A THEORETICAL STUDY OF THE MOMENT ON A BODY IN A COMPRESSIBLE FLUID

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SUMMARY

The extension to a compressible fluid of Lagally's theorem on the moment on a body in an incompressible fluid and Poggi's method of treating the flow of compressible fluids are employed for the determination of the effect of compressibility on the moment on an arbitrary body. Only the case of the two-dimensional subsonic flow of an ideal compressible fluid is considered.

As examples of the application of the general theory, two well-known systems of profiles are treated; namely, the elliptic profile and the symmetrical Joukowski profiles with sharp trailing edges.

The effect of compressibility on the position of the center of pressure is also discussed. In order to determine this effect, it is necessary to calculate the additional circulation induced by the compressibility of the fluid for both the elliptic and the Joukowski profiles. For these two types of profile, the centers of pressure in the compressible and the incompressible fluids are found to coincide for a definite and fairly small angle of attack, which is essentially dependent on the thickness coefficients. For angles of attack less than this neutral angle, the center of pressure in the compressible fluid is farther from the nose and, for angles of attack greater than the neutral angle, nearer to the nose than the center of pressure in the incompressible fluid.

Several numerical examples of both the elliptic and the Joukowski profiles are given. The results show that, although the effect of compressibility on the moment and on the lift may be large, the effect on the center of pressure for conventional profiles is negligible. Thus, for a Joukowski profile, the maximum thickness of which is equal to 15 percent of the chord, the center of pressure moves toward the nose a distance equal to only 0.19 percent of the chord, where the angle of attack is 6° and $c_s/c_o = 0.70$.

NOTATION

GENERAL SYMBOLS

$r, \theta$, polar coordinates in the plane of the circle.

$R$, radius vector of a point far removed from the obstacle and also of a point far removed from the corresponding circle.

$r_o$, radius of circle into which the profile is mapped.

$\lambda = \frac{r_o}{r}$

$v_r, v_\theta$, components of the velocity in the radial and the circumferential directions in the $z$ plane.

$v$, magnitude of the velocity in the plane of the obstacle.

c, magnitude of the local velocity of sound.

$\rho$, density of the fluid.

$p_o$, static pressure of the fluid.

$v_o, c_o, \rho_o, p_o$, corresponding magnitudes in the undisturbed part of the fluid.

$\gamma$, ratio of the specific heats ($c_p/c_v$).

$\mu = \frac{\rho v^2}{c^2}$, square of the Mach number.

$\beta$, angle of attack.

$\Gamma$, circulation about the obstacle.

$K = \frac{\Gamma}{2\pi r_o \rho o}$

$L = \rho_o v o \Gamma$, lifting force on the obstacle.

$M$, moment on the obstacle.

$c_o, c_i$, centers of pressure, respectively, in the compressible and the incompressible fluids.

Subscripts $c$ and $i$ refer, respectively, to the compressible and the incompressible fluids.

$\Delta \nu_c, \Delta \nu_i$, additional velocity components due to compressibility at a point $P(R, \theta)$ far removed from the circle in the $z$ plane.

$\omega$, complex velocity potential of the fluid.

$\frac{d\omega}{dz}$, complex velocity in the plane of the obstacle.

$\Delta \omega_c, \Delta \omega_i$, complex and conjugate complex coefficients, respectively, in the power series development of $\frac{d\omega}{dz}$ (see equation (12)).

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SYMBOLS PERTAINING TO ELLIPTIC PROFILE

$c$, semifocal distance.
$t$, thickness ratio of ellipse (ratio of semiminor and semimajor axes).

\[
\sigma = \frac{1 - t}{1 + t}, \quad \text{where} \quad \sigma = \frac{c}{2r_0}.
\]

\[
\tanh \alpha = t
\]

$M$, $N$, functions of the thickness coefficient $t$ only (see equation (24)).

SYMBOLS PERTAINING TO JOUKOWSKI PROFILES

$\varepsilon$, thickness coefficient (see fig. 4).

\[
h = \frac{1 - \varepsilon}{1 + \varepsilon}
\]

\[
k = \frac{1 - \varepsilon}{1 + \varepsilon} = 1 - 2h
\]

$I$, $J$, (see equation (26)).

$M$, $N$, functions of the thickness coefficient $\varepsilon$ only (see equation (29)).

DERIVATION OF THE FORMULA FOR THE MOMENT

Theodorsen's method (reference 1) of extending Lagally's formula for the force on a body in an incompressible fluid to a compressible fluid may be used to obtain the corresponding formula for the moment. The body is fixed in an infinite two-dimensional stream of a frictionless compressible fluid flowing uniformly in the far field. Then the moment exerted on the body with respect to the origin of the coordinate system is given by (fig. 1)

\[
M = \int f [r \cdot n] p \, ds
\]

where the positive direction of the unit normal vector $n$ is taken from the boundary into the fluid, $p$ is the pressure of the fluid, $[r \cdot n]$ is the vector product of the radius vector $r$ and the unit normal vector $n$, and $ds$ is the element of length along the profile taken positively in the direction of the unit tangent vector $t$. With the directions of the unit tangent vector $t$ and the unit normal vector $n$ so chosen that they form a right-hand system, a positive value for the moment corresponds to a counterclockwise rotation.

According to one of the generalizations from Gauss' theorem, it may be shown that

\[
\int \text{curl} \, p r \, ds = \int [r \cdot n] p \, ds + \int [r \cdot n] p \, ds
\]

where $ds$ is the element of surface in a region $S$ included between the obstacle $i$ and an arbitrary curve $o$ enclosing it.

Now

\[
\text{curl} \, pr = [\text{grad} \, p \cdot r]
\]

and, from the Euler equations of motion for steady flow (reference 1),

\[
\text{grad} \, p = -\frac{1}{2} \rho \text{grad} \, \varepsilon
\]

Therefore

\[
\text{curl} \, pr = -\frac{1}{2} \rho [\text{grad} \, \varepsilon \cdot r]
\]

Equation (1) then becomes

\[
M = \int f [r \cdot n] p \, ds - \frac{1}{2} \rho \int [r \cdot n] p \, ds
\]

Since the outer boundary is arbitrary, it may be chosen to be a large circle so that the vector product $[r \cdot n]$ equals zero. Hence

\[
M = \frac{1}{2} \int f [r \cdot n] p \, ds
\]

But

\[
\text{grad} \, \varepsilon = 2 (\text{grad} \, \varepsilon) \cdot v + 2 \varepsilon [\text{curl} \, v] r
\]

Therefore

\[
\int \text{div} \, p F \, ds = -\int f [r \cdot n] p \, ds - \int [r \cdot n] p \, ds
\]

But

\[
\text{div} \, p F = \text{div} \, p F + \rho \text{grad} \, p F
\]

Hence

\[
\int f [r \cdot n] p \, ds = -\int f [r \cdot n] p \, ds - \int [r \cdot n] p \, ds
\]

From the manner in which $F$ occurs in this formula, it is clear that the formula remains valid if $F$ is replaced by a vector point function, say $[v \cdot r]$. That is,

\[
\int f [r \cdot n] p \, ds = -\int f [r \cdot n] p \, ds + \int [r \cdot n] p \, ds
\]

Substituting from this equation into equation (2), it follows that

\[
M = \int f [r \cdot n] p \, ds + \int [r \cdot n] p \, ds
\]

where $\int f [r \cdot n] p \, ds = 0$ since $[v \cdot n] = 0$ everywhere on the obstacle.

In the problem considered herein, the compressible fluid is of uniform velocity $v_0$ in the undisturbed stream

\[\text{footnote 1}\]
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and, furthermore, is assumed to be irrotational and free of sources in the region of flow \( S \). The condition for irrotational motion is simply that \( \text{curl } \mathbf{v} = 0 \), and the absence of sources means that \( \text{div } \rho \mathbf{v} = 0 \). Therefore, equation (3) becomes

\[
M = \int_{\partial S} \rho [r \mathbf{v}] (v \mathbf{n}) \, ds
\]

where it is recalled that the outer boundary has been assumed to be a large circle of radius \( R \). The vector product \([r \mathbf{v}]\) is then equal to \(-\mathbf{v}_n\) since \( \mathbf{n} = -R/\mathbf{R} \). Therefore

\[
M = -R^2 \int_{\partial S} \rho \mathbf{v} \mathbf{v}_n \, d\varphi
\]  

This simple expression for the moment about the origin, of the resultant force acting on the body, can be directly obtained by considering the rate at which angular momentum passes out of the region included between the outer circle and the body. Thus, the momentum per unit time passing normally through the element \( ds \) is \( \rho R v_n \mathbf{v}_n \, d\varphi \) and the arm is \( R \sin \varphi \). The angular momentum is therefore \( \rho R^2 v_n \sin \varphi \mathbf{v}_n \, d\varphi \) or \( \rho R^2 v_n \mathbf{v}_n \, d\varphi \) and equation (4) follows.

GENERAL DEVELOPMENTS

It may be well to emphasize at this point that the main problem of this paper lies in obtaining useful expansions for the velocity components \( v_x \) and \( v_y \) of a compressible fluid.

These expansions are obtained by making use of Poggio's conception of compressible flow. Thus the basic differential equation for the steady flow of a compressible fluid may be written as (reference 2)

\[
\frac{\partial v_x}{\partial \xi} + \frac{\partial v_y}{\partial \eta} = \frac{1}{2 c^2} \left( \frac{\partial v_x^2}{\partial \xi} + \frac{\partial v_y^2}{\partial \eta} \right)
\]

The expression on the left-hand side is \( \text{div } \mathbf{v} \), so that the expression on the right-hand side may be considered to represent a source distribution of a strength given by

\[
\frac{1}{4\pi c^2} \left( \frac{\partial v_x^2}{\partial \xi} + \frac{\partial v_y^2}{\partial \eta} \right) d\xi \, d\eta
\]

In the plane of the circle into which the profile of the obstacle is mapped by a suitable conformal transformation, the strength of the source distribution may be written as

\[
\frac{1}{4\pi c^2} \left( \frac{\partial v_x^2}{\partial \xi} + \frac{\partial v_y^2}{\partial \eta} \right) R^2 d\lambda \, d\theta
\]

where \( r, \theta \) are the polar coordinates of a point in the plane \( z = z + i\theta \) of the circle.

\[
\lambda = \frac{r_0}{r}
\]

where \( r_0 \) is the radius of the circle into which the profile is mapped.

\[
v_r = -\frac{\partial \phi}{\partial r}, v_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}, \text{ where } \phi \text{ is the velocity potential of the flow.}
\]

\( v \), the magnitude of the velocity of the fluid in the plane of the profile.

\( c \), the magnitude of the local velocity of sound.

Poggio's method of approximating the flow of a perfect compressible fluid past an obstacle is based on the fact that the incompressible fluid is a good first approximation. A first approximation for the source distribution given by equation (6) is therefore obtained by substituting for \( v_x, v_y \), and \( v^2 \) the values belonging to the incompressible fluid. It is further assumed that the Mach number \( v_x/c_0 \) is small so that only terms involving the lowest order of \( v/c_0 \) are to be considered. This assumption then limits the application of the analysis to stream velocities small in comparison with the velocity of sound. The disturbances to the main flow due to the presence of a body in the fluid may, however, be large.

The density \( \rho \) and the pressure \( p \) of the fluid are then determined by the following equations (reference 2):

\[
\rho = \rho_0 \left[ 1 + \frac{7 \mu}{4} \left( 1 - \frac{c^2}{v_x^2} \right)^{\frac{1}{2}} \right]
\]

\[
\rho = \rho_0 \left[ 1 + 1/2 \mu \left( 1 - \frac{v_x^2}{v_y^2} \right) + \ldots \right]
\]

\[
p = \text{constant} - \frac{1}{4} \rho \mu \frac{v_x^2}{v_y^2} + \frac{1}{8} \mu \frac{v_x^2}{v_y^2} + \ldots
\]

where the adiabatic equation of state has been adopted. Also \( \mu = (v_x/c_0)^2 \) and \( v_x \) and \( v_y \) are, respectively, the velocities in the incompressible and the compressible fluids.

It follows from equation (5), with the stipulation that only terms involving the square of the Mach number are to be retained, that:

\[
M = -\rho \rho_0 \frac{1}{2} \int_{\partial S} \frac{v_x}{v_y} \left( \frac{\partial v_x}{\partial \xi} + \frac{\partial v_y}{\partial \eta} \right) \, d\varphi
\]

\[
-\frac{1}{2} \mu \rho \rho_0 \frac{1}{2} \int_{\partial S} \left( 1 - \frac{v_x^2}{v_y^2} \right) \left( \frac{\partial v_x}{\partial \xi} + \frac{\partial v_y}{\partial \eta} \right) \, d\varphi
\]

It is clear that the evaluation of the second integral presents no difficulties, for it involves only expressions of the well-known incompressible fluid-velocity components. The difficulty of the problem lies mainly in determining the velocity components \( v_x \) and \( v_y \) for a compressible fluid. Since the first integral is taken around a circle whose radius \( R \) may be infinite, it is obvious that the developments for \( v_x/v_0 \) and \( v_y/v_0 \) in the neighborhood of infinity need not go beyond the \( 1/R^2 \) terms.

In order to obtain the series for \( v_x/v_0 \) and \( v_y/v_0 \), it is expedient to consider first the effect of a single source of unit strength situated at a point \( Q(r, \theta) \) in the plane of a circle. In the presence of a circular boundary of radius \( r_0 \), the velocity induced by a unit source at any

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1 In the approximation made by Glencert (reference 3), \( \rho_0 \) is not small compared with \( \rho_0 \) but the disturbances to the main flow \( v_x \) are assumed to be small.
point \( P(R, \delta) \) external to the boundary is given by (fig. 2):

\[
\frac{dw}{dz} = -\left( \frac{1}{z_p - z_0} + \frac{1}{z_p - z_0} \right) = -w_v + iw_x
\]

where \( S \) is the point inverse to \( Q \) with respect to the circle:

\[
z_p = Re^{i\theta}; \quad z_0 = Re^{i\theta}; \quad \text{and} \quad z_x = \frac{r_0^2}{r} e^{i\theta}.
\]

Then

\[
v_x - iv_y = \frac{1}{Re^{i\theta} - Re^{i\theta}} + \frac{1}{Re^{i\theta} - Re^{i\theta}} - \frac{1}{Re^{i\theta} - Re^{i\theta}}
\]

or since

\[
(-w_x + iw_y)e^{-i\theta} = -w_x + iw_y
\]

for \( r_0 < r < R \)

\[
v_x = \frac{1}{2\pi} \int_0^\infty \frac{1 - \lambda \lambda^2}{1 - 2\lambda \lambda \cos (\theta - \delta) + \lambda^2 \lambda^2} \sin (\theta - \delta) \, d\theta
\]

and for \( R < r < \infty \)

\[
v_x = \frac{1}{2\pi} \int_0^\infty \frac{1 - \lambda \lambda^2}{1 - 2\lambda \lambda \cos (\theta - \delta) + \lambda^2 \lambda^2} \sin (\theta - \delta) \, d\theta
\]

where \( \lambda = r_0/r \) and \( \lambda_p = r_0/R \).

Then, making use of the expansion

\[
\frac{1 - \delta^2}{1 - 2\delta \cos (\theta - \delta) + \delta^2} = 1 + 2\sum_{n=1}^\infty \delta^n \cos n(\theta - \delta)
\]

it follows that the components of the velocity induced at any point \( P(R, \delta) \) by the source distribution given by equation (6) are:

\[
\begin{align*}
\Delta v_x &= -\frac{\mu}{4\pi} \int_0^{2\pi} \int_0^R \frac{1}{F} \lambda^2 \lambda^2 \left[ \frac{1}{2} + \sum_{n=1}^\infty \left( \frac{\lambda^2}{\lambda} \right)^n \cos n(\theta - \delta) \right] d\lambda \, d\theta \\
\Delta v_y &= \frac{\mu}{4\pi} \int_0^{2\pi} \int_0^R \frac{1}{F} \lambda^2 \lambda^2 \left[ \frac{1}{2} + \sum_{n=1}^\infty \left( \frac{\lambda^2}{\lambda} \right)^n \cos n(\theta - \delta) \right] d\lambda \, d\theta
\end{align*}
\]

and

\[
\begin{align*}
\Delta v_x &= -\frac{\mu}{4\pi} \int_0^{2\pi} \int_0^R \frac{1}{F} \lambda^2 \lambda^2 \left[ \frac{1}{2} + \sum_{n=1}^\infty \left( \frac{\lambda^2}{\lambda} \right)^n \sin n(\theta - \delta) \right] d\lambda \, d\theta \\
\Delta v_y &= \frac{\mu}{4\pi} \int_0^{2\pi} \int_0^R \frac{1}{F} \lambda^2 \lambda^2 \left[ \frac{1}{2} + \sum_{n=1}^\infty \left( \frac{\lambda^2}{\lambda} \right)^n \sin n(\theta - \delta) \right] d\lambda \, d\theta
\end{align*}
\]

where

\[
F = \left( \frac{v_x}{v_0} \right) \left( \frac{\partial v_x}{\partial \delta} - \frac{v_0}{v_0} \frac{\partial v_0}{\partial \delta} \right)
\]

The next step is to obtain the Fourier series expansion of the function \( F \) for an arbitrary profile. The profile is derived from a circle by means of the general conformal transformation

\[
t' = t' + \frac{at_1}{t} + \frac{at_2}{t^2} + \ldots
\]

where the coefficients \( a_i \) are, in general, complex. In order to obtain the velocity of the fluid in the plane of the profile, it is first necessary to determine the velocity potential of the flow past a circle (fig. 3). Thus, sup-
pose that the undisturbed flow is of velocity \( v_0 \) inclined at an angle \( \beta \) to the negative direction of the real axis and that the circulation is denoted by \( \Gamma \). In terms of the complex coordinate \( z \), with origin at the center of the circle, the potential function of the flow past the circle is:

\[
\omega = v_0 \left( z e^{i\theta} + r_0^2 \right) + \frac{i\Gamma}{2\pi} \log \frac{z e^{i\theta}}{r_0} \tag{11}
\]

Also, the variables \( z \) and \( z' \) are connected by the equation

\[
z' = z + m e^{i\theta}.
\]

Therefore

\[
\frac{dw}{dz} = \frac{dz'}{dz} = \frac{v_0 e^{i\theta} - r_0 e^{-i\theta}}{z^2} + \frac{i\Gamma}{2\pi z} \left( 1 - \frac{2r_0}{z^2} \right) + \ldots
\]

Substituting for \( z' \) in terms of \( z \) and expanding in descending powers of \( z \), this expression becomes:

\[
\frac{1}{v_0} \frac{dz}{dz} = e^{i\theta} + \frac{i\Gamma}{2\pi x} \left( 1 - \frac{2r_0}{x^2} \right) + \ldots \tag{12}
\]

It follows then, that:

\[
\frac{1}{v_0} \frac{dz}{dz} = \sum_{n=0}^{\infty} A_n \left( \frac{z^n}{z^2} \right)
\]

where the \( A_n \)'s are, in general, complex.

In order to introduce the angle \( \theta - \delta \), the general term of this series is multiplied by \( e^{-i\theta \delta} / e^{-i\theta \delta} \).

Then

\[
\frac{1}{v_0} \frac{dz}{dz} = \sum_{n=0}^{\infty} A_n \left( e^{-i\theta \delta} \right)
\]

and

\[
\frac{1}{v_0} \frac{dz}{dz} = \sum_{n=0}^{\infty} \frac{A_n e^{i\theta \delta}}{2\pi r_0^2 \theta^2 e^{-i(\theta - \delta)}}
\]

Hence

\[
\frac{\gamma^2}{v_0^2} = \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \lambda^2 \sum_{k=0}^{n} \lambda^{2k}}{2\pi r_0^2 \theta^2 e^{-i(\theta - \delta)}}
\]

The real and the imaginary parts of \( A_n A_n e^{i(\theta - \delta)} \) are, respectively:

\[
\frac{\sum_{k=0}^{n} \lambda^2 A_n A_n e^{i(\theta - \delta)}}{2}
\]

and

\[
\frac{\sum_{k=0}^{n} \lambda^2 A_n A_n e^{i(\theta - \delta)}}{2}
\]

Let

\[
\delta - k = n
\]

where \( n \) changes sign when \( j \) and \( k \) are interchanged and therefore takes values from \( -\infty \) to \( \infty \).

Then

\[
\delta + k = n + 2k
\]

and the expression for \( \gamma / v_0^2 \), a real quantity, becomes:

\[
\frac{\gamma^2}{v_0^2} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[ A_n \cos (\delta - \delta) + B_n \sin (\delta - \delta) \right] \lambda^{n+2k} \tag{13}
\]

where

\[
A_n = \frac{A_n A_n e^{i(\theta - \delta)} + A_n A_n e^{-i(\theta - \delta)}}{2\pi r_0^2 \theta^2}
\]

and

\[
B_n = \frac{A_n A_n e^{i(\theta - \delta)} - A_n A_n e^{-i(\theta - \delta)}}{2\pi r_0^2 \theta^2}
\]

From the definition of \( n \), it follows that:

\[
A_{n-k} = A_{n,k}
\]

and

\[
B_{n-k} = -B_{n,k}
\]

The terms of equation (13) can therefore be grouped in pairs, thus:

\[
\frac{\gamma^2}{v_0^2} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \lambda^{2k} A_n \cos (\delta - \delta) + 2\sum_{n=0}^{\infty} \sum_{k=0}^{n} B_n \sin (\delta - \delta) \sum_{k=0}^{n} A_n \lambda^{n+2k} \tag{14}
\]

Also, from equation (11),

\[
\frac{\gamma}{v_0} = \frac{(1 - \lambda^2) \cos (\delta - \delta) \cos (\delta + \beta) + \lambda \gamma^2}{v_0} + \frac{(1 + \lambda^2) \sin (\delta - \delta) \sin (\delta + \beta)}{v_0}
\]

and

\[
\frac{\gamma}{v_0} = \frac{(1 + \lambda^2) \cos (\delta - \delta) \sin (\delta + \beta) + (1 + \lambda^2) \sin (\delta - \delta) \cos (\delta + \beta) + \lambda K}{v_0}
\]

where

\[
K = \frac{\Gamma}{2\pi r_0 v_0}
\]
Then, by means of equations (14) and (15), it becomes a simple matter to obtain the Fourier series for the function $F$. When the Fourier series for $F$ is substituted into equations (9) and (10) and all terms containing powers of $\lambda_r$ higher than the second are neglected, it follows, after some rather tedious but elementary integrations, that:

\[
\frac{\Delta v_r}{v_0} = -\frac{\mu}{2} \lambda_r \left[ A_{1,0} \cos (\alpha + \beta) - B_{1,0} \sin (\alpha + \beta) \right] \\
- \frac{\mu}{2} \left[ A_{0,1} \cos (\alpha + \beta) + KB_{1,0} \lambda_r^2 \log \lambda_r \right] \\
+ \mu \lambda_r \left[ \frac{3}{2} A_{1,0} \sin (\alpha + \beta) + \frac{1}{2} A_{1,0} \sin (\alpha + \beta) + \frac{1}{2} KB_{1,0} \right] \\
+ \frac{\mu K}{4} \lambda_r \sum_{n=0}^{\infty} \frac{n+1}{n(n+2)} B_{1,n} \\
+ \frac{\mu}{2} \lambda_r \sum_{n=1}^{\infty} \frac{A_{n+1} \cos (\alpha + \beta) + B_{n+1} \sin (\alpha + \beta)}{n+1}.
\] (16)

and

\[
\frac{\Delta v_\theta}{v_0} = \frac{\mu}{2} \lambda_r \left[ A_{1,0} \sin (\alpha + \beta) - B_{1,0} \cos (\alpha + \beta) \right] \\
- \frac{\mu}{2} \left[ A_{0,1} \sin (\alpha + \beta) + KB_{1,0} \lambda_r^2 \log \lambda_r \right] \\
+ \mu \lambda_r \left[ \frac{3}{2} A_{1,0} \sin (\alpha + \beta) + \frac{1}{2} A_{1,0} \sin (\alpha + \beta) + \frac{1}{2} \lambda_r B_{1,0} \lambda_r \right] \\
+ \frac{\mu K}{4} \lambda_r \sum_{n=0}^{\infty} \frac{n+1}{n(n+2)} B_{1,n} \\
+ \frac{\mu}{2} \lambda_r \sum_{n=1}^{\infty} \frac{A_{n+1} \sin (\alpha + \beta) + B_{n+1} \cos (\alpha + \beta)}{n+1}.
\] (17)

Use has been made of the following definite integrals in obtaining $\Delta v_r/v_0$:

\[
\int_0^{\pi} \frac{\sin (\theta - \beta) \sin (\theta - \beta) \, d\theta}{1 - 2h \cos (\theta - \beta) + h^2} = \begin{cases} 0 & \text{if } n = 0 \\ \pi h^{-n} & \text{if } n \geq 1 \end{cases}
\]

The velocity of the compressible fluid in the plane of the circle then has the following components:

\[
\frac{v_\theta}{v_0 \cos (\alpha + \beta) + \frac{\Delta v_\theta}{v_0}} = -\left(1 - \frac{1}{\lambda_r^2} \right) \cos (\alpha + \beta) + \frac{\Delta v_\theta}{v_0}
\]

and

\[
\frac{v_\theta}{v_0 \cos (\alpha + \beta) + \frac{\Delta v_\theta}{v_0}} = \left(1 + \frac{1}{\lambda_r^2} \right) \sin (\alpha + \beta) + \lambda_r K + \frac{\Delta v_\theta}{v_0}
\]

The calculation of the moment is facilitated by expressing the velocity of the compressible fluid in complex form. Thus, as in the case of an incompressible fluid, a complex velocity is defined in the following way:

\[
\frac{1}{v_0}(\frac{dw}{dz}) = \left( -\frac{v_\theta}{v_0} + \frac{v_\theta}{v_0} \right) e^{-i\alpha}
\]

Then equation (8) for the moment assumes the Blasius form:

\[
M_c = \frac{1}{2} v_0^2 v_\theta^2 R. P. \int \left( \frac{1}{v_0} \frac{\partial \lambda}{\partial z} \right) \frac{dz}{dz} dz
\]

where the integrals are taken around a circle whose radius $R$ approaches infinity.

Next, from equations (13), (16), (17), and (18), the expression for $1/v_0 \frac{dw}{dz}$, may be written as:

\[
\frac{1}{v_0}(\frac{dw}{dz}) = e^{i\alpha} + \frac{iK_r}{z} - \frac{r_0 e^{i\alpha}}{z} + \frac{4}{z^2} \left( A_{1,1} e^{-i\alpha} + K_{A_{1,1} e^{-i\alpha}} \right)
\]

Also, from equation (12) for $1/v_0$:

\[
A_0 = e^{i\alpha}, A_1 = iK_r, A_2 = b e^{i(\pi r_0)} - r_0 e^{-i\alpha}
\]

where $a_1 = b e^{i\pi r_0}$

Therefore:

\[
\frac{1}{v_0}(\frac{dw}{dz}) = e^{i\alpha} + \frac{iK_r}{z} - \frac{r_0 e^{-i\alpha}}{z} + \frac{4}{z^2} \left( A_{1,1} e^{-i\alpha} + K_{A_{1,1} e^{-i\alpha}} \right)
\]

\[
+ \frac{4K_r e^{-i\alpha}}{z^2} \log \frac{r_0}{R} + \frac{2K_r e^{i\alpha}}{z} + \frac{b e^{i(\pi r_0)} - r_0 e^{-i\alpha}}{z^2} \frac{2K_r e^{-i\alpha}}{z^2}
\]

\[
- \frac{1}{z^2} \sum_{n=1}^{\infty} \frac{A_{n+1}}{n+1} \frac{A_{n+1}}{n+1} + \frac{2}{z^2} \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{A_{n+1}}{n+1}
\]

\[
- \frac{2}{z^2} \sum_{n=1}^{\infty} \frac{A_{n+1}}{n+1} \frac{A_{n+1}}{n+1} + \frac{2}{z^2} \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{A_{n+1}}{n+1}
\]
It may be noted that this expression is not an analytic function of \( z \), since powers of \( R \) occur in some of the terms.

The first integral on the right-hand side of equation (19) for \( M_z \) then becomes:

\[
-\frac{1}{2} \rho_0 b \int \frac{1}{z_0} \frac{d\omega}{dz} \frac{dz}{d\xi} \frac{dz}{dz}
\]

\[
= -\frac{1}{2} \rho_0 b \int \frac{1}{z_0} \frac{d\omega}{dz} \left( 1 + \frac{m \omega' e^{i \theta}}{\omega^2} + \cdots \right) dz
\]

\[
= 2\pi b^3 \rho_0 b \sin (\beta + \gamma) + \rho_0 \Gamma m \cos (\beta + \gamma)
\]

\[
+ \frac{1}{2} \int 3 \pi b^3 \rho_0 b \sin (\beta + \gamma)
\]

\[
+ \frac{1}{2} \rho_0 b \Gamma \int \frac{1}{z_0} \frac{d\omega}{dz} \frac{dz}{d\xi} \frac{dz}{dz}
\]

\[
= 2\pi b^3 \rho_0 b \sin (\beta + \gamma) + \rho_0 \Gamma m \cos (\beta + \gamma)
\]

\[
+ \frac{1}{2} \rho_0 b \Gamma \int \frac{1}{z_0} \frac{d\omega}{dz} \frac{dz}{d\xi} \frac{dz}{dz}
\]

\[
= 2\pi b^3 \rho_0 b \sin (\beta + \gamma) + \rho_0 \Gamma m \cos (\beta + \gamma)
\]

Therefore

\[
M_z = M_0 \left( 1 + \frac{\mu}{2} \right) + \rho_0 \Gamma m \cos (\beta + \gamma)
\]

\[
+ \frac{1}{2} \rho_0 \Gamma \int \frac{1}{z_0} \frac{d\omega}{dz} \frac{dz}{d\xi} \frac{dz}{dz}
\]

\[
= 2\pi b^3 \rho_0 b \sin (\beta + \gamma) + \rho_0 \Gamma m \cos (\beta + \gamma)
\]

\[
+ \frac{1}{2} \rho_0 \Gamma \int \frac{1}{z_0} \frac{d\omega}{dz} \frac{dz}{d\xi} \frac{dz}{dz}
\]

\[
= 2\pi b^3 \rho_0 b \sin (\beta + \gamma) + \rho_0 \Gamma m \cos (\beta + \gamma)
\]

Equation (20) for the moment \( M_z \) is applicable to arbitrary profiles, but it contains two infinite series that cannot, in general, be replaced by closed forms. In the case of a profile for which the conformal transformation to a circle contains a finite number of terms, the two infinite series may, however, sometimes be replaced by elementary functions. In the following, two such systems of profiles will be discussed, the elliptic and the Joukowski profiles.

**THE ELLIPTIC PROFILE**

It is well known that the Joukowski transformation

\[
\zeta = z + \frac{c^2}{4z}
\]

maps the circle of radius \( c/2 \) with its center at the origin of the \( z \) plane into a line segment \((-c, 0; +c, 0)\) in the \( \zeta \) plane. Also, circles concentric with the base circle are transformed into a family of confocal ellipses with common focii at \((-c, 0)\) and \((+c, 0)\). If \( r_o > c/2 \) denotes the radius of one of these circles, then the semimajor and the semiminor axes of the ellipse into which it is transformed are, respectively, \( r_o + c^2/4r_o \) and \( r_o - c^2/4r_o \). The thickness ratio \( t \) is then defined as

\[
t = \frac{r_o - c^2}{r_o + c^2} = \frac{1 - \sigma^2}{1 + \sigma^2}
\]

or

\[
\sigma = \frac{c}{2r_o}
\]

Now, for the case of the elliptic cylinder, equation (12) becomes:

\[
\rho_0 \int \frac{1}{z_0} \frac{d\omega}{dz} \frac{dz}{d\xi} \frac{dz}{dz}
\]

\[
= \left( e^{i \theta} + \frac{i r_o}{2} \frac{1}{z_0} \frac{r_o e^{i \theta}}{z^2} \right) \sum_{n=1}^{\infty} \frac{A_n}{z^n}
\]

or

\[
A_0 = e^{i \theta}, \quad A_{n+1} = i K r_o e^{i (n+1) \theta}, \quad A_{2n} = -e^{-i \theta} r_o e^{i n \theta} (e^{2i \theta} - 1)
\]

where

\[
K = \frac{r_o}{2 \pi r_o}
\]

Then

\[
e^{i \theta} \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)(n+2)} \frac{A_n}{r_o^{n+2}} = e^{i \theta} \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)(n+2)} \frac{A_n}{r_o^{n+2}}
\]

\[
+ \frac{4n+3}{(n+1)(n+2)} \frac{A_{n+2}}{r_o^{n+2}}
\]

\[
= i K r_o (e^{2i \theta} - 1) \sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)} \frac{1}{2n+2} \sigma^n
\]

and

\[
R. P. e^{i \theta} \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)(n+2)} \frac{A_n}{r_o^{n+2}} = K r_o \sin 2 \beta \sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)} \frac{1}{2n+2} \log 1 + \sigma^2 + \frac{1}{1 - \sigma^2}
\]

Similarly

\[
e^{i \theta} \sum_{n=1}^{\infty} \frac{A_n}{r_o^{n+2}} = e^{i \theta} \sum_{n=1}^{\infty} \frac{A_n}{r_o^{n+2}} + i e^{i \theta} \sum_{n=1}^{\infty} \frac{A_n}{r_o^{n+2}}
\]

\[
= i e^{i \theta} r_o (1 - 2 \sigma \cos 2 \beta + \sigma^2) \sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)} + i e^{i \theta} K r_o \sum_{n=1}^{\infty} \frac{1}{2n+2}
\]
and

\[ R.P. = \sum_{n=0}^{\infty} \frac{A_n A_{n+2}}{(n+1)!} r^{n+2} \]

\[ = -r_0^2 \sin 2\theta (1 - 2r_0^2 \cos 2\theta + r_0^4) \left( \frac{1}{2r_0^2} \log \frac{1 + \frac{r_0^2}{1 - \frac{r_0^2}{\sigma^2}}}{1 + \frac{1}{\sigma^2}} \right) \]

\[ - K_{r_0} r_0^2 \sin 2\beta \left( \sum_{n=0}^{\infty} \frac{a^n}{2^n} \right) \]

Therefore, since \( m=0 \) and \( v=0 \) for an elliptic profile, equation (20) becomes:

\[ M_r = M_1 \left[ 1 + \frac{2}{3} (1 - 2r_0^2 \cos 2\theta + r_0^4) \left( \frac{1}{2r_0^2} \log \frac{1 + \frac{r_0^2}{1 - \frac{r_0^2}{\sigma^2}}}{1 + \frac{1}{\sigma^2}} \right) \right] \]

\[ + \frac{\mu K_r}{2} \int \left[ \frac{1 + \frac{r_0^2}{1 - \frac{r_0^2}{\sigma^2}} - 1 - \frac{1}{2r_0^2} - \frac{1}{\sigma^2} \log (1 - \sigma^2) }{4r_0^2} \right] \] \( \text{(21)} \)

where \( M_r = \frac{\pi}{2} \rho_0 r_0^2 \sigma^3 \sin 2\beta \).

For thick profiles, or for small values of \( \sigma \), this formula may be expressed as a power series in \( \sigma \). Thus, by making use of the expansions

\[ \log \frac{1 + \sigma^2}{1 - \sigma^2} = 2 \left( \frac{\sigma^2}{2} + \frac{\sigma^4}{4} + \frac{\sigma^6}{6} + \cdots \right) \]

and

\[ \log (1 - \sigma^2) = - \left( \frac{\sigma^2}{2} + \frac{\sigma^4}{4} + \frac{\sigma^6}{6} + \cdots \right) \]

it follows, from equation (21), that

\[ M_r = M_1 \left[ 1 + \frac{2}{3} (1 - 2r_0^2 \cos 2\theta + r_0^4) \left( \frac{1}{2r_0^2} \log \frac{1 + \frac{r_0^2}{1 - \frac{r_0^2}{\sigma^2}}}{1 + \frac{1}{\sigma^2}} \right) \right] \]

\[ + \frac{\mu K_r}{2} \int \left[ \frac{5}{3} \frac{23}{30} \sigma^4 + \frac{53}{105} \sigma^6 + \frac{95}{252} \sigma^{12} + \cdots \right] \] \( \text{(21)} \)

It is seen from this equation that the value of \( \frac{M_r - M_1}{M_1} \) for the limiting case of a circle, for which \( t=1 \) or \( \sigma=0 \), is:

\[ \frac{M_r - M_1}{M_1} = \frac{\mu}{2} 4 + 5K \]

On the other hand, equation (21) shows that, for the limiting case of a straight-line segment for which \( t=0 \) or \( \sigma=1 \) and the angle of attack \( \beta \) is finite,

\[ \frac{M_r - M_1}{M_1} = \infty \]

It is to be noted that, although \( M_1 \) and therefore \( M_r \) is zero when the angle of attack \( \beta \) is zero, the ratio \( \frac{M_r - M_1}{M_1} \) possesses the limiting value

\[ \left( \frac{M_r - M_1}{M_1} \right)_{\sigma=0} = \frac{4}{3} \frac{\mu}{2} \] \( \text{(22)} \)

where \( K_r = 2 \sin \beta \), according to the condition that the rear stagnation point occurs at the end of the major axis of the ellipse. For the two limiting cases of the circle and the straight-line segment, this ratio becomes, respectively:

\[ \left( \frac{M_r - M_1}{M_1} \right)_{\sigma=0} = \frac{4}{3} \frac{\mu}{2} \]

\[ \left( \frac{M_r - M_1}{M_1} \right)_{\sigma=1} = \frac{\mu}{2} \] \( \text{(23)} \)

In appendix A, an independent derivation of equation (21), with \( K_r = 0 \), is obtained by a direct integration of the pressures over the surface of an elliptic cylinder. It shows clearly the superiority of the present general method, in which all integrations are performed along a circle at infinity.

It is of interest to investigate the effect of compressibility on the center of pressure. If the rear stagnation point is supposed to be at the end of the major axis, it is immediately seen from the second of equations (18) that, for \( \delta = \pi \) and \( \lambda_r = 1 \),

\[ \left( \frac{M_r}{\rho_0} \right)_{\sigma=0} = -2 \sin \beta + K_r + \Delta K + \left( \frac{\Delta \psi}{\psi} \right)_{\sigma=1} = 0 \]

Since \( K_r = 2 \sin \beta \), it follows that the additional circulation \( \Delta K \) is given by

\[ \Delta K = \left( \frac{\Delta \psi}{\psi} \right)_{\sigma=1} \]

The problem of determining the additional circulation thus amounts to finding an expression for \( \Delta \psi/\psi \) at the stagnation point \( \delta = \pi \). This calculation is given in appendix B and it is shown there that

\[ \frac{\Delta \Gamma}{\Gamma_t} = \frac{\Delta K}{K_r} = \frac{\mu}{2} \frac{M + N \sin^2 \beta}{2} \] \( \text{(24)} \)

where \( M \) and \( N \) are functions of the thickness coefficient \( t \) only. Table I presents values of \( M \) and \( N \) for various values of \( t \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( M )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.00000</td>
<td>0.0</td>
</tr>
<tr>
<td>0.1</td>
<td>1.09320</td>
<td>3.45779</td>
</tr>
<tr>
<td>0.2</td>
<td>1.21923</td>
<td>2.49505</td>
</tr>
<tr>
<td>0.3</td>
<td>1.39257</td>
<td>1.96138</td>
</tr>
<tr>
<td>0.4</td>
<td>1.67620</td>
<td>1.63790</td>
</tr>
<tr>
<td>0.5</td>
<td>1.95923</td>
<td>1.35814</td>
</tr>
<tr>
<td>0.6</td>
<td>2.23143</td>
<td>1.10008</td>
</tr>
<tr>
<td>0.7</td>
<td>2.50332</td>
<td>0.90784</td>
</tr>
<tr>
<td>0.8</td>
<td>2.78816</td>
<td>0.85974</td>
</tr>
<tr>
<td>0.9</td>
<td>3.08812</td>
<td>0.79704</td>
</tr>
<tr>
<td>1.0</td>
<td>3.60035</td>
<td>0.68067</td>
</tr>
</tbody>
</table>

For the straight-line segment, \( \Delta \Gamma/\Gamma_t \) equals infinity if the value of \( \beta \) is other than zero but, for very thin profiles and vanishingly small angles of attack,

\[ \frac{\Gamma_r + \Delta \Gamma}{\Gamma_t} = 1 + \frac{\mu}{2} \]

This result agrees with that given by Glauert (reference 3), namely:

\[ \frac{\Gamma_r + \Delta \Gamma}{\Gamma_t} = \frac{1}{\sqrt{1 - \mu}} = 1 + \frac{\mu}{2} + \cdots \]

In reference 3, Glauert has also shown that the lifting force on a body in a compressible fluid is given by

\[ L_\infty = \rho_0 \frac{\Delta \psi}{\psi} \]

or

\[ L_\infty = L \left( 1 + \frac{\mu}{2} \right) \] \( \text{(25)} \)
THEORETICAL STUDY OF THE MOMENT ON A BODY IN A COMPRESSIBLE FLUID

If \( c_c \) and \( c_t \) denote, respectively, the centers of pressure in the compressible and the incompressible fluids, then according to equations (23) and (25)

\[
\frac{c_c}{c_t} = 1
\]

That is, for a very thin profile and for vanishingly small angles of attack, the center of pressure is unaffected by the compressibility of the fluid. For the general elliptic profiles, it follows from equations (21) and (24) that

\[
\frac{c_c}{c_t} = 1 + \frac{\mu}{2}
\left[ 1 + \left( 1 - 2\sigma^2 \cos 2\beta + \sigma^4 \right) \left( \frac{1}{2\sigma^3} \log \frac{1 + \sigma}{1 - \sigma} - \frac{1}{\sigma^4} \right) \right] + 4 \sin^2 \beta \left[ \frac{1 + \sigma}{4\sigma^3} \log \frac{1 + \sigma}{1 - \sigma} - \frac{1}{2\sigma^2} - \frac{1}{4\sigma^4} \log (1 - \sigma) \right]
\]

\[
1 + \frac{\mu}{2} (M + N \sin^2 \beta)
\]

It may be shown from this expression that, for any given ellipse, there exists an angle of attack \( \beta \), independent of the stream velocity \( v_0 \), to the first order of \( \mu \), for which the ratio \( c_c/c_t \) equals unity. Furthermore, if the angle of attack is [greater] than this neutral value of \( \beta \), then \( c_c/c_t \) is [greater] than unity and the center of pressure \( c_c \) in the compressible fluid is [farther from] the origin than the center of pressure \( c_t \) in the incompressible fluid. Table II presents the neutral values of \( \beta \) for the entire range of ellipses including the straight-line profile and the circle.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \cos \beta )</th>
<th>( \beta_{neutral} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>1.00000</td>
<td>0.0000</td>
</tr>
<tr>
<td>( .1 )</td>
<td>0.99983</td>
<td>0.0177</td>
</tr>
<tr>
<td>( .2 )</td>
<td>0.99969</td>
<td>0.0356</td>
</tr>
<tr>
<td>( .3 )</td>
<td>0.99956</td>
<td>0.0534</td>
</tr>
<tr>
<td>( .4 )</td>
<td>0.99943</td>
<td>0.0713</td>
</tr>
<tr>
<td>( .5 )</td>
<td>0.99930</td>
<td>0.0891</td>
</tr>
<tr>
<td>( .6 )</td>
<td>0.99917</td>
<td>0.1069</td>
</tr>
<tr>
<td>( .7 )</td>
<td>0.99904</td>
<td>0.1247</td>
</tr>
<tr>
<td>( .8 )</td>
<td>0.99891</td>
<td>0.1425</td>
</tr>
<tr>
<td>( .9 )</td>
<td>0.99878</td>
<td>0.1603</td>
</tr>
<tr>
<td>( 1.0 )</td>
<td>0.99865</td>
<td>0.1781</td>
</tr>
</tbody>
</table>

As a numerical example, consider an elliptic cylinder with a thickness coefficient \( t = \frac{1}{6} \) and with \( v_0/c_t = 0.0024 \). In this case, the neutral angle of attack \( \beta \) is given by \( \cos 2\beta = 0.0024 \), or \( \beta = 12^\circ 40' \). If the angle of attack is increased to \( 15^\circ \), say, then

\[
\frac{M_c}{M_t} = 1.20776, \quad \frac{L_c}{L_t} = 1.19345
\]

and

\[
\frac{c_c}{c_t} = 1.012
\]

Now

\[
c_t = \frac{c}{2} (1 - t)
\]

Therefore

\[
\frac{c_c - c_t}{2a} = 0.0015
\]

or the motion of the center of pressure toward the origin is 0.0015 percent of the chord. Again, if the angle of attack is vanishingly small, then cos \( \beta \approx 1 \) and sin \( \beta \approx \beta \), and

\[
\frac{M_c}{M_t} = 1.14486, \quad \frac{L_c}{L_t} = 1.18174
\]

Then

\[
\frac{c_c - c_t}{2a} = -0.0039
\]

or the motion of the center of pressure away from the origin is 0.0039 percent of the chord. In general, when the angle of attack is vanishingly small, the center of pressure \( c_c \) in the compressible fluid is nearer to the origin than the center of pressure \( c_t \) in the incompressible fluid. Table III gives values for \( \frac{c_c - c_t}{2a} \) for various values of \( t \) and for the critical values of \( \mu \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \frac{\mu}{2} )</th>
<th>( \frac{(c_c - c_t)}{2a} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>1.00000</td>
<td>1.0000</td>
</tr>
<tr>
<td>( .1 )</td>
<td>0.99983</td>
<td>0.0000</td>
</tr>
<tr>
<td>( .2 )</td>
<td>0.99969</td>
<td>0.0000</td>
</tr>
<tr>
<td>( .3 )</td>
<td>0.99956</td>
<td>0.0000</td>
</tr>
<tr>
<td>( .4 )</td>
<td>0.99943</td>
<td>0.0000</td>
</tr>
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<td>( .5 )</td>
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<td>( .9 )</td>
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<td>0.0000</td>
</tr>
<tr>
<td>( 1.0 )</td>
<td>0.99865</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

THE SYMMETRICAL JOUKOWSKI PROFILE

The Joukowski profiles are derived by means of the conformal transformation

\[
\zeta = z' + \frac{a^2}{z'}
\]

which maps the circle of radius \( a \) with its center at the origin of the \( z' \) plane into a line segment \((-2a, 0; 2a, 0)\) in the \( \zeta \) plane. Any other circle of radius \( r_0 (> a) \) with its center lying on the real axis at a distance \( ae \) from \( O \) (see fig. 4) and touching the base tide at \((-a, 0)\) is mapped into a symmetrical Joukowski profile with a sharp trailing edge. The variables \( z \) and \( z' \) are connected by the equation

\[
z' = ea + z
\]

The circulation \( \Gamma_z \) is chosen in accordance with the Kutta-Joukowski hypothesis that the rear stagnation point of the flow occurs at the point \( A \) of the circle. In

\[\text{[Footnote 1]}\]
terms of the complex coordinate \( z \), the potential function of the flow past the circle is

\[
w = w_0 \left( z e^{i\theta} + \frac{r_0^2}{z e^{i\theta}} \right) + 2\pi i \log \frac{z e^{i\theta}}{r_0}
\]

The complex velocity at the point \( A \) is then given by

\[
\frac{dv}{dz} = 0 = v_0 (e^{i\theta} - e^{-i\theta}) - \frac{i\Gamma}{2\pi r_0}
\]

Therefore

\[
\Gamma = 4\pi r_0 \sin \beta \quad \text{or} \quad K = 2 \sin \beta
\]

The complex velocity in the \( \zeta \) plane is given by

\[
\frac{1}{v_0} \frac{dw}{dz} = \left( 1 + \frac{x\zeta}{z} \right) \left( \frac{e^{i\theta} - \frac{a\zeta}{z}}{1 - a(1 - e)} \right)
\]

Then, since

\[
z' + a = z + a(1 + e) = z + r_0 \quad \text{and} \quad z' - a = z - a(1 - e)
\]

it follows that

\[
\frac{1}{v_0} \frac{dw}{dz} = \left( 1 + \frac{x\zeta}{z} \right) \left( \frac{e^{i\theta} - \frac{a\zeta}{z}}{1 - a(1 - e)} \right)
\]

Putting \( \frac{e}{1 + e} = h \) and \( \frac{1 - e}{1 + e} = k \),

\[
\frac{1}{v_0} \frac{dw}{dz} = \sum_{n=0}^{\infty} A_n e^{i\theta} \sum_{m=0}^{n} \left( \frac{r_0}{z} \right)^m + (2he^{i\theta} - e^{i\theta}) \sum_{n=0}^{\infty} \left( \frac{r_0}{z} \right)^{n+1}
\]

\[
+ (h^2 e^{i\theta} - 2he^{i\theta} + 2) \sum_{n=0}^{\infty} \left( \frac{r_0}{z} \right)^{n+2} - h^3 e^{i\theta} \sum_{n=0}^{\infty} \left( \frac{r_0}{z} \right)^{n+3}
\]

Therefore

\[
A_0 = e^{i\theta}, \quad A_1 = 2ir_0 \sin \beta, \quad A_2 = a^2 e^{i\theta} - r_0 e^{i\theta},
\]

and

\[
A_n = a^2 r_0 a^{n-2} (e^{i\theta} - e^{-i\theta}) \quad \text{for} \quad n \leq 3
\]

Equation (20) giving the moment \( M_\alpha \) then may be written as

\[
M_\alpha = \frac{\Delta \Gamma}{2\pi} \left[ 1 - \frac{e(1 + e)}{2} + \frac{(1 + e)^2}{2} (I + J) \right]
\]

where

\[
I = R. P. \int \frac{1 - e^{i\theta}}{2} \sin \frac{2\beta}{2} \sum_{n=0}^{(n+1)} \frac{A_n e^{i\theta}}{r_0^{n+1}}
\]

\[
J = - R. P. \int \frac{e^{i\theta}}{2} \sin \frac{2\beta}{2} \sum_{n=0}^{(n+1)} \frac{A_n e^{i\theta}}{r_0^{n+1}}
\]

and

\[
M_\alpha = 2\pi r_0 a^3 [1 + (1 + e)] \sin 2\beta
\]

Figure 4.—Transformation of a symmetrical Joukowsky profile into a circle.

In appendix C, closed expressions are derived for \( I \) and \( J \). They may be written as follows:

\[
(1 + e)^2 (I + J) = \frac{(1 + k)^2}{8k^4} (1 - 2k \cos 2\beta + k^2)
\]

\[
\int \left[ \frac{(1 + k)^2}{k} \log \frac{1}{1 - k} - (1 + 2k + \frac{2}{6}k^2 + k^2 + \frac{5}{6}k^2 + \frac{2}{7}k^4) \right] \]

\[
\frac{1}{6}(3 + 2k)(1 - 2k \cos 2\beta + k^2) + \frac{1}{6}(9 + 5k)(1 - \cos 2\beta)
\]

\[
\frac{1}{2}k(1 - k) + \frac{1}{48}(15 + 5k + 4k^2)(1 - k)^3
\]

For small values of \( k \), or thick profiles,

\[
(1 + e)^2 (I + J) = \frac{k(1 + k)^2}{4} (1 - 2k \cos 2\beta + k^2)
\]

\[
\frac{9}{40}k^2 + \frac{3}{4}k^2 + \frac{11}{6}k^2 + \frac{5}{4}k^2 + \ldots
\]

\[
\frac{1}{6}(3 + 2k)(1 - 2k \cos 2\beta + k^2) + \frac{1}{6}(9 + 5k)(1 - \cos 2\beta)
\]

\[
\frac{1}{2}k(1 - k) + \frac{1}{48}(15 + 5k + 4k^2)(1 - k)^3
\]

It remains to evaluate the ratio of the additional circulation \( \Delta \Gamma \) and the circulation \( \Gamma \) of the incompressible fluid. This ratio may be obtained from equation
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(22) of reference 2 by evaluating $\Delta m/\rho_0$ for $\delta=\pi$ and is written as follows (see appendix D):

$$\frac{\Delta m}{\rho_0} = 2 \sin \left( \frac{\Delta m}{\rho_0} \right)$$

where $M$ and $N$ depend only on the thickness coefficient $\epsilon$ of the profile. Table IV lists values of $M$ and $N$ for several values of $\epsilon$.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$M$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00000</td>
<td>=</td>
</tr>
<tr>
<td>0.05</td>
<td>1.05462</td>
<td>2.76919</td>
</tr>
<tr>
<td>0.10</td>
<td>1.11285</td>
<td>2.96285</td>
</tr>
<tr>
<td>0.20</td>
<td>1.21809</td>
<td>3.77996</td>
</tr>
<tr>
<td>0.30</td>
<td>1.32317</td>
<td>4.06522</td>
</tr>
<tr>
<td>0.40</td>
<td>1.40028</td>
<td>4.65486</td>
</tr>
<tr>
<td>0.50</td>
<td>1.49404</td>
<td>1.36592</td>
</tr>
</tbody>
</table>

It is interesting to note that $M=1$ for a straight-line profile but that $N$ is infinite on account of a term containing $\log \epsilon$. For very thin profiles and vanishingly small angles of attack, however,

$$\frac{\Gamma_t + \Delta \Gamma}{\Gamma_t} = 1 + \frac{\mu}{2}$$

This result agrees with Glaucut's well-known formula (see section on the elliptic cylinder)

$$\frac{\Gamma_t + \Delta \Gamma}{\Gamma_t} = \frac{1}{\sqrt{1 - \mu}} = 1 + \frac{\mu}{2} + \ldots$$

It follows from equations (26) and (27) that, for an infinitely thin profile with an angle of attack so small that $\sin \beta \approx \beta$ and $\cos \beta \approx 1$,

$$M_t = 1 + \frac{\mu}{2}$$

With $L_c = \rho_0 \beta_0 (\Gamma_t + \Delta \Gamma)$ and $L_t = \rho_0 \beta \Gamma_t$, it therefore follows from equations (30) and (31) that

$$\frac{M_t}{L_c} = \frac{M_t}{L_c}$$

or

$$\frac{c_t}{c_t} = 1$$

(as in the case of the ellipse). In general, according to equations (26) and (29),

$$\frac{c_t}{c_t} = \frac{M_t}{L_c} \left[ 1 + \frac{\epsilon}{1 + \epsilon} \left( \frac{3}{2} + M_t + N \sin^2 \beta \right) + \frac{(1 + \epsilon)^2}{1 + \epsilon (1 + \epsilon)} \left( I + J \right) \right]$$

In order to gain some idea as to the order of magnitude of the movement of the center of pressure due to compressibility, consider the case of a thin profile, say $\epsilon=0.05$. Then equation (32) becomes

$$\frac{c_t}{c_t} = \frac{1 + \frac{\mu}{2} (7.00142 - 5.95991 \cos 2\beta)}{1 + \frac{\mu}{2} (2.92162 - 1.86666 \cos 2\beta)}$$

Letting $c_s$ and $c_t$ coincide, i.e., $c_s/c_t=1$, yields

$$\cos 2\beta = 0.99671$$

or a neutral angle

$$\beta = 2^\circ 20'$$

If the angle of attack is taken to be $0^\circ$, then

$$\frac{c_t}{c_t} = 1 + 0.52075 \mu$$

which shows that $c_t$ is nearer the origin than $c_t$. It is seen that the movement of the center of pressure is very small. Thus, even for a large value of $\omega_0/c_0$, say 0.70, the center of pressure moves only about 0.07 percent of the chord toward the origin.

Again, if the angle of attack is increased to $4^\circ$,

$$\frac{c_t}{c_t} = 1 + 0.54975 \mu$$

which shows that $c_s$ is nearer the nose of the profile than $c_t$. The center of pressure in this case, with $\omega_0/c_0=0.70$, moves about 0.13 percent of the chord toward the nose.

This numerical example indicates that, although the effect of compressibility on the lift and the moment of a thin airfoil may be large, namely

$$\frac{L_c - L_s}{L_t} = 0.25517, \quad \frac{M_c - M_s}{M_t} = 0.25847$$

for

$$\beta = 0^\circ, \quad \frac{\epsilon_0}{c_0} = 0.70$$

and

$$\frac{L_c - L_s}{L_t} = 0.26938, \quad \frac{M_c - M_s}{M_t} = 0.26291$$

for

$$\beta = 4^\circ, \quad \frac{\epsilon_0}{c_0} = 0.70$$

the effect on the center of pressure may be considered negligible.

As an example of a thicker airfoil, let $\epsilon=0.10$. The thickness of Joukowski airfoils is proportional to $\epsilon$ and a value of $\epsilon$ of 0.10 gives a maximum thickness of about 0.13 times the chord, a value rarely exceeded in practice.

For this case, equation (32) becomes:


equation (32) becomes:

$$\frac{c_t}{c_t}$$

again, as in the case of elliptic cylinders, a neutral value for $\beta$ is obtained when the centers of pressure in the compressible and the incompressible fluids coincide. It may be shown by means of numerical examples that, when the angle of attack is less than the neutral value of $\beta$, the center of pressure $c_s$ in the compressible fluid moves from its position $c_t$ in the incompressible fluid toward the origin and that, when the angle of attack is greater than the neutral value of $\beta$, the center of pressure $c_s$ moves toward the nose of the profile.
Putting \( \frac{c_d}{c_1} = 1 \) yields
\[
\cos 2\beta = 0.99161
\]
or a neutral angle
\[
\beta = 3^\circ 42'
\]
If the angle of attack is taken to be 0° and \( \nu_0/c_0 = 0.70 \), then
\[
\frac{c_s}{c_1} = \frac{1.28636}{1.27232} = 0.99631
\]
or the center of pressure is displaced toward the origin a distance equal to 0.12 percent of the chord.

If the angle of attack is increased to 6° with \( \nu_0/c_0 = 0.70 \), then
\[
\frac{c_s}{c_1} = \frac{1.28962}{1.28005} = 1.00748
\]
or the center of pressure is displaced toward the nose of the airfoil a distance equal to 0.19 percent of the chord.

**CONCLUDING REMARKS**

It has been shown that, for a thin airfoil and for small angles of attack, the moment is given by equation (31):
\[
M_c = M_t \left( 1 + \frac{\mu}{2} \right)
\]
This result is, however, limited to small values of \( \mu \). On the other hand, Glaucert (reference 3) has shown that the lift on an airfoil in a compressible fluid is given by
\[
L_s = L_t \left[ 1 + \frac{\mu}{2} + \cdots \right]
\]
The validity of this formula depends on the assumption that the velocity at the surface of the airfoil does not differ appreciably from the undisturbed velocity \( \nu_0 \), although the velocity in the undisturbed stream may be large. This assumption means that the airfoil must be thin and the angle of attack small. The similarity of the two preceding formulas, insofar as the t power of \( \nu \) is concerned, strongly suggests that the expression for the moment on a thin airfoil at small angles of attack and for large stream velocities is given by:
\[
M_c = \frac{M_t}{\sqrt{1 - \mu}}
\]
This result is indeed implied in Glaucert's work, where it is stated that the lift distribution along the chord remains unaltered but that the strength of each elementary vortex is increased by the factor \( (1 - \mu)^{-1} \).

In view of this discussion, it would appear that a more accurate expression for the moment \( M_c \) may be obtained by introducing the factor \( (1 - \mu)^{-1} \) into equation (20). Thus, equation (20) is replaced by
\[
M_c = \frac{1}{\sqrt{1 - \mu}} \left[ M_t + \rho_0 \Delta \Gamma m \cos (\beta + \sigma) \right.
\]
\[
+ \frac{\mu}{2} \left[ \frac{3}{2} \rho_0 \Delta \Gamma m \cos (\beta + \sigma) \right.
\]
\[
+ \frac{1}{2} \rho_0 \Delta \Gamma r \sum_{n=1}^{n-1} \frac{A_{n+1}}{r_n^{2n}} \left[ \frac{A_n}{r_n^{2n}} \right]
\]
\[
- 2 \pi \rho_0 c^2 \sum_{n=1}^{n-1} \frac{A_n}{r_n^{2n}} \right] \right]
\]
This formula differs from equation (20) in that it is valid for large stream velocities and differs from equation (33) in that it estimates the effect of large disturbances to the main flow \( \nu_0 \) because of the presence of an obstacle.

As an example, consider the case of symmetrical Joukowski airfoils with sharp trailing edges. Equation (26) is then replaced by
\[
M_c = \frac{M_t}{\sqrt{1 - \mu}} \left[ 1 + \frac{\mu}{2} \cos (\beta + \sigma) \right.
\]
\[
+ \frac{\mu}{2} \left[ \frac{3}{2} \rho_0 \Delta \Gamma m \cos (\beta + \sigma) \right.
\]
\[
+ \frac{1}{2} \rho_0 \Delta \Gamma r \sum_{n=1}^{n-1} \frac{A_{n+1}}{r_n^{2n}} \left[ \frac{A_n}{r_n^{2n}} \right]
\]
\[
- 2 \pi \rho_0 c^2 \sum_{n=1}^{n-1} \frac{A_n}{r_n^{2n}} \right] \right]
\]
In an analogous manner, the formula for the lift becomes
\[
L_s = L_t \left[ 1 + \frac{\mu}{2} \right] \left[ -1 + M + N \sin^2 \beta \right]
\]
As a numerical example, consider an airfoil for which \( \epsilon = 0.10 \). For this case
\[
M = 1.11153
\]
\[
N = 2.88585
\]
\[
\frac{\Delta \Gamma}{\Gamma_t} = \frac{\mu}{2} (1.11153 + 2.88585 \sin^2 \beta)
\]
and
\[
(1 + \epsilon)^2 (I + J) = 4.80317 - 4.66366 \cos 2\beta
\]
Table V gives the values of \( \frac{2 \Delta \Gamma}{\mu \Gamma_t} \) and of \( (1 + \epsilon)^2 (I + J) \) for several angles of attack.

<table>
<thead>
<tr>
<th>( \epsilon ) (deg)</th>
<th>( \frac{2 \Delta \Gamma}{\mu \Gamma_t} )</th>
<th>( (1 + \epsilon)^2 (I + J) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.11153</td>
<td>0.13881</td>
</tr>
<tr>
<td>2</td>
<td>1.16205</td>
<td>0.16293</td>
</tr>
<tr>
<td>4</td>
<td>1.20607</td>
<td>0.18458</td>
</tr>
<tr>
<td>6</td>
<td>1.24359</td>
<td>0.20111</td>
</tr>
<tr>
<td>8</td>
<td>1.27748</td>
<td>0.21808</td>
</tr>
<tr>
<td>10</td>
<td>1.30554</td>
<td>0.23078</td>
</tr>
</tbody>
</table>

Table VI lists the values of the ratios \( M_s/M_t \) and \( L_s/L_t \) as given by equations (35) and (36), respectively, for various values of \( \nu_0/c_0 \). The last column gives the values of \( M_s/M_t \) (or \( L_s/L_t \)) calculated from equation (38).
A THEORETICAL STUDY OF THE MOMENT ON A BODY IN A COMPRESSIBLE FLUID

TABLE VI

<table>
<thead>
<tr>
<th>( \frac{\alpha}{\alpha_0} )</th>
<th>( \frac{M}{M_1} )</th>
<th>( \frac{L}{L_1} )</th>
<th>( \frac{1}{\sqrt{1-\mu}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>.40</td>
<td>1.0671</td>
<td>1.0694</td>
<td>1.0700</td>
</tr>
<tr>
<td>.60</td>
<td>1.1270</td>
<td>1.1294</td>
<td>1.1300</td>
</tr>
<tr>
<td>.70</td>
<td>1.2690</td>
<td>1.2713</td>
<td>1.2718</td>
</tr>
<tr>
<td>.75</td>
<td>1.3480</td>
<td>1.3507</td>
<td>1.3512</td>
</tr>
<tr>
<td>.80</td>
<td>1.4382</td>
<td>1.4402</td>
<td>1.4407</td>
</tr>
<tr>
<td>.85</td>
<td>1.5495</td>
<td>1.5527</td>
<td>1.5532</td>
</tr>
<tr>
<td>1.00</td>
<td>1.7183</td>
<td>1.7205</td>
<td>1.7210</td>
</tr>
<tr>
<td>1.05</td>
<td>1.8613</td>
<td>1.8633</td>
<td>1.8638</td>
</tr>
</tbody>
</table>

Figures 5 and 6 show graphs based on table VI with \( \frac{M}{M_1} \) and \( \frac{L}{L_1} \) respectively, as ordinates and the Mach number \( \frac{v_b}{c_0} \) as the abscissa. The dashed curves represent the Glauert approximation:

\[
\frac{M}{M_1} = \frac{L}{L_1} = \frac{1}{\sqrt{1-\mu}}
\]

It is seen from an examination of the table and the curves that, below the neutral angle of attack (in this case \( 3^\circ 42' \)), the Glauert approximation is better for the moment than for the lift but that, above this angle, the approximation is better for the lift than for the moment. In any case, it appears that, at least for thin airfoils and for small angles of attack, the Glauert approximation for both the moment and the lift is sufficiently good. It then follows that the effect of compressibility on the center of pressure is negligible.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., March 24, 1939.
APPENDIX A

EFFECT OF COMPRESSIBILITY ON THE COUPLE ABOUT AN ELLIPTIC CYLINDER

The moment on a body due to the fluid motion is, according to equation (1), given by

\[ M = - \int [\mathbf{r} \cdot \mathbf{n}] p \, ds \]

where the positive direction of the normal vector \( \mathbf{n} \) is taken from the body into the fluid. The components of the vectors \( \mathbf{r} \) and \( \mathbf{n} \) are, respectively, \((x, y)\) and \((\cos m \xi, \cos n \eta)\).

Therefore

\[ [\mathbf{r} \cdot \mathbf{n}] = x \cos ny - y \cos nx \]

From figure 1, it is seen that

\[ dx = ds \cos ny \text{ and } dy = -ds \cos nx \]

Hence

\[ M = -\frac{1}{2} \int p (x \, dx + y \, dy) \]

Now, the equations of transformation from Cartesian to elliptic coordinates are:

\[ z = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta \]

or

\[ z = c \cosh (\xi + i\eta) \]

where \( \xi \) takes on all values from zero to infinity and \( \eta \) takes on all values from \( 0 \) to \( 2\pi \). Then \( \xi = \text{constant} \) and \( \eta = \text{constant} \) represent confocal ellipses and hyperbolas, respectively, the distance between the foci being \( 2c \).

For any given ellipse \( \xi = \alpha \), say,

\[ z = c \cosh \alpha \cos \eta, \quad y = c \sinh \alpha \sin \eta \]

or

\[ r^2 = c^2 (\cosh^2 \alpha \cos^2 \eta + \sinh^2 \alpha \sin^2 \eta) \]

Then

\[ ds^2 = -c^2 \sin 2\eta \, d\eta \]

Therefore

\[ M = -\frac{1}{2} c^2 \int_0^{2\pi} p \sin 2\eta \, d\eta \]

Now, according to equation (7), the pressure \( p \) in a compressible fluid is given by

\[ p = \text{constant } \frac{1}{2} \rho c^2 \frac{1}{\gamma} \frac{d}{\mu c^2} + \frac{1}{8} \frac{\mu c^2}{v_0} \]

so that

\[ M \approx \frac{1}{4} \rho c^2 \int_0^{2\pi} \sin 2\eta \, d\eta + \frac{1}{8} \frac{\mu c^2}{v_0} \int_0^{2\pi} v_0^2 \sin 2\eta \, d\eta \]

\[ -\frac{1}{10} \frac{\mu c^2}{v_0} \int_0^{2\pi} v_0^4 \sin 2\eta \, d\eta \]

The last two integrals in this expression are easily obtained since they involve a knowledge only of the velocity \( v_0 \) in the incompressible fluid. Thus, suppose the elliptic cylinder to be in a flow of velocity \( v_0 \) inclined at an angle \( \beta \) to the negative direction of the real axis and the circulation to be taken as zero. Then, if \( r_0 \) is the radius of the circle into which the ellipse \( \xi = \alpha \) is mapped by the Joukowski transformation,

\[ z = z' + \frac{c^2}{4d^2} \]

and the complex potential of the flow past the circle is

\[ w = \eta \left( z' e^{i\theta} + \frac{r_0^2}{z' e^{i\theta}} \right) \]

it follows that the complex velocity is given by

\[ \frac{1}{v_0} \frac{dw}{dz} = \frac{e^{i\theta} - \frac{r_0^2}{z' e^{i\theta}}}{1 - \frac{c^2}{4d^2}} \]

On the surface of the cylinder, \( z' = r e^{i\eta} \). Therefore,

\[ \frac{1}{v_0} \frac{dw}{dz} = \frac{e^{i\theta} - e^{-i(\theta + \beta)}}{1 - \frac{c^2}{4r_0^2} e^{-2i\beta}} \]

and

\[ \frac{v_0}{v_0} = \frac{1 - \cos 2(\theta + \beta)}{1 - 2a^2 \cos 2\theta + \sigma^2} \]

where

\[ \sigma = \frac{c}{2r_0} \]

Now, from the Joukowski transformation,

\[ z' = z + \frac{\sqrt{a^2 - \sigma^2}}{2} \]

where the positive sign of the radical has been chosen in order that the regions at infinity of the \( z \) and \( z' \) planes shall coincide.

Then if \((x, y)\) is replaced by the elliptic coordinates \((\xi, \eta)\), i. e., \( z = c \cosh (\xi + i\eta) \), it follows that:

\[ z' = \frac{c}{2} e^{i+\eta} \]

or, on the surface of the cylinder,

\[ r e^{i\eta} = \frac{c}{2} e^{i\theta} \]

Therefore

\[ \sigma = e^{-\pi} \text{ and } \theta = \eta \]

and

\[ \frac{v_0^2}{\eta^2} = 2 \frac{1 - \cos 2(\pi + \beta)}{1 - 2e^{-2\pi} \cos 2\eta + e^{-4\pi}} = \frac{e^{i\pi}(1 - \cos 2(\pi + \beta))}{\cosh 2\pi - \cos 2\pi} \]
With this expression for $v_i^2/n_o^2$, it is very easy to calculate the last two integrals in the expression for $M_c$. Thus

$$
\frac{1}{8\mu_o \rho_0 c^2} \int_{0}^{2\pi} v_i^2 \sin 2\eta d\eta = \frac{1}{8\mu_o \rho_0 c^2} \int_{0}^{2\pi} \frac{1-\cos 2(\eta+\beta)}{\cosh 2\alpha - \cos 2\eta} \sin 2\eta d\eta
$$

Now

$$
\int_{0}^{2\pi} \sin 2\eta d\eta = 0
$$

and

$$
\int_{0}^{2\pi} \frac{\cos 2\eta d\eta}{\cosh 2\alpha - \cos 2\eta} = 2\pi \sinh 2\alpha \cosh 2\alpha
$$

Therefore

$$
\frac{1}{8\mu_o \rho_0 c^2} \int_{0}^{2\pi} v_i^2 \sin 2\eta d\eta = \frac{1}{4\mu_o \rho_0 c^2} \int_{0}^{2\pi} \frac{1-\cos 2(\eta+\beta)}{\cosh 2\alpha - \cos 2\eta} \sin 2\eta d\eta
$$

Similarly,

$$
\int_{0}^{2\pi} \sin 2\eta d\eta = 0
$$

and

$$
\int_{0}^{2\pi} \frac{\cos 2\eta d\eta}{\cosh 2\alpha - \cos 2\eta} = 2\pi \sinh 2\alpha \cosh 2\alpha
$$

Hence,

$$
\frac{1}{16\nu_o^2} \int_{0}^{2\pi} v_i^2 \sin 2\eta d\eta = \frac{1}{16\mu_o \rho_0 c^2} \int_{0}^{2\pi} \frac{1-\cos 2(\eta+\beta)}{\cosh 2\alpha - \cos 2\eta} \sin 2\eta d\eta
$$

$$
= \frac{1}{4\mu_o \rho_0 c^2} \int_{0}^{2\pi} \frac{\cosh 2\alpha - \cos 2\eta}{\sinh 2\alpha \cosh 2\alpha} \sin 2\eta d\eta
$$

The sum of the two integrals becomes

$$
\frac{1}{4\mu_o \rho_0 c^2} \int_{0}^{2\pi} \frac{\cosh 2\alpha - \cos 2\eta}{\sinh 2\alpha \cosh 2\alpha} \sin 2\eta d\eta
$$

It is much more difficult to calculate the first integral in the expression for $M_c$ for it is necessary in this case to know the velocity $v_i$ at the surface of the elliptic cylinder in a compressible fluid. For this purpose, it is convenient to make use of equation (13) of reference 2. This equation, when applied to the case of an elliptic profile, becomes

$$
\frac{\Delta \phi}{\nu_o} = \frac{\mu}{2} \left[ \frac{\cos 2\beta - \cos^2 \beta}{\sin^2 (\eta + \beta)} \right]
$$

$$
+ \sum_{n=0}^{\infty} \left[ (2n+1) \sin (2n+1) \eta + \beta \right] \int_{0}^{1} \lambda^{2n+1} a_{2n+3} d\lambda
$$

$$
- \sum_{n=0}^{\infty} \left[ (2n+1) \sin (2n+1) \eta - \beta \right] \int_{0}^{1} \lambda^{2n+1} a_{2n+3} d\lambda
$$

+ 4 \sin \beta \sum_{n=0}^{\infty} \cos 2n \int_{0}^{1} \lambda^{2n+1} a_{2n+3} d\lambda
$$

$$
- \sin 2\beta \sum_{n=0}^{\infty} \cos^{2n-2} \cos [(2n+1) \eta + \beta]
$$

$$
+ \sin 2\beta \sum_{n=0}^{\infty} \cos^{2n-2} \cos [(2n+1) \eta - \beta]
$$

$$
+ 2 \sin 2\beta \sum_{n=0}^{\infty} \cos^{2n-2} \sin 2n \eta
$$

where $\lambda = \eta/r_o$, $\sigma = c/2r_0$, and the $a_{2n}'s$ are the coefficients of the cosine terms in the Fourier development of $v_i^2/n_o^2$ in the region of flow. For the elliptic cylinder

$$
\frac{v_i^2}{v_o^2} = \frac{1-2\lambda^2 \cos 2(\beta + \beta) + \lambda^4}{1-2\sigma^2 \lambda^2 \cos 2\beta + \sigma^4 \lambda^4}
$$

so that

$$
a_{2n} = \frac{2^{n+1}(1+\lambda)^{2n+1}}{\sigma^2(1-\sigma^4 \lambda^4)^2} \cos 2\beta (\lambda \lambda^4)
$$

$$
b_{2n} = \frac{2^{n}(1+\lambda)^{2n} \cos 2\beta (\lambda \lambda^4)}{\sigma^2(1-\sigma^4 \lambda^4)^2}
$$

Now, it is recalled that $\Delta \phi/\nu_o$ refers to the plane of the circle into which the elliptic profile is mapped by the Joukowski transformation. Therefore

$$
\frac{v_i^2}{v_o^2} = \frac{1}{1-2\sigma^2 \cos 2\eta + \sigma^4 \sin 2\eta}
$$

or

$$
\frac{v_i^2}{v_o^2} = \frac{\sin 2\eta}{1-2\sigma^2 \cos 2\eta + \sigma^4}
$$

But

$$
\frac{\sin 2\eta}{1-2\sigma^2 \cos 2\eta + \sigma^4} = \sum_{n=0}^{\infty} \sigma^{2n-2} \sin 2n \eta
$$

Therefore, the first integral in the expression for $M_c$ becomes

$$
\frac{1}{4\mu_o \rho_0 c^2} \int_{0}^{2\pi} \frac{\sin 2\eta d\eta}{1-2\sigma^2 \cos 2\eta + \sigma^4}
$$

$$
+ \frac{1}{2\sigma^2} \sum_{n=1}^{\infty} \left[ (2n-1) \sigma^{2n-2} \int_{0}^{1} \lambda^{2n-1} a_{2n+1} d\lambda
$$

The last two integrals in this expression may be written as

$$
2 \sum_{n=0}^{\infty} \sigma^{2n-2} \int_{0}^{1} \lambda^{2n} (a_{2n-3} - a_{2n+3}) d\lambda
$$

$$
- \sum_{n=0}^{\infty} \sigma^{2n-2} \int_{0}^{1} \lambda^{2n-1} (a_{2n-3} + a_{2n+3}) d\lambda
$$

$$
= 2 \sum_{n=0}^{\infty} \sigma^{2n-2} \int_{0}^{1} \left[ \frac{a(1+\lambda)^{2n-2}}{1-\sigma^4 \lambda^4} \cos 2\beta (\lambda \lambda^4) d\lambda
$$

$$
- \sum_{n=0}^{\infty} \sigma^{2n-2} \int_{0}^{1} \lambda^{2n} (a(1+\lambda)^{2n-2} d\lambda
$$

$$
= 2 \int_{0}^{1} \frac{a(1+\lambda)^{2n-2}}{1-\sigma^4 \lambda^4} \cos 2\beta (\lambda \lambda^4) d\lambda
$$

$$
- \int_{0}^{1} \lambda^{2n} (a(1+\lambda)^{2n-2} d\lambda
$$

$$
= \int_{0}^{1} \lambda^{2n} (a(1+\lambda)^{2n-2} d\lambda
$$

$$
= 1-2\sigma^2 \cos 2\beta + \frac{1-\sigma^2 \cos 2\beta}{2\sigma^2} \log \frac{1+\sigma^2}{1-\sigma^2}
$$
Therefore
\[
\frac{1}{2} \rho_\infty c^2 \int_0^{2\pi} \int_0^\pi \sin 2\gamma d\gamma = \frac{\pi \rho_\infty c^2}{2} \sin 2\beta \left( 1 + \mu \frac{\cos 2\beta - \sigma^2}{\sigma^4} \right)
\]
\[
+ 2 \left( 1 - \sigma^2 \cos 2\beta - \frac{1 - \sigma^2 \cos 2\beta}{\sigma^4} \log \frac{1 + \sigma^2}{1 - \sigma^2} \right)
\]
Finally
\[
M_0 = M \left( \frac{1}{2} \frac{\cos 2\beta - \sigma^2}{\sigma^4} + 2 \frac{1 - \sigma^2 \cos 2\beta}{\sigma^4} \log \frac{1 + \sigma^2}{1 - \sigma^2} \mu \cos (2\sigma - \cos 2\beta) \right)
\]
\[
+ \frac{1}{2} \frac{1 - 2\sigma^2 \cos 2\beta + \sigma^4}{\sigma^2} \log \frac{1 + \sigma^2}{1 - \sigma^2} \frac{1}{2} \frac{\sinh 2\beta}{\sinh 2\alpha}
\]

where
\[
M_0 = \frac{\pi \rho_\infty c^2}{2} \sin 2\beta
\]

Replacing \(\epsilon^{2n}\) by \(\sigma^2\):
\[
M_0 = M_0 \left( 1 + \frac{\mu}{2} \right)
\]
\[
+ \frac{1}{2} \left( 1 - 2\sigma^2 \cos 2\beta + \sigma^4 \right) \left( \frac{1}{2\sigma^2} \log \frac{1 + \sigma^2}{1 - \sigma^2} \right)
\]

This expression agrees with equation (21) when the circulation is taken to be zero.

**APPENDIX B**

**THE CALCULATION OF \(\Delta V/\nu_0\) FOR AN ELLIPTIC CYLINDER AT \(\delta = \pi\)**

Equation (13) of reference 2 gives an expression for \(\Delta V/\nu_0\) independent of the shape of the obstacle. In order to evaluate \(\Delta V/\nu_0\), it is necessary to know the Fourier development of \(\sigma^2\). For an elliptic cylinder, with the circulation chosen so that the rear stagnation point occurs at the end of the major axis, it may be easily shown by means of the Joukowski transformation and the complex velocity potential of the flow about a circular cylinder that:

\[
\sigma^2 = \frac{(1 + 2\lambda \cos \theta + \lambda^2)(1 - 2\lambda \cos (\theta + 2\beta) + \lambda^2)}{1 - 2\sigma^4 \lambda^2 \cos 2\theta + \sigma^4 \lambda^2}
\]

where
\[
\lambda = r_o/r, \quad \epsilon = c/2r_o, \quad \beta \quad \text{the angle of attack, and} \quad r, \theta \quad \text{are the polar coordinates of a point in the region of flow.}
\]

The Fourier series for \(\sigma^2/\nu_0^2\) is obtained by making use of the expansion

\[
\frac{1}{1 - 2\sigma^4 \lambda^2 \cos 2\theta + \sigma^4 \lambda^2} = \frac{1}{1 - \sigma^4 \lambda^2} \left[ 1 + 2 \sum_{n=1}^\infty (\sigma^2 \lambda)^n \cos 2n\theta \right]
\]

and it may be shown without difficulty that

\[
a_n = \frac{2(1 + \lambda^2)^2}{1 - \sigma^4 \lambda^2} \frac{4\lambda^2 \cos 2\beta}{1 - \sigma^4 \lambda^2}
\]
\[
a_{2n} = \left( \frac{(1 + \lambda^2)^2}{1 - \sigma^4 \lambda^2} - \frac{1 + \sigma^4 \lambda^2}{\sigma^2(1 - \sigma^4 \lambda^2)} \cos 2\beta \right) (\sigma^2 \lambda)^{2n}
\]
\[
a_{2n+1} = \frac{4\lambda(1 + \lambda^2) \sin^2 \theta}{1 - \sigma^4 \lambda^2} (\sigma^2 \lambda)^{2n}
\]
\[
b_{2n} = 2\lambda^2 (\sigma^2 \lambda)^{2n} \sin 2\beta
\]
\[
b_{2n+1} = 2\lambda(1 + \lambda^2) \sin 2\beta (\sigma^2 \lambda)^{2n}
\]

If the expressions for \(a_n\) and \(b_n\) are inserted and the integrations performed, it follows after considerable but straightforward labor that

\[
\left( \frac{\Delta V}{\nu_0} \right)_{\delta = \pi} = \frac{\mu}{2} \sin \beta + \frac{1}{\sigma^4} \sin 3\beta + \frac{1}{\sigma^4} \sin \beta + \frac{1}{\sigma^4} \sin 3\beta + \frac{1}{\sigma^4} [1 + \sigma^2 \log (1 + \sigma^2) - (1 + \sigma^2)^2 \log (1 + \sigma^2)] \sin \beta
\]

\[
+ \frac{8}{\sigma^4} \sin \beta \frac{12}{\sigma^4} \sin \beta \log (1 + \sigma^2)
\]
The additional circulation $\Delta K$ is given by

$$\Delta K = -\left(\frac{\Delta \psi}{\sigma_0}\right)_{x=x_0}$$

(See section on the elliptic profile.)

Therefore, with $K_i = 2 \sin \beta$, it follows that

$$\frac{\Delta \Gamma}{\Gamma_i} \frac{\Delta K}{K_i} = \frac{\mu}{2} (M + N \sin^3 \beta)$$

where

$$M = \frac{2 - \sigma^2}{2 \sigma^2} + \frac{2}{\sigma^2} \log (1 + \sigma^2) + \frac{1 + 6 \sigma^2 + \sigma^4}{4 \sigma^4} \log \frac{1 + \sigma^2}{1 - \sigma^2} - \frac{1 + \sigma^2}{\sigma^2} \log \frac{1 + \sigma}{1 - \sigma}$$

$$N = \frac{1 + \sigma^2}{\sigma^2} \log \frac{1 + \sigma}{1 - \sigma} \frac{2}{\sigma^2} (1 + \log (1 + \sigma^2))$$

APPENDIX C

EVALUATION OF $I$

$$I = \frac{2 + n(1+1)}{2 \sigma_0^2 + \sigma_0^2} \sum_{n=1}^{\infty} A_n A_{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{2 + n(1+1)}{2 \sigma_0^2 + \sigma_0^2} \frac{A_n A_{n+1}}{r_n^a}$$

$$= \frac{a^2}{r_0^a} \frac{1 + k^2}{k^2} (1 - 2k \cos 2\beta + k^2) \log (1 - k^2)$$

$$- \frac{a^2}{r_0^a k^2} (1 - 2k \cos 2\beta + k^2) \left(1 + \frac{3 k^2}{2} + \frac{5 k^4}{6} \right)$$

$$\approx 3 \sin \beta \left(\frac{a^2}{r_0^a} \frac{1 + k^2}{k^2} \log (1 - k^2) \right)$$

$$+ \frac{5 a^2}{r_0^a k^2} \frac{1 + k^2}{k^2} (1 - k^2)$$

Therefore

$$I = R. \Phi \cdot \frac{1 - e^{i \phi}}{2 \sin \beta} \sum_{n=1}^{\infty} \frac{2 + n(1+1)}{2 \sigma_0^2 + \sigma_0^2} \frac{A_n A_{n+1}}{r_n^a}$$

$$= \frac{a^2}{r_0^a} \frac{1 + k^2}{k^2} (1 - 2k \cos 2\beta + k^2) \log (1 - k^2)$$

$$- \frac{a^2}{2 r_0^a k^2} (1 - 2k \cos 2\beta + k^2) \left(1 + \frac{3 k^2}{2} + \frac{5 k^4}{6} \right)$$

$$+ \frac{a^2}{6 r_0^a} (9 + 5k)(1 - \cos 2\beta) + \frac{5 a^2}{12 r_0^a} (1 - k^2)$$

Now

$$a = \frac{1}{1 + e} \quad \text{and} \quad k = \frac{1 - e}{1 + e}$$

It is interesting to note that, for the limiting case of a circular cylinder $\sigma = 0$, the foregoing equation yields

$$\left(\frac{\Delta \Gamma}{\Gamma_i}\right)_{\text{circle}} = \frac{\mu}{2} \left(\frac{11}{6} + \frac{2}{3} \sin^2 \beta\right)$$

and compares with Pogg's result (reference 5)

$$\left(\frac{\Delta \Gamma}{\Gamma_i}\right)_{\text{circle}} = \frac{11}{12} \mu$$

For the straight-line profile $\sigma = 1$, it is seen that $M = 1$ and $N = \infty$. For an infinitely thin profile and a vanishingly small angle of attack, however,

$$\left(\frac{\Delta \Gamma}{\Gamma_i}\right)_{\text{limit}} = \frac{1}{2} \mu$$

and compares with Glaubert's formula (reference 3):

$$\left(\frac{\Delta \Gamma}{\Gamma_i}\right) = -1 + \frac{1}{\sqrt{1 - \mu}} = \frac{1}{2} \mu + \ldots$$

For small values of $k$, or thick airfoils,

$$(1 + e)^2 I = \left(1 + k^2\right)^2 \left(1 - 2k \cos 2\beta + k^2\right) \left[1 + \frac{1}{2} k^2 \log \frac{1}{1 - k^2}\right]$$

$$(1 + e)^2 I = \left(1 + k^2\right)^2 \left(1 - 2k \cos 2\beta + k^2\right) \left[\frac{1}{2} k^2 \log \frac{1}{1 - k^2}\right]$$

$$+ \frac{5}{48} (3 + k)(1 - k)^2$$

EVALUATION OF $J$

$$J = \sum_{n=1}^{\infty} \frac{A_n A_{n+1}}{r_n^a}$$

$$= \frac{a^2}{r_0^a} \frac{1 + k^2}{k^2} (1 - 2k \cos 2\beta + k^2) \log (1 - k^2)$$

$$- \frac{a^2}{2 r_0^a k^2} (1 - 2k \cos 2\beta + k^2) \left(1 + \frac{3 k^2}{2} + \frac{5 k^4}{6} \right)$$

$$+ \frac{a^2}{6 r_0^a} (9 + 5k)(1 - \cos 2\beta) + \frac{5 a^2}{12 r_0^a} (1 - k^2)$$

Therefore

$$a = \frac{1}{1 + e} \quad \text{and} \quad k = \frac{1 - e}{1 + e}$$

$$(1 + e)^2 I = \left(1 + k^2\right)^2 \left(1 - 2k \cos 2\beta + k^2\right) \left[1 + \frac{1}{2} k^2 \log \frac{1}{1 - k^2}\right]$$

$$(1 + e)^2 I = \left(1 + k^2\right)^2 \left(1 - 2k \cos 2\beta + k^2\right) \left[\frac{1}{2} k^2 \log \frac{1}{1 - k^2}\right]$$

$$+ \frac{5}{48} (3 + k)(1 - k)^2$$

EVALUATION OF $J$

$$J = \sum_{n=1}^{\infty} \frac{A_n A_{n+1}}{r_n^a}$$

$$= \frac{a^2}{r_0^a} \frac{1 + k^2}{k^2} (1 - 2k \cos 2\beta + k^2) \log (1 - k^2)$$

$$- \frac{a^2}{2 r_0^a k^2} (1 - 2k \cos 2\beta + k^2) \left(1 + \frac{3 k^2}{2} + \frac{5 k^4}{6} \right)$$

$$+ \frac{a^2}{6 r_0^a} (9 + 5k)(1 - \cos 2\beta) + \frac{5 a^2}{12 r_0^a} (1 - k^2)$$

Now

$$a = \frac{1}{1 + e} \quad \text{and} \quad k = \frac{1 - e}{1 + e}$$
Therefore
\[
J = -R \cdot P \cdot \frac{ie^{2\pi s}}{\sin 2\beta \pi i(n+1)\theta_{t=0}^{\infty}
\]
\[
= \frac{a^4(1-2k \cos 2\beta + k^2)}{r_0 k^4} \log (1-k^2)
\]
\[
- \frac{a^4(1-2k \cos 2\beta + k^2)}{r_0 k^4} \left(1 + \frac{k^2}{2} + \frac{k^4}{3}\right)
\]
\[
+ \frac{a^2 k}{2r_0^2} (1-2k) (1-2k \cos 2\beta + k^2)
\]
\[
+ \frac{a^2 k}{2r_0^2} (1-k) + \frac{a^2 k}{r_0^2} (a^2 - k)
\]

and
\[
(1+e)^s J = \frac{(1+k)^2}{4k^4} \left(1-2k \cos 2\beta + k^2\right) \left[ \frac{1}{k^3} \log \frac{1}{1-k^2}\right]
\]
\[
- \left(1 + \frac{k^2}{2} + \frac{k^4}{5}\right)\left[ \frac{1}{6} (3+2k) (1-2k \cos 2\beta + k^2)\right]
\]
\[
+ \frac{1}{12} k(1-k) + \frac{1}{12} k^4 (1-k)^2
\]

For small values of \(k\) or thick airfoils,
\[
(1+e)^s J = \frac{(1+k)^2}{4k^4} \left(1-2k \cos 2\beta + k^2\right) \left[ \frac{1}{16} k^4\right]
\]
\[
+ \frac{1}{20} k^4 + \frac{1}{24} k^4 + \ldots\right] + \frac{1}{6} (3+2k) (1-2k \cos 2\beta + k^2)
\]
\[
+ \frac{1}{12} k(1-k) + \frac{1}{12} k^4 (1-k)^2
\]

APPENDIX D

EVALUATION OF \((\Delta v/\nu)_\Delta\) FOR THE CASE OF A JOUKOWSKI PROFILE

According to equation (22) of reference 2, \((\Delta v/\nu)_\Delta\) takes the following form:
\[
(\Delta v/\nu)_\Delta = \frac{\mu}{2} \sin \beta \left[ -\frac{1}{2} + h \left( h + 2k + 2k h^2 - \frac{5}{6} h^3 \right)\right]
\]
\[
+ \frac{h}{k^3} \left[ -3 - 3k h - \frac{3}{2} - k h^2 + \frac{5}{6} k^2 \right] \log (1-k^2)
\]
\[
+ \frac{4 h k^2 (1-k)^2}{k^2} \left[ k h^2 - \frac{1}{2} - h \right] \log (1+k)
\]
\[
+ \frac{4 h (1-k)^3}{k^3} \left[ \log (1-k) + 4 h^3 \log (1+k) - k^2 \right]
\]
\[
+ \frac{\mu}{2} \sin \beta \left[ \frac{4 (h+k)^2}{k^2} \left( h + 1 + \frac{1}{3} k h^2 \right)\right]
\]
\[
+ \frac{4 h k^2}{k^3} \left[ -1 + k - \frac{k^3}{2} h - \frac{2}{k^2} \log (1-k^2)\right]
\]
\[
+ \frac{8 (1-k)^3}{k^4} \left[ k h^2 - \frac{1}{2} - (1-h)^2 + 2 h (1-h) \log (1+k)\right]
\]
\[
+ \frac{4 (1-h)^4}{k^4} \left[ \log (1-k) + h^3 \log (1+k) - k^2 \right]
\]

where
\[
h = \frac{e}{1+e} \text{ and } k = \frac{1-e}{1+e}
\]

When this expression for \((\Delta v/\nu)_\Delta\) was obtained, a slight error was found in equations (19) of reference 2. The expressions for \(a_1 - a_1\) and \(b_1 - b_1\) should be as follows:
\[
a_1 - a_1 = -\frac{2 \lambda \cos 2\beta}{k^2} \left[ \frac{h^2 + 2 k h^2}{k^2} \right]
\]
\[
- \frac{2 \lambda k^2 (1-2k \cos 2\beta)}{1-k^2}
\]

and
\[
b_1 - b_1 = -\frac{2 \lambda \sin 2\beta}{k^2} \left[ \frac{h^2 + 2 k h^2}{k^2} \right]
\]
\[
- \frac{2 \lambda k^2 (1-2k \cos 2\beta)}{1-k^2}
\]

It is to be noted in the expression for \((\Delta v/\nu)_\Delta\) that most of the terms contain powers of \(k\) in the denominator. It appears at first, then, that the coefficients of \(\sin \beta\) and \(\sin \beta\) may become infinite for \(k=0\). This apparent difficulty disappears, however, when \((\Delta v/\nu)_\Delta\) is expressed as a power series in \(k\). It is then found that the terms involving reciprocal powers of \(k\) cancel and the following expression results:
\[
(\Delta v/\nu)_\Delta
\]
\[
= \frac{\mu}{2} \sin \beta \left[ -\frac{689}{102} - \frac{89}{160} k^2 - \frac{280}{72} k^4 - \frac{7}{32} k^6 - \frac{23}{640} k^8 + \ldots\right]
\]
\[
+ \frac{\mu}{2} \sin \beta \left[ -\frac{41}{24} - \frac{689}{240} k^2 - \frac{1373}{560} k^4 - \frac{183}{1080} k^6 + \ldots\right]
\]

REFERENCES