.

Let $x \in X_{\infty}$ and $g(x) = y$ in $Y_{\infty}$. Let $U$ be a basic open neighborhood of $y$ in $Y_{\infty}$. Then $U = \overline{\pi_{\alpha}}^{-1}(U_{\alpha})$ where $U_{\alpha}$ is open in $X_{\alpha}$ and $\alpha \in E$. Now $\overline{\pi_{\alpha}}(y) = \pi_{\alpha}(x)$. Therefore, $U_{\alpha}$ is open in $X_{\alpha}$ and $\pi_{\alpha}(x) \in U_{\alpha}$. Hence, $x \in \pi_{\alpha}^{-1}(U_{\alpha})$ which is open in $X_{\infty}$. Let $\pi_{\alpha}^{-1}(U_{\alpha}) = U'$. If $s \in U'$, $(g(s))_{\alpha} \in U_{\alpha}$. Therefore, $g(s) \in \pi_{\alpha}^{-1}(U_{\alpha})$. Hence $g(U') \subseteq U$. Consequently, $g$ is continuous.

Let $y \in Y_{\infty}$. Then $g^{-1}(y) = x \in X_{\infty}$. Let $U = \pi_{\alpha}^{-1}(U_{\alpha})$ be a basic neighborhood of $x$ in $X_{\infty}$. Then, $\pi_{\alpha}(x) \in U_{\alpha}$. But, by the way in which an inverse element was found in proving $g$ onto, $\pi_{\alpha}(x) = f_{\alpha'}(\overline{\pi_{\alpha}}(y))$ where $a \leq a'$ and $a' \in E$. Therefore, $\overline{\pi_{\alpha}}(y) \in f_{\alpha'}\pi_{\alpha}^{-1}(U_{\alpha})$. Let

$$f_{\alpha'}\pi_{\alpha}^{-1}(U_{\alpha}) = \pi_{\alpha}^{-1}(U_{\alpha}).$$

Hence, $y \in \pi_{\alpha}^{-1}(U_{\alpha})$. Since $a' \in E$, $\pi_{\alpha}^{-1}(U_{\alpha})$ is open in $Y_{\infty}$. Let $s \in \pi_{\alpha}^{-1}(U_{\alpha})$. Then $g^{-1}(s) \in X_{\infty}$ and $\overline{\pi_{\alpha}}(s_{\alpha'}) \in U_{\alpha'}$. Hence,

$$f_{\alpha'}\pi_{\alpha}^{-1}(\overline{\pi_{\alpha}}(s_{\alpha'})) = \pi_{\alpha}(g^{-1}(s)) \in U_{\alpha}.$$ 

Therefore, $g^{-1}(s) \in \pi_{\alpha}^{-1}(U_{\alpha})$. Now, $g^{-1}(\pi_{\alpha}^{-1}(U_{\alpha})) \subseteq \pi_{\alpha}^{-1}(U_{\alpha})$. Hence, $g^{-1}$ is continuous. Therefore, $g$ is a homeomorphism.

Hence, $Y_{\infty}$ is homeomorphic to $X_{\infty}$.

**Definition 1.3:** A continuous function $f_{\alpha}$ from $Y$ onto $Z$, is monotone if and only if for each $p \in Z$, $f_{\alpha}^{-1}(p)$ is connected.
Lemma 1.3: A continuous function $f$ from $Y$ onto $Z$, where $Y$ is compact and Hausdorff and $Z$ is Hausdorff, is monotone if and only if for each connected set $C$ in $Z$, $f^{-1}(C)$ is connected in $Y$.

Proof: Suppose that $f$ is monotone. Let $C$ be a set connected in $Z$. Suppose that $f^{-1}(C)$ is not connected in $Y$. Then, $f^{-1}(C) = A \cup B$ where $A \cap B = \emptyset$, $A \neq \emptyset \neq B$, and $A$ and $B$ are closed in $f^{-1}(C)$. Let $p \in C$. Suppose that

$$f^{-1}(p) \cap A \neq \emptyset \neq f^{-1}(p) \cap B.$$ 

Then, 

$$f^{-1}(p) = (f^{-1}(p) \cap A) \cup (f^{-1}(p) \cap B).$$

But, since $A$ and $B$ are closed in $f^{-1}(C)$, $A = H \cap f^{-1}(C)$ and $B = K \cap f^{-1}(C)$ where $H$ and $K$ are closed in $Y$. Therefore, 

$$f^{-1}(p) \cap A = f^{-1}(p) \cap H,$$

$$f^{-1}(p) \cap B = f^{-1}(p) \cap K$$

and each is closed in $f^{-1}(p)$. Hence,

$$f^{-1}(p) = (f^{-1}(p) \cap H) \cup (f^{-1}(p) \cap K).$$

But $(f^{-1}(p) \cap H)$ and $(f^{-1}(p) \cap K)$ are disjoint and nonempty. Hence, $f^{-1}(p)$ is not connected. This is a contradiction to the supposition that $f$ is monotone. Therefore, $f^{-1}(p) \subseteq A$ or $f^{-1}(p) \subseteq B$.

Let

$$M = \{p \in C | f^{-1}(p) \subseteq A\}$$

and

$$N = \{p \in C | f^{-1}(p) \subseteq B\}.$$
Then $M \cap N = \emptyset$, $C = M \cup N$ and $M \neq \emptyset \neq N$. Suppose that $M$ is not closed in $C$. Then, there exists $p \in C \setminus M$ such that, if $U$ is open in $C$ and $p \in U$ then $U \cap M \neq \emptyset$. Since $Z$ is Hausdorff, for each $y \in Z$ such that $y \neq p$, there exist disjoint sets $U_y$ and $V_y$, open in $Z$ such that $p \in U_y$ and $y \in V_y$. Since $f$ is continuous, $f^{-1}(V_y)$ is open for each $y$. Therefore,

$$U = \{f^{-1}(V_y) \mid y \neq p\}$$

covers $Y \setminus f^{-1}(p)$. Since $p \notin M$, $f^{-1}(p) \subseteq B$. Suppose that $y \in f^{-1}(p) \cap H$. Then, $y \in f^{-1}(C) \cap H$. Hence, $y \in A$. But $y \notin f^{-1}(p) \subseteq B$. Therefore, $A \cap B \neq \emptyset$. This is a contradiction to $A \cap B = \emptyset$. Hence $f^{-1}(p) \cap H = \emptyset$. Therefore, $H \subseteq Y \setminus f^{-1}(p)$. Hence, $U$ covers $H$. Since $H$ is closed, it is compact. Therefore, there exists a finite subcollection of $U$ covering $H$. Let

$$\gamma = \{f^{-1}(V_{y_i}) \mid 1 \leq i \leq n\}$$

be this finite cover. Furthermore, let $U = \bigcap_{i=1}^{n} U_{y_i}$.

Then, $p \in U$ and $U$ is open in $Z$. Hence, $U \cap C$ is open in $C$ and contains $p$. Let $W = U \cap C$. If $W \cap M \neq \emptyset$, then there is $q \in W \cap M$. But $q \in M$. Therefore, $f^{-1}(q) \subseteq A$. Since $q \in W$,

$$q \in \bigcap_{i=1}^{n} U_{y_i},$$

it follows that $f^{-1}(q) \subseteq f^{-1}(U_{y_i})$ for each $i$. Hence,

$$f^{-1}(q) \subseteq \bigcap_{i=1}^{n} f^{-1}(U_{y_i}).$$
But,

\[ \bigcap_{i=1}^{n} f^{-1}(U_{i}) \bigcap A = \phi. \]

This contradicts the fact that \( f^{-1}(q) \subseteq A \). Therefore, \( W \cap M = \phi \). Hence, \( M \) is closed in \( C \). Similarly, \( N \) is closed in \( C \). Therefore, \( C \) is not connected. This contradicts the supposition that \( C \) is connected. Hence, \( f^{-1}(C) \) is connected in \( Y \).

Suppose that \( f^{-1}(C) \) is connected in \( Y \) for each \( C \) connected in \( Z \). Since \( \{ p \} \) is connected in \( Z \) for each \( p \), \( f^{-1}(p) \) is connected in \( Y \). Hence, \( f \) is monotone.

**Lemma 1.4:** Let \( X_{\infty} = \lim(X_{\alpha}, f_{\beta}, D) \) where each \( X_{\alpha} \) is compact and Hausdorff. If \( A \cap B = \phi \) and each of \( A \) and \( B \) are closed in \( X_{\infty} \), then there exists \( \delta \in D \) such that

\[ \pi_{\delta}(A) \cap \pi_{\delta}(B) = \phi. \]

**Proof:** Let \( x \in A \). Then, for each \( y \in B \) there exists \( \lambda \in D \) such that \( x_{\lambda} \neq y_{\lambda} \). Since \( X_{\infty} \) is Hausdorff, there exist disjoint sets \( U_{y_{\lambda}} \) and \( V_{y_{\lambda}} \), open in \( X_{\lambda} \), such that \( x_{\lambda} \in U_{y_{\lambda}} \) and \( y_{\lambda} \in V_{y_{\lambda}} \). Let

\[ \gamma = \{ \pi_{y_{\lambda}}^{-1}(V_{y_{\lambda}}) | y \in B \}. \]

Then, \( \gamma \) covers \( B \). Since \( B \) is closed in \( X_{\infty} \), \( B \) is compact. Therefore, there exists a finite subcollection of \( \gamma \) covering \( B \). Let

\[ \varphi = \{ \pi_{y}^{-1}(V_{y}) | y \in F \} \]

where \( F \) is a finite subset of \( D \), be this finite cover.
There exists $\alpha \in D$ such that $\alpha \geq \gamma$ for each $\gamma \in F$. Let

$$H_\alpha = \bigcup\{f_{\alpha \lambda}^{-1}(V_\lambda) \mid \lambda \in F\}$$

and

$$K_\alpha = \bigcup\{f_{\alpha \lambda}^{-1}(U_\lambda) \mid \lambda \in F\}.$$  

Then each of $H_\alpha$ and $K_\alpha$ is open in $X_\alpha$ and $H_\alpha \cap K_\alpha = \emptyset$.

However, $x_\alpha \in K_\alpha$ and $\pi_\alpha(B) \subseteq H_\alpha$. Hence, for each $x \in A$, there is $\beta \in D$ such that there exists $M_{x_\beta}$ and $N_{x_\beta}$ disjoint and open in $X_\beta$, such that $x_\beta \in M_{x_\beta}$ and $\pi_\beta(B) \subseteq N_{x_\beta}$. Let

$$\mathcal{L} = \{\pi_\beta^{-1}(M_{x_\beta}) \mid x \in A\}.$$  

Then $\mathcal{L}$ covers $A$. Therefore, there exists a finite sub-collection $\theta$, of $\mathcal{L}$ covering $A$. Then, there exists $\delta \in D$ such that, if $\sigma \in D$ and $\pi_\sigma^{-1}(M_{x_\sigma}) \in \theta$, then $\delta \geq \sigma$. Let

$$U = \bigcap\{f_{\delta \sigma}^{-1}(N_{x_\sigma}) \mid \pi_\sigma^{-1}(M_{x_\sigma}) \in \theta\}.$$  

Then $U$ is open in $X_\delta$ and $\pi_\delta(B) \subseteq U$. Let

$$V = \bigcup\{f_{\delta \alpha}^{-1}(M_{x_\alpha}) \mid \pi_\alpha^{-1}(M_{x_\alpha}) \in \theta\}.$$  

Then, $V$ is open in $X_\delta$ and $\pi_\delta(A) \subseteq V$. If $s \in U$, then $f_{\delta \sigma}(s) \in N_{x_\sigma}$ for each $N_{x_\sigma}$ such that $\pi_\sigma^{-1}(M_{x_\sigma}) \in \theta$. Therefore, $f_{\delta \sigma}(s) \notin M_{x_\sigma}$ for each $M_{x_\sigma}$ such that $\pi_\sigma^{-1}(M_{x_\sigma}) \in \theta$.

Hence,

$$f_{\delta \sigma}^{-1}(f_{\delta \sigma}(s)) \cap f_{\delta \sigma}^{-1}(M_{x_\sigma}) = \emptyset$$

for each $M_{x_\sigma}$ such that $\pi_\sigma^{-1}(M_{x_\sigma}) \in \theta$. Therefore, $s \notin V$.  

Similarly, if \( s \in V, s \notin U \). Hence, \( U \cap V = \phi \). But, \( \pi_\delta(B) \subseteq U \) and \( \pi_\delta(A) \subseteq V \). Therefore,

\[
\pi_\delta(A) \cap \pi_\delta(B) = \phi.
\]

**Theorem 1.11:** Let \( (X_\alpha, f_{\beta \alpha}, D) \) be an inverse system of compact Hausdorff spaces such that each \( f_{\beta \alpha} \) is monotone. Then, each \( \pi_\alpha \) is monotone.

**Proof:** Suppose that there exists \( \alpha \in D \) such that \( \pi_\alpha \) is not monotone. Then, there exists \( x_\alpha \in X_\alpha \) such that \( \pi_\alpha^{-1}(x_\alpha) = A \cup B \) where each of \( A \) and \( B \) is closed in \( \pi_\alpha^{-1}(x_\alpha) \) and \( A \cap B = \phi \) and \( A \neq \phi \neq B \). Since \( A \) and \( B \) are closed in \( \pi_\alpha^{-1}(x_\alpha) \), they are closed in \( X_\infty \) and \( \prod X_\alpha \). By Lemma 1.4, there exists \( \delta \in D \) such that \( \pi_\delta(A) \cap \pi_\delta(B) = \phi \). Since \( X_\delta \) is Hausdorff, there exists disjoint sets \( H_\delta \) and \( K_\delta \), open in \( X_\delta \), such that \( \pi_\delta(A) \subseteq H_\delta \) and \( \pi_\delta(B) \subseteq K_\delta \). There exists \( \sigma \in D \) such that \( \delta \leq \sigma \) and \( \alpha \leq \sigma \). Let

\[
H = f_{\sigma \delta}^{-1}(H_\delta),
\]

and

\[
K = f_{\sigma \delta}^{-1}(K_\delta).
\]

Then, \( H \cap K = \phi \) and each of \( H \) and \( K \) is open in \( X_\sigma \). If \( s \in A \),

\[
s_\sigma \in f_{\sigma \delta}^{-1}(s_\delta) \subseteq f_{\sigma \delta}^{-1}(H_\delta).
\]

If \( s \in B \),

\[
s_\sigma \in f_{\sigma \delta}^{-1}(s_\delta) \subseteq f_{\sigma \delta}^{-1}(K_\delta).
\]

If \( p_\sigma \in f_{\sigma \alpha}^{-1}(x_\alpha) \) then there is \( p \in X_\infty \) such that \( \pi_\sigma(p) = p_\sigma \). Now,

\[
p_\alpha = f_{\sigma \alpha}(p_\sigma) = f_{\sigma \alpha}(x_\sigma) = x_\alpha.
\]
Therefore, \( p \in \pi^{-1}_a(x_\alpha) \). It then follows that either \( p \in A \) or \( p \in B \). Hence, \( p_\alpha \in H \) or \( p_\alpha \in K \). But, this means that \( f^{-1}_\alpha(x) \) is separated by the nonempty, disjoint, open sets \( H \) and \( K \). This contradicts the supposition that \( f_\alpha \) is monotone. Hence, \( \pi_\alpha \) must be monotone.

**Definition 1.4:** A space \( Y \) is **locally connected** if for each \( y \in Y \) and each open set \( U \) containing \( y \), there exists an open connected set \( V \) such that \( y \in V \cup U \).

**Definition 1.5:** A space \( Y \) is **semi-locally connected** if for each \( y \in Y \) and each open set \( U \) containing \( y \), there exists an open set \( V \) such that \( y \in V \cup U \) and \( Y \setminus V \) has only finitely many components.

For the remaining theorems of Chapter 1, it is assumed that each \( f_{\beta_\alpha} \) is monotone.

The following theorem returns to the examination of properties passed to \( X_\infty \) by the factor spaces.

**Theorem 1.12:** Let \( (X_\alpha, f_{\beta_\alpha}, D) \) be an inverse system of compact Hausdorff spaces. If for all \( \beta \) and \( \alpha \) in \( D \), \( f_{\beta_\alpha} \) is monotone and \( x_\alpha \) is locally connected (semi-locally connected), then \( X_\infty \) is locally connected (semi-locally connected).

**Proof:** Suppose that for all \( \beta \) and \( \alpha \) in \( D \) such that \( \beta \geq \alpha \), \( f_{\beta_\alpha} \) is monotone and \( x_\alpha \) is locally connected. Let \( x \in X_\infty \) and \( U \) be open in \( X_\infty \) such that \( x \in U \). Then, there exists a basic neighborhood of \( x \), \( \pi^{-1}_\lambda(U_\lambda) \), such that \( \pi^{-1}_\lambda(U_\lambda) \subseteq U \). Since \( x_\lambda \) is locally connected and \( x_\lambda \in U_\lambda \), there exists \( V_\lambda \), connected and open in \( X_\lambda \) such that
\( x_\lambda \in V_\lambda \subseteq U_\lambda \). Since \( \pi_\lambda \) is monotone, \( \pi_\lambda^{-1}(V_\lambda) \) is connected.

But, \( x \in \pi_\lambda^{-1}(V_\lambda) \subseteq \pi_\lambda^{-1}(U_\lambda) \subseteq U_\lambda \). Hence, \( X_\infty \) is locally connected.

Suppose that for all \( \beta \) and \( \alpha \) in \( D \) such that \( \beta \geq \alpha \), 

\( f_{\beta \alpha} \) is monotone and \( X_\alpha \) is semi-locally connected. Let \( y \in U \) where \( U \) is open in \( X_\infty \). Then, there exists a basic neighborhood, \( \pi_\alpha^{-1}(U_\alpha) \), of \( y \) such that \( \pi_\alpha^{-1}(U_\alpha) \subseteq U \). Therefore, \( y \in U_\alpha \) and \( U_\alpha \) is open in \( X_\alpha \). Since \( X_\alpha \) is semi-locally connected, there exists \( V_\alpha \) open in \( X_\alpha \) such that \( y \in V_\alpha \subseteq U_\alpha \) and \( X_\alpha \setminus U_\alpha \) has only finitely many components.

Let \( C_1, C_2, \ldots, C_n \) denote these components. Now,

\[ y \in \pi_\alpha^{-1}(V_\alpha) \subseteq U. \]

Suppose that there exists more than finitely many components of \( X_\infty \setminus \pi_\alpha^{-1}(V_\alpha) \). Then, there exists \( C \), a component of \( X_\infty \setminus \pi_\alpha^{-1}(V_\alpha) \), such that \( C \neq \pi_\alpha^{-1}(C_i) \) for any \( i \). Since \( \pi_\alpha^{-1}(C_i) \) is connected for each \( i \), if \( C \subseteq \pi_\alpha^{-1}(C_j) \), for some \( j \), then

\[ C = \pi_\alpha^{-1}(C_j). \]

But \( C \neq \pi_\alpha^{-1}(C_i) \) for any \( i \). It then follows that \( C \not\subseteq \pi_\alpha^{-1}(C_i) \) for any \( i \). However, \( \pi_\alpha(C) \) is connected in \( X_\alpha \) and \( \pi_\alpha(C) \subseteq X_\alpha \setminus V_\alpha \). Hence, \( \pi_\alpha(C) \subseteq C_j \) for some \( j \). But,

\[ C \subseteq \pi_\alpha^{-1}(\pi_\alpha(C)) \subseteq \pi_\alpha^{-1}(C_j). \]

Therefore, \( C \subseteq \pi_\alpha^{-1}(C_j) \). Consequently,

\[ C = \pi_\alpha^{-1}(C_j). \]

This contradicts \( C \neq \pi_\alpha^{-1}(C_i) \). Hence, \( X_\infty \) is semi-locally connected.
The following example illustrates that Theorem 1.12 will not be true if each $f_\alpha$ is not monotone.

**Example 1.3:** For each $n \in \mathbb{Z}^+$ let $X_n$ be the unit circle in the complex plane. Give $X_n$ the relative topology. An element of $X_n$, $e^{i\theta}$, will be denoted, in degree, by $\theta$.

Let $f_{n+1} : X_{n+1} \to X_n$ be defined by

$$f_{n+1}(\theta) = 2\theta$$

and

$$f_{mn}(\theta) = \theta.$$

Then each $f_{n+1}$ is continuous. For $m > n$, let $f_{mn} : X_m \to X_n$ be defined by

$$f_{mn}(\theta) = f_{n+1} \circ f_{n+2} \circ \cdots \circ f_{m-1}(\theta).$$

Then $(X_n, f_{mn}, \mathbb{Z}^+)$ is an inverse sequence. Hence, an inverse system. Let

$$X_\infty = \lim(X_n, f_{mn}, \mathbb{Z}^+).$$

Suppose that $X_\infty$ is locally connected. Let $U$ be an open, connected, nonempty subset of $X_n$ not containing $0^\circ$. Then,$$
U = \{\alpha \in X_n | \beta < \alpha < \delta\}$$
where $0^\circ \leq \beta$ and $\delta \leq 360^\circ$. Let

$$H = \{\gamma \in X_{n+1} | \frac{\beta}{2} < \gamma < \frac{\delta}{2}\}$$
and

$$K = \{\gamma \in X_{n+1} | \frac{\beta}{2} + 180^\circ < \gamma < \frac{\delta}{2} + 180^\circ\}.$$
Then $H$ and $K$ are disjoint regions and $0^\circ \notin H \cup K$. If $\theta \in U$, then $\beta < \theta < \sigma$. But $f_{n+1}^{-1}(\theta) = \{\frac{\theta}{2}, \frac{\theta}{2} + 180^\circ\}$. 
Hence, $\beta < \frac{\theta}{2} < \frac{\delta}{2}$ and $\beta + 180^\circ < \frac{\theta}{2} + 180^\circ < \frac{\delta}{2} + 180^\circ$. Therefore, 

$$f_{n+1}^{-1}(\theta) \subseteq H \cup K.$$ 

Consequently, $f_{n+1}^{-1}(U) \subseteq H \cup K$. Furthermore, $f_{n+1}^{-1}(\theta)$ meets both $H$ and $K$. If $\sigma \in H \cup K$, then $\sigma \in H$ or $\sigma \in K$.

But if $\sigma \in H$, then $\beta < \sigma < \frac{\delta}{2}$. Therefore, $\beta < 2\sigma < \delta$. Hence, 

$$\beta < f_{n+1}^{-1}(\sigma) < \delta.$$ 

Consequently, $f_{n+1}^{-1}(\sigma) \in U$. Also, if $\sigma \in K$, then $\beta + 180^\circ < \sigma < \frac{\delta}{2} + 180^\circ$. It then follows that 

$$2(\frac{\beta}{2} + 180^\circ) < 2\sigma < 2(\frac{\delta}{2} + 180^\circ).$$ 

Hence, $\beta < 2\sigma < \delta$. Therefore, 

$$f_{n+1}^{-1}(\sigma) \in U.$$ 

It has now been shown that 

$$f_{n+1}^{-1}(U) = H \cup K.$$ 

By induction, $f_{n+1}^{-1}(U)$ is the disjoint union of $2^{n-1}$ mutually disjoint regions. Further, if $\lambda \in U$, $f_{n+1}^{-1}(\lambda)$ meets each region. Let $z \in X_\infty$. If $p \in \pi_n^{-1}(z_1)$, then 

$$\pi_n^{-1}(p) \cap \pi_n^{-1}(z_1) \neq \emptyset.$$ 

Therefore, there exists $s \in X_\infty$ such that $s \in \pi_n^{-1}(p) \cap \pi_n^{-1}(z_1)$. Now $s_n = p$ and $s_1 = z_1$.

Hence, $f_{n+1}(p) = z_1$. Therefore, $p \in f_{n+1}^{-1}(z_1)$. It then follows that 

$$\pi_n(\pi_n^{-1}(z_1)) \subseteq f_{n+1}^{-1}(z_1).$$ 

Let $s \in X_\infty$ such that $s_n = p$. Since $s_1 = f_{n+1}(p) = z_1$, 

$s \in \pi_n^{-1}(z_1)$. Consequently, $\pi_n(s) \in \pi_n^{-1}(z_1)$. But $s_n = p$. Hence, $p \in \pi_n^{-1}(\pi_n^{-1}(z_1))$. Therefore, 

$$\pi_n(\pi_n^{-1}(z_1)) = f_{n+1}^{-1}(z_1).$$ 

Now, $U$ can be chosen such that $180^\circ \in U$. Then, 

$$\pi_1^{-1}(180^\circ) \subseteq \pi_1^{-1}(U).$$ 

Since $X_\infty$ is locally connected for each $z \in \pi_1^{-1}(180^\circ)$, there
exists \( V_z \), an open connected subset of \( X_\infty \) such that \( z \in V_z \subseteq \pi_1^{-1}(U) \). Then,

\[
\mathcal{V} = \{ V_z | z \in \pi_1^{-1}(180^\circ) \}
\]
is an open cover of \( \pi_1^{-1}(180^\circ) \). Since \( \pi_1^{-1}(180^\circ) \) is closed in \( X_\infty \), it is compact. Therefore, there is a finite subcollection \( \mathcal{W} \) of \( \mathcal{V} \), covering \( \pi_1^{-1}(180^\circ) \). Let

\[
\mathcal{W} = \{ V_1, V_2, \ldots, V_m \}.
\]

There exists \( n \in \mathbb{Z}_+ \) such that \( 2^{n-1} > m \). Now, \( f_n^{-1}(U) \) is the union of \( 2^{n-1} \) disjoint regions and \( f_n^{-1}(180^\circ) \) meets each region. For each \( i \), \( \pi_n(V_i) \) is connected. Therefore, \( \pi_n(V_i) \) is a subset of one of the regions. Since there are more than \( m \) regions, there exists at least one region \( R \), such that \( \pi_n(V_i) \cap R = \emptyset \) for each \( i \). But

\[
\pi_1^{-1}(180^\circ) \subseteq \bigcup_{i=1}^{m} V_i.
\]

Therefore,

\[
\pi_n(\pi_1^{-1}(180^\circ)) \cap R = \emptyset.
\]

However,

\[
\pi_n(\pi_1^{-1}(180^\circ)) = f_n^{-1}(180^\circ)
\]

and \( f_n^{-1}(180^\circ) \) meets each region. Consequently, it meets \( R \). Therefore, there exists some \( j \) such that \( \pi_n(V_j) \cap R \neq \emptyset \). This contradicts the fact that \( \pi_n(V_i) \cap R = \emptyset \) for each \( i \).

Hence, \( X_\infty \) is not locally connected.

**Theorem 1.13:** Let \( (X_\alpha, f_{\beta \alpha}, D) \) be an inverse system of compact Hausdorff spaces. Let \( C \) be a closed subset of \( X_\infty \) and \( a \) and \( b \) be distinct points of \( X_\infty \setminus C \). For \( \alpha \in D \), let
$C_{\alpha} = \pi_{\alpha}(C)$. If, for $\alpha \in D$, $X_{\alpha} \setminus C_{\alpha} = A_{\alpha} \cup B_{\alpha}$ where $A_{\alpha}$ and $B_{\alpha}$ are disjoint regions containing $a_{\alpha}$ and $b_{\alpha}$ respectively, then there exists disjoint regions $A$ and $B$ in $X_{\infty}$ containing $a$ and $b$ respectively, such that $X_{\infty} \setminus C = A \cup B$.

**Proof:** By Theorem 1.8,

$$X_{\infty} \setminus C = \bigcup \{ \pi_{\alpha}^{-1}(X_{\alpha} \setminus C_{\alpha}) | \alpha \in D \}.$$  

But $X_{\alpha} \setminus C_{\alpha} = A_{\alpha} \cup B_{\alpha}$. Hence,

$$X_{\infty} \setminus C = \bigcup \{ \pi_{\alpha}^{-1}(A_{\alpha} \cup B_{\alpha}) | \alpha \in D \}.$$  

Since

$$\pi_{\alpha}^{-1}(A_{\alpha} \cup B_{\alpha}) = \pi_{\alpha}^{-1}(A_{\alpha}) \cup \pi_{\alpha}^{-1}(B_{\alpha}),$$  

$$X_{\infty} \setminus C = \bigcup \{ \pi_{\alpha}^{-1}(A_{\alpha}) \cup \pi_{\alpha}^{-1}(B_{\alpha}) | \alpha \in D \} = \{ \bigcup_{\alpha \in D} \pi_{\alpha}^{-1}(A_{\alpha}) \} \cup \{ \bigcup_{\alpha \in D} \pi_{\alpha}^{-1}(B_{\alpha}) \}.$$  

Let

$$M = \bigcup_{\alpha \in D} \pi_{\alpha}^{-1}(A_{\alpha})$$  

and

$$N = \bigcup_{\alpha \in D} \pi_{\alpha}^{-1}(B_{\alpha}).$$  

Then, $a \in M$ and $b \in N$.

Let $x \in M \cap N$. Then there exists $\lambda$ and $\delta$ in $D$ such that $x \in \pi_{\lambda}^{-1}(A_{\lambda})$ and $x \in \pi_{\delta}^{-1}(B_{\delta})$. There is $\sigma \in D$ such that $\lambda \leq \sigma$ and $\delta \leq \sigma$. It then follows that $x_{\lambda} \in A_{\lambda}$ and $x_{\delta} \in B_{\delta}$. Suppose that $z_{\lambda} \in A_{\lambda}$ and $f_{\sigma\lambda}^{-1}(z_{\lambda}) \notin A_{\sigma}$. Then, there exists $z_{\sigma} \in X_{\sigma}$ such that $z_{\sigma} \notin A_{\sigma}$ and $f_{\sigma\lambda}(z_{\sigma}) = z_{\lambda}$. Hence, $z_{\sigma} \in B_{\sigma}$ or $z_{\sigma} \in C_{\sigma}$. If $z_{\sigma} \in C_{\sigma}$, there exists $z \in C$ such that $P_{\sigma}(z) = z_{\sigma}$. Since $z \in C$, $z \in X_{\infty}$. Therefore,

$$f_{\sigma\lambda}(z_{\sigma}) = z_{\lambda} = \pi_{\lambda}(z).$$
Hence, $z_\lambda \in C_\lambda$. But $z_\lambda \in C_\lambda$ and $z_\lambda \in A_\lambda$. Consequently, $z_\lambda \in A_\lambda \cap C_\lambda$. However, $A_\lambda \cap C_\lambda = \emptyset$. Therefore, $z_\sigma \notin C_\sigma$. It must follow that $z_\sigma \in B_\sigma$. Since each $f_{\beta \alpha}$ is monotone, $f_{\sigma \lambda}^{-1}(z_\lambda)$ is connected. Thus, $f_{\sigma \lambda}^{-1}(z_\lambda) \subseteq B_\sigma$. But, since $A_\lambda$ is connected and $z_\lambda \in A_\lambda$, $f_{\sigma \lambda}^{-1}(A_\lambda) \subseteq B_\sigma$. However, $a_\sigma \in f_{\sigma \lambda}^{-1}(A_\lambda)$. Hence, $a_\sigma \in B_\sigma$. This contradicts the hypothesis. Therefore, $f_{\sigma \lambda}^{-1}(A_\lambda) \subseteq A_\sigma$. Similarly, $f_{\sigma \lambda}^{-1}(B_\lambda) \subseteq B_\sigma$. Since $x_\lambda \in A_\lambda$ and $x_\delta \in B_\delta$, $x_\sigma \in f_{\sigma \lambda}^{-1}(x_\lambda) \subseteq A_\sigma$ and $x_\sigma \in f_{\sigma \delta}^{-1}(x_\delta) \subseteq B_\sigma$. Hence, $A_\sigma \cap B_\sigma = \emptyset$. This contradicts the hypothesis. Therefore, $M \cap N = \emptyset$.

Suppose that $M = H \cup K$ where $H \cap K = \emptyset$, $H \neq \emptyset \neq K$ and each of $H$ and $K$ are open in $M$. Then, $a \in H$ or $a \in K$. Suppose that $a \in H$. Suppose also that for some $\alpha \in D$, $y \in \pi_\alpha^{-1}(A_\alpha)$. Now, $\pi_\alpha^{-1}(A_\alpha) \subseteq H \cup K$ and $\pi_\alpha^{-1}(A_\alpha)$ is connected. If $y \in K$, $\pi_\alpha^{-1}(A_\alpha) \subseteq K$. However, $a \in \pi_\alpha^{-1}(A_\alpha)$ and $a \notin K$. Therefore, for each $\alpha \in D$, $\pi_\alpha^{-1}(A_\alpha) \subseteq H$.

Hence, $K = \emptyset$. This contradicts the supposition that $H$ and $K$ disconnect $M$. Consequently, $M$ is connected. If $a \in K$ then, by a similar argument, $H = \emptyset$. The supposition that $M$ is not connected is again denied. Hence, $M$ is connected.

Clearly, by substituting $N$ for $M$, $N$ is also connected. Now, $X \setminus C = M \cup N$ where $M$ and $N$ are disjoint regions containing $a$ and $b$, respectively.

It is noteworthy to point out that if $D$ is countable and each $X_\alpha$ is a separable metric space, then $X_\infty$ is a separable metric space.
Definition 1.6: Any space homeomorphic to $[0,1]$ is an arc.

Definition 1.7: A connected space is irreducibly connected between two points if no proper connected subset contains both points.

The following two theorems are beyond the scope of this paper and will be assumed. Theorem A has been proven by Wilder (5) and Theorem B by Whyburn (4).

Theorem A: A space $Y$ is irreducibly connected between a and $b$ if and only if for $y \in Y \setminus \{a,b\}$, $Y \setminus y = A \cup B$ where $A$ and $B$ are disjoint regions such that $a \in A$, $b \in B$.

Theorem B: A separable metric continuum $Y$ is an arc if and only if $Y$ is irreducibly connected between some two points.

The following two lemmas and Theorem 1.15 are proven in preparation for showing that if $X_\alpha$ is an arc for each $\alpha$, then $X_\omega$ is an arc.

Lemma 1.5: If a continuum $X$ is irreducibly connected between $a$ and $b$ and $X$ is irreducibly connected between $c$ and $d$, then

$$\{a,b\} = \{c,d\}.$$

Proof: Suppose that $a \neq y \neq b$. Suppose that $X$ is irreducibly connected between $a$ and $y$. Then $X \setminus b = H \cup K$ where $a \in H$, $y \in K$, $H \cap K = \emptyset$ and each of $H$ and $K$ is a region. But, since $X$ is irreducibly connected between $a$ and $b$, $X \setminus y = M \cup N$, where $M$ and $N$ are disjoint regions
containing a and b, respectively. There exists \( r \in K \) such that \( r \neq y \). If not, \( \{y\} \) is both open and closed, contradicting the supposition that \( X \) is connected. Since \( r \neq y \), \( r \in M \) or \( r \in N \). If \( r \in M \), then \( r \in M \cap K \). Hence, \( M \cup K \) is open and connected. Therefore,

\[
X = (M \cup K) \cup N.
\]

But \( (M \cup K) \cap N \neq \emptyset \). If not, \( X \) is not connected. Since \( M \cap N = \emptyset \), \( K \cap N \neq \emptyset \). Hence, \( K \cup N \) is connected and open. But, \( (K \cup N) \cup H = X \). Therefore, \( (K \cup N) \cap H \neq \emptyset \). Otherwise, \( X \) is not connected. However, \( H \cap K = \emptyset \). Consequently, \( N \cap H \neq \emptyset \). It follows that \( N \cup H \) is connected and open.

Since \( y \in N \) and \( y \in H \), \( N \cup H \) is a proper subset of \( X \). But, \( a \in N \) and \( b \in H \). This contradicts \( X \) being irreducibly connected between a and b. Therefore, \( X \) is not irreducibly connected between a and y. A similar argument shows that \( X \) is not irreducibly connected between b and y. Hence, if \( \{a, b\} \cap \{c, d\} \neq \emptyset \), then,

\[
\{a, b\} = \{c, d\}.
\]

Suppose that \( \{a, b\} \cap \{c, d\} = \emptyset \). Then \( a \neq c \neq b \) and \( a \neq d \neq b \). By the previous argument, \( X \) is not irreducibly connected between a and c or between a and d. By Definition 1.7, there must be a proper connected subset \( H \) containing a and c. There must also be a proper connected subset \( K \) containing a and d. Now, if \( X \neq H \cup K \) then \( H \cup K \) is a proper connected subset of \( X \). But \( c \in H \cup K \) and \( d \in H \cup K \). There-
fore, $X$ is not irreducibly connected between $c$ and $d$. This
contradicts the hypothesis. Hence, $X = H \cup K$. However,
b $\in X$. Consequently, $b \in H$ or $b \in K$. But, $a \in H \cap K$. It
then follows that $X$ is not irreducibly connected between
$a$ and $b$. This also contradicts the hypothesis. As a result,
$$\{a, b\} = \{c, d\}.$$ 

Therefore, Lemma 1.5 guarantees that if a continuum
$X$ is irreducibly connected between two points they are
unique.

**Lemma 1.6:** If $f$ is a monotone function from a con-
tinuum $X$ onto a continuum $Y$ and $X$ is irreducibly connected
between $a$ and $b$, then if $Y$ is irreducibly connected between
c and $d$,
$$\{f(a), f(b)\} = \{c, d\}.$$ 

**Proof:** Suppose that $f(a) = f(b)$. Then, $f^{-1}(f(a))$
is connected and contains both $a$ and $b$. Hence, $f^{-1}(f(a)) = X$.
Therefore, $Y = f(a)$. Consequently, $c = d$ and $Y$ is not
irreducibly connected between $c$ and $d$. Consequently,
f(a) $\neq f(b)$. Suppose that $Y$ is not irreducibly connected
between $f(a)$ and $f(b)$. Then there exists a proper connected
subset $U$ of $Y$ such that $f(a) \in U$ and $f(b) \in U$. Since $f$
is monotone, $f^{-1}(U)$ is connected in $X$. But $a$ and $b$ are both
elements of $f^{-1}(U)$. Hence, $f^{-1}(U) = X$. However, $U$ is a
proper subset of $Y$. Therefore, there exists $x \in Y$ such
that $x \notin U$. Consequently, $f^{-1}(x) \cap f^{-1}(U) = \emptyset$. But
$f^{-1}(U) = X$. Hence, $f^{-1}(x) = \emptyset$. This contradicts $f$ being
onto. Therefore, $Y$ is irreducibly connected between $f(a)$ and $f(b)$. By Lemma 1.5, $\{f(a), f(b)\} = \{c, d\}$.

**Theorem 1.14:** If, for each $\alpha \in D$, $X_\alpha$ is a continuum irreducibly connected between two points, then $X_\infty$ is a continuum irreducibly connected between two points.

**Proof:** By Theorem 1.5, $X_\infty$ is a continuum. Let $\alpha \in D$. Let

$$E = \{\beta \in D | \alpha \leq \beta\}.$$ 

Then $E$ is cofinal in $D$. Hence, by Theorem 1.10,

$$Y_\infty = \liminf_{\alpha \in E} (X_\alpha, f_{\beta\alpha}, E)$$

is homeomorphic to $X_\infty$. For $\lambda \in E$, let $\pi_\lambda$ be the $\lambda$-th projection function of $\Pi_{\alpha \in E} X_\alpha$ restricted to $Y_\infty$. Let $X_\alpha$ be irreducibly connected between $a_\alpha$ and $b_\alpha$. By Lemma 1.6, for each $\beta \in E$, there exists $a_\beta$ and $b_\beta$ such that $X_\beta$ is irreducibly connected between $a_\beta$ and $b_\beta$ where $f_{\beta\alpha}(a_\beta) = a_\alpha$ and $f_{\beta\alpha}(b_\beta) = b_\alpha$. If $\lambda \leq \beta$ in $E$, let $X_\lambda$ be irreducibly connected between $a_\lambda$ and $b_\lambda$. Then

$$f_{\beta\lambda}(a_\beta) = a_\lambda.$$ 

If not, $f_{\beta\lambda}(a_\beta) = b_\lambda$ ($a_\lambda$ and $b_\lambda$ are unique). Therefore,

$$f_{\beta\alpha}(a_\beta) = f_{\lambda\alpha} \circ f_{\beta\lambda}(a_\beta) = f_{\lambda\alpha}(b_\lambda) = b_\lambda.$$ 

But,

$$f_{\beta\alpha}(a_\beta) = a_\alpha.$$ 

This contradicts $a_\alpha \neq b_\alpha$. Similarly,

$$f_{\beta\lambda}(b_\beta) = b_\lambda.$$ 

For $\delta \in E$, let $H_\delta = \pi_\delta^{-1}(a_\delta)$ and $K_\delta = \pi_\delta^{-1}(b_\delta)$. Each of
$H_6$ and $K_6$ is closed, compact, connected, non-empty and Hausdorff. Let

$$\mathcal{U} = \{H_6 | \delta \in E\}$$

and

$$\mathcal{V} = \{K_6 | \delta \in E\}.$$ 

Let $H_6$ and $H_\lambda$ be elements of $\mathcal{U}$. There exists $\sigma \in E$ such that $\beta \leq \sigma$ and $\lambda \leq \sigma$. If $s \in H_\beta$, then $s_\sigma = a_\sigma$ and

$$s_\beta = f_{\sigma \beta}(a_\sigma) = a_\beta.$$ 

Hence, $s \in H_\beta$. Similarly, $s \in H_\lambda$. Therefore, $H_\sigma \subseteq H_\lambda \cap H_\beta$.

By Lemma 1.1, $\cap \mathcal{U} \neq \emptyset$. Similarly, $\cap \mathcal{V} \neq \emptyset$. Let $a \in \cap \mathcal{U}$ and $b \in \cap \mathcal{V}$. For each $\delta \in E$, $\overline{\pi}_\delta(a) = a_\delta$ and $\overline{\pi}_\delta(b) = b_\delta$.

Let $t \in Y_\infty \setminus \{a, b\}$. Then, there exists $\eta \in E$ such that $a_\eta \neq t_\eta \neq b_\eta$. Let

$$F = \{\delta \in E | \eta \leq \delta\}.$$ 

Then $F$ is cofinal in $E$. Then $Z_\infty = \lim(X_\infty, f_{\beta \alpha}, F)$ is homeomorphic to $Y_\infty$. Let $f : X_\infty \rightarrow Y_\infty$ and $g : Y_\infty \rightarrow Z_\infty$ denote these homeomorphisms. For each $\lambda \in F$,

$$X_\lambda \setminus (g(t))_\lambda = M_\lambda \cup N_\lambda$$

where $M_\lambda$ and $N_\lambda$ are disjoint regions containing $(g(a))_\lambda$ and $(g(b))_\lambda$ respectively. Recall that from Theorem 1.10, $(g(a))_\lambda = a_\lambda$. By Theorem 1.13, $Z_\infty \setminus g(t) = M \cup N$ where $M$ and $N$ are disjoint regions containing $g(a)$ and $g(b)$, respectively. Therefore,

$$Y_\infty \setminus t = g^{-1}(M) \cup g^{-1}(N).$$

Hence, $Y_\infty$ is irreducible between $a$ and $b$. Consequently, $X_\infty$ is irreducible between some two points.
Theorem 1.15: If \( D \) is countable and for each \( \alpha \in D \), \( X_\alpha \) is an arc, then \( X_\infty \) is an arc.

Proof: By Theorem B, for each \( \alpha \in D \), \( X_\alpha \) is irreducibly connected between some two points. By Theorem 1.14, \( X_\infty \) is a continuum irreducibly connected between some two points. As was previously noted, \( X_\infty \) is a separable metric space. Therefore, by Theorem B, \( X_\infty \) is an arc.

Definition 1.8: Any space homeomorphic to \( S' \), the unit circle in the plane, is a simple closed curve.

Definition 1.9: A space \( Y \) has property \( Q \) if it is connected, non-degenerate and for every pair of distinct points \( x \) and \( y \), \( Y \setminus \{x,y\} = A \cup B \) where \( A \) and \( B \) are nonempty, disjoint regions.

This paper will assume the following theorem by Whyburn (4).

Theorem C: A separable metric continuum \( Y \) is a simple closed curve if and only if \( Y \) has property \( Q \).

Theorem 1.16: If, for each \( \alpha \in D \), \( X_\alpha \) is a continuum having property \( Q \), then \( X_\infty \) is a continuum having property \( Q \).

Proof: By Theorem 1.5, \( X_\alpha \) is a continuum. Let \( x \neq y \in X_\infty \). Then there exists \( \alpha \in D \) such that \( x_\alpha \neq y_\alpha \). Let \( E = \{ \lambda \in D | \alpha \leq \lambda \} \).

Since \( X_\beta \) has property \( Q \) for all \( \beta \in D \), \( X_\alpha \setminus \{x_\alpha,y_\alpha\} = A_\alpha \cup B_\alpha \) where \( A_\alpha \) and \( B_\alpha \) are disjoint nonempty regions of \( X_\alpha \). Let \( \delta \in E \). Then \( x_\delta \neq y_\delta \). Hence, \( X_\delta \setminus \{x_\delta,y_\delta\} = H \cup K \) where \( H \) and \( K \) are disjoint nonempty regions of \( X_\delta \). Let \( t \in f_\delta^{-1}(A_\alpha) \).
If \( t = x_\alpha \), then \( f_\delta (t) = x_\alpha \). Hence, \( x_\alpha \in A_\alpha \). But \( x_\alpha \notin A_\alpha \). Therefore, \( t \neq x_\alpha \). Similarly, \( t \neq y_\delta \). Hence,

\[
f_\delta^{-1}(A_\alpha) \subseteq \emptyset \cup K.
\]

In a like manner,

\[
f_\delta^{-1}(B_\alpha) \subseteq \emptyset \cup K.
\]

Suppose that \( f_\delta^{-1}(A_\alpha) \neq \emptyset \) and \( f_\delta^{-1}(A_\alpha) \neq K \). Then \( f_\delta^{-1}(A_\alpha) \) meets both \( H \) and \( K \). But, \( f_\delta \) is assumed to be monotone. Hence \( f_\delta^{-1}(A_\alpha) \) is connected. Therefore,

\[
f_\delta^{-1}(A_\alpha) \subseteq H
\]

or

\[
f_\delta^{-1}(A_\alpha) \subseteq K.
\]

Similarly,

\[
f_\delta^{-1}(B_\alpha) \subseteq H
\]

or

\[
f_\delta^{-1}(B_\alpha) \subseteq K.
\]

For \( \delta \in E \), denote the region containing \( f_\delta^{-1}(A_\alpha) \) as \( A_\delta \) and the region containing \( f_\delta^{-1}(B_\alpha) \) as \( B_\delta \).

Now, \( E \) is cofinal in \( D \). Hence,

\[
Y_\infty = \lim_{\alpha \in E} (X_\alpha, f_\beta, E)
\]

is homeomorphic to \( X_\infty \). Let \( \pi_\alpha \) be the \( \alpha \)-th projection function of \( \Pi X_\alpha \) restricted to \( Y_\infty \). Let \( g \) be the homeomorphism from \( X_\infty \) onto \( Y_\infty \) as described in Theorem 1.10. Then \( (g(x))_\beta = x_\beta \) and \( (g(y))_\beta = y_\beta \) for all \( \beta \in E \).

Let \( t \in Y_\infty \setminus \{g(x), g(y)\} \). Then, there exists \( \lambda \in E \) such that \( (g(x))_\lambda = x_\lambda \neq t_\lambda \neq y_\lambda = (g(y))_\lambda \). Therefore,
t_\lambda \in A_\lambda \text{ or } t_\lambda \in B_\lambda. \text{ Hence, } t \in \pi^{-1}_\lambda(A) \text{ or } t \in \pi^{-1}_\lambda(B). \\
Consequently, \\
t \in (\bigcup_{\beta \in E} \pi^{-1}_\beta(A_\beta)) \cup (\bigcup_{\beta \in E} \pi^{-1}_\beta(B_\beta)). \\
Let \\
A = \bigcup_{\beta \in E} \pi^{-1}_\beta(A_\beta) \\
and \\
B = \bigcup_{\beta \in E} \pi^{-1}_\beta(B_\beta). \\
Then \\
Y_\infty \setminus \{g(x), g(y)\} \subseteq A \cup B. \\
Let t \in A \cup B. \text{ If } t \notin Y_\infty \setminus \{g(x), g(y)\}, \text{ then } t = g(x) \text{ or } t = g(y). \text{ Suppose that } t = g(x), \text{ then } t_\beta = x_\beta \text{ for all } \beta \in E. \text{ Hence, } t_\beta \notin A_\beta \text{ and } t_\beta \notin B_\beta \text{ for any } \beta \in E. \text{ Therefore, } t \notin A \cup B. \text{ Similarly, if } t = g(y), t \notin A \cup B. \text{ This contradicts } t \in A \cup B. \text{ Hence, } \\
t \in Y_\infty \setminus \{g(x), g(y)\}. \\
Consequently, \\
A \cup B = Y_\infty \setminus \{g(x), g(y)\}. \\
Since A_\alpha \text{ is nonempty, there exists } v_\alpha \in A_\alpha. \text{ Now each } f_{\beta \alpha} \text{ is monotone. Therefore each } f_{\beta \alpha} \text{ is onto. Hence, } \pi_\alpha \text{ is onto for each } \alpha \in E. \text{ Therefore, it follows that there exists } v \in Y_\infty \text{ such that } \pi_\alpha(v) = v_\alpha. \text{ Since } v_\alpha \in A_\alpha, \\
f_{\beta \alpha}^{-1}(v_\alpha) \subseteq A_\beta \text{ for each } \beta \in E. \text{ Hence, } \\
v \in \bigcap_{\alpha \in E} \pi^{-1}_\alpha(A_\alpha). \\
Therefore, A \text{ is connected. Similarly, } B \text{ is connected.} \\
Since A_\alpha \text{ and } B_\alpha \text{ are open in } X_\alpha \text{ for each } \alpha \in E, \pi^{-1}_\alpha(A_\alpha) \text{ and }
\( \pi^{-1}(B_\alpha) \) are open in \( Y_\infty \). Consequently, \( A \) and \( B \) are open in \( Y_\infty \). Suppose that \( w \in A \cap B \). Then \( w \in \pi^{-1}_\delta(A_\delta) \) and \( w \in \pi^{-1}_\sigma(B_\sigma) \) for some \( \delta \) and \( \sigma \) in \( E \). Hence, \( w_\delta \in A_\delta \) and \( w_\sigma \in B_\sigma \). There exists \( \lambda \in E \) such that \( \sigma \leq \lambda \) and \( \delta \leq \lambda \). Therefore, \( w_\lambda \in A_\lambda \) and \( w_\lambda \in B_\lambda \). This contradicts the fact that \( A_\beta \cap B_\beta = \phi \) for all \( \beta \in E \). Hence, \( A \cap B = \phi \). Therefore,

\[
Y_\infty \backslash \{g(x), g(y)\} = A \cup B,
\]

where \( A \) and \( B \) are disjoint regions of \( Y_\infty \). Hence, \( Y_\infty \) has property \( Q \). As a result, \( X_\infty \) has property \( Q \).

An obvious result of Theorem 1.16 is that if \( D \) is countable and if for each \( \alpha \in D \), \( X_\alpha \) is a simple closed curve, then \( X_\infty \) is a simple closed curve.
CHAPTER BIBLIOGRAPHY


CHAPTER II

MAPPINGS BETWEEN INVERSE LIMITS

Before attempting to characterize compact metric spaces, it will be helpful to investigate the following situations.

Let \((X, f_\alpha, D)\) and \((Y, g_\alpha, D)\) be inverse systems where for each \(\alpha \leq \beta\) in \(D\), \(f_\beta\) and \(g_\alpha\) are continuous. By a mapping \(Q: (X, f_\beta, D) \rightarrow (Y, g_\alpha, D)\) is meant a collection of functions \(\{Q_\alpha\}_{\alpha \in D}\) such that \(Q_\alpha: X_\alpha \rightarrow Y_\alpha\) and for \(\alpha \leq \beta\) in \(D\),

\[Q_\alpha \circ f_\beta = g_\beta \circ Q_{\beta}.\]

Define a relation \(Q_\infty: X_\infty \rightarrow Y_\infty\) by

\[Q_\infty(\{x_\alpha\}_{\alpha \in D}) = \{Q_\alpha(x_\alpha)\}_{\alpha \in D}.\]

For \(\prod X_\alpha\), let \(\pi_\alpha\) be the \(\alpha\)-th projection function restricted to \(X_\infty\) and for \(\prod Y_\alpha\), let \(\overline{\pi}_\alpha\) be the \(\alpha\)-th projection function restricted to \(Y_\infty\). Let \(x \in X_\infty\) such that \(x = \{x_\alpha\}_{\alpha \in D}\). Then,

\[Q_\infty(x) = \{Q_\alpha(x_\alpha)\}_{\alpha \in D}.\]

Let \(\alpha \leq \beta\) in \(D\). Now,

\[\overline{\pi}_\alpha(Q_\infty(x)) = Q_\alpha(x_\alpha).\]

But

\[g_\beta(Q_\alpha(x_\beta)) = Q_\alpha \circ f_\beta(x_\beta).\]

So that

\[g_\beta(Q_\alpha(x_\beta)) = Q_\alpha(x_\alpha).\]
Therefore,
\[ \pi_{\alpha}(Q_{\infty}(x)) = g_{\beta\alpha}(\pi_{\beta}(Q_{\infty}(x))). \]
Hence, \( Q_{\infty}(x) \in Y_{\infty} \). It then follows that \( Q_{\infty}(X_{\infty}) \subseteq Y_{\infty} \).

Let \( x = y \) in \( X_{\infty} \). Then, for each \( \alpha \in D, x_{\alpha} = y_{\alpha} \).
Since \( Q \) is a function, \( Q_{\alpha}(x_{\alpha}) = Q_{\alpha}(y_{\alpha}) \). Therefore,
\[ \{Q_{\alpha}(x_{\alpha})\}_{\alpha \in D} = \{Q_{\alpha}(y_{\alpha})\}_{\alpha \in D}. \]
Consequently,
\[ Q_{\infty}(x) = Q_{\infty}(y). \]
Now \( Q_{\infty} \) is a function.

To say that \( Q \) has a certain property will mean that \( Q_{\alpha} \) has that property for each \( \alpha \in D \). By placing the proper restrictions on \( Q \), \( Q_{\infty} \) can be forced to possess certain properties, as is shown by the following theorems.

**Theorem 2.1:** Let each of \( (X_{\alpha}, f_{\beta\alpha}, D) \) and \( (Y_{\alpha}, g_{\beta\alpha}, D) \) be inverse systems and \( Q \) be a mapping from \( (X_{\alpha}, f_{\beta\alpha}, D) \) to \( (Y_{\alpha}, g_{\beta\alpha}, D) \). Then, if \( Q \) is continuous (one to one and onto) (a homeomorphism), then \( Q_{\infty} \) is continuous (one to one and onto) (a homeomorphism).

**Proof:** Suppose that \( Q \) is continuous. Let \( s \in X_{\infty} \) and \( Q_{\infty}(s) = w \). Let \( U \) be open in \( Y_{\infty} \) such that \( w \in U \). Then, there exists a basic neighborhood, \( \pi_{\delta}^{-1}(U_{\delta}) \), of \( w \) such that \( w \in \pi_{\delta}^{-1}(U_{\delta}) \subseteq U \). Now, \( w_{\delta} \in U_{\delta} \) and
\[ Q_{\infty}(s) = \{Q_{\alpha}(s_{\alpha})\}_{\alpha \in D} = \{w_{\alpha}\}_{\alpha \in D}. \]
Since \( Q_{\delta} \) is continuous and \( Q_{\delta}(s_{\delta}) = w_{\delta} \in U_{\delta} \), there exists \( V_{\delta} \), open in \( X_{\delta} \), such that \( Q_{\delta}(V_{\delta}) \subseteq U_{\delta} \) and \( s_{\delta} \in V_{\delta} \). Also, \( s \in \pi_{\delta}^{-1}(V_{\delta}) \) which is open in \( X_{\infty} \). Let \( p \in \pi_{\delta}^{-1}(V_{\delta}) \).
Then, \( p_\alpha \in V_\delta \). Hence, \( Q_\delta(p_\delta) \in U_\delta \). But \( Q_\infty(p) \in Y_\infty \) and
\[
Q_\infty(p) = \{Q_\alpha(p_\alpha)\}_{\alpha \in D}.
\]
Since \( Q_\delta(p_\delta) \in U_\delta \), \( Q_\infty(p) \in \overline{\pi_\delta^{-1}(U_\delta)} \subseteq U \). Consequently, \( Q_\infty(\pi_\delta^{-1}(V_\delta)) \subseteq U \). As a result, \( Q_\infty \) is continuous.

Suppose that \( Q \) is one to one and onto. Also, suppose that \( Q_\infty(x) = Q_\infty(y) \). Then,
\[
\{Q_\alpha(x_\alpha)\}_{\alpha \in D} = \{Q_\alpha(y_\alpha)\}_{\alpha \in D}.
\]
Hence, \( Q_\alpha(x_\alpha) = Q_\alpha(y_\alpha) \) for all \( \alpha \). Since each \( Q_\alpha \) is one to one, \( x_\alpha = y_\alpha \) for all \( \alpha \). Therefore, \( x = y \). It then follows that \( Q_\infty \) is one to one. Let \( t \in Y_\infty \). Then for each \( \alpha \), \( t_\alpha \in Y_\alpha \). Each \( Q_\alpha \) is one to one and onto. Consequently, for each \( \alpha \), there is a unique \( p_\alpha \) in \( X_\alpha \) such that \( Q_\alpha(p_\alpha) = t_\alpha \).

Let \( p \) be the element in \( \Pi X_\alpha \) such that the \( \alpha \)-th projection of \( p \) is \( p_\alpha \) for each \( \alpha \). Let \( \alpha \leq \beta \) in \( D \) and suppose that \( f_\beta\alpha(p_\beta) \neq p_\alpha \). Since \( Q \) is one to one, \( Q_\alpha(f_\beta\alpha(p_\beta)) \neq Q_\alpha(p_\alpha) \).

Hence, \( Q_\alpha(f_\beta\alpha(p_\beta)) \neq t_\alpha \). However,
\[
Q_\alpha(f_\beta\alpha(p_\beta)) = g_\beta\alpha \circ Q_\beta(p_\beta) = g_\beta\alpha(t_\beta) = t_\alpha.
\]
This contradicts the previous statement. Therefore,
\[
f_\beta\alpha(p_\beta) = p_\alpha.
\]
Hence, \( p \in X_\infty \). But
\[
Q_\infty(p) = \{Q_\alpha(p_\alpha)\}_{\alpha \in D} = t.
\]

Consequently, \( Q_\infty \) is onto.

Suppose that \( Q \) is a homeomorphism. Then, \( Q \) is continuous, one to one and onto. By the first two arguments, \( Q_\infty \) is continuous, one to one and onto. Let \( y \in Y_\infty \) and \( Q_\infty^{-1}(y) = x \). Let \( U \) be open in \( X_\infty \) such that \( x \in U \). Then,
there exists \( \alpha \) in \( D \) such that \( U_\alpha \) is open in \( X_\alpha \) and \( x \in \pi^{-1}_\alpha(U_\alpha) \subseteq \mathcal{U} \). Since \( Q_\alpha \) is a homeomorphism, \( Q_\alpha(U_\alpha) = V_\alpha \) is open in \( Y_\alpha \). Therefore, \( \pi^{-1}_\alpha(V_\alpha) \) is open in \( Y_\alpha \). Since \( x \in \pi^{-1}_\alpha(U_\alpha), x_\alpha \in U_\alpha \). Consequently, \( Q_\alpha(x_\alpha) \in V_\alpha \). However, \( Q_\alpha(x_\alpha) = y_\alpha \). Hence, \( y \in \pi^{-1}_\alpha(V_\alpha) \). Let \( w \in \pi^{-1}_\alpha(V_\alpha) \). Then \( w_\alpha \in V_\alpha \). As a result, \( Q^{-1}_\alpha(w_\alpha) \in Q^{-1}_\alpha(V_\alpha) = U_\alpha \). It then follows that \( Q^{-1}_\alpha(w) \in \pi^{-1}_\alpha(U_\alpha) \). From this it is observed that \( Q^{-1}_\alpha \) is continuous. Hence, \( Q_\alpha \) is a homeomorphism.

**Definition 2.1**: If \( f \) is a function from a space \( X \) to \( X, x \in X \) and \( n \in \mathbb{Z}^+ \), then \( f^n(x) \) means \( f \) composed with itself \( n \) times acting on \( x \).

**Definition 2.2**: A function \( f \) from \( X \) to \( X \) is periodic if there exists some positive integer \( n \) such that, for each \( x \in X, f^n(x) = x \). The least such \( n \) is the period of \( f \).

If \( (X,d) \) is a metric space then \( f \) is called almost periodic if for each \( c > 0 \), there is a positive integer \( n \) such that for all \( x \in X, d(x,f^n(x)) < c \).

**Theorem 2.2**: Let \( ((X_n,d_n),f_{mn},\mathbb{Z}_+) \) be an inverse sequence of metric spaces and

\[
Q:((X_n,d_n),f_{mn},\mathbb{Z}_+) \to ((X_n,d_n),f_{mn},\mathbb{Z}_+)
\]

be continuous. Then, if \( Q \) is periodic, \( Q_\infty \) is almost periodic.

**Proof**: It will be helpful to recall that, if for each \( n \in \mathbb{Z}_+, (X_n,d_n) \) is a metric space, then a metric \( d \)
for \( \bigcap_{n=1}^{\infty} (X_n, d_n) \) is given by

\[ d(x, y) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n[1 + d_n(x_n, y_n)]} \]

where

\[ x = \{x_n\}_{n=1}^{\infty} \]

and

\[ y = \{y_n\}_{n=1}^{\infty} \]

For each \( i \in Z_+ \), let \( Q_i \) have periodicity \( n_i \) and let \( c > 0 \). Then, there exists \( k \in Z_+ \) such that \( \sum_{p=k+1}^{\infty} \frac{1}{2^p} < c \).

Let \( m \) be the product of the \( n_i \)'s for \( 1 \leq i \leq k \). Let \( z \in X_\infty \) and let

\[ d_p(Q^m_p(z_p), z_p) = s_p. \]

Then

\[ d(Q^m(z), z) = \sum_{p=1}^{\infty} \frac{s_p}{2^p[1 + s_p]} = \sum_{p=1}^{k} \frac{s_p}{2^p[1 + s_p]} + \sum_{p=k+1}^{\infty} \frac{s_p}{2^p[1 + s_p]} \]

For \( p \leq k \), let

\[ n_1 \cdot n_2 \cdot \ldots \cdot n_{p-1} \cdot n_{p+1} \cdot \ldots \cdot n_k = r_p. \]

Then \( Q^m_p = Q^r_p \). Hence, \( Q^m_p(z_p) = z_p \). Therefore, for \( p \leq k \), \( s_p = 0 \). Consequently,

\[ d(Q^m(z), z) = \sum_{p=k+1}^{\infty} \frac{s_p}{2^p[1 + s_p]} < \sum_{p=k+1}^{\infty} \frac{1}{2^p} < c. \]

As a result, \( Q_\infty \) is almost periodic.
The following example illustrates that requiring each $Q_n$ to be periodic will not force $Q_\infty$ to be periodic.

**Example 2.1:** For each $n \in \mathbb{Z}^+$, let $X_n$ be the unit circle in the plane. Give each $X_n$ the relative metric topology, $d_n$. An element, $e^{i\theta}$, of $X_n$ will be denoted by $\theta$, expressed in degrees. Define $Q_n : X_n \rightarrow X_n$ by

$$Q_n(\theta) = \theta + \frac{360^\circ}{n}.$$ 

Clearly, each $Q_n$ is continuous and of period $n$. Define $f_{nn} : X_n \rightarrow X_n$ to be the identity map and $f_{n+1} : X_{n+1} \rightarrow X_n$ by

$$f_{n+1}(\theta) = \frac{n+1}{n}(\theta).$$

Obviously, each $f_{n+1}$ is continuous and onto. For $m > n$ in $\mathbb{Z}^+$, let $f_{mn} : X_m \rightarrow X_n$ be $f_{n+1} \circ f_{n+2} \circ \cdots \circ f_{m}$.

Then $f_{mn}$ is continuous and $((X_n, d_n), (X_n, Z_+), f_{mn}, Z_+)$ is an inverse sequence of metric spaces and $Q_n$ is onto for each $n$. Also,

$$f_{mn} \circ Q_n(\theta) = f_{mn}(\theta + \frac{360^\circ}{m})$$

$$= \frac{n+1}{n} \cdot \frac{n+2}{n+1} \cdots \frac{m}{m-1}(\theta + \frac{360^\circ}{n})$$

$$= \frac{m}{n}\theta + \frac{360^\circ}{n}.$$ 

However,

$$Q_n \circ f_{mn}(\theta) = Q_n\left(\frac{n+1}{n} \cdot \frac{n+2}{n+1} \cdots \frac{m}{m-1}(\theta)\right)$$

$$= Q_n\left(\frac{m}{n}\theta + \frac{360^\circ}{n}\right).$$

Therefore,

$$f_{mn} \circ Q_n = Q_n \circ f_{mn}.$$
Now, suppose that $Q_\alpha$ is of period $s$. Then, $Q_\alpha^S(x) = x$ for each $x \in X_\alpha$ and $Q_\alpha^S(x) = \{Q_\alpha^S(x_n)\}_{n \in \mathbb{Z}^+}$. If $p > s$, then there exists $y \in X_p$ such that $Q_p^S(y) \neq y$. Let $x \in \pi_p^{-1}(y)$. Then,

$$Q_\alpha^S(x) = \{Q_\alpha^S(x_n)\}_{n \in \mathbb{Z}^+} = x.$$  

Therefore, $Q_p^S(y) = y$. This is a contradiction to $Q_p^S(y) \neq y$. Hence, $Q_\alpha$ is not of period $s$.

**Theorem 2.3:** Let $((X_n,d_n),f_{mn},Z^+)$ be an inverse sequence and $Q:((X_n,d_n),f_{mn},Z^+) \rightarrow ((X_n,d_n),f_{mn},Z^+)$ be continuous. If $Q_n$ is almost periodic and $X_n$ is compact for each $n$, then $Q_\alpha$ is almost periodic.

**Proof:** Let $a > 0$. Then, there exists $b$ such that $0 < b < \text{minimum } \{1,a\}$. There exists $k \in \mathbb{Z}^+$ such that $k > b$ and $\sum_{p=k+1}^{\infty} \frac{1}{p} < \frac{b}{2}$. Let $0 < \delta < \frac{b}{k-b}$. Since each $X_n$ is compact, each $f_{mn}$ is uniformly continuous. Therefore, for each $i$ such that $i \leq k$, there exists $\delta_i$ such that if $U$ is set of $X_k$ of diameter less than $\delta_i$, the diameter of $f_{ki}(U)$ is less than $\delta$. There exists $c$ such that $0 < c < \text{minimum } \{\delta_i | 1 \leq i \leq k\}$.

But $Q_k$ is almost periodic. Consequently, there exists $m$ such that for $x \in X_k$, $d_k(Q_k^m(x),x) < \frac{c}{2}$. If $x \in X_k$, then there is an open set $U_x$ of diameter less than $c$ such that $\{x,Q_k^m(x)\} \subseteq U_x$. But $f_{ki}(U_x)$ has diameter less than $\delta$. 
Hence, $d_1(f_{k_1}(x), f_{k_1}(Q^m(x))) < \delta$. Therefore, $d_1(Q^m_1(f_{k_1}(x)), f_{k_1}(x)) < \delta$. Let $y \in X_\infty$ and $d_p(Q^m(y_p), y_p) = s_p$. Then,

$$d(Q^m(y), y) = \sum_{p=1}^{k} \frac{s_p}{2^p[1 + s_p]} + \sum_{p=k+1}^{\infty} \frac{s_p}{2^p[1 + s_p]}.$$  

But for $p \leq k$, $s_p < \delta < \frac{b}{k-b}$. Hence,

$$s_p < \frac{b}{k(1 - \frac{b}{k})} = \frac{b}{1 - \frac{b}{k}}.$$  

Consequently, $s_p[1 + \frac{b}{k}] < \frac{b}{k}$, so that $s_p < \frac{b}{k} + s_p(\frac{b}{k})$. Now,

$$s_p < \frac{b}{k}[1 + s_p].$$  

Thus, $\frac{s_p}{1 + s_p} < \frac{b}{k}$. Therefore,

$$\frac{k}{\sum_{p=1}^{k} \frac{s_p}{2^p[1 + s_p]}} < \frac{k}{\sum_{p=1}^{k} \frac{1}{2^p(k)}} \leq \frac{\frac{k}{2}}{k} = \frac{1}{2}(k)(\frac{b}{k}) = \frac{b}{2}.$$  

As a result,

$$d(Q^m(y), y) < \frac{b}{2} + \sum_{p=k+1}^{\infty} \frac{s_p}{2^p[1 + s_p]} < \frac{b}{2} + \sum_{p=k+1}^{\infty} \frac{1}{2^p} < \frac{b}{2} + \frac{b}{2} < a.$$  

It is then evident from this that $Q_\infty$ is almost periodic.

**Theorem 2.4**: Let $(X_\alpha, f_{\beta \alpha}, D)$ and $(y_\alpha, g_{\beta \alpha}, D)$ be inverse systems and $Q: (X_\alpha, f_{\beta \alpha}, D) \to (Y_\alpha, g_{\beta \alpha}, D)$ be continuous. If $Q_\alpha$ is monotone and $X_\alpha$ is compact and Hausdorff for each $\alpha$, then $Q_\infty$ is monotone.

**Proof**: Let $y \in Y_\infty$ such that $Q_\infty^{-1}(y) \neq \phi$. For each $\alpha$, let $Q_\infty^{-1}(y) = A_\alpha$. Since $Q_\alpha$ is monotone, $A_\alpha$ is a continuum. Hence, $\prod A_\alpha$ is a continuum. For $\alpha \leq \beta$ in $D$,

$$J_{\beta} = f_{\beta \alpha}|_{A_\beta}.$$  

Then $j_{\beta \alpha}: A_\beta \to A_\alpha$. For if $x \in A_\beta$,
\( Q_\beta(x) = y_\beta \). Hence,
\[ g_{\beta\alpha} \circ Q_\beta(x) = g_{\beta\alpha}(y_\beta) = y_\alpha. \]

However,
\[ g_{\beta\alpha} \circ Q_\beta(x) = Q_\alpha \circ f_{\beta\alpha}(x) = y_\alpha. \]

Therefore,
\[ f_{\beta\alpha}(x) \in Q_\alpha^{-1}(y_\alpha) = A_\alpha. \]

Now, \((A_\alpha, j_{\beta\alpha}, D)\) is an inverse system and \( A_\infty = \lim(A_\alpha, j_{\beta\alpha}, D) \) is a continuum.

Let \( s \in A_\infty \). Then, for \( \alpha \leq \beta \) in \( D \), \( j_{\beta\alpha}(s_\beta) = s_\alpha \).

Therefore, \( f_{\beta\alpha}(s_\beta) = s_\alpha \). Consequently, \( s \in X_\infty \). Since \( A_\infty \subseteq \Pi A_\alpha \), \( A_\alpha \subseteq \Pi A_\alpha \cap X_\infty \). Let \( s \in \Pi A_\alpha \cap X_\infty \). Then, \( s \in X_\infty \). As a result, for \( \alpha \leq \beta \) in \( D \), \( f_{\beta\alpha}(s_\beta) = s_\alpha \). But \( s \in \Pi A_\alpha \). Hence, \( s_\beta \in A_\beta \). So that \( j_{\beta\alpha}(s_\beta) = s_\alpha \). Therefore, \( s \in A_\infty \). Now,
\[ A_\infty = X_\infty \cap \Pi A_\alpha. \]

Let \( s \in A_\infty \). Then, \( s \in X_\infty \) and for all \( \alpha \in D \), \( s_\alpha \in A_\alpha \).

Consequently,
\[ Q_\alpha(s_\alpha) = y_\alpha. \]

As a result, \( Q_\infty(s) = y \). It then follows that \( A_\infty \subseteq Q_\infty^{-1}(y) \).

Let \( s \in Q_\infty^{-1}(y) \). Then \( s \in X_\infty \) and, for \( \alpha \in D \), \( s_\alpha \in Q_\alpha^{-1}(y_\alpha) \).

Hence, \( s_\alpha \in A_\alpha \). Therefore, \( s \in \Pi A_\alpha \), so that \( s \in \Pi A_\alpha \cap X = A_\infty \).

Hence, \( A_\infty = Q_\infty^{-1}(y) \). Since \( A_\infty \) is a continuum, \( Q_\infty^{-1}(y) \) is connected. It then follows that \( Q_\infty \) is monotone.

**Theorem 2.5:** Let
\[ ((X_n, \sigma_n), f_{mn}, Z_+) \]
and
\[ ((Y_n, \alpha_n), g_{mn}, Z_+) \]
be inverse sequences and
\[ Q: ((X_n, \sigma_n, f_{mn}, Z_+) \rightarrow ((Y_n, \alpha_n, g_{mn}, Z_+)) \]
be continuous and open. Suppose that if \( g_{n+1} n(y_{n+1}) = y_n \)
and \( Q_n(x_n) = y_n \), then
\[ Q_{n+1}^{-1}(y_{n+1}) \cap f_{n+1} n^{-1}(x_n) \neq \emptyset. \]
Then \( Q_\infty \) is open.

**Proof:** Let \( \pi_n \) be the n-th projection function of
\( \Pi X_n \) restricted to \( X_\infty \) and \( \overline{\pi}_n \) be the n-th projection
function of \( \Pi Y_n \) restricted to \( Y_\infty \). Let \( U = \pi_n^{-1}(U_n) \)
where \( U_n \) is open in \( X_n \). If \( x \in U \), then \( x_n \in U_n \). Hence,
\[ Q_n(x_n) \in Q_n(U_n). \]
Since \( Q_\infty(x) = \{Q_n(x_n)\}_{n=1}^\infty \),
\[ Q_\infty(x) \in \overline{\pi}_n^{-1}(Q_n(U_n)). \]
Therefore,
\[ Q_\infty(U) \subseteq \overline{\pi}_n^{-1}(Q_n(U_n)). \]
Let \( x \in \overline{\pi}_n^{-1}(Q_n(U_n)) \). Then, \( x_n \in Q_n(U_n) \). Hence, there
exists \( k_n \in U_n \) such that \( Q_n(k_n) = x_n \). But \( x \in Y_\infty \). Conse-
quently, \( x_m \in Y_m \) for each \( m \) and \( g_{mn}(X_m) = x_n \) for all \( n \leq m \).
By the hypothesis, since \( Q_n(k_n) = x_n \) and \( g_{n+1} n(x_{n+1}) = x_n \),
\[ Q_{n+1}^{-1}(x_{n+1}) \cap f_{n+1} n^{-1}(k_n) \neq \emptyset. \]
Let \( k_{n+1} \) be an element of this intersection. By induction,
for each \( m \in Z_+ \) such that \( m > n \), there exists \( k_m \in X_m \) such
that \( Q_m(k_m) = x_m \) and \( f_m m^{-1}(k_m) = k_{m-1} \). Let \( k \in \Pi X_n \)
that for \( n \leq i \) in \( Z_+ \), \( \pi_1(k) = k_i \) and for \( n < i \),
\[ \pi_i(k) = f_{ni}(k_n). \]
Then \( k \in X_\infty \) and \( Q_\infty(k) = \{Q_n(k_n)\}_{n=1}^\infty = x \). Since \( k_n \in U_n \), \( k \in \pi_n^{-1}(U_n) \). Therefore,

\[
Q_\infty(k) = x \in Q_\infty(U).
\]

Hence, \( \pi_n^{-1}(Q_n(U_n)) \subseteq Q_\infty(U) \). It follows that

\[
Q_\infty(U) = \pi_n^{-1}(Q_n(U_n)).
\]

Consequently, \( Q_\infty(U) \) is open in \( Y \). Now, \( Q_\infty \) is open.

The following example proves that requiring each \( Q_n \) to be open is not sufficient to guarantee that \( Q_\infty \) will be open.

**Example 2.2:** Consider Example 1.3. Let \( H \) be

\[
\{f_{nl}^{-1}(\{0^\circ, 180^\circ\})|n \in Z_+ \text{ and } 1 < n\}.
\]

For each \( n \in Z_+ \), let \( X_n \) and \( Y_n \) be the unit circle with \( H \) deleted. Give \( X_n \) and \( Y_n \) the relative topology from the plane. An element of \( X_n \) or \( Y_n \) will be represented the same as in Example 1.3. Let \( f_{n+1}^n: X_{n+1} \to X_n \) be defined by

\[
f_{n+1}^n(\theta) = 2\theta. \quad \text{For } m > n, \text{ let } f_{mn}^n: X_m \to X_n \text{ be defined as } f_{n+1}^n \circ f_{n+2}^n \circ \ldots \circ f_m^m. \text{ Then } (X_n, f_{mn}, Z_+) \text{ is an inverse sequence. Define } g_{n+1}^n: Y_{n+1} + Y_n \text{ by } g_{n+1}^n(\theta) \text{ is } 0 \text{ if } 0^\circ < \theta < 180^\circ \text{ and } 0 \text{ if } 0^\circ < \theta < 360^\circ. \text{ Then } g_{n+1}^n \text{ is continuous. For } m > n \text{ let } g_{mn}: Y_m \to Y_n \text{ be } g_{n+1}^n \circ g_{n+2}^n \circ \ldots \circ g_m^m. \text{ Then } g_{mn} \text{ is continuous. Now } (Y_n, g_{mn}, Z_+) \text{ is an inverse sequence. Define } Q_n: X_n + Y_n \text{ by } Q_n(\theta) \text{ is } 360^\circ - \theta \text{ if } 0^\circ < \theta < 180^\circ \text{ and } 0 \text{ if } 180^\circ < \theta < 360^\circ. \text{ Then each } Q_n \text{ is continuous and open for each } n.
\]
If $0^\circ < \theta < 90^\circ$ then $0^\circ < 2\theta < 180^\circ$ and $270^\circ < 360^\circ - \theta < 360^\circ$.

Therefore,
\[ Q_n \circ f_{n+1} n(\theta) = Q_n(2\theta) = 360^\circ - 2\theta \]

and
\[ g_{n+1} n \circ Q_{n+1} n(\theta) = g_{n+1} n(360^\circ - \theta) = 2(360^\circ - \theta) = 360^\circ - 2\theta. \]

If $90^\circ < \theta < 180^\circ$, then $180^\circ < 2\theta < 360^\circ$ and $180^\circ < 360^\circ - \theta < 270^\circ$.

Therefore,
\[ Q_n \circ f_{n+1} n(\theta) = Q_n(2\theta) = 2\theta \]

and
\[ g_{n+1} n \circ Q_{n+1} n(\theta) = g_{n+1} n(360^\circ - \theta) = 360^\circ - 2(360^\circ - \theta) = 2\theta. \]

If $180^\circ < \theta < 270^\circ$, then $0^\circ < 2\theta < 180^\circ$. Hence,
\[ Q_n \circ f_{n+1} n(\theta) = Q_n(2\theta) = 360^\circ - 2\theta \]

and
\[ g_{n+1} n \circ Q_{n+1} n(\theta) = g_{n+1} n(\theta) = 360^\circ - 2\theta. \]

If $270^\circ < \theta < 360^\circ$, then $180^\circ < 2\theta < 360^\circ$. Therefore,
\[ Q_n \circ f_{n+1} n(\theta) = Q_n(2\theta) = 2\theta \]

and
\[ g_{n+1} n \circ Q_{n+1} n(\theta) = g_{n+1} n(\theta) = 2\theta. \]

Now $Q$ is a mapping.

Let $\pi_1$ be the 1-th projection of $\Pi X_n$ restricted to $X$ and $\pi_1'$ be the 1-th projection of $\Pi Y_n$ restricted to $Y_\infty$.

Let $V = \pi_1'^{-1}(U)$ where $U$ is open in $X_1$ and $140^\circ \in U$. Then,
\[ x = (140^\circ, 70^\circ, 35^\circ, \ldots) \in V. \]
\[ Q_\infty(x) = (220^\circ, 290^\circ, 325^\circ, \ldots) \in Q_\infty(V). \]
Let $Q_\infty(x) = y$. For each open set $H$ of $Y_\infty$ such that $y \in H$, there exists $H_1$ open in $Y_1$ such that $y \in \pi_1^{-1}(H_1) \subseteq H$.

Since $y \in \pi_1^{-1}(H_1)$, $y_1 \in H_1$. But $y_1$ is such that $180^\circ < y_1 < 360^\circ$. Therefore, there is $0 \in Y_{i+1}$ such that $0^\circ < \theta < 90^\circ$ and $4\theta = y_1$. Hence, $g_{i+1}(\theta) = y_1$. But $0^\circ < \theta < 90^\circ$. Hence, $0^\circ < \frac{\theta}{4} < 90^\circ$. Therefore,

$$g_{i+2 \cdot i+1}(\frac{\theta}{4}) = \theta.$$  

By induction,

$$z = (220^\circ, 290^\circ, \ldots, y_1, \frac{y_1}{4}, \frac{y_1}{16}, \ldots) \in Y_\infty.$$  

But $z_1 = y_1$. Hence, $z \in \pi_1^{-1}(H_1) \subseteq H$. Now $z \notin Q_\infty(V)$. Therefore, $H \cap (Y_\infty \setminus Q_\infty(V)) \neq \emptyset$ for each $H$. Hence $Q_\infty(V)$ is not open in $Y_\infty$. Therefore, $Q_\infty$ is not open.

If each of $(X_\alpha, f_\beta\alpha, D)$ and $(Y_\alpha, g_\beta\alpha, D)$ is an inverse system of compact Hausdorff spaces and $Q$ is a continuous mapping from $(X_\alpha, f_\beta\alpha, D)$ to $(Y_\alpha, g_\beta\alpha, D)$ then $Q_\infty$ will be closed. Example 2.3 illustrates that requiring $Q$ to be closed will not guarantee that $Q_\infty$ is closed.

Example 2.3: For each $n \in \mathbb{Z}^+$, let $X_n = \mathbb{Z}^+$. Give $X_n$ the discrete topology. Define $f_{n+1} n : X_{n+1} \to X_n$ to be the identity function. For $m > n$, let $f_{mn} : X_m \to X_n$ be $f_{n+1} n \circ f_{n+2} n+1 \circ \ldots \circ f_m m-1$. Then $(X_n, f_{mn}, \mathbb{Z}^+)$ is an inverse sequence. For each $n \in \mathbb{Z}^+$, let

$$Y_n = \{\frac{1}{p} | 1 \leq p \leq n\}.$$  

Give $Y_n$ the discrete topology also. Define $g_{n+1} n : Y_{n+1} \to Y_n$ by $g_{n+1} n(\frac{1}{p}) = \frac{1}{p}$ if $\frac{1}{p} \leq \frac{1}{n}$, and $\frac{1}{n}$ if $\frac{1}{p} < \frac{1}{n}$. For $m > n$, let
$g_{m+n} : X_m \rightarrow X_n$ be $g_{n+1} \circ g_{n+2} \circ \cdots \circ g_m \circ m-1$. Then $g_{m+n}$ is continuous and $(Y_n, g_{m+n}, Z_+)$ is an inverse sequence. For each $n$, let $Q_n : X_n \rightarrow Y_n$ be defined by $Q_n(z) = \frac{1}{z}$ if \( z \leq n \) and $\frac{1}{n}$ if $z > n$. Then each $Q_n$ is continuous and closed. Let $t \in X_{n+1}$ such that $t < n+1$. Then, $t \leq n$ and $\frac{1}{n+1} < \frac{1}{t}$. Hence,

$$Q_n \circ f_{n+1} n(t) = Q_n(t) = \frac{1}{t}$$

and

$$g_{n+1} n \circ Q_n(t) = g_{n+1} n \left( \frac{1}{t} \right) = \frac{1}{n}.$$

If $t \in X_{n+1}$ and $t = n+1$ then,

$$Q_n \circ f_{n+1} n(t) = Q_n(t) = \frac{1}{n}.$$ 

Now,

$$g_{n+1} n \circ Q_n(t) = g_{n+1} n \left( \frac{1}{n} \right) = \frac{1}{n}.$$ 

If $t \in X_{n+1}$ and $t > n+1$, then $t > n$. Therefore,

$$Q_n \circ f_{n+1} n(t) = Q_n(t) = \frac{1}{n}.$$ 

But

$$g_{n+1} n \circ Q_n(t) = g_{n+1} n \left( \frac{1}{n+1} \right) = \frac{1}{n}.$$ 

Now $Q$ is a mapping.

An element $N$ of $X_\infty$ is described by $N = (n, n, n, \ldots)$, and $Q_\infty(N) = (1, \frac{1}{2}, \ldots, \frac{1}{n-1}, \frac{1}{n}, \frac{1}{n}, \ldots)$. Now

$$y = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots) \in Y_\infty.$$ 

A basic open set $U$ of $y$ is $\varpi_n^{-1}(U_n)$ where $\frac{1}{n} \in U_n$. But $N = (n, n, n, \ldots) \in X_\infty$, and $Q_\infty(n, n, n, \ldots) = (1, \frac{1}{2}, \ldots, \frac{1}{n-1}, \frac{1}{n}, \frac{1}{n}, \ldots)$ where $\frac{1}{n}$ is the $n$-th coordinate. Therefore $Q_\infty(N) \notin \varpi_n^{-1}(U_n)$. Consequently, $Q_\infty(X_\infty)$ is not closed. Hence $Q_\infty$ is not closed.
Definition 2.3: A space $X$ is totally disconnected provided it has no nondegenerate components.

Definition 2.4: A function $f$ from a space $X$ to a space $Y$ is called light if for each $y \in Y$, $f^{-1}(y)$ is totally disconnected.

Theorem 2.6: Let $(X_{\alpha}, f_{\alpha}, D)$ and $(Y_{\alpha}, g_{\beta_{\alpha}}, D)$ be inverse systems and $Q$ be a continuous mapping from $(X_{\alpha}, f_{\beta_{\alpha}}, D)$ to $(Y_{\alpha}, g_{\beta_{\alpha}}, D)$. If $Q$ is light then $Q_{\infty}$ is light.

Proof: Let $\pi_{\alpha}$ and $\overline{\pi}_{\beta}$ be the respective projection functions of $X_{\infty}$ and $Y_{\infty}$. Suppose that there exists $t \in Y_{\infty}$ such that $Q_{\infty}^{-1}(t)$ is not totally disconnected. Then there exists $H$, a nondegenerate connected subset of $Q_{\infty}^{-1}(t)$. For each $\alpha$, $\pi_{\alpha}(H)$ is connected. If $x \in H$, then $x_{\alpha} \in \pi_{\alpha}(H)$.

But, $Q_{\alpha}(x_{\alpha}) = t_{\alpha}$. Consequently, $x_{\alpha} \in Q_{\alpha}^{-1}(t_{\alpha})$. Since $Q_{\alpha}$ is light, $Q_{\alpha}^{-1}(t_{\alpha})$ contains no nondegenerate connected subsets. Hence, $\pi_{\alpha}(H)$ must be a single point. Let

$$\pi_{\alpha}(H) = k_{\alpha}.$$ 

If $x$ and $y$ are elements of $H$ such that $x \neq y$, then, for some $\alpha$, $x_{\alpha} \neq y_{\alpha}$. As a result, $\pi_{\alpha}(H)$ is not a single point. This is a contradiction to $\pi_{\alpha}(H) = k_{\alpha}$, a single point.

It then follows that $Q_{\infty}$ is light.

Although Theorem 2.7 breaks the tradition of investigating $Q_{\infty}$, it is interesting and will be useful later.

Theorem 2.7: Let $(X_{\alpha}, f_{\beta_{\alpha}}, D)$ be an inverse system. If $X_{\alpha}$ is totally disconnected for each $\alpha$, then $X_{\infty}$ is totally disconnected.
Proof: Suppose that there exists \( H \), a nondegenerate connected subset of \( X_\infty \). Then, \( \tau_\alpha(H) \) is connected for each \( \alpha \). Hence, \( \pi_\alpha(H) \) must be a one point set. But, \( H \) is nondegenerate. Consequently, there exists \( x \) and \( y \) in \( H \) such that \( x \neq y \). Therefore, there exists \( \beta \in D \) such that \( x_\beta \neq y_\beta \). As a result, \( \{x_\beta, y_\beta\} \subseteq \pi_\beta(H) \). This is a contradiction to \( \pi_\beta(H) \) being a single point. It then follows that \( X_\infty \) is totally disconnected.

Definition 2.5: A function \( f \) from a space \( X \) to a space \( Y \) is compact if for each compact subset \( K \) of \( Y \), \( f^{-1}(K) \) is compact.

Theorem 2.8: Let \((X_\alpha, f_\beta, D)\) and \((Y_\alpha, \varepsilon_\beta, D)\) be inverse systems of Hausdorff spaces. If \( Q: (X_\alpha, f_\beta, D) \rightarrow (Y_\alpha, \varepsilon_\beta, D) \) is continuous and compact, then \( Q_\infty \) is compact.

Proof: Let \( K \) be a compact subset of \( Y_\infty \). Let \( \pi_\alpha \) and \( \tau_\alpha \) be the respective projection functions of \( X_\infty \) and \( Y_\infty \). Since \( \tau_\alpha \) is continuous for each \( \alpha \), \( \tau_\alpha(K) \), denoted by \( K_\alpha \), is compact. However, \( Q_\alpha \) is compact for each \( \alpha \). Therefore, each \( Q_\alpha^{-1}(K_\alpha) \) is a compact subset of \( X_\alpha \). Hence, \( \Pi(Q_\alpha^{-1}(K_\alpha)) \) is a compact subset of \( \Pi X_\alpha \). Let \( t \in Q_\alpha^{-1}(K_\alpha) \). Then, there exists \( w \in K \) such that \( Q_\infty(t) = w \). For each \( \alpha \),

\[ Q_\alpha(t_\alpha) = w_\alpha \in K_\alpha. \]

Consequently, \( t_\alpha \in Q_\alpha^{-1}(K_\alpha) \). As a result, \( t \in \Pi(Q_\alpha^{-1}(K_\alpha)) \). It then follows that \( Q_\alpha^{-1}(K) \subseteq \Pi(Q_\alpha^{-1}(K_\alpha)) \). Since \( K \) is compact and \( Y_\infty \) is Hausdorff, \( K \) is closed in \( Y_\infty \). Therefore,
$Q^{-1}_\infty(K)$ is closed. But, $Q^{-1}_\infty(K) \subseteq \Pi(Q^{-1}_\infty(K'))$ which is compact. Hence, $Q^{-1}_\infty(K)$ is compact. It is then evident that $Q^{-1}_\infty$ is compact.

Now that it is understood how certain properties can be built into $Q^{-1}_\infty$, it will be helpful, in characterizing compact metric spaces, to examine the following seemingly unrelated material.

**Definition 2.6:** If $x$ and $p$ are elements of a space $X$, then $x$ and $p$ are **separate** in $X$ if there exist open disjoint sets $H$ and $K$ such that $x \in H$, $y \in K$ and $X = H \cup K$.

**Definition 2.7:** For an element $p$ in a space $X$, $C_p = \{x \in X \text{ such that } x \text{ and } p \text{ cannot be separated in } X\}$ is called the **quasi-component** of $p$.

**Theorem 2.9:** If $p \in X$, then $C_p = \cap \mathcal{K}$ where

$$\mathcal{K} = \{U | p \in U \text{ and } U \text{ is both open and closed in } X\}.$$

**Proof:** Let $x \in C_p$. If $U \in \mathcal{K}$ and $x \notin U$, then $U$ is open, $X \setminus U$ is open, $p \in U$ and $x \in X \setminus U$. Since $X = U \cup (X \setminus U)$, $U$ and $X \setminus U$ separate $x$ and $p$. This contradicts $x \in C_p$. Hence, $x \in U$. Therefore, $C_p \subseteq \cap \mathcal{K}$.

Let $x \in \cap \mathcal{K}$. If $x \notin C_p$, then there exist disjoint sets $W$ and $V$, open and closed in $X$, such that $p \in W$, $x \in V$ and $X = W \cup V$. Now, $p \in W$ and $W$ is open and closed in $X$. Hence, $W \in \mathcal{K}$. Since $x \in V$, $x \notin W$. Therefore, $x \notin \cap \mathcal{K}$. This contradicts the supposition that $x \in \cap \mathcal{K}$. Consequently, $x \in C_p$. It then follows that $\cap \mathcal{K} \subseteq C_p$. As a result, $C_p = \cap \mathcal{K}$. 
It is now revealed in Theorem 2.10 that in a compact Hausdorff space components and quasi-components are identical.

Theorem 2.10: If $p$ is an element of $X$, a compact Hausdorff space, then $C_p$ is a component.

Proof: Let $z \in X\setminus C_p$. Then, there exists disjoint sets $U$ and $V$, open in $X$, such that $X = U \cup V$, $z \in V$ and $p \in U$. Suppose that $w \in V \cap C_p$. Then $p \in U$ and $w \in V$. Hence, $w \notin C_p$. Therefore, $V \cap C_p = \emptyset$. Now, $z \in V \setminus X \setminus C_p$. Consequently, $X\setminus C_p$ is open. As a result, $C_p$ is closed in $X$. It then follows that $C_p$ is compact.

Suppose that $C_p$ is not connected. Then, there exists $H$ and $K$, disjoint, closed, nonempty subsets of $C_p$, such that $C_p = H \cup K$. Since $H$ and $K$ are closed in $C_p$, $H$ and $K$ are both compact. Therefore, there exist disjoint sets $U$ and $V$, open in $X$, such that $H \subseteq U$ and $K \subseteq V$. But $X \setminus (U \cup V)$ is closed and compact. If $x \in X \setminus (U \cup V)$, then $x \notin C_p$. Hence, there exist disjoint sets $U_x$ and $V_x$, open and closed in $X$, such that $X = U_x \cup V_x$, $p \in V_x$ and $x \in U_x$. Also, $C_p \subseteq V_x$. Now, $U = \{U_x \mid x \in X \setminus (U \cup V)\}$ covers $X \setminus (U \cup V)$. Consequently, there is a finite subcollection of $U$ covering $X \setminus (U \cup V)$. Let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be this finite cover, $M = \bigcap_{i=1}^{n} V_i$. Then, $C_p \subseteq M$. Since
each \( V_i \) is open and closed in \( X \), \( M \) is open and closed in \( X \).

Let \( N = \bigcup_{i=1}^{n} U_i \). It then follows that \( X \setminus (U \cup V) \subseteq N \) and \( N \) is open and closed in \( X \). Clearly, \( M \cap N = \emptyset \). If \( z \in M \), \( z \notin N \). Consequently, \( z \notin X \setminus (U \cup V) \). As a result, \( z \in (U \cup V) \).

Therefore, \( M \subseteq (U \cup V) \). Since \( C_p \subseteq M \), \( C_p \subseteq (M \cap U) \cup (M \cap V) \).

Each of \( M \) and \( U \) is open in \( X \). Consequently, \( M \cap U \) is open in \( X \). Suppose that \( M \cap U \) is not closed in \( X \). Then, there exists \( z \in X \setminus (M \cap U) \) such that if \( z \in H \) and \( H \) is open in \( X \), then \( H \cap (M \cap N) \neq \emptyset \). But \( V \) is open and \( U \cap V = \emptyset \). Hence, \( V \cap (M \cap U) = \emptyset \). Therefore, \( z \notin V \). If \( z \notin U \), then \( z \notin U \cup V \).

As a result, \( z \in X \setminus (U \cup V) \). It then follows that \( z \in N \).

However, \( N \) is open and \( N \cap M = \emptyset \). Consequently, \( N \cap (M \cap U) = \emptyset \).

This contradicts the statement that if \( H \) is open and \( z \in H \) then \( H \cap (M \cap U) \neq \emptyset \). Therefore, \( z \notin N \). It then follows that \( z \in U \).

If \( z \in N \), then \( z \in M \cap U \). However, \( z \notin X \setminus (M \cap U) \).

Hence, \( z \notin M \). Hence, \( z \in X \setminus M \) which is open in \( X \). But \( (X \setminus M) \cap M = \emptyset \). Therefore, \( (X \setminus M) \cap (M \cap U) = \emptyset \). This is also a contradiction to the previous statements about \( z \).

It now follows that \( M \cap U \) is closed in \( X \). Now
\[
X = (M \cap U) \cup (X \setminus (M \cap U)).
\]

Hence, \( C_p \subseteq M \cap U \) or \( C_p \subseteq X \setminus (M \cap U) \). If \( C_p \subseteq M \cap U \), then \( C_p \subseteq U \). Since \( U \cap V = \emptyset \), \( C_p \cap V = \emptyset \). However, \( C_p = H \cup K \) and \( K \subseteq V \). Consequently, \( K = \emptyset \). This contradicts the supposition that \( K \neq \emptyset \). Therefore, \( C_p \subseteq X \setminus (M \cap U) \). Hence,
\( C_p \cap (M \cap U) = \emptyset \). But \( C_p \subseteq M \). Thus, \( C_p \cap U = \emptyset \). Since \( H \subseteq U \), \( H = \emptyset \). This contradicts \( H \neq \emptyset \). It now follows that \( C_p \) is connected.

If \( C \) is a connected subset of \( X \) and \( C_p \) is a proper subset of \( C \), then there exists \( x \in C \) such that \( x \notin C_p \).

Since \( x \notin C_p \), there exists disjoint sets \( U_x \) and \( V_x \), open and closed in \( X \), such that \( x \in U_x \), \( p \in V_x \) and

\[ X = U_x \cup V_x. \]

But \( U_x \) and \( V_x \) separate \( C \). Therefore, \( C \) is not connected.

This contradicts the supposition that \( C \) is connected.

Consequently, \( C_p \) is a component.

**Theorem 2.11:** A compact Hausdorff space \( X \) is totally disconnected if and only if for distinct points \( x \) and \( y \) in \( X \), there is a set \( U \), open and closed in \( X \), such that \( x \in U \) and \( y \notin U \).

**Proof:** Let \( x \) and \( y \) be distinct points of a totally disconnected, compact Hausdorff space \( X \). Since \( X \) is compact and Hausdorff, \( C_x \) is a component. But, since \( X \) is totally disconnected, \( C_x = \{x\} \). Now \( C_x = \bigcap \mathcal{K} \) where

\[ \mathcal{K} = \{U \subseteq X| x \in U \text{ and } U \text{ is open and closed in } X \}. \]

Since \( y \notin C_x \), there exists \( U \in \mathcal{K} \) such that \( y \notin U \). Hence, \( x \in U \), \( y \notin U \) and \( U \) is open and closed in \( X \).

Let \( X \) be a compact Hausdorff space such that for distinct points \( x \) and \( y \) in \( X \), there exists a set \( U \), open and closed in \( X \), such that \( x \in U \) and \( y \notin U \). Suppose that \( X \) is not totally disconnected. Then, there exists \( H \), a
nondegenerate component of $X$. Let $x$ and $y$ be distinct elements of $H$. By supposition, there exists an open and closed subset $U$ of $X$ such that $x \in U$ and $y \notin U$. Hence, $y \in X \setminus U$ and $X \setminus U$ is open in $X$. However, $U \cap H$ is open in $H$ and $(X \setminus U) \cap H$ is open in $H$. Consequently,

$$H = (U \cap H) \cup ((X \setminus U) \cap H),$$

$x \in U \cap H$, $y \in (X \setminus U) \cap H$ and $(U \cap H) \cap ((X \setminus U) \cap H) = \emptyset$. Therefore, $H$ is not connected. This contradicts the statement that $H$ is a component. Hence, $X$ is totally disconnected.

Theorem 2.12: If $(X,d)$ is a compact metric space and $\mathcal{U}$ is finite open cover of $X$, then there exists $c > 0$ such that if $V$ is an open set of $X$ having diameter less than $c$, then $V$ is a subset of some element of $\mathcal{U}$.

Proof: Let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a finite open cover of $X$. Suppose that for each $c > 0$, there exists $V_c$, an open set of diameter less than $c$, such that $V_c \subseteq U_i$ for $1 \leq i \leq n$. For each $m \in Z_+$, let $x_m \in V_m$ where $V_m$ is an open set of diameter less than $\frac{1}{m}$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ has a cluster point $x$. Since $x \in X$, $x \in U_i$ for some $i$. There exists $S(x,\delta)$, a spherical open neighborhood of $x$ of diameter less than $\delta$, such that $S(x,\delta) \subseteq U_i$. However, there is $m \in Z_+$ such that $\frac{1}{m} < \frac{\delta}{4}$. Since $\{x_n\}_{n=1}^{\infty}$ clusters at $x$, there exists $p \in Z_+$ such that $p > m$ and $x_p \in S(x,\frac{\delta}{4})$. Let $y \in V_p$. Then $d(y,x) \leq d(y,x_p) + d(x_p,x)$. But, $d(y,x_p) < \frac{1}{p}$ and $d(x_p,x) < \frac{\delta}{4}$. Consequently,

$$d(y,x) < \frac{1}{p} + \frac{\delta}{4} < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2} < \delta.$$
As a result, \( y \in S(x, \delta) \). Hence, \( V_p \subseteq S(x, \delta) \subseteq U_1 \). This contradicts the supposition that \( V_c \notin U_1 \) for any 1. Therefore, the theorem is proven.

Definition 2.8: Let \( U \) and \( V \) be covers of \( X \), then \( U \) refines \( V \), written as \( U \subseteq V \), means that for each \( A \in U \), there exists \( B \in V \) such that \( A \subseteq B \).

The following theorem establishes a connection between inverse systems and theorems 2.9 through 2.12.

Theorem 2.13: If \( (X, d) \) is a totally disconnected, compact, metric space, then \( X \) is homeomorphic to the inverse limit of an inverse sequence of finite spaces.

Proof: Let \( U \) be open in \( X \) and \( x \in U \). Since \( X \setminus U \) is closed in \( X \), it is compact. For each \( y \in X \setminus U \), there exists \( V_y \), open and closed in \( X \), such that \( y \in V_y \) and \( x \notin V_y \). Then \( \mathcal{V} = \{ V_y \mid y \in X \setminus U \} \) covers \( X \setminus U \). Hence, there exists a finite subcollection of \( \mathcal{V} \) covering \( X \setminus U \). Let \( \mathcal{V} = \{ V_1, V_2, \ldots, V_n \} \) be this finite cover. Now, \( \bigcup \mathcal{V} \) is open and closed in \( X \) and does not contain \( x \). Let \( z \in X \setminus \bigcup \mathcal{V} \). Then, \( z \notin \bigcup \mathcal{V} \). Therefore, \( z \notin X \setminus U \). Consequently, \( z \in U \). Hence, \( X \setminus \bigcup \mathcal{V} \subseteq U \). As a result, \( x \in X \setminus \bigcup \mathcal{V} \subseteq U \). Now, \( X \) has a base of open and closed sets.

Let \( n \in \mathbb{Z}_+ \). Then, for each \( x \in X \), there exists \( H_x \) such that \( x \in H_x \) and \( H_x \) is open and closed in \( X \) and has a diameter less than \( \frac{1}{2^n} \). Therefore, \( \mathcal{V} = \{ H_x \mid x \in X \} \) covers \( X \).
Since $X$ is compact, there exists a finite subcollection $\mathcal{K}$ covering $X$. Let

$$\mathcal{K} = \{H_1, H_2, \ldots, H_m\}.$$ 

Let $U_1 = H_1$,

$$U_2 = H_2 \setminus H_1, \ldots, U_n = H_n \setminus \bigcup_{p=1}^{n-1} H_p.$$ 

Continue this process as long as the $U_i$'s remain nonempty. Let

$$\mathcal{U} = \{U_i | 1 \leq i \leq m\}$$ 

be this collection of nonempty $U_i$'s. For each $i$,

$$H_i \setminus \bigcup_{p=1}^{i-1} H_p = (X \setminus \bigcup_{p=1}^{i-1} H_p) \cap H_i.$$ 

Consequently, $U_i$ is both open and closed in $X$. Let $x \in X$. Then, $x \in H_i$ for some $i$. Let $j$ be the least such $i$. Now

$$x \in H_j \setminus \bigcup_{p=1}^{j-1} H_p = U_j.$$ 

Therefore, $\mathcal{U}$ covers $X$. Suppose that $1 \leq j \leq m$ and $1 \leq \ell \leq m$ and $j \neq \ell$. Either $j < \ell$ or $\ell < j$. Suppose that $j < \ell$. Then,

$$U_\ell = H_\ell \setminus \bigcup_{p=1}^{\ell-1} H_p.$$ 

So, if $x \in U_\ell$, $x \notin H_j$. Hence,

$$x \notin H_j \setminus \bigcup_{p=1}^{j-1} H_p = U_j.$$ 

If $x \in U_j$, since $j < \ell$, $x \in \bigcup_{p=1}^{\ell-1} H_p$. As a result, $x \in H_\ell \setminus \bigcup_{p=1}^{\ell-1} H_p$.

Hence, $x \notin U_\ell$. Therefore $U_j \cap U_\ell = \emptyset$. Similarly, if $\ell < j$, ...
Then $U\in\mathcal{U}$ is a cover of mutually disjoint, closed and open sets each having diameter less than $\frac{1}{2^n}$. By Theorem 2.12, there exists $\delta > 0$ such that if $V$ is a subset of $X$ of diameter less than $\delta$, then $V \subseteq U \in \mathcal{U}$. Let $\lambda$ be less than the minimum of $\{\delta, \frac{1}{2^{n+1}}\}$. By the previous construction, there exists a finite open cover of $X$ by mutually disjoint, open and closed sets of diameter less than $\lambda$.

Let $\mathcal{V}$ be this cover. Then, $\mathcal{V} \subset \mathcal{U}$. By induction, for each $n \in \mathbb{Z}^+$, there exists $\mathcal{U}_n$, a cover of $X$ by mutually disjoint, open and closed sets of diameter less than $\frac{1}{2^n}$ such that $\mathcal{U}_{n+1} \subset \mathcal{U}_n$. Let $\{\mathcal{U}_n\}_{i=1}^\infty$ be a sequence of such covers. For each $n$, let $Y_n$ be $\{\mathcal{U}_n\}$ with the discrete topology. Define $f_{n+1}:Y_{n+1} \rightarrow Y_n$ by $f_{n+1}(U) = V$ where $V$ is the unique element of $Y_n$ such that $U \subseteq V$. Then, each $f_{n+1}$ is a continuous function. Let $f_{mn}:X_n \rightarrow X_n$ be the identity map and for $m > n$, $f_{mn} = f_{n+1} \circ f_{n+2} \circ \cdots \circ f_m \circ f_{m-1}$.

Then, $(Y_n, f_{mn}, Z^+)$ is an inverse sequence. Let $Y_\infty = \lim_{n \to \infty}(Y_n, f_{mn}, Z^+)$ and $y \in Y_\infty$ where $y = \{U_i\}_{i=1}^\infty$ such that $U_i \in \mathcal{U}_i$. Define $F:Y_\infty \rightarrow X$ by $F(y) = F(\{U_i\}_{i=1}^\infty) = \bigcap_{i=1}^\infty U_i$.

Since $f_{n+1}(U_{n+1}) = U_n$, $U_{n+1} \subseteq U_n$. But $X$ is compact and
each $U_i$ is closed. Therefore, $U_i$ is compact. By Lemma 1.1,

$$\bigcap_{i=1}^{\infty} U_i \neq \emptyset.$$  

If $x$ and $y$ are elements of $\bigcap_{i=1}^{\infty} U_i$ such that $x \neq y$, then there is $n \in \mathbb{Z}_+^*$ such that $\frac{1}{2^n} < d(x, y)$. If $x \in U_n$, $y \notin U_n$. Hence, if $x \in \bigcap_{i=1}^{\infty} U_i$, $y \notin \bigcap_{i=1}^{\infty} U_i$. Therefore, $\bigcap_{i=1}^{\infty} U_i$ is a single point. Consequently, if $F(y) = p$ and $F(y) = q$, then $p = q$. As a result, $F$ is a function.

Suppose that $F(X) = F(y)$ where $x = \{U_i\}_{i=1}^{\infty}$ and $y = \{V_i\}_{i=1}^{\infty}$. If $U_i \neq V_i$ for some $i$, then $U_i \cap V_i = \emptyset$. Hence,

$$\bigcap_{i=1}^{\infty} U_i \neq \bigcap_{i=1}^{\infty} V_i.$$  

This contradicts $F(x) = F(y)$. Consequently, $U_i = V_i$ for each $i$. Therefore, $x = y$. It then follows that $F$ is one to one.

Let $x \in X$. Then, for each $n$, there exists $U_n \in \mathcal{U}_n$ such that $x \in U_n$ and $U_n$ is unique in $Y_n$. Let $y \in \Pi Y_n$ such that $P_n(y) = U_n$ for each $n$. Suppose that for some $m$, $f_{m+1} m(U_{m+1}) \neq U_m$. Then

$$f_{m+1} m(U_{m+1}) \cap U_m = \emptyset.$$  

But $x \in U_{m+1}$. Therefore, $x \in f_{m+1} m(U_{m+1})$. Since $x \in U_m$,

$$f_{m+1} m(U_{m+1}) \cap U_m \neq \emptyset.$$  

This is a contradiction to $f_{m+1} m(U_{m+1}) \cap U_m = \emptyset$. Consequently,

$$f_{m+1} m(U_{m+1}) = U_m.$$
As a result, \( y \in Y_\infty \). But
\[
F(y) = \bigcap_{i=1}^{\infty} U_i = x.
\]
Hence, \( F \) is onto.

Let \( t = \{w_i\}_{i=1}^{\infty} \in Y \) and \( Z \) be an open set of \( X \) such that \( F(t) \subseteq Z \). Then, there exists a basic open neighborhood \( U = S(F(t), \delta) \), the spherical neighborhood about \( F(t) \) of radius \( \delta \), such that \( F(t) \subseteq U \subseteq Z \). Let
\[
H = \{y = \{U_i\}_{i=1}^{\infty} \in Y_\infty | \text{there exists } n_y \in \mathbb{Z}_+ \text{ such that } U_{n_y} \subseteq U\}.
\]
Since \( t = \{w_i\}_{i=1}^{\infty} \) and \( F(t) = \{ \bigcap_{i=1}^{\infty} W_i \} \subseteq U \), by Lemma 1.2, there exists \( n \in \mathbb{Z}_+ \) such that \( w_n \subseteq U \). Therefore, \( t \in H \).

Hence, \( H \neq \emptyset \). Let \( y = \{U_i\}_{i=1}^{\infty} \in H \). Then, for some \( j \), \( U_j \subseteq U \). But \( F(y) = \bigcap_{i=1}^{\infty} U_i \subseteq U_j \). As a result, \( F(y) \subseteq U \).

Consequently, \( F(H) \subseteq U \). Let
\[
z = \{V_i\}_{i=1}^{\infty} \in H.
\]
Then for some \( n \), \( V_n \subseteq U \). For each \( i \), \( \{V_i\} \) is open in \( Y_i \).

Hence, \( \pi^{-1}_n(\{V_n\}) \) is open in \( Y_\infty \) and contains \( z \). If
\[
s = \{s_i\}_{i=1}^{\infty} \in \pi^{-1}_n(\{V_n\}),
\]
then \( \{s_n\} \subseteq \{V_n\} \). Therefore, \( s_n = V_n \). But \( V_n \subseteq U \). Since \( s_n = V_n \), \( s_n \subseteq U \). It then follows that \( s \in H \). Now, \( H \) is open, \( t \in H \) and \( F(H) \subseteq U \subseteq Z \). Consequently, \( F \) is continuous.

Now, \( F \) is a continuous function from a compact space onto a Hausdorff space. Hence, \( F \) is a homeomorphism. Therefore, \( X \) is homeomorphic to \( Y_\infty \), the inverse limit of an inverse sequence of finite spaces.
**Definition 2.9:** A space $X$ is perfect if each element of $X$ is a limit point of $X$.

**Theorem 2.14:** If $U$ is a nonempty open subset of $X$, a compact, Hausdorff, totally disconnected, perfect space and if $n$ is a positive integer, then $U$ is the union of exactly $n$ open, mutually disjoint, nonempty sets.

**Proof:** Since $U$ is nonempty, there exists $x \in X$ such that $x \in U$. Since $X$ is perfect, $x$ is a limit point of $X$. Therefore, there exists $y \in U$ such that $y \neq x$. Since $X$ is compact, Hausdorff and totally disconnected by Theorem 2.11, there exists a set $V$, open and closed in $X$, such that $x \in V$ and $y \notin V$. Now, $x \in V \cap U$ which is open in $X$ and $y \in (X \setminus V) \cap U$ which is also open in $X$. Let $V \cap U = U_1$ and $(X \setminus V) \cap U = U_2$. Then $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$, and $U_1 \neq \emptyset \neq U_2$. By induction, for any $n \in \mathbb{Z}^+$, $U$ is the union of $n$ disjoint, open, nonempty sets.

**Theorem 2.15:** Any two compact, perfect, totally disconnected, metric spaces are homeomorphic.

**Proof:** Let $X$ and $Y$ satisfy the above hypothesis. For a set $H$, let $|H|$ denote the cardinality of $H$. For each $n \in \mathbb{Z}^+$, let $\mathcal{U}_n$ and $\mathcal{V}_n$ be finite covers of $X$ and $Y$, respectively, of mutually disjoint, closed and open sets such that each element of $\mathcal{U}_n$ and each element of $\mathcal{V}_n$ has diameter less than $\frac{1}{2^n}$ and $\mathcal{U}_{n+1} \subset \mathcal{U}_n$ and $\mathcal{V}_{n+1} \subset \mathcal{V}_n$. By Theorem 2.14, $|\mathcal{U}_n|$ can be made the same as $|\mathcal{V}_n|$ for each $n$. For each
Let $X_n$ be $\{U_n\}$ with the discrete topology and $Y_n$ be $\{V_n\}$ with the discrete topology. Let

$$X_\infty = \lim_{n \to \infty} (X_n, f_{mn}, Z_+)$$

and

$$Y_\infty = \lim_{n \to \infty} (Y_n, g_{mn}, Z_+)$$

be the same as in Theorem 2.13. Then, $X$ is homeomorphic to $X_\infty$ and $Y$ is homeomorphic to $Y_\infty$. Define $Q_n: X_n \to Y_n$ by

$$Q_n(U_i) = V_1,$$

for each $n$. Then, each $Q_n$ is a homeomorphism. Now,

$$g_{n+1} n o Q_{n+1}(U_{n+1}) = g_{n+1} n (V_{n+1}) = V_n$$

and

$$Q_n o f_{n+1} n (U_{n+1}) = Q_n(U_n) = V_n.$$

Therefore, by Theorem 2.1, $Q_\infty: X_\infty \to Y_\infty$ is a homeomorphism. Hence, $X$ is homeomorphic to $Y$.

**Theorem 2.16:** Every compact metric space is the continuous image of the Cantor set $C$.

**Proof:** Let $(X,d)$ be a compact metric space. Let

$$\mathcal{U}_1 = \{A_{11}, A_{21}, \ldots, A_{n_1 1}\}$$

be a cover of $X$ by closed sets of diameter less than $\frac{1}{2}$.

For each $i$ such that $1 \leq i \leq n_1$, let $K_i$ be a finite cover of $A_{i1 1}$ by closed sets of diameter less than $\frac{1}{4}$. Since $A_{i1 1}$ is closed in $X$, it can be assumed that each element of $K_i$ is a closed subset of $A_{i1 1}$. Let $\mathcal{U}_2 = \bigcup_{i=1}^{n_1} K_i$ be denoted by

$$\mathcal{U}_2 = \{A_{12}, A_{22}, \ldots, A_{n_2 2}\}. $$
Then $\mathcal{U}_2 \subsetneq \mathcal{U}_1$ and if $x \in A_{i1}$, there exists $A_{j2} \in \mathcal{U}_2$ such that $x \in A_{j2} \subseteq A_{i1}$. By continuing this process, a sequence $\{ \mathcal{U}_i \}_{i=1}^{\infty}$ of finite covers of $X$ by closed sets can be obtained. An element of $\mathcal{U}_n$ has diameter less than $\frac{1}{2^n}$. Each $\mathcal{U}_n$ has the properties that $\mathcal{U}_{n+1} \subsetneq \mathcal{U}_n$ and if $x \in A_{in} \in \mathcal{U}_n$ then there exists $A_{j_{n+1}} \in \mathcal{U}_{n+1}$ such that $x \in A_{j_{n+1}} \subseteq A_{in}$.

For each $m$, let

$$\mathcal{U}_m = \{ A_{1m}, A_{2m}, \ldots, A_{nm} \}.$$ 

For each $i$ such that $1 \leq i \leq n_1$, let

$$B(i,1) = \{ (x,i,1) \mid x \in A_{i1} \}.$$ 

Let $K$ be open in $B(i,1)$ if and only if $\{ x \mid (x,i,1) \in K \}$ is open in $A_{i1}$. Let

$$H_1 = \bigcup_{i=1}^{n_1} B(i,1).$$

Define a set $V$ to be open in $H_1$ if and only if $V \cap B(j,1)$ is open in $B(j,1)$ for each $j$.

For each $A_{j2}$ such that $A_{j2} \subseteq A_{p1}$, let

$$B(j,p,2) = \{ (x,j,p,2) \mid x \in A_{j2} \}.$$ 

Let

$$H_2 = \bigcup_{p=1}^{n_1} (B(j,p,2) \mid A_{j2} \subseteq A_{p1}).$$

Define open sets in the same manner as for $B(i,1)$ and $H_1$.

For each $A_{k3}$ such that $A_{k3} \subseteq A_{j2} \subseteq A_{p1}$, let

$$B(k,j,p,3) = \{ (x,k,j,p,3) \mid x \in A_{k3} \}.$$ 

Let

$$H_3 = \bigcup_{p=1}^{n_1} (B(k,j,p,3) \mid A_{k3} \subseteq A_{j2} \subseteq A_{p1}).$$
Define open sets in the same manner as for $B(j,2)$ and $H_2$. By continuing this process, a sequence of sets $\{H_n\}_{i=1}^{\infty}$ is obtained. Now,

$$H_n = \bigcup_{i=1}^{n}(\bigcup B(j_n,j_{n-1},\ldots,j_1,n) \mid A_{j_n} \subseteq A_{j_{n-1}} \subseteq \ldots \subseteq A_{j_1}).$$

The method used to define open sets of $H_n$ guarantees that each $B(i_n,i_{n-1},\ldots,i_1,n)$ is both open and closed in $H_n$. It is also apparent that the B's are disjoint. Since $H_n$ is the union of the B's and the B's are open, $H_n$ is the union of the open sets defined on $H_n$. Let $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in D}$ be a collection of open sets of $H_n$ indexed by D. Let

$$K = B(i_n,i_{n-1},\ldots,i_1,n).$$

Since each $U_{\alpha}$ is open in $H_n$,

$$M_{\alpha} = \{x \mid (x,j_n,j_{n-1},\ldots,j_1,n) \in K \cap U_{\alpha}\}$$

is open in $A_{j_n}$, Therefore $\bigcup_{\alpha \in D} M_{\alpha}$ is open in $A_{j_n}$. But

$$\bigcup_{\alpha \in D} M_{\alpha} = \{x \mid (x,j_n,j_{n-1},\ldots,j_1,n) \in K \cap (\bigcup \mathcal{U})\}.$$ 

Hence $\bigcup \mathcal{U}$ is open in $K$. Therefore $\bigcup \mathcal{U}$ is open in $H_n$.

Let $U_1$ and $U_2$ be open in $H_n$. Then for $i \in \{1,2\}$,

$$M_i = \{x \mid (x,j_n,j_{n-1},\ldots,j_1,n) \in U_i \cap K\}$$

is open in $A_{j_n}$. Therefore $M_1 \cap M_2$ is open in $A_{j_n}$. But

$$M_1 \cap M_2 = \{x \mid (x,j_n,j_{n-1},\ldots,j_1,n) \in (U_1 \cup U_2) \cap K\}.$$ 

Hence, $U_1 \cap U_2$ is open in $K$. Therefore, $U_1 \cap U_2$ is open in $H_n$. By induction, this can be extended to the intersection of any finite number of open sets. Now the open sets of $H_n$ form a topology for $H_n$. 
For each $n$, let $Q_n : H_n \to X$ be defined by

$$Q_n((x,j_n,j_{n-1},\ldots,j_1,n)) = x.$$ 

Clearly $Q_n$ is a function from $H_n$ onto $X$. Let

$$y = (x,j_n,j_{n-1},\ldots,j_1,n)$$

be an element of $H_n$ and $V$ be an open set of $X$ containing $x$. Since $V$ is open in $X$, $K = V \cap A_{j_n}^n$ is open in $A_{j_n}^n$. Let

$$U = \{(w,j_n,j_{n-1},\ldots,j_1,n) \, | \, w \in K\}$$

and

$$T = B(j_n,j_{n-1},\ldots,j_1,n).$$

Then, $U \subseteq T$. Since $\{y \, | \, (y,j_n,j_{n-1},\ldots,j_1,n) \in U\} = K$, $U$ is open in $T$. Since the $B$'s are disjoint in $H_n$ and $U$ is an open subset of $T$, $U$ is open in $H_n$. Clearly, $y \in U$. If

$$h = (v,j_n,j_{n-1},\ldots,j_1,n) \in U$$

then $v \in K$. Hence, $v \in V$. Therefore $Q_n(U) \subseteq V$. Consequently, $Q_n$ is continuous.

Define $f_{n+1} : H_{n+1} \to H_n$ by

$$f_{n+1}((x,j_{n+1},j_n,\ldots,j_1,n+1)) = (x,j_n,j_{n-1},\ldots,j_1,n).$$

Let $f_n$ be the identity function. Clearly, $f_{n+1}$ is a function. Let

$$t = (x,j_n,j_{n-1},\ldots,j_1,n) \in H_n.$$ 

Then $x \in A_{j_n}^n$. There exists $A_{j_{n+1}n+1} \in \mathcal{U}_{n+1}$ such that $x \in A_{j_{n+1}n+1} \subseteq A_{j_n}^n$. Then

$$s = (x,j_{n+1},j_n,\ldots,j_1,n+1) \in H_{n+1}$$

and
and \( f_{n+1} \) \( n(s) = t. \) Hence \( f_{n+1} \) \( n \) is onto. Let
\[
y = (x, j_{n+1}, j_n, \ldots, j_1, n+1) \in H_{n+1}
\]
and \( V \) be an open set of \( H_n \) containing \( f_{n+1} \) \( n(y). \) Let
\[
N = B(j_n, j_{n-1}, \ldots, j_1, n).
\]
Then \( f_{n+1} \) \( n(y) \in N. \) Since \( V \) is open in \( H_n, \)
\[
M = \{ z \mid (z, j_n, j_{n-1}, \ldots, j_1, n) \in V \cap N \}
\]
is open in \( A_{j_n} \). Since \( A_{j_n+1} \subseteq A_{j_n}, \) \( K = A_{j_n+1} \cap M \)
is open in \( A_{j_n+1}. \) Also, \( x \in K. \) Let
\[
U = \{ (w, j_{n+1}, j_n, \ldots, j_1, n+1) \mid w \in K \}.
\]
Now \( U \subseteq N \) and \( U \) is open in \( N. \) Hence \( U \) is open in \( H_{n+1}. \) If
\[
s = (t, j_{n+1}, j_n, \ldots, j_1, n+1) \in U,
\]
then \( t \in K. \) Hence, \( t \in A_{j_n+1} \cap M. \) Therefore, \( t \in V. \)

Hence, \( f_{n+1} \) \( n(s) \in V. \) Now, \( f_{n+1} \) \( n(U) \subseteq V. \) Since \( y \in U \)
and \( f_{n+1} \) \( n(y) = x \in V, \) \( f_{n+1} \) \( n \) is continuous. For \( m > n, \)
let \( f_{mn} : H_m \to H_n \) be defined by
\[
f_{mn}(x) = f_{n+1} \circ f_{n+2} \circ \cdots \circ f_{m}(x).
\]
Then \( f_{mn} \) is continuous and onto. Therefore, \( (H_n, f_{mn}, Z_+) \)
is an inverse sequence.

For each \( n \) in \( Z_+, \) let \( X_n = X \) and \( i_{n+1} \) be the identity
function from \( X_{n+1} \) onto \( X_n. \) For \( m > n, i_{mn} : X_m \to X_n \) be
\[
i_{n+1} \circ i_{n+2} \circ \cdots \circ i_{m-1}. \]
Then \( (X_n, i_{mn}, Z_+) \) is an inverse sequence. Further, \( Q_n : H_n \to X_n \) is continuous and
onto. If
\[
y = (v, j_{n+1}, j_n, \ldots, j_1, n+1) \in H_{n+1}
\]
then

$$Q_n \circ f_{n+1}(y) = Q_n((v, j_n, j_{n-1}, \ldots, j_1, n)) = v.$$ 

But

$$i_{n+1} \circ Q_{n+1}(y) = i_{n+1}(v) = v.$$ 

Therefore, by Theorem 2.1, $Q_\infty : H_\infty \rightarrow X_\infty$ is continuous.

Let $w = (x, x, \ldots) \in X_\infty$. Then $x \in A_{i_1}$ for some $i$.

Hence $(x, 1_1, 1) \in H_1$. There exists $A_{i_2} \in \mathcal{U}_2$ such that

$$x \in A_{i_2} \subseteq A_{i_1}.$$ 

Hence $(x, i_2, 1, 2) \in H_2$ and

$$f_{i_1}(x, i_2, 1, 2) = (x, i_1, 1).$$ 

By induction, there is an element $s$ of $H_\infty$ such that the $n$-th projection of $s$ is $(x, 1_n, 1_{n-1}, \ldots, 1_1, n)$. Hence,

$$Q_\infty(s) = w.$$ 

Therefore, $Q_\infty$ is onto. Since $X_\infty$ is homeomorphic to $X$, $X$ is the continuous image of $H_\infty$.

To complete the theorem it will be sufficient to show that $H_\infty$ is the continuous image of $H_\infty \times C$ and that $H_\infty \times C$ is homeomorphic to $C$. Let

$$s = (x, j_n, j_{n-1}, \ldots, j_1, n)$$ 

and

$$t = (y, i_n, i_{n-1}, \ldots, i_1, n)$$ 

be elements of $H_\infty$ such that $s \neq t$. If there is $p$ such that $1 \leq p \leq n$ and $j_p \neq i_p$, then $s$ and $t$ are in different $B$'s.

Hence, there exists disjoint open sets

$$B(i_n, i_{n-1}, \ldots, i_1, n) = T.$$
and
\[ S = B(j_n, j_{n-1}, \ldots, j_1, n) \]
containing \( t \) and \( s \), respectively. If \( j_p = i_p \) for all \( p \)
then \( x \neq y \) and \( s \) and \( t \) are in \( T \). Therefore, \( x \) and \( y \) are
elements of \( A_{j_n} \). Since \( x \neq y \), there exist open disjoint
sets \( M \) and \( N \) of \( A_{j_n} \) such that \( x \in M \) and \( y \in N \). Let
\[ J = \{(v, j_n, j_{n-1}, \ldots, j_1, n) | v \in M \} \]
and
\[ I = \{(w, j_n, j_{n-1}, \ldots, j_1, n) | w \in N \} \].
Then \( s \in J \) and \( t \in I \) and \( J \cap I = \emptyset \). By the method of de-
fining \( I \) and \( J \), each of \( I \) and \( J \) is open in \( T \). Further,
\( J \subseteq T \) and \( I \subseteq T \). Hence, \( I \) and \( J \) are open in \( H_n \). Therefore,
\( H_n \) is Hausdorff.

Let \( \mathcal{V} = \{V_\alpha \}_{\alpha \in D} \) be an open cover of \( H_n \). Let
\[ R = B(j_n, j_{n-1}, \ldots, j_1, n) \]
be one of the disjoint collections of \( H_n \). For each \( \alpha \in D \),
\( V_\alpha \cap R \) must be open in \( R \). Therefore,
\[ M_{\alpha R} = \{x | (x, j_n, \ldots, j_1, n) \in V_\alpha \cap R \} \]
is open in \( A_{j_n} \). If \( y \in A_{j_n} \), then \( s = (y, j_n, \ldots, j_1, n) \in R \).
Therefore, there exists \( \delta \in D \) such that \( s \in V_\delta \). Hence,
\( s \in M_{\delta R} \). Now, \( \{M_{\alpha R} | \alpha \in D \} \) covers \( A_{j_n} \). Since \( A_{j_n} \) is
closed in \( X \), \( A_{j_n} \) is compact. Hence, there exists a finite
subset \( F \) of \( D \), such that \( \{M_{\alpha R} | \alpha \in F \} \) covers \( A_{j_n} \). Let
\[ z = (v, j_n, \ldots, j_1, n) \in R \].
Then \( v \in A_{n}^{1} \). Therefore, \( v \in M_{\alpha R} \) for some \( \alpha \in F \). Hence, 
\( z \in V_{\alpha} \). Therefore \( \{V_{\alpha}\}_{\alpha \in F} \) covers \( R \). Hence, \( H_{n} \) is compact.

Now, \( H_{n} \) is compact and Hausdorff for each \( n \). Therefore, 
\( \Pi H_{n} \) is compact and Hausdorff. Hence, \( H_{\infty} \) is compact.

Let \( w \not= y \) be elements of \( X_{\infty} \). Then there exists an \( n \) in \( Z_{+} \) such that \( \pi_{n}(w) \neq \pi_{n}(y) \). Let 
\[
\pi_{n}(w) = w_{n} = (x, j_{n}, j_{n-1}, \ldots, j_{1}, n)
\]
and 
\[
\pi_{n}(y) = y_{n} = (v, i_{n}, i_{n-1}, \ldots, i_{1}, n).
\]

If for some \( p \) such that \( 1 \leq p \leq n \), \( i_{p} \neq j_{p} \), then 
\[
w_{n} \in B(j_{n}, u_{n-1}, \ldots, j_{1}, n) = W
\]
and 
\[
y_{n} \in B(i_{n}, i_{n-1}, \ldots, i_{1}, n) = T
\]
and \( W \neq T \). Therefore, \( W \cap T = \emptyset \). Hence, \( \pi_{n}^{-1}(W) \) is open and closed in \( X_{\infty} \), contains \( w \) and does not contain \( y \). If \( j_{p} = i_{p} \) for all \( p \), then \( x \neq v \). There exists \( m \in Z_{+} \) such that 
\[
\frac{1}{2^{m}} < \frac{d(x,y)}{2}
\]
Therefore, \( x \in A_{i_{m}}^{1} \in \mathcal{U}_{m} \) and \( x \in A_{j_{m}}^{1} \in \mathcal{U}_{m} \)
where \( i_{m} \neq j_{m} \). Further, \( A_{i_{m}}^{1} \cap A_{j_{m}}^{1} = \emptyset \). Then 
\[
\pi_{m}(w) = (x, j_{m}, j_{m-1}, \ldots, j_{1}, m)
\]
and 
\[
\pi_{m}(y) = (v, i_{m}, i_{m-1}, \ldots, i_{1}, m).
\]
Since \( j_{m} \neq i_{m} \), by the above argument there exists an open and closed set of \( X_{\infty} \) containing \( w \) and not \( y \). Therefore by Theorem 2.10, \( X_{\infty} \) is totally disconnected.
Since $X_\infty$ is compact and Hausdorff, it is regular. If $X_\infty$ is second countable then by Urysohn's metrization theorem, $X_\infty$ is metrizable. Now, if $H_n$ has a countable base for each $n$, then $\Pi H_n$ will be second countable. Hence, $X_\infty$ will be second countable and metrizable. Since $X$ is a compact metric space, $X$ is second countable. Let $\mathcal{V} = \{V_i\}_{i=1}^\infty$ be a countable base for $X$. For each $B(j_n,j_{n-1},\ldots,j_1,n)$, let \[ M_1(j_n,j_{n-1},\ldots,j_1,n) = \{(x,j_n,j_{n-1},\ldots,j_1,n) | x \in V_i \cap A_j\}. \]
Then, \[ M_1(j_n,j_{n-1},\ldots,j_1,n) \]
is an open subset of \[ B(j_n,j_{n-1},\ldots,j_1,n) \]
and for each $B$ there are only countably many such sets. Let \[ B(j_n,j_{n-1},\ldots,j_1,n) = R \]
and $U$ be open in $R$. Then \[ \{x | (x,j_n,j_{n-1},\ldots,j_1,n) \in U \cap R\} = P \]
is open in $A_{jn}$. But $\mathcal{V}$ covers $X$. Hence, there is a sub-collection $\mathcal{C}$ of $\mathcal{V}$ such that $(\cup \mathcal{C}) \cap A_{jn} = P$. Let \[ \mathcal{C} = \{V_{p_1}, V_{p_2}, \ldots\}. \]
Then, \[ M_{p_1}(j_n,j_{n-1},\ldots,j_1,n) \subseteq U. \] If \[ t = (w,j_n,j_{n-1},\ldots,j_1,n) \in U, \]
then $w \in P$. Therefore, there is $V_{p_j} \in \mathcal{C}$ such that $w \in V_{p_j}$.
Hence, $t \in M_{p_j}(j_n,j_{n-1},\ldots,j_1,n)$. Consequently,
\[ \{M_i \cup \{j_n, j_{n-1}, \ldots, j_1, n\} \}_{i=1}^{\infty} \] is a base for \( \mathbb{B}(j_n, j_{n-1}, \ldots, j_1, n) \).

As a result, \( H_n \) is second countable. Therefore, \( H_\infty \) is metrizable.

Let \( K = H_\infty \times C \). An element of \( K \) can be written as \( (h, c) \) where \( h \in H_\infty \) and \( c \in C \). Since the projection function from \( K \) to \( H_\infty \) is continuous and onto, \( H_\infty \) is the continuous image of \( K \).

Since each of \( H_\infty \) and \( C \) is a compact metric space, \( K \) is a compact metric space. If \( (h_1, c_1) \neq (h_2, c_2) \) in \( K \), then \( h_1 \neq h_2 \) or \( c_1 \neq c_2 \). If \( h_1 \neq h_2 \), then since \( H_\infty \) is totally disconnected, there exists \( U \) open and closed in \( H_\infty \) such that \( h_1 \in U \) and \( h_2 \notin U \). Consequently, \( (h_1, c_1) \in U \times C \) and \( (h_2, c_2) \) is not. A similar argument holds for the assumption that \( c_1 \neq c_2 \). As a result, \( K \) is totally disconnected. Let \( (h, c) \in K \) and \( U \) be a basic open set of \( K \) containing \( (h, c) \). Then \( U = U_1 \times U_2 \) where \( U_1 \) is open in \( H_\infty \) and \( U_2 \) is open in \( C \). Since \( C \) is perfect, there exists \( c \in U_2 \) such that \( c \neq c_1 \). Therefore, \( (h, c_1) \in U \) and \( (h, c) \neq (h, c_1) \). It then follows that \( K \) is perfect. Now, by Theorem 2.15, \( K \) is homeomorphic to \( C \). Since \( H_\infty \) is the continuous image of \( K \) and \( X \) is the continuous image of \( H_\infty \), \( X \) is the continuous image of \( C \).
CHAPTER BIBLIOGRAPHY


CHAPTER III

AN INVERSE LIMIT CHARACTERIZATION OF

COMPACT HAUSDORFF SPACES

To characterize compact Hausdorff spaces, it will be necessary to establish the proper background.

Definition 3.1: Let $E^n$ be euclidean n-space. Then, a point $a_i$ in $E^n$ is expressed by $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$. Let $\{a_0, a_1, \ldots, a_r\}$ be $r + 1$ linearly independent points in $E^n$. Define $[a_0, a_1, \ldots, a_r]$ to be

$$\{(x_1, x_2, \ldots, x_n) | x_i = \sum_{j=0}^{r} \lambda_j a_{ij}, 1 \leq i \leq n, \text{ and } \sum_{j=1}^{r} \lambda_j = 1 \text{ and for each } j, 0 \leq \lambda_j \leq 1\}.$$  

Then, $[a_0, a_1, \ldots, a_r]$ is called an $r$-simplex and each $a_i$ is called a vertex. If $x$ is such that $x_i = \sum_{j=0}^{r} \lambda_j a_{ij}$ where $\lambda_j = \frac{1}{r+1}$ for each $j$, then $x$ is called the barycentre of $[a_0, a_1, \ldots, a_r]$.

To simplify notation, whenever $x = (x_1, x_2, \ldots, x_n)$ is expressed by $x_i = \sum_{j=0}^{r} \lambda_j a_{ij}$, for $1 \leq i \leq n$, the superscript $i$ will be deleted if no confusion results.
Definition 3.2: If \( a_0, a_1, \ldots, a_s \) are \( s + 1 \) points of \( \{ a_0, a_1, \ldots, a_r \} \), then \([a_0, a_1, \ldots, a_s]\) is called an s-face of \([a_0, a_1, \ldots, a_r]\).

Definition 3.3: Let \( K \) be a finite collection of simplices in \( E^n \) such that every face of a simplex of \( K \) is in \( K \) and the intersection of any two simplices of \( K \) is either empty or a face of each. Then \( K \) is called a complex and \( \bigcup K \) is a polyhedron.

It is important to notice that each simplex of \( K \) is closed and compact. As a consequence, \( \bigcup K \) is compact.

Theorem 3.1: Let \( X^n \) be an \( n \)-simplex in \( K \) and \( A(X^n) \) be the barycentre of \( X^n \). Let \( n_0 > n_1 > \ldots > n_1 \) be a finite descending sequence of positive integers and \( n_0, n_1, \ldots, n_1 \) be such that each \( X_{n_j} \) is an \( n_j \)-simplex where \( X_{n_k} \) is a face of \( X_{n_{k-1}} \). Then, \([A(X^n_0), \ldots, A(X^n_1)]\) is a simplex.

Proof: For notational purposes, let
\[
X_{n_j} = X_{a_j} = [a_0, a_1, \ldots, a_j]
\]
and
\[
A(X_{a_j}^n) = B_{n_j}.
\]

Suppose that \( \{ B_{n_0}, B_{n_1}, \ldots, B_{n_1} \} \) is not linearly independent.
Since $X_{n_j}$ is a face of $X_{n_{j-1}}$, the following arrangement can be made. Let

$B_{n_0}$ be the barycentre of $[a_0, a_1, \ldots, a_{n_0}]$,

$B_{n_1}$ be the barycentre of $[a_0, a_1, \ldots, a_{n_1}]$,

$\vdots$ 

$B_{n_{n-1}}$ be the barycentre of $[a_0, a_1, \ldots, a_{n_{n-1}}]$.

Since the barycentres are not linearly independent, there exists $B_{n_p}$ such that

$$B_{n_p} = c_0 B_{n_0} + c_1 B_{n_1} + \ldots + c_{n_p-1} B_{n_{p-1}} + c_{n_p} B_{n_p-1}$$

where all the $c_j$'s are not all 0. Therefore,

$$\frac{1}{n_{p+1}} \sum_{s=0}^{n_p} a_s = \frac{c_0}{n_0+1} \sum_{s=0}^{n_0} a_s + \ldots + \frac{c_{p-1}}{n_{p-1}+1} \sum_{s=0}^{n_{p-1}} a_s + \frac{c_p}{n_{p+1}} \sum_{s=0}^{n_p} a_s$$

As a result,

$$0 = \frac{c_0}{n_0+1} \sum_{s=0}^{n_0} a_s + \ldots + \frac{c_{p-1}}{n_{p-1}+1} \sum_{s=0}^{n_{p-1}} a_s - \frac{1}{n_{p+1}} \sum_{s=0}^{n_p} a_s$$

$$+ \frac{c_p}{n_{p+1}} \sum_{s=0}^{n_p} a_s + \ldots + \frac{c_1}{n_{1}+1} \sum_{s=0}^{n_1} a_s.$$
For $0 \leq j \leq 1$ and $j \neq p$, let $K_j = \frac{c_j}{n_j+1}$. Let $K_p = \frac{1}{n_p+1}$.

Then,

$$0 = K_0 \Sigma s=0^n a_s + \ldots + K_p \Sigma s=0^n a_s + \ldots + K_1 \Sigma s=0^n a_s.$$ 

Hence,

$$0 = \frac{1}{m=0^n} K_m a_0 + \frac{1}{m=0^n} K_m a_1 + \ldots + \frac{1}{m=0^n} K_m n_1 \ldots$$

$$+ \frac{1}{m=0^n} K_m a_{n_1} + \ldots + \frac{1}{m=0^n} K_m a_{n_1+1}$$

$$+ \ldots + K_0 a_{n_1+1} + K_0 a_{n_1+2} + \ldots + K_0 a_{n_0}.$$ 

However, $\{a_0, a_1, \ldots, a_{n_0}\}$ is linearly independent. Therefore each coefficient must be 0. Let $c_j = \frac{j}{m=0^n} K_m$. Now, $c_0 = K_0$.

If $K_p = K_0$, then $K_0 = -\frac{1}{n_p+1} \neq 0$. This is a contradiction to each coefficient being 0. As a result, $K_p \neq K_0$.

Therefore, $K_0 = \frac{c_0}{n_0+1} = 0$. Hence $c_0 = 0$. But,

$$0 = c_1 = K_0 + K_1 = 0 + K_1.$$ 

If $n_p = n_1$, $K_1 = -\frac{1}{n_1+1} \neq 0$. This contradicts each coefficient being 0. Therefore, $n_p \neq n_1$. Hence, $K_1 = 0$. Consequently, $\frac{c_1}{n_1+1} = 0$. Thus, $c_1 = 0$. By induction, $n_p \neq n_1$.
for any $i$ such that $0 \leq i \leq n_0$. This contradicts the assumption that $\{B_{n_0}, B_{n_1}, \ldots, B_{n_1}\}$ was not linearly independent.

Hence $\{B_{n_0}, \ldots, B_{n_1}\}$ is linearly independent. Now, $[A(x_{a_0}^0), \ldots, A(x_{a_1}^{n_1})]$ is a simplex.

**Theorem 3.2:** Let $K$ be a complex and $X^n$ be an $n$-simplex of $K$. Let $n_0 > n_1 > \ldots > n_1$ be a finite descending sequence of positive integers and $X_{a_0}^0, X_{a_1}^1, \ldots, X_{a_1}^{n_1}$ be such that each $X_{a_j}^n$ is an $n_j$-simplex where $X_{a_k}^{n_k}$ is a face of $X_{a_k}^{n_{k-1}}$. Let $B_{a_k}$ be the barycentre of $X_{a_k}^{n_k}$. Then $[B_{a_0}, B_{a_1}, \ldots, B_{a_1}]$ is a simplex. If $K_1$ is the collection of all such simplices formed from $X^n$ for each $X^n$ in $K$, then $K_1$ is a complex.

**Proof:** Since $K$ is a complex, it contains only finitely many simplices. Each simplex of $K$ gives rise to only finitely many simplices of $K_1$. Consequently, $K_1$ has only finitely many simplices.

Let $X_{n_1} = [B_0, B_1, \ldots, B_{n_1}]$ be a simplex in $K_1$ where

- $B_0$ is the barycentre of $[b_0, b_1, \ldots, b_{n_0}]$,
- $B_1$ is the barycentre of $[b_0, b_1, \ldots, b_{n_1}]$,
- $\vdots$
- $B_{n_1}$ is the barycentre of $[b_0, b_1, \ldots, b_{n_1}]$. 

Let \([B_{a_0}, B_{a_1}, \ldots, B_{a_i}]\) be a face of \(X_{n_i}\). Then

\[B_{a_0}\] is the barycentre of \([b_0, b_1, \ldots, b_{a_0}]\),

\[B_{a_1}\] is the barycentre of \([b_0, b_1, \ldots, b_{a_1}]\),

\[\vdots\]

\[B_{a_j}\] is the barycentre of \([b_0, b_1, \ldots, b_{a_j}]\).

However, since for each \(i\), \([b_0, b_1, \ldots, b_{a_i}]\) is a face of \([b_0, b_1, \ldots, b_{n_0}]\), it is a simplex of \(K\). Therefore, \([B_{a_0}, B_{a_1}, \ldots, B_{a_j}] \in K_1\).

The following lemma will help to show that if the intersection of elements of \(K_1\) is nonempty, then the intersection is a face of each.

**Lemma 3.1:** Let \(X = [B_0, B_1, \ldots, B_p]\) be an element of \(K\) where

\[B_0\] is the barycentre of \([b_0, b_1, \ldots, b_{m_0}]\),

\[B_1\] is the barycentre of \([b_0, b_1, \ldots, b_{m_1}]\),

\[\vdots\]

\[B_p\] is the barycentre of \([b_0, b_1, \ldots, b_{m_p}]\).

Let \(S = [a_0, a_1, \ldots, a_n]\) be a face of \([b_0, b_1, \ldots, b_{m_0}]\) such that \(X \cap S \neq \emptyset\). Then \(X \cap S\) is a face of \(X\).
Suppose that there does not exist \( i \) such that \( 0 \leq i \leq p \) and \( B_i \in S \). Then for each \( i \) such that \( 0 \leq i \leq p \), there exists \( b_j \) such that \( 0 \leq j \leq m_i \) and \( b_j \notin \{a_0, a_1, \ldots, a_n\} \).

Let \( x \in X \cap S \). Then,

\[
x = \lambda_0 B_0 + \lambda_1 B_1 + \ldots + \lambda_p B_p = \delta_0 a_0 + \delta_1 a_1 + \ldots + \delta_n a_n.
\]

Since each \( B_i \) is the barycentre of \([b_0, b_1, \ldots, b_{m_i}]\),

\[
x = \frac{\lambda_0}{m_0 + 1} \sum_{s=0}^{m_0} b_s + \ldots + \frac{\lambda_p}{m_p + 1} \sum_{s=0}^{m_p} b_s = \delta_0 a_0 + \ldots + \delta_n a_n.
\]

For each \( j \) such that \( 0 \leq j \leq p \), let \( K_j = \frac{\lambda_j}{m_j + 1} \). Then,

\[
x = \left( \sum_{j=0}^{p} K_j b_0 \right) + \ldots + \left( \sum_{j=0}^{p} K_j b_{m_p} \right) + \left( \sum_{j=0}^{p-1} K_j b_{m_p+1} \right) + \ldots + \left( \sum_{j=0}^{p-1} K_j b_{m_p-1} \right) + \ldots + K_0 b_{m_1+1} + \ldots + K_0 b_{m_0} = \delta_0 a_0 + \ldots + \delta_n a_n.
\]

However, there is \( b_j \) such that \( b_j \neq a_1 \) for any \( i \) such that \( 0 \leq i \leq n \). Hence, the coefficient of \( b_j \) is 0. Since the coefficient of \( b_j \) is \( \sum_{j=0}^{p} K_j \),

\[K_0 + K_1 + \ldots + K_p = 0.\]
Therefore,
\[ \lambda_0 = \lambda_1 = \ldots = \lambda_p = 0. \]
This contradicts the statement that if \( x \in X \) then the sum of the coefficients must be 1. Consequently, there exists \( i \) such that \( 0 \leq i \leq p \) and \( B_i \in S \).

Let \( \{ B_0, B_1, \ldots, B_p \} \) be the collection of \( B_i \)'s such that \( 0 \leq i \leq p \) and \( B_i \in S \). Let \( x \in X \cap S \). Since \( x \in S \),
\[ x = \delta_0 a_0 + \delta_1 a_1 + \ldots + \delta_n a_n. \]
Since \( x \in X \),
\[ x = \lambda_0 B_0 + 1 B_1 + \ldots + \lambda_p B_p. \]
For each \( i \) such that \( 0 \leq i \leq p \), let \( K_1 = \frac{\lambda_1}{m_i+1} \). Then,
\[
\begin{align*}
  x &= \left( \sum_{m=0}^{p} K_m b_0 \right) + \ldots + \left( \sum_{m=0}^{p} K_m b_m \right) \\
  &= \left( \sum_{m=0}^{p-1} K_m b_m + 1 \right) + \ldots + \left( \sum_{m=0}^{p-1} K_m b_{m-1} \right) \\
  &= \ldots + K_0 b_{n_1+1} + \ldots + K_0 b_{n_0}.
\end{align*}
\]
Suppose that \( 0 \leq i \leq p \) and \( B_i \neq B_j \) for any \( j \) such that \( 0 \leq j \leq t \) and \( \lambda_1 \neq 0 \). Let \( s \) be the largest such \( i \). Then there exists \( b_r \in \{ b_0, \ldots, b_m \} \) such that \( b_r \notin \{ a_0, \ldots, a_n \} \).

Now, for each \( j \leq s \), \( a_r \in \{ b_0, \ldots, b_m \} \). The coefficient of \( b_r \) is \( \sum_{j=0}^{s} K_j = 0 \). Hence, \( \lambda_0 = \lambda_1 = \ldots = \lambda_s = 0 \).
By assumption, \( \lambda_s \neq 0 \). Therefore, \( x \in [B_{a_0}, \ldots, B_{a_t}] \).

Clearly, if \( x \in [B_{a_0}, B_{a_1}, \ldots, B_{a_t}] \), \( x \in X \cap S \). Therefore,

\[ X \cap S = [B_{a_0}, B_{a_1}, \ldots, B_{a_t}] \]

which is a face of \( X \).

Now to complete the proof of Theorem 3.2, let \( X \) and \( T \) be simplices of \( X_1 \) such that \( X \cap T \neq \emptyset \). Let

\[ X_s = [B_0; B_1, \ldots, B_p] \]

and

\[ X_t = [H_0; H_1, \ldots, H_t] \]

be the smallest faces of \( X \) and \( T \) respectively such that \( X \cap T \subseteq X_s \) and \( X \cap T \subseteq X_t \). Let

\[ B_0 \] be the barycentre of \( [b_0; b_1, \ldots, b_n_0] = P \),

\[ B_1 \] be the barycentre of \( [b_0; b_1, \ldots, b_n_1] \),

\[ \vdots \]

\[ B_p \] be the barycentre of \( [b_0; b_1, \ldots, b_n_p] \) and \( H_0 \) be the barycentre of \( [h_0; h_1, \ldots, h_m_0] = L \),

\[ H_1 \] be the barycentre of \( [h_0; h_1, \ldots, h_m_1] \),

\[ \vdots \]

\[ H_t \] be the barycentre of \( [h_0; h_1, \ldots, h_m_t] \).

Now, let

\[ [b_0; b_1, \ldots, b_n_0] \cap [h_0; h_1, \ldots, h_m_0] = [a_0; a_1, \ldots, a_p] = R. \]
The $X \cap T \subseteq R$ and $R$ is a face of $P$ and $L$. By Lemma 3.1, $X_s \cap R$ is a face of $X_s$. But $X \cap T \subseteq X_s \cap R$, hence, $x_s \cap R = X_s$. Also, $X_t \cap R$ is a face of $X_t$ and $X \cap T \subseteq X_t \cap R$. Hence, $X_t \cap R = X_t$. It can be assumed that $R$ is the smallest face of $P$ and $L$ containing $X \cap T$. If $a_i \in \{a_0, a_1, \ldots, a_p\}$, then $a_i \in \{b_0, b_1, \ldots, b_{n_0}\} \cap \{h_0, h_1, \ldots, h_m\}$. For each $j$ such that $0 \leq j \leq p$, there exists $x \in X_s$ such that

$$x = \lambda_0 b_0 + \lambda_1 b_1 + \ldots + \lambda_j b_j + \ldots + \lambda_p b_p,$$

and $\lambda_j \neq 0$. For each $j$, let $K_j = \frac{\lambda_j}{n_j+1}$. Then,

$$x = K_0 (\sum_{s=0}^{n_0} b_s) + K_1 (\sum_{s=0}^{n_1} b_s) + \ldots + K_p (\sum_{s=0}^{n_p} b_s)$$

$$= (\sum_{s=0}^{p} K_s) b_0 + \ldots + (\sum_{s=0}^{p} K_s) b_{n_p} + (\sum_{s=0}^{p-1} K_s) b_{n_p+1}$$

$$+ \ldots + (\sum_{s=0}^{p-1} K_s) b_{n_p-1} + \ldots + K_0 b_{n_0} + \ldots + K_0 b_{n_0}^{-1}.$$
By a similar argument,
\[ \{h_0, h_1, \ldots, h_{m_0}\} = \{a_0, a_1, \ldots, a_p\}. \]

Therefore, \( B_0 = H_0 \).

If \( B_i \neq H_j \) for some \( j \) such that \( b \leq j \leq t \), then either:

Case 1: there exists \( b \in \{b_0, b_1, \ldots, b_{n_1}\} \) such that
\[ b \notin \{h_0, h_1, \ldots, h_{m_j}\} \]
or

Case 2: there exists \( h \in \{h_0, h_1, \ldots, h_{m_j}\} \) such that
\[ h \notin \{b_0, b_1, \ldots, b_{n_1}\} \].

Suppose that \( B_1 \neq H_1 \). Let \( x \in [B_0, B_1, \ldots, B_p] \) where
\[ x = \lambda_0 B_0 + \ldots + \lambda_p B_p \]
and \( \lambda_1 \neq 0 \). If \( x \in [H_0, H_1, \ldots, H_t] \), then,
\[ x = \frac{\lambda_0}{n_0 + 1} \sum_{s=0}^{n_0} b_s + \ldots + \frac{\lambda_p}{n_p + 1} \sum_{s=0}^{n_p} b_s \]
\[ = \frac{\delta_0}{m_0 + 1} \sum_{s=0}^{m_0} h_s + \ldots + \frac{\delta_t}{m_t + 1} \sum_{s=0}^{m_t} h_s. \]

Let \( K_1 = \frac{\lambda_1}{n_1 + 1} \) and \( L_1 = \frac{\delta_1}{m_1 + 1} \). Therefore,
\[ x = (\sum_{m=0}^{p} K_m) b_0 + \ldots + (\sum_{m=0}^{p} K_m) b_{n_1} + (\sum_{m=0}^{p-1} K_m) b_{n_1 + 1} + \ldots + K_0 b_{n_0} \]
\[ = (\sum_{m=0}^{t} L_m) h_0 + \ldots + (\sum_{m=0}^{t} L_m) h_{m_1} + (\sum_{m=0}^{t-1} L_m) h_{m_1 + 1} + \ldots + L_0 h_{m_0}. \]
If Case 1 is true, then the coefficient of $b$ is $\sum_{m=0}^{1} K_m$ for $i \geq 1$. However, since $b \notin \{h_0, h_1, \ldots, h_m\}$, the coefficient of $b$ must be $L_0$. Therefore,

$$L_0 = K_0 + K_1 + \ldots + K_i.$$ 

If $L_0 \neq K_0$, then $K_0 = \sum_{m=0}^{p} L_m$ for some $p \geq 1$. Hence,

$$L_0 = L_0 + L_1 + \ldots + L_p + K_1 + \ldots + K_i.$$ 

As a result, $K_1 = 0$. It must follow that $\lambda_1 = 0$. This is contrary to the supposition that $\lambda_1 \neq 0$. Therefore, $K_0 = L_0$. But, $L_0 = K_0 + K_1 + \ldots + K_i$. Hence, $K_1 = 0$. It then follows that $\lambda_1 = 0$. This contradicts $\lambda_1 \neq 0$. Hence, Case 1 is not true. Therefore, Case 2 must be true.

Now, the coefficient of $h$ is $\sum_{m=0}^{1} L_m$ for some $i \geq 1$ and it is also $K_0$. If $K_0 \neq L_0$, then $L_0 = K_0 + K_1 + \ldots + K_j$ for some $j \geq 1$. But, $K_0 = L_0 + L_1 + \ldots + L_1$ for some $i \geq 1$.

Consequently,

$$K_0 = K_0 + K_1 + \ldots + K_j + L_1 + \ldots + L_1.$$ 

Therefore, $K_1 = 0$. Hence, $\lambda_1 = 0$. This contradicts $\lambda_1 \neq 0$. As a result, $K_0 = L_0$. It then follows that since $K_0 = L_0 + L_1 + \ldots + L_1$, $L_1 = 0$. Now, $\delta_1 = 0$.

Suppose that if $B_1 \neq H_r$ for $1 \leq r \leq \lambda$, then $K_0 = L_0$ and $L_1 = L_2 = \ldots = L_\lambda = 0$.

Suppose that $B_1 \neq H_{\lambda+1}$. Then, if Case 1 is true, the
coefficient of b is \( \frac{1}{j} \sum_{s=0}^{j} k_s \), for \( i \geq 1 \), and it is also \( \sum_{s=0}^{j} L_s \) for some \( j \) such that \( 0 < j < \ell + 1 \). Therefore, \( K_1 = 0 \).

Hence, \( \lambda_1 = 0 \). But, \( \lambda_1 \neq 0 \). Consequently, Case 2 must be true. As a result, the coefficient of h is \( K_0 \) and also

\( \frac{1}{s} \sum_{i=0}^{s} \delta_i \) for some \( i \geq \ell + 1 \). Then,

\[
K_0 = L_0 + L_1 + \ldots + L_{\ell+1} + \ldots + L_t.
\]

Since \( K_0 = L_0 \), \( L_{\ell+1} = 0 \). Consequently, \( \delta_{\ell+1} = 0 \). Hence, by induction, if \( B_1 \neq H_p \), then \( \delta_p = 0 \). However, on the contrary, \( \sum_{i=0}^{t} \delta_i = 1 \). Therefore, \( \delta_0 = 1 \). Since \( \delta_0 = \lambda_0 \), \( \lambda_1 = 0 \). This contradicts \( \lambda_1 \neq 0 \). It then follows that

\( x \in [H_0, \ldots, H_t] \). Consequently, \( B_1 \) is not a necessary vertex of \( \{B_0, B_1, \ldots, B_p\} \). But, by the supposition, \( B_1 \) is necessary. Therefore, \( B_1 = H_j \) for some \( j \) such that \( 0 < j \leq t \).

By a similar argument, \( H_1 = B_1 \) for some \( i \) such that \( 0 < i \leq p \).

Suppose that \( B_1 = H_j \) and \( j > 1 \). Then

\[
\{b_0, b_1, \ldots, b_{n_1}\} \subseteq \{h_0, h_1, \ldots, h_m\}
\]

which is a proper subset of \( \{h_0, h_1, \ldots, h_m\} \). But

\[
\{h_0, h_1, \ldots, h_m\} = \{b_0, b_1, \ldots, b_{n_1}\}.
\]

Therefore, \( \{b_0, b_1, \ldots, b_{n_1}\} \) is a proper subset of \( \{b_0, b_1, \ldots, b_{n_1}\} \).

However, \( i \geq 1 \). This is a contradiction. Hence, \( H_1 = B_1 \).

There exists \( b \in \{0, \ldots, b_{n_0}\} \) such that \( b \notin \{b_0, \ldots, b_{n_1}\} \).
Since $B_i = H_i$, $b \in \{h_0, \ldots, h_m\}$ and $b \notin \{h_0, \ldots, h_{m+1}\}$.

Therefore, if $x \in X_x \cap X_t$ and
\[ x = \lambda_0 B_0 + \ldots + \lambda_p B_p = \delta_0 H_0 + \ldots + \delta_t H_t, \]
then $\lambda_0 = \delta_0$.

Suppose that for each $i < j$, $B_i = H_i$ and if $x \in X_s \cap X_t$ such that
\[ x = \lambda_0 B_0 + \ldots + \lambda_p B_p = \delta_0 H_0 + \ldots + \delta_t H_t, \]
then $\lambda_{i-1} = \delta_{i-1}$. Suppose that $B_j \neq H_j$. Let $x \in X_s$ such that $x = \lambda_0 B_0 + \ldots + \lambda_p B_p$ and $\lambda_j \neq 0$. Then,
\[ x = (\sum_{m=0}^{p-1} K_m) b_0 + \ldots + (\sum_{m=0}^{p-1} K_m) b_{n_p} + (\sum_{m=0}^{p-1} K_m) b_{n_p+1} + \ldots + K_0 b_{n_1+1} + \ldots + K_0 b_n. \]

If $x \in X_t$, then $x = \delta_0 H_0 + \ldots + \delta_t H_t$. Hence,
\[ x = (\sum_{m=0}^{t} L_m) h_0 + \ldots + (\sum_{m=0}^{t} L_m) h_m + (\sum_{m=0}^{t} L_m) h_{m+1} + \ldots + L_0 h_{m+1} + \ldots + L_0 h_{m_0}. \]

If Case 1 is true, then the coefficient of $b$ is $\sum_{m=0}^{i} K_m$ for some $i \geq j$ and $\sum_{m=0}^{s} L_m$ for some $s < j$. Hence,
\[ K_0 + K_1 + \ldots + K_j + \ldots + K_i = L_0 + L_1 + \ldots + L_s. \]
Since \( s < j \), \( K_j = 0 \). Therefore, \( \lambda_j = 0 \). But \( \lambda_j \neq 0 \).

Consequently, Case 2 must be true. Hence, the coefficient of \( b \) is \( \frac{1}{m} \sum L_m \) for some \( i \geq j \) and \( \sum K_m \) for some \( s < j \).

Therefore,
\[
K_0 + \ldots + K_s = L_0 + \ldots + L_j + \ldots + L_{j+1}.
\]

Since \( s < j \), \( L_j = 0 \). Hence, \( \delta_j = 0 \). By a similar argument, if \( B_j \neq H_{j+1} \), then \( \delta_{j+1} = 0 \).

Suppose that if \( B_j \neq H_{j+s} \) for \( 0 \leq s < i \) then \( \delta_{j+s} = 0 \).

Suppose that \( B_j \neq H_{j+1} \). If Case 1 is true, then the coefficient of \( b \) is \( \sum K_p \) for some \( l > j \) and \( \sum L_p \) for some \( k < j + 1 \). Hence, \( K_j = 0 \). Therefore, \( \lambda_j = 0 \). But \( \lambda_j \neq 0 \).

Then, Case 2 must be true. Consequently the coefficient of \( b \) must be \( \sum L_{s} \) for some \( k \geq j + 1 \) and \( \sum K_{s} \) for \( k < j + 1 \).

Therefore,
\[
L_0 + L_1 + \ldots + L_{j+1} + \ldots + L_k = K_0 + K_{j+1} + \ldots + K_k.
\]

Hence, \( L_{j+1} = 0 \). As a result, \( \delta_{j+1} = 0 \). Therefore, by induction, for all \( s \geq j \), \( \delta_{s} = 0 \). Hence, \( \delta_0 + \ldots + \delta_{j-1} = 1 \).

Therefore, \( \lambda_0 + \lambda_1 + \ldots + \lambda_{j-1} = 1 \). Hence \( \lambda_j = 0 \). This is a contradiction to \( \lambda_j \neq 0 \). Therefore, \( x \notin [H_0, H_1, \ldots, H_t] \).

Consequently, \( B_j \) is not a necessary vertex of \( [B_0, B_1, \ldots, B_p] \).

This is a contradiction. Therefore, \( B_j = H_{k^i} \) for some \( k^i \). The same
argument that proved $H_j = B_j$ will hold for $H_j = B_j$.

Therefore,

$$[B_0, B_1, \ldots, B_p] = [H_0, H_1, \ldots, H_t].$$

It now follows that $K_1$ is a complex.

Now, $K_1$ is called the barycentric subdivision of $K$.

It is noteworthy to point out that if $K_1$ is the barycentric subdivision of $K$, then $\bigcup K_1 = \bigcup K$.

**Theorem 3.3:** If $T = [A_0, A_1, \ldots, A_p]$ is a simplex in $E^n$ where $d$ is the usual metric on $E^n$, then the diameter of $T$ is the maximum $\{d(A_i, A_j) | 0 \leq i \leq p, 0 \leq j \leq p\}$.

**Proof:** Let

$$D = \text{maximum } \{d(A_i, A_j) | 0 \leq i \leq p \text{ and } 0 \leq j \leq p\}.$$

Let $x$ and $y$ be in $T$. Let $A_1$ be a vertex farthest from $y$ and $d(y, A_1) = \delta$. Then $A_j \in S(y, \delta)$, the spherical neighborhood around $y$ of radius $\delta$, for each $j$. Consequently, $T \subseteq S(y, \delta)$. If $x \in T$, $d(x, y) \leq d(y, A_1)$. Let $A_j$ be vertex farthest from $A_1$ and $\lambda = d(A_1, A_j)$. Then $T \subseteq S(A_1, \lambda)$. Therefore, $d(y, A_1) \leq d(A_1, A_j) \leq D$. Consequently,

$$d(x, y) \leq d(y, A_1) \leq d(A_1, A_j) \leq D.$$

Hence, the theorem is proven.

**Theorem 3.4:** If $K = [A_0, A_1, \ldots, A_m]$ is a simplex of diameter $D$ in $E^n$, $K_1$ is the barycentric subdivision of $K$ and $D_1$ is the diameter of $K_1$, then $D_1 \leq \frac{m}{m+1} D$.

**Proof:** Let $d$ be the usual metric on $E^n$. By Theorem 3.3, $D_1$ is the maximum of the distances between the vertices
of $K_i$. Let $D_1 = d(B_i, B_j)$ where $1 \leq j$, $B_i$ is the barycentre of $[A_0, A_1, \ldots, A_i]$ and $B_j$ is the barycentre of $[A_0, A_1, \ldots, A_j]$.

Then, $B_i^r = \frac{1}{1+i} \sum_{p=0}^{i} A_p^r$ and $B_j^r = \frac{1}{j+1} \sum_{p=0}^{j} A_p^r$ for $1 \leq r \leq n$. Therefore,

$$d(B_i, B_j) = \left( \frac{n}{1+i} \sum_{r=1}^{n} \frac{1}{1+i} \sum_{p=0}^{i} A_p^r - \frac{1}{j+1} \sum_{p=0}^{j} A_p^r \right) \frac{1}{2}$$

$$= \left( \frac{n}{1+i} \sum_{r=1}^{n} \left( \frac{1}{1+i} - \frac{1}{j+1} \right) \sum_{p=0}^{i} A_p^r - \frac{1}{j+1} \sum_{p=0}^{j} A_p^r \right) \frac{1}{2}$$

$$= \left( \frac{n}{1+i} \sum_{r=1}^{n} \left( \frac{j-1}{j+1} \sum_{p=0}^{i} A_p^r - \frac{j}{j+1} \sum_{p=0}^{j} A_p^r \right) \right) \frac{1}{2}$$

$$= \left( \frac{n}{1+i} \sum_{r=1}^{n} \left( \frac{j-1}{j+1} \sum_{p=0}^{i} A_p^r - \frac{j}{j+1} \sum_{p=0}^{j} A_p^r \right)^2 \right) \frac{1}{2}$$

$$= \left( \frac{n}{1+i} \sum_{r=1}^{n} \left( \frac{j-1}{j+1} \sum_{p=0}^{i} A_p^r - \frac{j}{j+1} \sum_{p=0}^{j} A_p^r \right)^2 \right) \frac{1}{2}$$

However, $\frac{j-1}{j+1} = 1$, $(\frac{1}{j-1})(j-1)$ and $A_1$ is a vertex of $K$ for each $i \leq j$. Consequently, $x = (x_1, x_2, \ldots, x_n)$ where

$$x^r = \frac{1}{1+i} \sum_{p=0}^{i} A_p^r$$

and $y = (y_1, y_2, \ldots, y_n)$ where $y^r = \frac{1}{j+1} \sum_{p=0}^{j} A_p^r$

are elements of $K$. Hence,

$$\left( \frac{n}{1+i} \sum_{r=1}^{n} \frac{1}{1+i} \sum_{p=0}^{i} A_p^r - \frac{1}{j+1} \sum_{p=0}^{j} A_p^r \right)^2 \leq D.$$
Therefore, \( d(B_i, B_j) \leq \frac{j-i}{j+1} D \). But, \( j-i \leq j \) and \( j \leq m \). As a result, \( d(B_i, B_j) \leq \frac{i}{j+1} D \leq \frac{m}{m+1} D \).

**Definition 3.4:** If \( K \) is a complex in \( E^n \), then the mesh of \( K \) is the maximum of the diameters of simplices of \( K \).

**Theorem 3.5:** If \( \{K_i\}_{i=1}^\infty \) is a sequence of complexes such that for each \( i \), \( K_{i+1} \) is the barycentric subdivision of \( K_i \) and \( M_i \) is the mesh of \( K_i \), then \( \{M_i\} \to 0 \) as \( i \to \infty \).

**Proof:** Suppose that \( \{K_i\}_{i=1}^\infty \) is a sequence of complexes satisfying the hypothesis and that the mesh of \( K_i \) is \( M_i \).

Let \( k_i = [A_0, A_1, \ldots, A_m] \) be a simplex of \( K_i \) of diameter \( M_i \) such that \( k_i \) has a maximum number of vertices. Then, if \( k_2 \) is a simplex of \( K_2 \), by Theorem 3.4, the diameter of \( k_2 \leq \frac{m}{m+1} M_i \). Hence, \( M_2 \leq \frac{m}{m+1} M_1 \). Furthermore, \( k_2 \) has at most \( m \) vertices. If \( k_3 \) is a simplex of \( K_3 \), then the diameter of \( k_3 \) is less than or equal to \( \frac{m}{m+1} M_2 \). Hence, \( M_3 \leq \frac{m}{m+1} M_2 \). By induction, \( M_n \leq (\frac{m}{m+1})^{n-1} M_1 \). Since \( M_1 \) is a constant and \((\frac{m}{m+1})^n \to 0 \) as \( n \to \infty \), \( \{M_i\}_{i=1}^\infty \to 0 \) as \( i \to \infty \).

This paper will assume without proof that if \( N \) is a finite indexing set, then \( I^N = \bigcap_{\alpha \in N} I_\alpha \) where \( I_\alpha \) is the closed unit interval for each \( \alpha \), is a polyhedron.

**Theorem 3.6:** If \( N \) is a finite indexing set, \( F \) is a closed subset of \( I^N \) and \( U \) is an open subset of \( I^N \) such that \( F \subseteq U \subseteq I^N \), then there exists a polyhedron \( P \) such that \( F \subseteq P \subseteq P^o \subseteq P \subseteq U \).
Proof: Since $F \subseteq U$ and $I^N$ is a compact metric, there exists an open subset $V$ of $I^N$ such that $F \subseteq V \subseteq \bar{V} \subseteq U$. Now, since $I^N$ is assumed to be a polyhedron, there exists a complex $H$ such that $I^N = \bigcup H$. Let $\{H_i\}_{i=1}^\infty$ be a sequence of complexes such that $H_1 = H$ and $H_{i+1}$ is the barycentric subdivision of $H_i$. For each $i$, let $M_i$ denote the mesh of $H_i$. Since $\bar{V} \subseteq U$, $\bar{V} \cup (I^N \setminus U) = \emptyset$ and each of $\bar{V}$ and $I^N \setminus U$ is closed in $I^N$. Let $(I^N \setminus U) = W$. There exists $\delta > 0$ such that if $x \in \bar{V}$ and $y \in W$, $d(x,y) \geq \delta$, where $d$ is the usual metric on $I^N$. There exists $j \in \mathbb{Z}_+$ such that $M_j < \delta$. Let 
\[ R = \{ S | S \text{ is a simplex of } H_j \text{ and } S \cap \bar{V} \neq \emptyset \}. \]
Then $R$ covers $\bar{V}$. Let 
\[ K = \{ s | s \text{ is a face of some } S \in R \}. \]
Now, $K$ is complex, $\cup K$ is a polyhedron and $\bar{V} \subseteq \cup K$. Suppose that $x \in \cup K$ and $x \notin U$. Then $x \in W$ and there exists $s \in K$ such that $x \in s$. Since $s \in K$, $s$ is a face of some $S \in R$. Therefore, $S \cap \bar{V} \neq \emptyset$ and $x \in S$. Let $y \in \bar{V} \cap S$. Then, since $y \in \bar{V}$ and $x \in W$, $d(x,y) \geq \delta$. However, $x \in S$, $y \in S$ and $S \in R$. Since $S \in R$, $S$ is a simplex of $H_j$. Therefore, the diameter of $S$ is less than $\delta$. Hence, $d(x,y) < \delta$. This contradicts the statement that $d(x,y) \geq \delta$. Consequently, $x \in U$. As a result, $\cup K \subseteq U$. Now, $F \subseteq V \subseteq \bar{V} \subseteq \cup K \subseteq U$. It now follows that $F \subseteq \cup K \subseteq \cup K \subseteq U$.

Theorem 3.7: If $Y$ is a compact Hausdorff space, then $Y$ is homeomorphic to the inverse limit of an inverse system of polyhedra.
Proof: Let $Y$ be a compact Hausdorff space. Then $Y$ is homeomorphic to a subspace of $\prod_{\alpha \in A} I_\alpha$, denoted by $I^A_\alpha$, where $I_\alpha$ is the closed unit interval and $A$ is some indexing set. Denote the homeomorphic image of $Y$ in $I^A$ as $X$. Then, $X$ is a closed compact subset of $I^A$. Let

$\mathcal{B} = \{ B | B$ is a nonempty finite subset of $A \}.$

Now, for $B$ and $D$ in $\mathcal{B}$, $B \neq D$ if and only if $I^B \cap I^D = \emptyset$.

For each $B \in \mathcal{B}$, let $f_{AB} : I^A \to I^B$ be defined by

$$f_{AB}(\{ x_\alpha \}_{\alpha \in A}) = \{ x_\alpha \}_{\alpha \in B}.$$ 

Then, $f_{AB}$ is continuous and onto for each $B \in \mathcal{B}$ and $f_{AB}(X)$ is a closed subset of $I^B$. By Theorem 3.6, there exists $P$, a polyhedron, such that $f_{AB}(X) \subseteq P \subseteq I^B$. Let

$$\mathcal{P} = \{ P | P$ is a polyhedron such that for some $B \in \mathcal{B},$

\[ f_{AB}(X) \subseteq P \subseteq I^B \}.$$

Let $\mathcal{B}$ be an indexing set for $\mathcal{P}$. If $\lambda \in \mathcal{B}$, and $B$ and $B'$ are elements of $\mathcal{B}$ and

$$f_{AB}(X) \subseteq P_\lambda \subseteq P \subseteq I^B$$

and

$$f_{AB}(X) \subseteq P_\lambda \subseteq P \subseteq I^B',$$

then $B = B'$. If not, $I^B \cap I^{B'} = \emptyset$. Therefore, for each $\lambda \in \mathcal{B}$, there is a unique $B \in \mathcal{B}$ such that $f_{AB}(X) \subseteq P_\lambda \subseteq P \subseteq I^B$.

Let

$$E = \{ (\lambda, B) | (\lambda, B) \in \mathcal{B} \times \mathcal{B} \}$$

and $f_{AB}(X) \subseteq P_\lambda \subseteq P \subseteq I^B$. 

Then $X$ is homeomorphic to $\mathcal{B}$.
Then, $E$ is an indexing set for $\varnothing$. Let $(\lambda, B) \in E$, be denoted by $\lambda_B$.

If $C \subseteq B \subseteq A$, where $C$ and $B$ are elements of $\mathcal{B}$, then define $f_{BC} : I^E \to I^C$ by

$$f_{BC}(x_\alpha) = x_\alpha.$$ 

Then each $f_{BC}$ is continuous. If $x = \{x_\alpha\}_{\alpha \in A} \in I^A$, then

$$f_{AC}(x) = \{x_\alpha\}_{\alpha \in C}.$$ 

However, $f_{AB}(x) = \{x_\alpha\}_{\alpha \in B}$ and $f_{BC}(\{x_\alpha\}_{\alpha \in B}) = \{x_\alpha\}_{\alpha \in C}$. Hence,

$$f_{AC} = f_{BC} \circ f_{AB}.$$ 

Direct $E$ in the following manner. Let $\beta_B$ and $\delta_D$ be elements of $E$. Then, $B$ and $D$ are unique in $\mathcal{B}$ such that

$$f_{AB}(x) \leq \mu_{\beta_B} \leq \mu_{\delta_D} \leq I^B.$$ 

and

$$f_{AD}(x) \leq \mu_{\delta_D} \leq \mu_{\delta_D} \leq I^D.$$ 

Now, define $\beta_B \leq \delta_D$ if and only if $B \subseteq D$ and $f_{DB}(\mu_{\delta_D}) \leq \mu_{\beta_B}$.

Suppose that $\lambda_D \in E$. Then $D \subseteq D$ and

$$f_{DD}(\mu_{\lambda_D}) = \mu_{\lambda_D}.$$ 

Therefore, $\lambda_D \leq \lambda_D$.

Suppose that $\lambda_F \leq \beta_B$ and $\beta_B \leq \delta_D$. Then,

$$f_{AB}(x) \leq \mu_{\lambda_F} \leq \mu_{\lambda_F} \leq I^F,$$

and

$$f_{AB}(x) \leq \mu_{\beta_B} \leq \mu_{\beta_B} \leq I^B.$$ 

and

$$f_{AD}(x) \leq \mu_{\delta_D} \leq \mu_{\delta_D} \leq I^D.$$
Now, since \( F \subseteq B \) and \( B \subseteq D \), \( F \subseteq D \). Also, \( f_{DB}(P_{\delta D}) \subseteq P_{\beta B} \) and 
\( f_{DB}(P_{\delta D}) \subseteq P_{\lambda F} \). Consequently, \( f_{DF}(P_{\delta D}) \subseteq P_{\lambda F} \). It then 
follows that \( \lambda F \leq \delta D \).

The remaining property, necessary to show that " \( \leq \) "
is a direction for \( E \), is that of a common follower for two 
elements of \( E \). The following lemma will make this easy 
and be helpful later.

**Lemma 3.2:** If \( \beta B \) is in \( E \) and \( D \) is in \( B \) such that 
\( B \subseteq D \), then \( f_{AD}(X) \subseteq f_{DB}^{-1}(P_{\beta B}) \).

**Proof:** Suppose that \( x \in f_{AD}(X) \) and \( x \notin f_{DB}^{-1}(P_{\beta B}) \).
Now, \( f_{DB}(x) \in f_{DB}(f_{AD}(X)) \). Therefore, \( f_{DB}(x) \in f_{AB}(X) \).
But, \( x \notin f_{DB}^{-1}(P_{\beta B}) \). Hence, \( f_{DB}(x) \notin P_{\beta B} \). Consequently,
\( f_{DB}(x) \notin f_{AB}(X) \). This is a contradiction to \( f_{DB}(x) \in f_{AB}(X) \).
As a result, \( x \in f_{DB}^{-1}(P_{\beta B}) \). It then follows that

\[ f_{AD}(X) \subseteq f_{DB}^{-1}(P_{\beta B}) \]

Now, to complete the proof that " \( \leq \) " is a direction
for \( E \), let \( \lambda F \) and \( \beta B \) be elements of \( E \). There exists \( D \in B \)
such that \( B \subseteq D \) and \( F \subseteq D \). By Lemma 3.2,
\[ f_{AD}(X) \subseteq f_{DB}^{-1}(P_{\beta B}) \cap f_{DF}^{-1}(P_{\lambda F}) \]
which is open in \( I^{D} \). By Theorem 3.6, there exists a poly-
hedron \( P \) such that

\[ f_{AD}(X) \subseteq P_{\delta D} \subseteq f_{DB}^{-1}(P_{\beta B}) \cap f_{DF}^{-1}(P_{\lambda F}) \subseteq I^{D} \].
Now, $P \in \mathcal{D}$ and must be indexed by some $\delta_M$ in $E$. Since $P \subseteq I^D$, $M = D$. Therefore, $\delta_D \in E$. But, $f_{DB}(P_{\delta}) \subseteq P^D_D^{\delta}$, $f_{DB}(P_{\delta}) \subseteq P^D_D^{\delta}$, $P \subseteq D$ and $B \subseteq D$. Therefore, $\lambda_P \leq \delta_D$ and $\beta_B \leq \delta_D$. Consequently, "$\leq"$ is a direction for $E$.

Let $\beta_B \leq \delta_D$ in $E$. Then $B \subseteq D$. If $B = D$, let $f_{\delta_D^B} : P_{\delta}^D \to P_{\delta}^D$ be the inclusion map. If $B$ is a proper subset of $D$, let

$$f_{\delta_D^B} : P_{\delta}^D \to P_{\delta}^D$$

Now, $(P_{\beta}^B, f_{\beta_B^D}, E)$ is an inverse system.

Let $x = \{x_\alpha\}_{\alpha \in A} \in X$. For $P_{\delta}^D \in \mathcal{D}$,

$$f_{AD}(X) \subseteq P_{\delta}^D \subseteq P_{\delta}^D \subseteq I^D$$

and

$$f_{AD}(x) = \{x_\alpha\}_{\alpha \in D} \in P_{\delta}^D \subseteq P_{\delta}^D$$

Let $x'$ be the element in $\Pi P_\lambda$ such that for each $\delta_D \in E$, $\lambda_P \subseteq \lambda_D$ the $\delta_D$-th projection of $x'$, $\mathcal{F}_{\delta}^D(x')$, is $f_{AD}(\{x_\alpha\}_{\alpha \in A})$.

If $\beta_B \leq \delta_D$ in $E$, then $\mathcal{F}_{\beta_B^D}(x') = f_{AB}(\{x_\alpha\}_{\alpha \in A})$ and $\mathcal{F}_{\delta_D^B}(x') = f_{AD}(\{x_\alpha\}_{\alpha \in A})$. Consequently, $\mathcal{F}_{\delta_D^B}(x') = f_{\delta_D^B}(f_{AD}(\{x_\alpha\}_{\alpha \in A}))$.

Since $f_{AD}(\{x_\alpha\}_{\alpha \in A}) \in P_{\delta}^D$,
Therefore, \( x' \in X_\infty = \lim(F_\delta, E) \).

Let \( g \) be a relation from \( X \) to \( X_\infty \) defined by \( g(x) = x' \)
and \( \pi_\delta = F_\delta | X_\infty \). Suppose that \( x = y \) in \( X \) where \( x = \{x_\alpha\}_{\alpha \in A} \)
and \( y = \{y_\alpha\}_{\alpha \in A} \). Then, \( g(x) = x' \) is such that
\[
\pi_{B_B}(x') = f_{AB}(\{x_\alpha\}_{\alpha \in A})
\]
and \( g(y) = y' \) such that
\[
\pi_{B_B}(y') = f_{AB}(\{y_\alpha\}_{\alpha \in A})
\]
for each \( B_B \) in \( E \). Since \( x = y \), \( x_\alpha = y_\alpha \) for each \( \alpha \in A \).
Hence,
\[
f_{AB}(\{x_\alpha\}_{\alpha \in A}) = f_{AB}(\{y_\alpha\}_{\alpha \in A})
\]
for each \( B \in B \). Therefore, \( \pi_{B_B}(x') = \pi_{B_B}(y') \) for each \( B_B \).
Consequently, \( x' = y' \). As a result, \( g \) is a function.

Suppose that \( x \neq y \) in \( X \). Then there exists \( \alpha \in A \)
such that \( x_\alpha \neq y_\alpha \). Let \( B \in B \) such that \( \alpha \in B \). Then,
\[
f_{AB}(x) \neq f_{AB}(y).
\]
There exists \( \lambda_B \in \mathcal{E} \) such that
\[
f_{AB}(x) \subseteq P^X_\lambda_B \subseteq P_{\lambda_B} \subseteq I^B.
\]
Consequently, \( \pi_{\lambda_B}(x') \neq \pi_{\lambda_B}(y') \). Therefore, \( g(x) \neq g(y) \).
Hence, \( g \) is one to one.

Let \( x \in X_\infty \) and \( \beta_B \in \mathcal{E} \). Suppose that \( \pi_{\beta_B}(x) \neq f_{AB}(X) \).
Then there exist disjoint open sets \( U \) and \( V \) of \( I^B \) such that
\( f_{AB}(X) \subseteq U \) and \( \pi_{\beta_B}(x) \in V \). Hence, \( f_{AB}(X) \subseteq U \cap P_{\beta_B}^0 \) which is open in \( I^B \). Consequently, there is \( \beta_B' \in E \) such that
\[ f_{AB}(X) \subseteq P_{\beta_B'}^0 \subseteq P_{\beta_B'}^0 \subseteq U \cap P_{\beta_B}^0. \]

Now, \( \pi_{\beta_B}(x) \notin P_{\beta_B} \). However, \( B \subseteq B \) and \( f_{BB}(P_{\beta_B'}) = P_{\beta_B'} \subseteq P_{\beta_B} \).

Therefore, \( \beta_B \leq \beta_B' \). Since \( x \in X \) and \( \beta_B \leq \beta_B' \),
\[ \pi_{\beta_B}(x) = f_{\beta_B'}(\pi_{\beta_B}(x)). \]

Since \( B = B \), \( f_{\beta_B'}^B \) is the inclusion map. Consequently,
\[ f_{\beta_B'}^B(\pi_{\beta_B}(x)) = \pi_{\beta_B}(x). \]

Therefore,
\[ \pi_{\beta_B}(x) = \pi_{\beta_B}(x) \in P_{\beta_B}' . \]

Hence, \( \pi_{\beta_B}(x) \in P_{\beta_B}' \subseteq U \cap P_{\beta_B}^0 \). Now, \( \pi_{\beta_B}(x) \in U \) and \( \pi_{\beta_B}(x) \in V \). This contradicts the statement that \( U \cap V = \phi \).

Therefore, if \( x \in X \), \( \pi_{\beta_B}(x) \in f_{AB}(X) \) for each \( \beta_B \in E \).

Suppose that there does not exist \( y \in X \) such that \( g(y) = x \).

Then for each \( y \in X \), there exists \( \beta_B \in E \) such that \( f_{AB}(y) \neq \pi_{\beta_B}(x) \). Therefore, there exists \( U_{y_{\beta_B}} \) and \( V_{y_{\beta_B}} \) open and disjoint in \( I^B \) such that \( f_{AB}(y) \in U_{y_{\beta_B}} \) and \( \pi_{\beta_B}(x) \in V_{y_{\beta_B}} \). Hence,
\[ U = \{ f_{AB}^{-1}(U_{y_{\beta_B}}) \mid y \in X \} \] covers \( X \).
There exists a finite subcollection \( \mathcal{H} \) of \( \mathcal{U} \) covering \( X \).

Let

\[
\mathcal{H} = \{ f_{AD}^{-1}(U_{\delta_D}) | \delta_D \in \mathcal{J} \}
\]

where \( \mathcal{J} \) is a finite subset of \( E \). There exists \( M \in \mathcal{H} \) such that \( B \subseteq M \) for each \( \beta_B \in \mathcal{J} \). By Lemma 3.2, \( f_{AM}(X) \subseteq f_{MB}^{-1}(P_{\beta_B}^0) \) for each \( \beta_B \in \mathcal{H} \). Let \( R = \bigcap_{\beta_B \in \mathcal{J}} f_{MB}^{-1}(P_{\beta_B}^0) \) which is open in \( \mathcal{I} \). Then, \( f_{AM}(X) \subseteq R \). There exists \( \sigma_M \in E \) such that

\[
f_{AM}(X) \subseteq P_{\sigma_M}^0 \subseteq P_{\sigma_M} \subseteq R.
\]

Consequently, \( f_{MB}(P_{\sigma_M}^0) \subseteq P_{\beta_B}^0 \) and \( B \subseteq M \) for each \( \beta_B \in \mathcal{J} \).

Hence, \( \beta_B \leq \sigma_M \) for each \( \beta_B \in \mathcal{J} \). As a result,

\[
\pi_{\beta_B}(x) = f_{\sigma_M^{\beta_B}}(\pi_{\sigma_M}(x))
\]

for each \( \beta_B \in \mathcal{J} \). Now, since \( \pi_{\sigma_M}(x) \in f_{AM}(X) \), there exists \( z \in X \) such that \( f_{AM}(z) = \pi_{\sigma_M}(x) \). But \( z \in f_{AD}^{-1}(U_{\delta_D}) \) for some \( \delta_D \in \mathcal{J} \). Therefore, \( f_{AD}(z) \in U_{\delta_D} \). However,

\[
\pi_{\delta_D}(x) = f_{\sigma_M^{\delta_D}}(\pi_{\sigma_M}(x)) = f_{\sigma_M^{\delta_D}}(f_{AM}(z)) = f_{MD} \circ f_{AM}(z) = f_{AD}(z).
\]

Consequently, \( U_{\delta_D} \cap V_{\delta_D} \neq \emptyset \). This is a contradiction to \( U_{\delta_D} \cap V_{\delta_D} = \emptyset \). As a result, there is \( y \in X \) such that \( g(y) = x \).

Hence, \( g \) is onto.
Suppose that \( y \in X \) and that \( U \) is open in \( X_\infty \) such that \( g(y) \in U \). Then there exists a basic open neighborhood \( K \) such that \( g(y) \in K \subseteq U \). Let \( K = \pi_D^{-1}(U_D) \). Now \( \pi_D(g(y)) \in U_D \) and \( f_{AD} \) is continuous. Therefore, there exists \( V \) open in \( X \) such that \( y \in V \) and \( f_{AD}(V) \subseteq U_D \). If \( z \in V, f_{AD}(z) \in U_D \). Therefore, \( g(z) \in K \). Consequently, \( g \) is continuous. Since \( X \) is compact Hausdorff, \( X_\infty \) is Hausdorff and \( g \) is a one to one continuous function from \( X \) onto \( X_\infty \), \( g \) is a homeomorphism. Now, \( Y \) is homeomorphic to \( X_\infty \), the inverse limit of an inverse system of polyhedra.
CHAPTER BIBLIOGRAPHY


BIBLIOGRAPHY


