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POTENTIAL FLOW ABOUT ARBITRARY BIPLANE WING SECTIONS

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SUMMARY

A rigorous treatment is given of the problem of determining the two-dimensional potential flow around arbitrary biplane cells. The analysis involves the use of elliptic functions and is sufficiently general to include the effects of such elements as the section shapes, the chord ratio, gap, stagger, and decalage, which elements may be specified arbitrarily. The flow problem is resolved by making use of the methods of conformal representation. Thus the solution of the problem of transforming conformally two arbitrary contours into two circles is expressed by a pair of simultaneous integral equations, for which a method of numerical solution is outlined. It is pointed out that an inverse method of transforming conformally two circles into the wing profiles of a biplane arrangement leads readily to the development of related families of biplane combinations. Flow formulas are developed giving the velocity and pressure at any point of the surface of either profile of the arbitrary biplane arrangement, for any angle of attack. The theory of the monoplane wing section in potential flow is shown to be a degenerate case in which the elliptic functions reduce to trigonometric functions. The general method presented may be employed to determine the potential flow in any doubly connected region and hence may be applied to the single slotted wing or to the auxiliary-airfoil wing.

As an example of the numerical process, the pressure distribution over certain arrangements of the N. A. C. A. 4418 airfoil in biplane combinations is presented and compared with the monoplane pressure distribution.

INTRODUCTION

It is the purpose of this paper to develop a general theory of arbitrary biplane cells of infinite span in potential flow. No attempt is made here to treat the case of finite span or to consider viscosity; rather it is the object of this work to bring the two-dimensional theory of biplane cells in uniform, steady potential flow to the same degree of exactness and generality to which the two-dimensional monoplane airfoil theory has been brought. The analysis will be sufficiently general to include such elements as profile shapes, chord ratio, gap/chord, stagger, and decalage, and will contain as special cases the monoplane theory, as well as the theories of the slotted monoplane wing, of the auxiliary-airfoil wing, and of the influence of the ground or plane barriers on a monoplane airfoil in two-dimensional potential flow.

In order to arrive in a natural manner at a perspective of the biplane analysis it is advantageous to consider first the simpler case of the monoplane wing section and to keep in view the essential concepts that carry over to the biplane analysis. It is well known that by virtue of the methods of conformal representation the two-dimensional potential flow around a single obstacle can be obtained by the following process. In the first place, a standard contour is selected, the region about which is simply connected and the flow function of which in uniform potential flow is known or obtainable. The transformation must then be found that transforms conformally the region of the given obstacle into this standard region. This transformation, in combination with the known flow function, gives the desired flow function for the obstacle. In the case of monoplane wing profiles, the standard flow region may be chosen to be that about a circle and the theorem which states that it is possible to transform conformally the contour of the given obstacle into a circle is known as Riemann's theorem. (Cf., for example, reference 1.) In the case of two obstacles, the region is termed “doubly connected” and the process is again applicable except that the standard doubly connected region is chosen to be the region about two circles. The theorem that states the existence of a transformation function bringing the doubly connected region (region of the biplane contours) into the region of two circles is Koebe's theorem (reference 2).

The flow function giving the uniform potential flow for a circular cylinder is well known and, in determining the flow about a monoplane airfoil section, the main problem is the transforming of the airfoil contour into a circle. In order to attain this result in a simple manner it is necessary to perform a few intermediate transformations. The airfoil profile itself may be regarded as a contour described about a conveniently chosen line segment or chord. An
initial transformation of a simple type exists (the
so-called "Joukowsky transformation") that trans-
forms the chord into a unit circle and automatically
maps the airfoil contour into a nearly circular contour
described about the unit circle. There remains then
the final task of transforming the nearly circular
contour into a true circle, and this may be performed
by a method given by Theodorsen (reference 3). This
method leads directly to a simple integral equation
which can be solved by a process of iteration or suc-
cessive approximations and which converges with
extreme rapidity. (Cf. reference 1.) It is important
in regard to practical considerations to observe that
the method is so powerful that one step in the process
is quite sufficient in all ordinary cases.

The standard doubly connected region has been
chosen as the region about two circles; and it is worthy
of mention that only as recently as 1920 was the
complex flow potential for two circles rigorously
developed (by Lagally, reference 4). Dupont, Bonder,
and Müller (references 5, 6, and 7) have also con-
tributed to this problem but Lagally’s solution is
the more elegant. The flow function for two circles
being known, the main problem in finding the flow
about a biplane arrangement is the obtaining of the
transformation mapping the two contours into two
circles.

In a manner analogous to the case of the single air-
foil section, the contours of a biplane arrangement may
be considered to be described about a skeleton of two
conveniently placed mean lines or chords. Hence, to
maintain the analogy it is seen that initially it is desired
to find a transformation function which transforms
the two line segments into two circles. This prob-
lem has been touched upon by Kutta who has given the
uniform potential flow for the special case of two paral-
lel equal line segments (reference 8). The transfor-
mation function bringing two circles into any two parallel
line segments has been developed by C. Ferrari (ref-
ERENCE 9). In the first part of the present paper the
more general problem of decalage of the line segments
has been studied and a function developed that trans-
forms two circles into any two nonintersecting line seg-
ments in any relative positions. This function that
transforms the skeleton or chords of the biplane ar-
range ment into two circles also transforms the contours
themselves into two nearly circular contours described
about the skeleton circles. There remains then the
problem of transforming the two nearly circular con-
tours into two true circles. In order to accomplish
this task, the method of Theodorsen is generalized in
the present paper to apply to doubly connected regions
by employing the concentric circular ring region as a
standard region and by utilizing a Laurent series devel-
opment instead of a one-way power series. There is
obtained finally a pair of simultaneous integral equa-
tions expressing the conformal representation of the
two nearly circular contours into two circles. Just
as in the case of the single integral equation in the
monoplane case, there exists an analogous process of
successive approximations or iteration that converges
with the same remarkable degree of rapidity.

The general transformation from the biplane con-
tours to two circles together with the Lagally formula
for the flow about two circles yields an expression for
the velocity and pressure at each point of the surface of
either profile of the biplane arrangement. There are
two arbitrary circulations in the flow formula, viz, the
separate circulations around each contour, and these
are determined uniquely by applying the well-known
Kutta–Joukowsky condition to the trailing edges of both
contours, specifying thereby that the flow leaves these
edges smoothly.

In the case of monoplane wing theory it has been
shown (reference 1) that theoretical shapes can be
coveniently developed by an inverse method of
transforming conformally a circle into a wing profile.
The Joukowsky airfoils and the other so-called "theo-
retical" airfoils are special examples of this process.
In an analogous manner it is possible to develop theo-
retical biplane combinations by an inverse process of
transforming two circles into two contours resembling
wing profiles. A general and flexible method of obtain-
ing these shapes is presented; the results are especially
instructive in that, in this process, the integral equa-
tions referred to in a preceding paragraph reduce to
definite integrals.

Elliptic functions arise in a natural manner in the
analysis and the problem treated provides a good
illustration of the power and beauty of these remark-
able functions. The general theory of the single,
arbitrary wing section is shown to be a degenerate
case in which the imaginary period of the doubly
periodic elliptic functions becomes infinite, and hence
the elliptic functions reduce to ordinary trigonometric
functions. A few pages are devoted to the monoplane
theory in view of the light that it throws on the more
general biplane analysis.

Numerical results are presented only to furnish an
illustration of the theory. In particular, the pressure
distribution is determined for certain arrangements of
the N. A. C. A. 4412 airfoil in biplane combinations.
The elliptic functions that arise in the analysis and that
are to be evaluated in a numerical case may
fortunately, when necessary, be developed in rapidly
convergent expansions.

Statement of the problem.—The problem treated in
this paper may be restated as follows. Given, an
arbitrary biplane arrangement oriented in a specified
manner in a nonviscous, incompressible fluid medium
and translated with uniform velocity V. To determine
the velocity and pressure distribution in two-dimen-
sional potential flow in the field of motion for all angles of attack, particularly, at each point of the surface of the biplane profiles.

As has been pointed out, it is well recognized that the aforementioned problem may be treated in two stages. In the first place, the complex function expressing the conformal transformation of the region of the biplane into a standard doubly connected region must be obtained and, finally, the complex flow function for this standard region, which is chosen as the region about two circles, must be known. The region external to the two contours of a biplane arrangement will be brought into the region about two circles by the intermediate use of two nearly circular contours. Before this result can be accomplished, however, it is desirable to discuss several preliminary transformations.

**1. PRELIMINARY TRANSFORMATIONS**

First, the transformation bringing the region external to two nonintersecting circles (t plane) into the annular region between two concentric circles (w plane) will be obtained. This annular region will then be mapped into a rectangular region (s plane) and the rectangular region into the region about two line segments (u plane). (See fig. 1.)

![Diagram](image)

**Figure 1.**—Mapping of: (a) two coaxial circles in the t plane into (b) two concentric circles in the w plane, (c) rectangular region in the s plane, (d) two line segments in the u plane.

Transformation of a coaxial system of circles into a concentric system.—A coaxial system of circles may be described most simply by the use of bipolar coordinates. Consider a complex t plane where \( t = t_1 + it_2 \). Let \( Q_1 (0, ic) \) and \( Q_2 (0, ic) \) located on the t axis be the origins of two polar coordinate systems \( r_1, \gamma_1 \) and \( r_2, \gamma_2 \). The variable \( t \) may be written in the two forms:

\[
 t = ic + r_1 e^{it_1} = -ic + r_2 e^{it_2}
\]

Then in the relation

\[
 \frac{t + ic}{t - ic} = \frac{r_2}{r_1} e^{i(\gamma_2 - \gamma_1)}
\]

there are expressed in a convenient form, the bipolar coordinates \( \frac{r_2}{r_1} \) and \( \gamma_2 - \gamma_1 \). (See fig. 1 (a).)

The curves \( r_2/r_1 = \text{constant} \) are circles with centers lying along the \( t_2 \) axis (theorem of Apollonius). This family of circles contains the points \( Q_1 \) and \( Q_2 \) as limiting circles of zero radius. For points in the upper half plane \( (t_2 > 0) \) we have \( r_2/r_1 > 1 \), for points in the lower half plane \( r_2/r_1 < 1 \), while on the \( t_1 \) axis, \( r_2/r_1 = 1 \). The curves given by \( \gamma_2 - \gamma_1 = \text{constant} \) also form a family of circles (theorem of the constant angle subtended by the chord of a circle) which is orthogonal to the first system and each circle of which contains the limit points \( Q_1 \) and \( Q_2 \) on its circumference.

A new complex variable \( w = ce^{it} \) is now introduced by the following relation

\[
 \frac{w}{c} = e^{it} = \frac{t + ic}{t - ic} = \frac{r_2}{r_1} e^{i(\gamma_2 - \gamma_1)}
\]

Hence

\[
 \mu = \log \frac{r_2}{r_1} \quad \text{and} \quad \theta = \gamma_2 - \gamma_1
\]

Also

\[
 t = i \frac{wo + c}{wo - c}
\]

These equations transform conformally the coaxial system of circles of the t plane into a concentric system of circles in the w plane. In particular, two circles \( K_1 \) and \( K_2 \) in the t plane, \( K_1 \) located in the upper half plane and \( K_2 \) in the lower half plane and defined by \( \log r_2/r_1 = \alpha \) and \( \log r_2/r_1 = -\beta \), respectively \((\alpha > 0, \beta > 0)\), transform into two concentric circles \( B_1 \) and \( B_2 \) about the origin in the w plane, of radii \( ce^\alpha \) and \( ce^{-\beta} \), respectively (fig. 1 (b)). It is noted also that the \( t_1 \) axis transforms into the circle of radius c in the w plane, and that the region at infinity in the t plane maps into the neighborhood of the point \( w = c \). It may also be remarked that the circles orthogonal to \( K_1 \) and \( K_2 \) transform into radial lines through the origin.

Transformation of the circular systems into rectangular systems.—There is now introduced another variable \( s = \lambda + iv \) defined by the relation

\[
 s = i \log \frac{w}{c} = i \log \frac{t + ic}{t - ic}
\]
Separating into real and imaginary parts

\[
\begin{align*}
\lambda &= -(y_2 - y_1) = -\theta \\
v &= \log \frac{r_2}{r_1} = \mu
\end{align*}
\]  

(6)

Hence the variable \( s \) may hereafter be denoted by

\[ s = -\theta + i\mu \]

or by \( s = \lambda + i\mu \)

Also from (5) we have

\[ w = ce^{-it} \]

(7)

and

\[ t = i\left( \frac{1 + e^{in}}{1 - e^{in}} \right) = -c \cot \frac{s}{2} \]

(8)

The circles in the \( t \) plane, \( r_2/r_1 = \text{constant} \) (or the circles in the \( w \) plane \( ce^\mu = \text{constant} \)), correspond uniquely to the straight lines \( \mu = \text{constant} \) in the \( s \) plane. In particular, the limiting points \( Q_1 \) and \( Q_2 \) correspond to \( \mu = 0 \) and \( \mu = -\infty \), respectively. Also the \( t \) axis corresponds to the axis \( \mu = 0 \), the point at infinity in the \( t \) plane going into the origin \( s = 0 \). The circular arcs \( \gamma_1 - \gamma_2 \), \( \text{constant} \) between \( Q_1 \) and \( Q_2 \) correspond to the parallel lines \( \mu = \text{constant} + 2k\pi \), \( k \) is any integer (fig. 1 (c)).

The whole \( t \) plane has thus infinitely many values on the \( s \) plane, but there is a one-to-one correspondence between the whole \( t \) plane and a strip of width \( 2\pi \) bounded by two parallels to the \( \mu \) axis. In the following investigation, the strip in the \( s \) plane bounded by the lines \( \lambda = -\pi \) and \( \lambda = \pi \) will be considered as the representation of the \( t \) plane cut along the length \( Q_1Q_2 \).

Equation (5) thus defines a conformal transformation of the coaxial system of circles in the \( t \) plane, or of the concentric system of circles in the \( w \) plane, to a rectangular system in the \( s \) plane. In particular, consider again the two definite circles \( K_1 \) and \( K_2 \) of the coaxial pencil. The circle \( K_1 \) is defined by \( \log \frac{r_2}{r_1} = \mu = \alpha \) and the circle \( K_2 \) by \( \log \frac{r_2}{r_1} = \mu = -\beta \) where \( \alpha \) and \( \beta \) are positive constants. It is then noted that the region of the \( t \) plane external to the circles \( K_1 \) and \( K_2 \) (or the ring region within \( B_1 \) and \( B_2 \)) corresponds uniquely to the rectangular region bounded by the lines \( \mu = \pi \), \( \mu = -\beta \), \( \lambda = -\pi \), and \( \lambda = \pi \). (The two sides \( \lambda = -\pi \) and \( \lambda = \pi \) correspond to the right and left edges respectively of a cut along the \( t \) axis drawn between the two circles.) The rectangle contains, necessarily, the point \( s = 0 \) as an internal point.

Geometrical relations.—Attention may be momentarily diverted to some geometrical relations existing in the various planes. Let the radii of \( K_1 \) and \( K_2 \) be \( a \) and \( b \), respectively, and let the centers of \( K_1 \) and \( K_2 \) be situated at \( O_1 \) and \( O_2 \), respectively (fig. 1 (a)).

The quantities \( a \) and \( b \) may be expressed in terms of \( \alpha \) and \( \beta \). The equation of \( K_1 \) in bipolar coordinates is, by equation (2)

\[ r_1 = \frac{|t + ic|}{|t - ic|} = e^\mu \]

Writing \( t = t_1 + i\tilde{t}_2 \) there results upon expansion

\[ t_1^2 + t_2^2 - 2\sigma_2 \cosh \alpha + \sigma_2^2 = 0 \]

which is the equation of a circle whose center \( O_1 \) is situated at

\[ t_2 = c \coth \alpha \]

and whose radius is

\[ a = c \csch \alpha \]

Similarly, for the second circle \( K_2 \), the center \( O_2 \) is at

\[ t_2 = -c \coth \beta \]

and the radius is

\[ b = c \csch \beta \]

Denoting by \( d \) the center-to-center distance \( O_1O_2 \) (fig. 1), there may be written the equations:

\[
\begin{align*}
  a &= c \csch \alpha \\
  b &= c \csch \beta \\
  d &= c (\coth \alpha + \coth \beta)
\end{align*}
\]  

(9)

which suffice to fix \( a \), \( b \), and \( d \) in terms of \( \alpha \), \( \beta \), and \( c \).

Forming the auxiliary quantity \( d^2 - a^2 - b^2 \), it is found that \( d^2 - a^2 - b^2 = 2abcosh(\alpha + \beta) \) immediately follows that

\[
\begin{align*}
  c &= \frac{ab}{d} \sinh (\alpha + \beta) \\
  \sinh \alpha &= \frac{b}{d} \sinh (\alpha + \beta) \\
  \sinh \beta &= \frac{a}{d} \sinh (\alpha + \beta)
\end{align*}
\]  

(10)

We observe that the quantity \( \alpha + \beta \) on the right-hand side is expressed in terms of \( a \), \( b \), and \( d \) by the relation

\[
\cosh (\alpha + \beta) = \frac{d^2 - a^2 - b^2}{2ab}
\]

2. TRANSFORMATION OF TWO CIRCLES INTO TWO ARBITRARY LINE SEGMENTS

The transformation that maps the rectangular region in the \( s \) plane into the region external to two nonintersecting line segments in a \( u \) plane will now be derived. In combination with the preliminary transformations of the preceding section this result will then transform the region of the two circles \( K_1 \) and \( K_2 \) of the \( t \) plane, or the ring region of the \( w \) plane, into the region of the line segments. Let the \( u \) plane (fig. 1 (d)) contain the two line segments \( c_1 \) and \( c_2 \) and let \( u(t) = X + iY \) be the analytic function that transforms the circles \( K_1 \) and \( K_2 \) into the desired line segments. With no loss in generality, the system of coordinates in the \( u \) plane may be so chosen that the \( X \) axis is parallel to \( c_2 \). Let the line segment \( c_1 \) be inclined at an angle \(-\delta/2\) with respect to the \( X \) axis. (The negative sign before \( \delta \) is a matter of later convenience; fig. 1 (d)) may be
regarded as illustrating the definition of positive
decalage.) Let \( l_1 \) and \( l_2 \) denote the two lines \( \mu = \alpha \) and \( \mu = -\beta \) of the rectangular \( s \) plane that corre-


spond to \( K_1 \) and \( K_2 \), then it is evident that \( \frac{dY}{dX} = 0 \)


for points of \( K_1 \) or \( l_2 \). Then, it is evident that \( \frac{dY}{dX} = -\tan \frac{s}{2} \) for points along


\( K_1 \) or \( l_1 \). Let \( f(s) = \frac{du}{ds} \) be the derivative of the func-
tion \( u(s) \) that gives the desired correspondence between


the \( u \) and \( s \) planes. From the well-known property of


conformal mapping, viz., that tangents at corresponding


points in the two planes differ in direction by the


argument of the derivative function, it follows that


the argument of \( f(s) \) equals 0 (or \( \pi \)) along \( l_2 \) and equals


\(-\delta/2 \) (or \(-\delta/2 + \pi \)) along \( l_1 \), or


\( f(s) \) is a real quantity along \( l_2 \)


\( f(s)e^{i\alpha} \) is a real quantity along \( l_1 \)


By a principle of Schwarz the as yet undetermined


function \( f(s) \) has the property of being extended by


analytic continuation to the whole strip region in the


\( s \) plane (fig. 1(c)) for, since \( f(s) \) is real along \( l_2 \), its


values for a pair of reflected points mirrored in the


line \( l_2 \) are conjugate complex. Similarly the function


\( f(s)e^{i\alpha} \) may be reflected about the line \( l_1 \). With


successive alternate reflections in \( l_1 \) and \( l_2 \) \( f(s) \) takes


on values as shown in figure 2. For every two succes-


sive reflections the original values of \( f(s) \) are repeated


except for a multiplying factor \( e^{-i\alpha} \). Hence it is clear


that \( f(s) \) must satisfy the relation


\[ f(s + 2i(\alpha + \beta)) = f(s)e^{-i\alpha} \] (1)

Also, since \( f(s) \) is a single-valued function of \( t \), it


satisfies the condition


\[ f(s + 2\pi) = f(s) \] (2)

If \( \delta = 0 \), then \( e^{-i\alpha} = 1 \) and it is seen at once that \( f(s) \) is


then a doubly periodic function, hence an elliptic func-
tion (of the first kind), of real period \( 2\omega = 2\pi \) and of


imaginary period \( 2\omega' = 2i(\alpha + \beta) \). In the general case


where \( \delta \neq 0 \), the function \( f(s) \) is not a purely doubly


periodic function but, since one of its periods gives


rise to a multiplying factor, is an elliptic function of


the second kind. It is completely determined, except


for a constant, by its behavior at its poles, in the


neighborhood of which the function becomes infinite


(Hermite’s theorem). In the present analysis, we


shall consider the fundamental periodic rectangle as


formed by the original transformed rectangle and its


reflection in the line \( l_2 \) (fig. 2).


We now investigate the poles of the function


\( f(s) = \frac{du}{ds} \). We assume that at infinity \( |s| = |t| \), in order


that the regions at infinity in the \( u \) and \( t \) planes be


equally magnified and map into each other (except


for a possible change in direction), and we have


\[ \left| \frac{du}{dt} \right|_{|t| = 1} = 1 \]

or noting equation (1.8)


\[ \left| \frac{du}{dt} \right|_{|s| = 0} = \left| \frac{du}{ds} \right|_{|t| = 0} \times \left| \frac{du}{dt} \right|_{|s| = 0} \times \left| \frac{2c}{|s|} \right|_{|t| = 0} = 1 \] (3)

This relation shows at once that \( f(s) \) possesses a singu-


lar point at \( s = 0 \), and hence has one also at the point


obtained by reflection of \( s = 0 \) in \( l_2 \), viz, \( s = -2i\beta \). In


the neighborhood of these points \( f(s) \) becomes infinite


in the order of \( t^2 \) as \( t \to \infty \) or as \( \frac{1}{s^2} \) as \( s \to 0 \), i. e., has


poles of order two at the origin \( s = 0 \) and at \( s = -2i\beta \).


\[ \text{Figure 2.—Illustrating the properties of } \frac{du}{dt} f(t) \text{ in the strip region: } f(t) \text{ is the} \]


conjugate complex quantity of \( f(t) \).


\[ \text{This notation denotes equation (9) of sec. 1.} \]
The fundamental function having a single pole of first order (with residue unity) at the origin and satisfying the foregoing period requirements is (reference 10, p. 416, and reference 11, p. 369)

\[ A(s) = e^{-\frac{s}{s}} \frac{\sigma(s+\delta)}{\sigma(\delta) e^{\sigma(\delta)}} \]  

(4)

where \( \sigma \) denotes the sigma function of Weierstrass and possesses the following period properties

\[ \sigma(u+2\omega) = -e^{-2z(u+\omega)} \sigma(u) \]
\[ \sigma(u+2\omega') = -e^{-2z(u+i\gamma_\delta(u))} \sigma(u) \]  

(5)

The expression (4) for \( A(s) \) may also be written as follows

\[ A(s) = \frac{H'(0) H(s+\delta)}{H(\delta) H(s)} \]  

(4')

where the Jacobi \( H(\tau) \) function is defined by the equation (cf. references 11 and 12)

\[ H(\tau) = 2q^{1/2} \sin \frac{\pi u}{2\omega} - 2q^{1/4} \sin \frac{3\pi u}{2\omega} + 2q^{1/4} \sin \frac{5\pi u}{2\omega} \]  

(6)

and possesses the period properties

\[ H(u+2\omega) = -H(u) \]
\[ H(u+2\omega') = -q^{-1} e^{-\frac{\tau}{\omega}} H(u) \]

where

\[ q = e^{\frac{i\pi}{\omega}} \]

The relation existing between the \( \sigma \) and \( H \) functions is (reference 11, p. 488)

\[ \sigma(u) = e^{\frac{\pi u}{2\omega}} H(u) \]  

(7)

A function such as we are seeking, having a single pole of the second order at the origin and the required period properties, may be obtained by taking the negative derivative of \( A(s) \) with regard to \( s \). From equation (4') we have

\[ A'(s) = -\frac{dA(s)}{ds} = -\frac{H'(0) dH(s+\delta)}{H(\delta) dH(s)} \]  

(8)

The function \( f(s) \) is now determined except for constants \( a_1 \) and \( a_2 \) which is given by

\[ f(s) = \frac{du}{ds} = a_1 A'(s) + a_2 A'(s+2i\beta) + a_3 \]  

(9)

To determine the constants, observe that by means of equation (3), and by the fact that the expansion of \( A'(s) \) about the origin begins with the term \( \frac{1}{s} \), we have in the neighborhood of \( s = 0 \)

\[ \left| \frac{du}{ds} \right|_{s=0} = \left| a_1 \right| \left| \frac{2c}{s} \right| \left| e^{-i\gamma} \right| = 1 \]

and since from equation (1.8) \( |f|_{s=0} = \left| \frac{2c}{s} \right| \left| e^{-i\gamma} \right| \) there results \( |a_1| = 2c \). Thus the magnitude of \( a_1 \) is determined and, in general, we may put \( a_1 = 2ce^{i\gamma} \) where \( \gamma \) is an arbitrary real parameter that determines the stagger of the segments, and the significance of which will be seen shortly. It may be observed at this point that with \( a_1 = 2ce^{i\gamma} \) the following relation holds

\[ \left| \frac{du}{dx} \right|_{s=0} = e^{i\gamma} \left| \frac{du}{dx} \right|_{s=0} = e^{i\gamma} \]  

(10)

i.e., the regions at infinity in the \( u \) and \( s \) planes agree in magnitude but differ by angle \( \gamma \) in direction. In order to determine \( a_2 \) it is sufficient to recall that \( f(s) \) must remain real on \( l_1 \) hence it may be seen that \( a_2 \) must equal \( 2ce^{-i\gamma} \). Then finally equation (9) may be expressed as

\[ \frac{du}{ds} = 2c[A'(s)e^{i\gamma} + A'(s+2i\beta)e^{-i\gamma}] \]  

(11)

The general function relating the \( u \) and \( s \) planes is then by integration with regard to \( s \)

\[ u(s) = -2c[A(s)e^{i\gamma} + A(s+2i\beta)e^{-i\gamma} + k] \]  

(12)

where the function \( A(s) \) is given by (4) or (4') and \( k \) is an arbitrary constant that is independent of \( s \) but may contain the parameter \( \delta \).

The singular points of transformation (12) are given by the roots of the equation \( \frac{du}{ds} = 0 \). It is possible to draw at once certain conclusions with regard to the singular points. There exists a theorem on elliptic functions (reference 11, p. 366) which states that the number of zeros of an elliptic function (of the first or second kind) in a periodic rectangle is equal to the number of poles. Since \( f(s) = \frac{du}{ds} \) has 2 poles of second order in the periodic rectangle it follows that the equation \( \frac{du}{ds} = 0 \) possesses 4 roots in this rectangle. It is demonstrable without difficulty that 2 of the zeros are located on the boundary \( \mu = \alpha \) and the remaining 2 on the boundary \( \mu = -\beta \). These zeros correspond to the end points of the line segments \( c_1 \) and \( c_2 \) of the \( u \) plane. It may be stated for reference that on \( l_1 \) the singular values of \( \lambda \) are obtained from the equation

\[ \frac{du}{ds} = 0 = A'(\lambda - i\beta)e^{i\gamma} + A'(\lambda + i\beta)e^{-i\gamma} \]

or since \( A'(\lambda - i\beta) \) and \( A'(\lambda + i\beta) \) are conjugate complex quantities, the singular points are given by the solutions of

\[ \text{Re.} \ A'(\lambda - i\beta)e^{i\gamma} = 0 \]

where \( \text{Re.} \) denotes "real part of." On \( l_1 \), we employ the period property (1) of \( A(s) \) and obtain for the equation satisfied by the singular values of \( \lambda \)

\[ \text{Re.} \ A'(\lambda + i\gamma)e^{i(\gamma + \delta\pi)} = 0 \]

1 The remainder of this section is commentary to this equation. The reader may, without loss of continuity, proceed to sec. 3, p. 56.
Developments of $A(s)$ convenient for numerical purposes will be discussed shortly. It may be of value to consider first several useful special examples of equation (12).

(a) Parallel segments of zero stagger $^3$ ($\delta = 0, \gamma = 0$).—It is necessary to observe first the limiting form of $A(s)$ as $\delta \to 0$. From equations (4) or (4') (or cf. reference 10, p. 425) we have that

$$\lim_{\delta \to 0} \left( A(s) - \frac{1}{\delta} \right) = \frac{\sigma'(s)}{\sigma(s)} \frac{\eta s}{\omega} = \frac{H'(s)}{H(s)} = Z_1(s)$$

where the various forms are equivalent. The functions $\sigma$ and $\varsigma$ are Weierstrass elliptic functions, and $H$ and $Z_1$ (eta and zeta functions) are the elliptic functions of Jacobi and Hermite. Then putting for convenience the arbitrary constant $k = \frac{4c}{\Delta} - ic$ in equation (12) we obtain

$$\lim_{\delta \to 0} u(s) = U_1(s) = z + i\gamma$$

or (as given by Ferrari),

$$-2c[Z_1(s) + Z_1(s + 2i\beta)] - ic$$

Observing that the $Z_1$ function has the following period properties

$$Z_1(s + 2\omega) - Z_1(s) = 0$$

$$Z_1(s + 2\omega') - Z_1(s) = -i$$

it follows that for $\mu = \alpha$, $\gamma = c$ and for $\mu = -\beta$, $\gamma = -c$. Hence the gap $G$ between the line segments is $2c$.

The singular values of $\lambda$ are in this case given by

$$\text{Re. } Z_1'(s + i\alpha) = 0$$

$$\text{Re. } Z_1'(s - i\beta) = 0$$

The case of parallel line segments has been studied by Ferrari (reference 9). or we may put down the complete equation for reference as follows. Noting that

$$Z_1'(s) = -\frac{d}{ds} \left[ i(s) - \frac{\eta s}{\omega} \right] = \frac{p(s) + \eta s}{\omega}$$

where the Weierstrass $p$ function is defined by

$$p(s) = -\frac{d}{ds} i(s), \text{ and writing } s = \lambda + i\alpha$$

equation determining the singular points of $c_1$ is

$$p(\lambda + i\alpha) + p(\lambda - i\alpha) - \frac{2\eta}{\pi} = 0$$

The addition theorem (reference 10, p. 140) of the $p$ function may be written

$$p(\lambda + i\alpha) = \frac{[p'(\lambda) - \overline{p}'(\alpha)]^2}{4[p(\lambda) + \overline{p}(\alpha)]^2} - p(\lambda) + \overline{p}(\alpha)$$

Here, $\overline{p}(\alpha) = -p(\alpha)$ and the bar designates that the elliptic function $\overline{p}$ is based on periods $2\sigma$ and $2\omega'$ conjugate to the periods of $p$, i.e., $2\omega = \frac{2\omega}{\Delta}$ and $2\omega' = 2i\omega$ (reference 10, p. 32). Making use of (16) and of the differential equations for the $p$ and $\overline{p}$ functions:

$$p'(\lambda)^2 = 4p^2(\lambda) - g_2p(\lambda) - g_3$$

$$\overline{p}'(\alpha)^2 = 4\overline{p}^2(\alpha) - g_2\overline{p}(\alpha) + g_3$$

equation (15) becomes

$$Ap(\lambda)^3 + Bp(\lambda) + C = 0$$

where

$$A = \frac{4[\eta(\sigma - \overline{p}(\alpha))]}{\pi}$$

$$B = 4\eta[\sigma(\alpha)^3 - 8(\eta)(\overline{p}(\alpha) - g_3)$$

$$C = 4(\eta\pi)(\overline{p}(\alpha)^3 + g_2\overline{p}(\alpha) - 2g_3)$$

This equation determines the two singular points for the upper segment $c_1$. In general, there is only one positive value of the root $p(\lambda)$, hence the singular points, $\lambda = \pm \lambda_1$, are symmetrical with respect to the origin. The negative root does not give real values for $\lambda$. For the lower line segment $c_2$ it is only necessary to replace $\alpha$ by $\beta$. (Cf. reference 10, p. 272: Given $p(\lambda)$, to find $\lambda$.)

---

**Figure 3.** Illustrating the case of parallel line segments: (a) $t$ plane, (b) $U_1$ plane ($\delta = 0, \gamma = 0$), (c) $U_2$ plane ($\delta = 0, \gamma = i\beta$), (d) $U$ plane ($\delta = 0, \gamma$ arbitrary).
The line segments \( c_1 \) and \( c_2 \) given by transformation \( 14 \) are without “stagger” since the midpoint of each segment is located on the \( y \) axis (fig. 3 (b)). In order to obtain further insight into the general transformation let us consider the case \( \delta = 0, \gamma = 90^\circ \).

(b) Tandem parallel segments \( \left( \delta = 0, \gamma = \frac{\pi}{2} \right) \).—In this case let the arbitrary constant \( k = -c \) in transformation (12), and noting equation (13), we obtain

\[
\lim_{s \to \infty} u(s) = U_1(s) = U_2(s) = U(s) = -\frac{2ic}{\pi} \left[ Z_1(s) - Z_1(s + 2i\beta) \right] - c \quad (18)
\]

It can be shown directly that the singular values of \( \lambda \) are 0 and \( \pi \), for we have

\[
\frac{dU_2}{ds} = 0 = p(s) - p(s + 2i\beta) \quad (19)
\]

and if

\[
p(u) = p(\psi) \text{ we must have } u = \pm \psi + 2m\omega + 2m'\omega' \quad \text{where } m \text{ and } m'
\]

are any integers. Hence, writing \( s = \lambda + i\mu \) we have for the solutions of (19) in the fundamental periodic rectangle \( \lambda = 0 \) and \( \lambda = \pi \). Figure 3(c) shows a typical correspondence for this case.

(c) Parallel segments of arbitrary stagger.—Let the arbitrary constant \( k = \left( \frac{4c}{\delta} - ic \right) \cos \gamma \) in equation (12), and noting relation (13),

\[
\lim_{s \to \infty} u(s) = U(s) = U_1(s) = \cos \gamma + U_2(s) \sin \gamma \quad (20)
\]

The function \( U(s) \) is the general relation bringing the region about any two parallel line segments into a rectangular region in the \( s \) plane. In this transformation the values of the parameters \( \alpha, \beta, \) and \( \gamma \) suffice to fix uniquely the chord ratio \( c_1/c_2 \), the gap/chord \( G/c_2 \), and the stagger/chord \( S/c_2 \) of the parallel segments. The gap between the line segments is \( 2c \cos \gamma \). Figure 3(d) illustrates the definitions of the various quantities. In the general case of parallel segments of arbitrary stagger, there are given the three ratios \( c_1/c_2, G:S \) and the parameters \( \alpha, \beta, \) and \( \gamma \) to be determined. This problem involves the solution of transcendental relations; in this connection it is convenient to draw up charts, e. g., figure 4. This figure shows a cross plot that presents \( G/c_2, S/c_2 \) in terms of \( \omega' \) and \( \gamma \) in the case of equal chords \( c_1/c_2 = 1 \), i. e., \( \alpha = \beta \).

The singular points of equation (20) are defined by the relation

\[
\frac{dU}{ds} = 0 = p(s)e^{\gamma} + p(s + 2i\beta)e^{-\gamma} + \frac{2c}{\pi} \cos \gamma \quad (21)
\]

Writing \( s = \lambda + i\alpha \) in (21) we have

\[
\cos \gamma [p(\lambda + i\alpha) + p(\lambda - i\alpha) + 2\eta/\pi] + i \sin \gamma [p(\lambda + i\alpha) - p(\lambda - i\alpha)] = 0
\]

Employing the notation of equation (17) and the relations preceding equation (17), there results

\[
a_4p^4(\lambda) + a_3p^3(\lambda) + a_2p^2(\lambda) + a_1p(\lambda) + a_0 = 0 \quad (22)
\]

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
\text{Angle of stagger, } \gamma, \text{ degrees} & 0 & 20 & 40 & 60 & 80 & \text{Ratio of periods, } \omega'/\omega \\
\hline
\hline
\text{G/chord} & 1.25 & 1.00 & .75 & .50 & .25 & \text{Chart presents the angle of stagger } \gamma \text{ and the ratio of the periods } \omega'/\omega \text{ against gap/chord } G/c_2 \text{ and stagger/chord } S/c_2 \text{ in the special case of equal parallel line segments, } (\delta = 0, \gamma = \beta) \text{ and } \sqrt{S/c_2}. \text{ The singular } \end{array}
\]

where

\[
a_4 = A^2d^3
\]
\[
a_3 = 2ABd^2 - 4
\]
\[
a_2 = (B^2 + 2AC)d^3
\]
\[
a_1 = 2BD^3 + g_2
\]
\[
a_0 = C^2d^2 + g_3 \text{ and } d = \frac{1}{2} \cot \gamma
\]

This equation suffices to determine the values of \( \lambda \) corresponding to the end points of the upper line seg-
ments \(c_i\). In the case of the lower segment \(\alpha\) is to be replaced by \(\beta\). In general, there are only two positive solutions for \(p(\lambda)\); and it may be observed that the solutions for both angles of stagger \(\pm \gamma\) are contained in equation (22). (In sec. 4 approximations for the singular values of \(A\) are given by simple formulas.)

Jacobi series—In order to obtain further insight into the general relation (12) and to separate \(u(s) = X + iY\) into its real and imaginary parts it is necessary to revert to Jacobi expansions. These developments are especially useful where numerical evaluations are required.

By reference 10, page 416, we have the following expansion for the function \(A(s)\):

\[
A(s) = \frac{e^{-\frac{m}{2} s}}{A(s)}\int_0^s \frac{A(t)}{\sigma(t)} dt
\]

(20)

where

\[
\sigma(s) = \frac{\sin \alpha}{\cosh \alpha - \cos \lambda}
\]

(21)

The expression for \(Z_1(s)\) occurring in the case of parallel segments is

\[
\lim_{s \to \infty} \left( A(s) - \frac{1}{2} \cot \frac{s}{2} \right) = Z_1(s)
\]

(22)

To separate \(A(s)\) into real and imaginary parts, replace \(s\) by \(\lambda + i\mu\) and note that

\[
\cot \frac{\lambda + i\mu}{2} = \frac{\sin \lambda - i \sinh \mu}{\cosh \mu - \cos \lambda}
\]

(23)

Then

\[
A(s) = M(\lambda,\mu) + iN(\lambda,\mu) + \frac{1}{2} \cot \frac{s}{2}
\]

(24)

where

\[
M(\lambda,\mu) = -\frac{\sin \lambda}{2(\cosh \mu - \cos \lambda)} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q^{mn} \sin (m\lambda + n\phi) \cosh \mu
\]

\[
N(\lambda,\mu) = \frac{\sinh \mu}{2(\cosh \mu - \cos \lambda)} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q^{mn} \cos (m\lambda + n\phi) \sinh \mu
\]

(25)

\[\text{The value of}~q=s^{-\pi/2}~\text{may always be kept less than}~\tan^{-1} \frac{0.0422}{\text{by resorting when necessary,}~c,~\phi,~\alpha,~\beta \text{small, to transformations that interchange the real and imaginary periods of the elliptic functions (reference 10, p. 256). Thus the expressions can always be made to converge very rapidly. Indeed, there exist several other expansions for}~A(s) \text{which though less simple in form are more rapidly convergent than the formulas given here (reference 10, p. 422).}\]

Let us put the arbitrary constant \(k\) equal to \((2c, \cot \frac{\delta}{2} - i)e^{\gamma - \sin \gamma}\) in equation (12) and separate \(u(s)\) as follows

\[
u_i(s) = X + iY
\]

where

\[
u_i(s) = x + iy
\]

(26)

and

\[
u_j(s) = x' + iy'
\]

(27)

It is evident that \(u_1(s)\) and \(u_2(s)\) are generalizations of \(U_1(s)\) and \(U_2(s)\), given by equations (14) and (18) for the cases \((\delta = 0, \gamma = 0)\) and \((\delta = 0, \gamma = \pi/2)\), respectively.

Employing relation (25), equation (27) giving \(u_1(s)\) may be separated into

\[
\begin{align*}
x &= -2c[\lambda(\lambda, \mu) + M(\lambda, \mu - 2\phi)] + c \cos \gamma - c \sin \gamma \\
y &= -2c[N(\lambda, \mu) + N(\lambda, \mu - 2\phi)] - c
\end{align*}
\]

(27a)

It is observed that for \(\mu = -\beta\), the coordinates become

\[
x_0 = -2c\lambda \cos \gamma - c
\]

(27b)

Equation (27a) is most useful in the neighborhood of \(\mu = -\beta\). For values of \(\mu\) near \(\alpha\), relation (27) is first rewritten by making use of the period property \(A(s + 2\pi) = A(s)e^{6}\), and we have

\[
\begin{align*}
x &= -2c[\lambda(\lambda, \mu) + M(\lambda, \mu - 2\phi)] + c \cos \gamma - c \sin \gamma \\
y &= -2c[N(\lambda, \mu) + N(\lambda, \mu - 2\phi)] - c
\end{align*}
\]

(27c)

For \(\mu = \alpha\), equation (27c) becomes

\[
\begin{align*}
x_0 &= -2c[\lambda(\lambda, \alpha) + M(\lambda, \alpha - 2\phi)] + c \cos \gamma - c \sin \gamma \\
y_0 &= -2c[N(\lambda, \alpha) + N(\lambda, \alpha - 2\phi)] - c
\end{align*}
\]

(27d)

It may be remarked that equations (27), which hold also for the special case \(\delta = 0\), show immediately that in this case \(y_0 = -c\) and \(y_0 = c\), or that the gap of the parallel segments is 2c. In the general case \((\delta \neq 0, \gamma = 0)\) it is clear from (27d) that the point \((x_0, y_0) = (0, c)\) lies on \(c_1\). The "gap" as measured along the \(y\) axis (i.e., from \(x_0 = 0\) to \(x_0 = 0\)) is therefore again 2c. The effect of decalage may be considered to a first order to be a rotation of the segment \(c_1\) for the case \((\delta = 0, \gamma = 0)\) by the angle \(\delta/2\) about the point \((0, c)\).

Employing relation (25), equation (28) giving \(u_0(s)\) may be separated into

\[
\begin{align*}
x' &= -2c[\lambda(\lambda, \mu) + M(\lambda, \mu - 2\phi)] - c \\
y' &= -2c[N(\lambda, \mu) - M(\lambda, \mu + 2\phi)] - c
\end{align*}
\]

(28a)

It is observed that for \(\mu = -\beta\) the coordinates become

\[
x'_0 = -2c[N(\lambda, \beta) - c
\]

(28b)
In the neighborhood of \( \mu = 0 \), it may be preferable to express equation (28a) as follows
\[
\begin{align*}
x' &= -2c[-N(\lambda, \mu) + N(\lambda, \mu - 2\alpha) \cos \delta - M(\lambda, \mu - 2\alpha) \sin \delta + c \cos \delta] \\
y' &= -2c[M(\lambda, \mu) - M(\lambda, \mu - 2\alpha) \cos \delta - N(\lambda, \mu - 2\alpha) \sin \delta - c \sin \delta]
\end{align*}
\] (28c)

For \( \mu = 0 \) the coordinates are seen to be
\[
\begin{align*}
x &= 2c[N(\lambda, \alpha)(1 + \cos \delta) + M(\lambda, \alpha) \sin \delta] \\
y &= -2c[M(\lambda, \alpha)(1 - \cos \delta) + N(\lambda, \alpha) \sin \delta - c \sin \delta]
\end{align*}
\] (28d)

For \( \delta = 0 \) it is clear from (28b) and (28d) that \( y' = y'' = 0 \). In general, for \( \delta \neq 0 \) it can be seen that the coordinates \( (x', y') = (-c, 0) \) satisfy equation (28d), hence this point lies on \( c_1 \).

The general equation (12) or (26) may now be separated into
\[
u(s) = x + iY
\] (29)

where
\[
\begin{align*}
x &= x \cos \gamma + x' \sin \gamma \\
y &= y \cos \gamma + y' \sin \gamma
\end{align*}
\]

In particular, it is clear from the foregoing that the lower segment \( c_2 \) is situated at \( y = -2c \cos \gamma \) and that the point \( (x, y) = (-c \sin \gamma, c \cos \gamma) \) lies on the segment \( c_1 \).

If there are given any two line segments in position, the three ratios \( c_1 : c_2 : c_3 \) are known (in addition \( \delta \) is known), and the quantities \( \alpha, \beta, \gamma \) are to be determined. Equation (29) is transcendental and a direct solution for a given case is not available; however, an indirect procedure of building up charts similar to figure 4 (for which \( \delta = 0 \)) for different values of \( \delta \) may be resorted to. The case of parallel segments, as well as the degenerate monoplane case (cf. sec. 4), will prove helpful in this procedure.

For later reference, the derivative expression \( \frac{dv}{ds} \) may be put down. We have
\[
\frac{dv}{ds} = \frac{d
u_1}{ds} \cos \gamma + \frac{d
u_2}{ds} \sin \gamma
\] (30)

where
\[
\begin{align*}
\frac{d
u_1}{ds} &= A'(s) + A'(s + 2i\beta) \\
&= P(\lambda, \mu) + iQ(\lambda, \mu) \\
\frac{d
u_2}{ds} &= i[A'(s) - A'(s + 2i\beta)] \\
&= P'(\lambda, \mu) - iQ'(\lambda, \mu)
\end{align*}
\]

In order to determine \( P, Q, P', \) and \( Q' \), the following development is noted
\[
A'(s) = \frac{1}{4} \sin^2 \frac{s}{2} \sum_{m=1, n=1}^{\infty} m q^{2m} \cos (ms + nt)
\] (31)

where
\[
\begin{align*}
M'(\lambda, \mu) &= \frac{1}{2} \cos \lambda \cosh \mu \\
&= \sum_{m=1, n=1}^{\infty} m q^{2m} \cos (ms + nt)
\end{align*}
\]

Hence,
\[
\begin{align*}
P &= M'(\lambda, \mu) + M'(\lambda, \mu + 2\beta) \\
Q &= N'(\lambda, \mu) + N'(\lambda, \mu + 2\beta) \\
P' &= -N'(\lambda, \mu) + N'(\lambda, \mu + 2\beta) \\
Q' &= M'(\lambda, \mu) - M'(\lambda, \mu + 2\beta)
\end{align*}
\]

And finally
\[
\frac{dv}{ds} = 2c[P \cos \gamma + P' \sin \gamma + i(Q \cos \gamma + Q' \sin \gamma)]
\] (32)

The equations of this section may be simplified in the noteworthy special case in which \( \alpha = \beta \). The constant \( 2i\beta \) is in this case equal to half the imaginary period, i.e., \( 2i\beta = \omega' \) and, in particular, the line segments \( c_1 \) and \( c_2 \) are equal. By reference 10, page 422, we have
\[
A(s + \omega') = \frac{\sigma_1(s + \omega')}{\sigma_1(s)\sigma_1(\omega')} \frac{e^{2\omega'}}{e^{\omega'}}
\]

This expression may therefore replace \( A(s + 2i\beta) \) in equations (26), (27), and (28). Similarly in equation (30) \( A'(s + 2i\beta) \) may be replaced by \( A'(s + \omega') \) where
\[
A'(s + \omega') = -e^{-\frac{\omega'}{2}} \sum_{m=1, n=1}^{\infty} m q^{(m-n)\omega} \cos (ms + (n-1/2) \omega)
\]

3. TRANSFORMATION OF A NEARLY CIRCULAR RING REGION IN THE \( w \) PLANE INTO A TRULY CIRCULAR RING REGION IN THE \( z \) PLANE

In the foregoing sections, there have been obtained the equations transforming the region external to two circles (in the \( t \) plane); or the annular region between two concentric circles (in the \( w \) plane); or also a rectangular region (in the \( s \) plane); into the region external to any two nonintersecting line segments (in the \( u \) plane). It may now be imagined, for definiteness, that two airfoil profiles are generated about the two line segments as chords in the \( u \) plane (fig. 5 (a)). In the plane of the rectangle, the two profiles will correspond to curves of small amplitude extending...
from $\lambda = -\pi$ to $\lambda = \pi$ near the boundary lines $\mu = \alpha$ and $\mu = -\beta$, respectively. In the ring region, the profiles will correspond to two nearly circular contours forming an annular region (fig. 5 (c)). It is intended to show how this annular region may be transformed into a concentric circular ring region (fig. 5 (e)).

At present it is assumed that the nearly circular ring region in the $w$ plane corresponding to a given biplane cellule in the $u$ plane is known. It is observed that this knowledge implies that equation (2.12) may be inverted and the variables $\lambda$, $\mu$ solved for in terms of $z$ and $y$. How this task may be done is taken up in section 4. It is recalled here that the variable $\lambda$ corresponds to $-\theta$ (equation (1.6)) and that the neighborhood of the point $w = c$, which corresponds to the region at infinity in the $t$ or $u$ planes, must be an internal point of the annular region.

The annular region in the $w$ plane then contains two boundary contours, an outer contour $B_1$ and an inner contour $B_2$. Let the contour $B_1$ be defined by

$$w = ce^{\alpha + i\theta}$$

and the contour $B_2$ by

$$w = ce^{\beta + i\theta}$$

where the range of $\theta$ may be chosen as $0 \leq \theta \leq 2\pi$.

Consider now a $z$ plane (fig. 5 (e)) containing two concentric circles about the origin, an outer circle $C_1$ that corresponds to $B_1$ and an inner circle $C_2$ that corresponds to $B_2$. The circle $C_1$ may be defined by

$$z = ce^{\alpha + i\theta}$$

and the circle $C_2$ by

$$z = ce^{\beta + i\theta}$$

where the radii are respectively,

$$R_1 = ce^{\alpha}, \quad R_2 = ce^{\beta}, \quad \sigma_1 > 0, \quad \sigma_2 < 0$$

At times it will be found convenient to denote $\sigma_1$ by $\sigma'$ and $\sigma_2$ by $-\beta'$.

Let the function that transforms the $w$ plane conformally into the $z$ plane be written as

$$w = ze^{\alpha + i\theta}$$

where $z = ce^{\alpha + i\theta}$. It is intended (sometimes it will be found convenient to denote $u_1$ by $-\beta'$).

$$h(z) = A_0 + \sum_{n=1}^{\infty} (A_n R_1^n + A_{-n} R_2^{-n}) \cos n\theta$$

and the function that transforms the $w$ plane into a concentric circular ring region (fig. 5 (e)). Let the function that transforms the $w$ plane conformally into the $z$ plane be written as

$$h(z) = h(R_1, \phi) + h(R_2, \phi)$$

where $R_2 = |z| \leq R_1$, or with $z = Re^{i\phi}$

$$h(z) = f(R, \phi) + ig(R, \phi)$$

It is seen that on $C_1$,

$$\log \frac{z}{c} = h(z) = h(R_1, \phi)$$

or, in short,

$$h_1(\phi) = f_1(\phi) + ig_1(\phi) = \mu_1 - \sigma_1 + i(\theta - \phi)$$

where $\theta - \phi_1$ means that the quantity $\theta - \phi$ is evaluated around $C_1$. On $C_2$ similarly

$$h_2(\phi) = f_2(\phi) + ig_2(\phi) = \mu_2 - \sigma_2 + i(\theta - \phi)$$

Let the complex coefficients in equation (6) be expressed as

$$a_n = A_n + iB_n$$

Then, from equations (6) and (7), it is found that

$$f_1(\phi) = A_0 + \sum_{n=1}^{\infty} (A_n R_1^n + A_{-n} R_1^{-n}) \cos n\phi$$

$$- (B_2 R_1^n - B_{-n} R_1^{-n}) \sin n\phi$$

and

$$g_1(\phi) = B_0 + \sum_{n=1}^{\infty} (B_n R_1^n + B_{-n} R_1^{-n}) \cos n\phi$$

$$+ (A_n R_1^n - A_{-n} R_1^{-n}) \sin n\phi$$

FIGURE 5.—Mapping of: (a) two contours $A_1$ and $B_1$ of a biplane arrangement into (b) two curved lines $L_1$ and $L_2$ in the $s$ plane, (c) two nearly circular contours $B_1$ and $B_1$ of an annular region in the $w$ plane, (d) two nearly circular contours $K_1$ and $K_2$ in the $t$ plane, (e) two true circles $C_1$ and $C_2$ of the concentric ring region in the $z$ plane.
Similarly
\[ f_3(\phi) = A_0 + \frac{1}{2\pi} \int_0^{2\pi} f_1(\phi) \cos n\phi \, d\phi \]
\[ g_2(\phi) = B_0 + \frac{1}{2\pi} \int_0^{2\pi} f_1(\phi) \cos n\phi \, d\phi \]  
\[ + \frac{1}{2\pi} \int_0^{2\pi} f_1(\phi) \sin n\phi \, d\phi \]  
\[ + \frac{1}{2\pi} \int_0^{2\pi} f_1(\phi) \cos n\phi \, d\phi \]  
\[ + \frac{1}{2\pi} \int_0^{2\pi} f_1(\phi) \sin n\phi \, d\phi \]  
\[ \text{From equation (10),} \]
\[ A_0 R_{1a} + A_{-n} R_{1b} = a_{1a} = \frac{1}{\pi} \int_0^{2\pi} f_1(\phi) \cos n\phi \, d\phi \]
\[ - B_{-n} R_{1a} + B_{-n} R_{1b} = b_{1a} = \frac{1}{\pi} \int_0^{2\pi} f_1(\phi) \sin n\phi \, d\phi \]
\[ \text{and} \]
\[ A_0 = a_{1a} = \frac{1}{2\pi} \int_0^{2\pi} f_1(\phi) \, d\phi = \frac{1}{2\pi} \int_0^{2\pi} \eta d\phi = \sigma_1 \]  
\[ \text{Similarly from equation (12),} \]
\[ A_0 R_{n} + A_{-n} R_{n} = a_{2a} = \frac{1}{\pi} \int_0^{2\pi} f_2(\phi) \cos n\phi \, d\phi \]
\[ - B_{-n} R_{n} + B_{-n} R_{n} = b_{2a} = \frac{1}{\pi} \int_0^{2\pi} f_2(\phi) \sin n\phi \, d\phi \]
\[ \text{and} \]
\[ A_0 = a_{2a} = \frac{1}{2\pi} \int_0^{2\pi} f_2(\phi) \, d\phi = \frac{1}{2\pi} \int_0^{2\pi} \eta d\phi = \sigma_2 \]  

The equality \( a_{1a} = a_{2a} = A_0 \) is a condition of uniformity that is necessary since \( h(z) \) is a regular analytic function in the ring region. There is, in addition, an arbitrary element in equations (10) to (13) (which may be chosen in a number of ways) viz, there is at our disposal the choice of the point in the \( z \) plane that shall correspond to, say \( w = \epsilon \). This choice, which will be introduced at a later point (p. 14), is \( z = \epsilon \) when \( w = \epsilon \), and will fix the constants \( A_0 \) and \( B_0 \) in terms of \( R_1 \) and \( R_2 \) and the remaining coefficients.

From the first parts of equations (14) and (15),

Denoting the variable in equation (17) by \( \varphi' \) instead of \( \varphi \) and substituting by means of equations (14) and (15), it appears that

\[ g_1(\varphi') = B_0 + \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} f_1(\varphi') (\sin n\varphi \epsilon \varphi' - \cos n\varphi \epsilon \varphi') \, d\varphi' \]

or

\[ g_1(\varphi') = B_0 + \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} f_2(\varphi') (\sin n\varphi \epsilon \varphi' - \cos n\varphi \epsilon \varphi') \, d\varphi' \]  

In a similar manner,

\[ g_2(\varphi') = B_0 + \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} f_3(\varphi') (\sin n\varphi \epsilon \varphi' - \cos n\varphi \epsilon \varphi') \, d\varphi' \]  

there is obtained solving for \( A_n, A_{-n}, B_n, \) and \( B_{-n} \)

\[ A_n = \frac{a_{1n} R_{1a} - a_{2n} R_{1b}}{(R_1^a - R_1^b)} \]
\[ B_n = -\frac{b_{1n} R_{1a} + b_{2n} R_{1b}}{(R_1^a - R_1^b)} \]

Let

\[ R_2 = e^{-\tau} = q \]

where

\[ \tau = \sigma_1 - \sigma_2 (= \alpha' + \beta') \]

Substituting by means of equation (16) in equation (11), it is seen that

\[ g_1(\varphi) = B_0 + \sum_{n=1}^{\infty} \left( b_{1n} e^{n\tau} + b_{2n} e^{-n\tau} \right) \cos n\varphi \]
\[ + \sum_{n=1}^{\infty} \left( a_{1n} e^{n\tau} + a_{2n} e^{-n\tau} \right) \sin n\varphi \]

or also

\[ g_1(\varphi) = B_0 + \sum_{n=1}^{\infty} \left( b_{1n} \cosh n\tau + b_{2n} \csch n\tau \right) \cos n\varphi \]
\[ + \sum_{n=1}^{\infty} \left( a_{1n} \cosh n\tau - a_{2n} \csch n\tau \right) \sin n\varphi \]

Similarly, by substitution of equation (16) in equation (13),

\[ g_2(\varphi) = B_0 + \sum_{n=1}^{\infty} \left( -b_{1n} \csch n\tau + b_{2n} \cosh n\tau \right) \cos n\varphi \]
\[ + \sum_{n=1}^{\infty} \left( a_{1n} \csch n\tau - a_{2n} \cosh n\tau \right) \sin n\varphi \]
The two series expressions

\[ a) \sum_{n=1}^{\infty} \sin n(\varphi - \varphi') \coth n\tau \]
\[ b) \sum_{n=1}^{\infty} \sin n(\varphi - \varphi') \csch n\tau \]

that occur in equations (19) and (20) may be evaluated in terms of elliptic functions. Consider the expansion for  \( \xi(u) \) (reference 10, p. 403)

\[ \xi(u) = \frac{\pi}{2\omega_1} \cot \frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \sin \frac{n\pi u}{\omega_1} \]

In order not to confuse the periods occurring here with those of the preceding section, the real period is denoted by \( 2\omega_1 = 2\pi \), and the imaginary period by \( 2\omega_2 = 2\pi \),

\[ q = e^{2\pi i / \omega_1} = e^{-\tau} = \frac{R_1}{R_2} \]

Then

\[ Z_1(u) = \frac{1}{2} \cot \frac{\pi u}{2\omega_1} + 2 \sum_{n=1}^{\infty} \frac{e^{-\pi n}}{1+e^{-\pi n}} \sin nu \]

\[ = \sum_{n=1}^{\infty} \sin nu + \sum_{n=1}^{\infty} \left( \frac{1+e^{-\pi n}}{1-e^{-\pi n}} - 1 \right) \sin nu \]

\[ = \sum_{n=1}^{\infty} \coth n\tau \sin nu \]  \hspace{4cm} (21)

Consider the expansion for  \( \xi(u + \omega_2) \) (reference 10, p. 426)

\[ \xi(u + \omega_2) = \frac{\pi}{\omega_1} \xi(u) = 1 \frac{\Theta' \xi}{\Theta(u)} = \frac{1}{\pi} \int_{\Theta}^{\Phi} \frac{d\xi}{\Theta(u)} \]

\[ = \frac{2\pi}{\omega_1} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \sin \frac{n\pi u}{\omega_1} \]

For \( \omega_1 = \pi \) this expression becomes

\[ Z(u) = \sum_{n=1}^{\infty} \frac{2e^{-\pi n}}{1-e^{-\pi n}} \sin nu \]

\[ = \sum_{n=1}^{\infty} \frac{1}{\cosh n\tau} \sin nu \]  \hspace{4cm} (22)

Then replacing \( u \) by  \( \varphi - \varphi' \), equations (19) and (20) become

\[ g_1(\varphi') = B_0 + \frac{1}{2\pi} \int_{-\pi}^{\pi} f_2(\varphi) Z(\varphi - \varphi') d\varphi \\
- \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\varphi) Z_1(\varphi - \varphi') d\varphi \]  \hspace{4cm} (23)

\[ g_2(\varphi') = B_0 + \frac{1}{2\pi} \int_{-\pi}^{\pi} f_2(\varphi) Z(\varphi - \varphi') d\varphi \\
- \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\varphi) Z_1(\varphi - \varphi') d\varphi \]  \hspace{4cm} (24)

Since  \( Z(u) = \frac{\Theta' \xi}{\Theta(u)} \) and  \( Z_1(u) = \frac{\Theta' \xi}{\Theta(u)} \), there is obtained also by integration by parts:

\[ g_1(\varphi') = B_0 + \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\varphi) \log \Theta(\varphi - \varphi') d\varphi + \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(\varphi) \log H(\varphi - \varphi') d\varphi \]  \hspace{4cm} (23')

\[ g_2(\varphi') = B_0 + \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\varphi) \log H(\varphi - \varphi') d\varphi \\
+ \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(\varphi) \log \Theta(\varphi - \varphi') d\varphi \]  \hspace{4cm} (24')

where the logarithm operates only on the absolute value of the quantities  \( \Theta(\varphi - \varphi') \) and  \( H(\varphi - \varphi') \).

In a manner similar to the foregoing procedure it is possible to solve for the coefficients in equations (11) and (13), substitute in equations (10) and (12), and obtain as the reciprocal relations to (23) and (24) the following:

\[ f_1(\varphi') = A_0 - \frac{1}{2\pi} \int_{-\pi}^{\pi} g_1(\varphi) Z(\varphi - \varphi') d\varphi + \frac{1}{\pi} \int_{-\pi}^{\pi} g_1(\varphi) Z_1(\varphi - \varphi') d\varphi \]  \hspace{4cm} (25)

\[ f_2(\varphi') = A_0 - \frac{1}{2\pi} \int_{-\pi}^{\pi} g_2(\varphi) Z(\varphi - \varphi') d\varphi \\
+ \frac{1}{\pi} \int_{-\pi}^{\pi} g_2(\varphi) Z_1(\varphi - \varphi') d\varphi \]  \hspace{4cm} (26)

Equations (23) and (24), which essentially express a pair of boundary-value relations for a concentric ring region, permit the obtaining of the imaginary parts of a complex function  \( h(z) \) along the boundary circles of a ring region from a knowledge of the real parts along the boundaries. These equations are fundamental in a potential-theory study of ring regions; they have been developed in a different manner and for another purpose by Henri Villat in 1912 (reference 13, p. 147). It will be shown shortly that equations (23) and (24), when generalized and regarded as integral equations instead of being considered as definite integrals, make it possible to obtain the complete correspondence which we are seeking for doubly connected regions.
The function giving the value of \( h(z) \) at any point interior to the ring region may also be expressed in terms of the real parts of \( h(z) \) along the boundaries \( C_1 \) and \( C_2 \). From equation (10),

\[
a_n = A_n + iB_n = -\frac{1}{2\pi \sinh n\tau} \left[ R_2 \int_0^{2\pi} f_1(\phi)e^{-i\tau r}d\phi - R_1 \int_0^{2\pi} f_2(\phi)e^{-i\tau r}d\phi \right]
\]

Then equation (6)

\[
h(z) = a_0 + \sum_{n=1}^{\infty} \left( a_n z^n + a_{-n} z^{-n} \right)
\]

becomes

\[
h(z) = a_0 - \frac{1}{\pi} \int_0^{2\pi} f_1(\phi) M d\phi + \frac{1}{\pi} \int_0^{2\pi} f_2(\phi) N d\phi
\]

where

\[
M = \sum_{n=1}^{\infty} \frac{R_2^n z^{n-\varepsilon} e^{i\tau n\xi}}{2 \sinh n\tau}
\]

\[
N = \sum_{n=1}^{\infty} \frac{R_1^n z^{-n+\varepsilon} e^{-i\tau n\xi}}{2 \sinh n\tau}
\]

The quantities \( M \) and \( N \) may be readily expressed in terms of elliptic functions. Let

\[
e^{i\tau z} = R_2 e^{-\varepsilon} e^{i\tau \varepsilon}
\]

Then by equation (22),

\[
M = \sum_{n=1}^{\infty} \frac{i \sin n\tau}{\sinh n\tau} \frac{1}{i \Theta(v_2)} = iZ(v_2)
\]

Similarly let

\[
e^{i\tau z} = R_1 e^{\varepsilon} e^{-i\tau \varepsilon}
\]

Then

\[
N = \frac{i}{\Theta(v_1)} = iZ(v_1)
\]

Then finally

\[
h(z) = a_0 - i \int_0^{2\pi} f_1(\phi) Z(v_2) d\phi + i \int_0^{2\pi} f_2(\phi) Z(v_1) d\phi
\]

where

\[
v_1 = i \log \frac{z}{R_1} + \varphi
\]

\[
v_2 = i \log \frac{z}{R_2} + \varphi
\]

or writing

\[
z = e^{\varphi + i\psi},
\]

\[
v_2 = \varphi - \varphi' + i(\sigma - \sigma_2)
\]

\[
v_1 = \varphi - \varphi' + i(\sigma - \sigma_1)
\]

Determination of the constants \( A_n \) and \( B_n \).—It will be recalled that the neighborhood of the point \( w = c \) corresponds to the region at infinity in the \( t \) and \( u \) planes. In order to make the correspondence of the \( w \) and \( z \) planes unique the following condition is put down. Let \( z = c \) when \( w = c \), hence causing the region about \( z = c \) to correspond also to the region at infinity in the \( t \) and \( u \) planes. There is, however, an essential fact to be noted, viz., \( \frac{dw}{dz} \) evaluated for \( z = c \) is, in general, different from unity, hence generally a magnification and rotation of the regions near \( w = c \) and \( z = c \) exists in the two planes.

The conditions to be studied are

\[
w = c
\]

\[
\frac{dw}{dz} = r e^{i\xi}, \text{ evaluated for } z = c
\]

From equation (5)

\[
w = z e^{i\alpha}
\]

there is obtained

\[
\frac{dw}{dz} = e^{i\alpha}(1 + 2 \frac{dh(z)}{dz})
\]

The condition (29) then corresponds to

\[
|h(z)|_{z=c} = 0
\]

And, in view of the preceding relations, equation (29') corresponds to

\[
\left[ e^{i\tau z} \frac{dh(z)}{dz} \right]_{z=c} = r e^{i\xi} - 1 = p + iq
\]

where it is noted that \( r \) and \( \xi \) are given in terms of \( p \) and \( q \) as follows:

\[
\tau = (1+p)^2 + q^2
\]

\[
\xi = \tan^{-1} \frac{q}{1+p}
\]

By equations (6) and (9), it is found that equation (31) separates into

\[
A_0 + \sum_{n=1}^{\infty} (A_n e^{\sigma} + A_{-n} e^{-\sigma}) = 0
\]

\[
B_0 + \sum_{n=1}^{\infty} (B_n e^{\sigma} + B_{-n} e^{-\sigma}) = 0
\]

Also equation (31') becomes

\[
\sum_{n=1}^{\infty} n(A_n e^{\sigma} - A_{-n} e^{-\sigma}) = p
\]

\[
\sum_{n=1}^{\infty} n(B_n e^{\sigma} - B_{-n} e^{-\sigma}) = q
\]

These equations may also be expressed in other forms. For example, from equation (28), since \( z = c \) corresponds to \( \sigma = 0 \), \( \varphi' = 0 \), we have that

\[
h(c) = 0 = A_0 + iB_0 - \frac{i}{\pi} \int_0^{2\pi} f_1(\phi) Z(v_2) d\phi
\]

\[
+ \frac{i}{\pi} \int_0^{2\pi} f_2(\phi) Z(v_1) d\phi
\]

where

\[
v_2 = -i\sigma_2,
\]

\[
v_1 = \varphi - i\sigma_1
\]
Employing equation (22), this separates into

\[
A_0 - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sinh n\alpha_2}{\sinh n\tau} \int_0^{2\pi} f_1(\phi) \cos n\phi d\phi
\]

\[
+ \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sinh n\alpha_1}{\sinh n\tau} \int_0^{2\pi} f_2(\phi) \cos n\phi d\phi = 0 \quad (32')
\]

\[
B_0 - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cosh n\alpha_2}{\sinh n\tau} \int_0^{2\pi} f_1(\phi) \sin n\phi d\phi
\]

\[
+ \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cosh n\alpha_1}{\sinh n\tau} \int_0^{2\pi} f_2(\phi) \sin n\phi d\phi = 0 \quad (33')
\]

In these equations, \( f_1 \) and \( f_2 \) may each be altered by the addition of a constant without altering the values of the integrals. Hence, if \( f_1(\phi) - A_0, f_2(\phi) - A_0 \) are known (i.e., only the variational parts of \( f_1 \) and \( f_2 \) are known) and if there are given or known the quantities \( R_1 = \omega \tau, R_2 = \omega \tau \), equations (32') and (33') determine directly the constants \( A_0 \) and \( B_0 \) so that condition (31) is satisfied.

Since \( \mu_1 \) and \( \mu_2 \) differ from \( f_1 \) and \( f_2 \) by constants (cf. equations (7) and (8)) they may replace \( f_1 \) and \( f_2 \) in equations (32') and (33'). Also, by equations (14) and (15) it is recalled that

\[
A_0 = \frac{1}{2\pi} \int_0^{2\pi} \mu_1 d\phi - \sigma_1 = \frac{1}{2\pi} \int_0^{2\pi} \mu_2 d\phi - \sigma_2
\]

or that \( \sigma = \sigma_1 - \sigma_2 = \frac{1}{2\pi} \int_0^{2\pi} (\mu_1 - \mu_2) d\phi \). Hence, if there are given the functions \( \mu_1 \) and \( \mu_2 \) (therefore \( \tau \) is known), equation (32') determines the individual quantities \( \sigma_1 \) and \( \sigma_2 \) (i.e., the radii \( R_1 \) and \( R_2 \)). Equation (33') then again defines the value of the constant \( B_0 \).

By the use of equations (15) and (16), the conditions (34) and (35) may also be written in other forms. Thus

\[
p = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{n \cosh n\alpha_2}{\sinh n\tau} \int_0^{2\pi} f_1(\phi) \cos n\phi d\phi
\]

\[
- \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{n \cosh n\alpha_1}{\sinh n\tau} \int_0^{2\pi} f_2(\phi) \cos n\phi d\phi \quad (34')
\]

\[
q = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{n \sinh n\alpha_2}{\sinh n\tau} \int_0^{2\pi} f_1(\phi) \sin n\phi d\phi
\]

\[
- \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{n \sinh n\alpha_1}{\sinh n\tau} \int_0^{2\pi} f_2(\phi) \sin n\phi d\phi \quad (35')
\]

Further study of equations (23) and (24).—It has already been mentioned that there are two points of view from which the simultaneous equations (23) and (24) may be studied. In one, the equations are regarded as definite integral evaluations and the functions \( f_1(\phi) \) and \( f_2(\phi) \) are known as functions of the variable \( \phi \). In the other, the equations are regarded as integral equations and the functions are known in terms of \( \theta \), not \( \phi \). In the next few paragraphs the definite-integral viewpoint will first be employed and it will be shown how it may be used to develop biplane arrangements in an artificial or indirect manner. The results obtained by this method will also be of some interest and value when the more direct integral-equation point of view is investigated in the subsequent section.

Families of biplane arrangements.—When \( f_1(\phi) \) and \( f_2(\phi) \) are known functions, the evaluation of equations (23) and (24) determine the "conjugate" functions \( g_1(\phi) \) and \( g_2(\phi) \). It may be observed that the Fourier series expansions of \( f_1(\phi) \) and \( g_1(\phi) \) and of \( f_2(\phi) \) and \( g_2(\phi) \) are related by the peculiar interchange of coefficients as seen in equations (10) to (13). The existence of the integrals in equations (23) and (24) requires only that \( f_1(\phi) \) and \( f_2(\phi) \) be piecewise continuous and differentiable, and have no poles of order equal to or greater than one. In this paper, however, the only interest is in continuous, single-valued functions \( f_1 \) and \( f_2 \) of period \( 2\pi \), and satisfying the conditions of uniformity (cf. paragraph following equation (15)), such functions may always be associated with the conformal transformation of doubly connected regions bounded by continuous closed contours for a proper choice of coordinates.

When the functions \( g_1(\phi) \) and \( g_2(\phi) \) are known, the correspondence of \( \theta \) and \( \phi \) is immediately known along the boundary contours since \( g(\phi) = \theta - \phi \). Also, the functions \( f_1(\phi) \) and \( f_2(\phi) \) together with the constants \( \sigma_1 \) and \( \sigma_2 \) determine the functions \( \mu_1(\phi) \) and \( \mu_2(\phi) \). The quantities \( \mu_1 \) and \( \mu_2 \) expressed as functions of \( \theta(= - \lambda) \) then permit the defining of two contours in the \( u \) plane, the external region of which is in one-to-one correspondence with the ring region. Some specific examples will shortly be given. With an insight gained by experience, the functions \( \mu_1(\phi) \) and \( \mu_2(\phi) \) may be so chosen that certain desired classes of practical biplane arrangements may be obtained. It may be remarked here that once there is obtained a definite biplane arrangement by means of this process, the problem may immediately be considered reversed and thus insight is obtained into the solution of the associated integral equation. This notion will be later examined; in this section, some illustrations of the afore-mentioned process are briefly presented.

---

* There is an additional condition on \( g(\phi) \) or \( g(\phi) \) necessary for the contours to be free of double points (cf. reference 1, p. 10), viz.

\[-\pi < \arg g + \arg g < \pi \]

* It is understood that the minor equation (31) is to be satisfied.
By reference to the Fourier series developments for \( f_1(\phi) \) and \( f_2(\phi) \), equations (10) and (12), it may be observed that a particularly simple example is the following:

\[
\begin{align*}
\mu_1(\phi) &= \mu_1(\phi) - \sigma_1 = A_0 - 0.1 \sin \phi \\
\mu_2(\phi) &= \mu_2(\phi) - \sigma_2 = A_0 + 0.1 \sin \phi
\end{align*}
\]

(36a)

\[
\begin{align*}
\psi_1(\phi) &= \psi_1(\phi) - \sigma_1 = (B_1 R_1 - B_2 R_2^{-1}) \sin \phi \\
\psi_2(\phi) &= \psi_2(\phi) - \sigma_2 = (B_1 R_2 - B_2 R_2^{-1}) \sin \phi
\end{align*}
\]

(36b)

Also let \( \sigma_1 = \frac{\pi}{2} = 1.4708 \), and let \( \sigma_2 = -1.4708 \).

(Hence \( \tau = \sigma_1 - \sigma_2 = 2(1.4708) \) and \( 2\omega_1 = 2\tau = 5.8832i \).)

\[
\psi_1(\phi) = -0.1 \cosh \frac{\tau}{2} + 0.1 \sinh \frac{\tau}{2} \cos \phi = g_1(\phi)
\]

or finally,

\[
\begin{align*}
\mu_1(\phi) &= 1.4708 - 0.1 \sin \phi = -\mu_2(\phi) \\
g_1(\phi) &= -0.0485 \pm 0.1114 \cos \phi = g_2(\phi)
\end{align*}
\]

It is now possible to define the variable \( \theta = \phi + g(\phi) \) along each contour:

\[
\begin{align*}
\theta_1 &= -\lambda_1 = \phi + g_1(\phi) \\
\theta_2 &= -\lambda_2 = \phi + g_2(\phi)
\end{align*}
\]

In addition to the choice of \( f_1 \) and \( f_2 \), the line segments, or chords, and their relative positions (deter-
Here, also, choose $\alpha=\beta=\frac{\pi}{2}$, $\sigma_1=-\sigma_2=1.4708$. Then

$$g_1(\varphi) = (\theta-\varphi) = B_0 + 0.09625 \cos \varphi - 0.0450 \sin \varphi$$
$$g_2(\varphi) = (\theta-\varphi) = B_0 + 0.09625 \cos \varphi + 0.0450 \sin \varphi$$

The constants $A_0$ and $B_0$ determined by equations (32) and (33) are

$$A_0 = 0.0216, \quad B_0 = -0.0420$$

(Again, in this example $\frac{\partial w}{\partial z}$ is not shown.

Forming $\theta_1$ and $\varphi_1$, the rectangular coordinates of the contours are obtained as was shown in the preceding example. Figure 7 shows some arrangements for various values of the angle of stagger $\gamma$. Figure 7 (f) shows the arrangement obtained (for $\gamma=0$, cf. fig. 7 (a)) when an angle of decalage $a/2 = -3^\circ$ is further introduced in this numerical example.

In this manner, by employing appropriate values for $f_1(\varphi)$ and $f_2(\varphi)$ (or $g_1(\varphi)$ and $g_2(\varphi)$) and for the other parameters involved, it is possible readily to develop arrangements of a great variety of contour shapes, gap/chord values, chord ratios, stagger, and decalage. Indeed, if the process is considered in the light of a boundary-value problem of the concentric ring region, it is seen that it is sufficiently general to yield any biplane arrangement (more generally, any doubly connected region).

Equations (23) and (24) as integral equations.—It is desirable at this point to introduce the following notation. Frequent use will be made of subscripts. The first subscript will usually be 1 or 2 and will indicate that the designated quantity is to be evaluated at the boundary $C_1$ or $C_2$, respectively (or also $B_1$ and $B_2$, respectively). A second subscript will sometimes be employed to denote the variable in terms of which the quantity is expressed. Thus $\mu_{1,\gamma}$ represents the quantity $\mu$ evaluated around $C_1$ (or $B_1$) expressed as a function of $\theta$; also, $\mu_{2,\varphi}$ denotes the quantity $\mu$ evaluated on $C_2$ expressed as a function of $\varphi$.

It is recalled that by equations (7) and (8)

$$f_{1,\gamma} = \mu_{1,\gamma} - \sigma_1$$
$$f_{2,\varphi} = \mu_{2,\varphi} - \sigma_2$$

or $f_1$ and $\mu_1$, and $f_2$ and $\mu_2$ differ by constants. Then in the integrands of equations (23) and (24) the functions $f_1$ and $f_2$ may be replaced by $\mu_1$ and $\mu_2$, since the
additive constants do not contribute to the integrals. If a definite, biplane arrangement is preassigned or, what is essentially the same, if there is given a definite nearly circular annular region (in a $w$ plane; $w=\exp(\zeta)$), it is the functions $\mu_1,\varphi$ and $\mu_2,\varphi$ that may be considered directly defined or known. Thus, from an initial knowledge of $\mu_1,\varphi$ and $\mu_2,\varphi$ and with the aid of equations (23) and (24), it is desired to obtain a knowledge of the functions $\mu_1,\varphi$ and $\mu_2,\varphi$. From this point of view the expressions (23) and (24) then represent a pair of simultaneous integral equations, whose process of solution is more intricate than that involved in the process of evaluating the definite integrals. The problem may be restated more precisely:

Given two functions $\mu_1,\varphi$ and $\mu_2,\varphi$ that define two continuous contours of an annular region with respect to an origin contained by both contours. Then two pairs of functions $\mu_1,\varphi$ and $g_1,\varphi$ and $\mu_2,\varphi$ and $g_2,\varphi$ are to be obtained such that they are interrelated in the manner shown by the interchange of coefficients in the Fourier expansions, equations (10) to (13), which also satisfy certain local specified conditions at $z=c$ (equation (31)), and for which the relations $g_1=(\theta-\varphi)_1$, $g_2=(\theta-\varphi)_2$, which permit an interchange of the arguments $\theta$ and $\varphi$ on each boundary, are consistent with the given functions $\mu_1,\varphi$ and $\mu_2,\varphi$; that is to say, when $\mu_1,\varphi$ is expressed as a function of $\theta$ by means of the function $g_1,\varphi$ there results the original function $\mu_1,\varphi$ in symbols

$$\mu_1,\varphi = \mu_1,\varphi$$

when $\theta = \varphi + g_1,\varphi$ or $\varphi = \theta - g_1,\varphi$

and similarly

$$\mu_2,\varphi = \mu_2,\varphi$$

when $\theta = \varphi + g_2,\varphi$ or $\varphi = \theta - g_2,\varphi$

The process employed in this paper to obtain the desired solution of the simultaneous integral equations is one of successive approximations or iteration. The degree of convergence of most methods of successive approximations usually depends on how good the initial approximation is. In this regard it is rather fortunate that the contours of practically any biplane arrangement transform, under the transformations already developed (with proper choice of coordinates), into nearly circular contours in the $w$ plane. The nearness of these boundary contours to circular contours is very significant and enables the initial approximations to be so chosen that the process converges ordinarily with great rapidity, one step in the process being sufficient for most practical purposes.

Outline of the method of successive approximations.—The various steps in the process of successive approximations will be written down schematically. (Allowing for a difference of notation the process is essentially similar to that employed in reference 1.)

An extension in the use of subsequent notation must first be noted. The symbol $f_1,\varphi$ represents, as mentioned previously, the function $f_1$ evaluated on $C_1$ and expressed as a function of $\varphi$. The symbol $f_1,\varphi$ shall now be employed to denote the value of $f_1$ as given in the $k$th step in the process of successive approximations, expressed as a function of $\varphi$. Thus the symbol $g_{1.4},g_{2.3}$ denotes the value of $g_2$ given by the fourth step in the process, expressed as a function of $\varphi_2$ as given by the third step in the process.

We start with the two functions $\mu_1,\varphi$ and $\mu_2,\varphi$ that define the contours $B_1$ and $B_2$ completely. Employing $\theta$ instead of $\varphi$ in the integrals, equations (32') define the constants $c_{1.1}$ and $c_{2.1}$ and equation (33') determines the constant $B_{0.1}$. The simultaneous equations (23) and (24) then determine completely the functions $g_{1.1,\varphi}$ and $g_{1.1,\varphi}$ as follows:

$$\begin{align*}
g_{1.1,\varphi} &= \theta - g_{1.1,\varphi} \\
g_{2.1,\varphi} &= \theta - g_{2.1,\varphi}
\end{align*}$$

(38a)

The values $\mu_1,\varphi_{1.1}$ and $\mu_2,\varphi_{1.1}$ may now be defined and these functions may be considered as known. Employing $\varphi_1,\varphi$ as variables instead of $\varphi$, equations (32') and (33') determine $c_{1.2}$, $c_{2.2}$, and $B_{0.2}$. Equations (23) and (24) determine then the functions $g_{1,\varphi_{1.1}}$ and $g_{2,\varphi_{2.1}}$ which may be expressed as functions of $\varphi$ in view of equation (38a). The variables $\varphi_1,\varphi$ and $\varphi_2,\varphi$ are now given by

$$\begin{align*}
\varphi_1,\varphi &= \theta - g_{1.1,\varphi} \\
\varphi_2,\varphi &= \theta - g_{2.1,\varphi}
\end{align*}$$

(38b)

The functions $\mu_1,\varphi_{1.2}$ and $\mu_2,\varphi_{2.2}$ are now determined and the process may be continued as outlined for $\mu_1,\varphi_{1.1}$ and $\mu_2,\varphi_{1.1}$.

It is noteworthy that this process converges, in practice, with extreme rapidity, that is to say, the functions $\mu_1,\varphi_{1.2}$ and $\mu_2,\varphi_{2.2}$ approach identity with $\mu_1,\varphi_{1.1}$ and $\mu_2,\varphi_{2.1}$ for small values of $k$. Experience has shown (cf. also reference 1) that in ordinary cases one, or at most two, steps in the process are sufficient for great accuracy. It must be noted, however, that a completely rigorous discussion of the convergence process is lacking.

In order to illustrate the method, consider the biplane arrangements of figure 7 defined by the functions in equations (37a) and (37b). Forgetting for the moment that the various functions are known, assume only $\mu_1,\varphi$ and $\mu_2,\varphi$ to be given and attempt to obtain $\mu_1,\varphi$ and $\mu_2,\varphi$. Figure 8 shows the various functions described. It is seen that the set of curves $g_{1,\varphi_{1.1}}$, $g_{2,\varphi_{2.1}}$, $\mu_1,\varphi_{1.1}$, and $\mu_2,\varphi_{1.1}$ (obtained by a 20-point numerical process similar to that sketched in

---

* Denoting the initial approximation by a zero subscript, observe that the initial approximation employed here is $g_{0.1}=g_1$ and $g_{0.2}=g_2$, where $g_{0.1}=g_{0.2}=0$. More generally, the initial transformation may be better defined where $g_1$ and $g_2$ are arbitrary functions, so chosen that they are better approximations to the final solutions $g_1$ and $g_2$. Then employing $g_{1,\varphi}$ and $g_{2,\varphi}$ as variables instead of $\theta$ the functions $\mu_1,\varphi_{1.2}$ and $\mu_2,\varphi_{2.2}$ may be defined from which $g_{1,\varphi}$ and $g_{2,\varphi}$ are determined and the process continued as outlined. (Cf. reference 1, p. 12.)
the appendix of reference 1) are completely coincident with the known solutions \( g_{1,\nu}, g_{2,\mu}, \mu_{1,\nu}, \) and \( \mu_{2,\mu}. \) A further application of the process can cause no further noticeable change.

For future reference, the derivative expression \( \frac{dw}{dz} \) evaluated at the boundary circles will be required. From equation (30)

\[
\frac{dw}{dz} = \frac{w}{z} \left( 1 + z \frac{dh(z)}{dz} \right)
\]

On \( C_1 \)

\( h(z) = f_{1,\nu} + ig_{1,\mu} \)

Then

\[
\frac{dw}{dz} \bigg|_{C_1} = \frac{w}{z} \left( 1 + \frac{d\overline{g}_{1,\mu}}{dz} - i \frac{d\overline{f}_{1,\nu}}{dz} \right)
\]

Observe that \( d\phi = d(\theta - g) \) this may also be expressed as

\[
\frac{dw}{dz} \bigg|_{C_1} = \frac{w}{z} \left( 1 - \frac{d\overline{f}_{1,\nu}}{dz} \right) - i \frac{d\overline{g}_{1,\mu}}{dz} \tag{39'}
\]

For the boundary \( C_2, \) the subscript 1 is replaced by 2.

4. GENERAL MONOPLANE WING SECTION THEORY—A DEGENERATE CASE OF THE BIPLANE ANALYSIS

It may be of some interest to discuss in this section briefly the case of the single airfoil section considered as a special case of the biplane analysis. A biplane arrangement in a two-dimensional field of flow corresponds mathematically to a doubly connected region. The degenerate case in which one of the biplane contours reduces to a (regular) point leads to the monoplane-wing profile the external region of which is simply connected.\(^{10}\) It should therefore be possible to obtain the complete theory of monoplane wing sections in potential flow as limiting values of the formulas already developed for biplane wing sections, as will be outlined in the present section. Conversely, the monoplane case is of further significance in that it permits a more complete understanding of the biplane analysis and, too, considerably simplifies the practical evaluations. The numerical process employed in the monoplane case, it will appear, needs to be modified in a relatively minor way to yield the results for ordinary biplane combinations. Indeed, according to a method outlined by Koebe (reference 2), it is known that even the more general problem of transforming a multiply connected region (multiplane problem) into a region bounded by circles may be resolved by a process of successive approximations employing only the separate cases for the simply connected regions. The details of this problem remain for a future investigation; it is mentioned here merely as a further possible application of the degenerate case.

It may first be observed that as one of the circles in the \( t \) plane reduces to a point the value of \( \alpha \) or \( \beta \) (according as the upper or lower circle degenerates) becomes infinite. Hence in the degenerate case, the elliptic functions introduced in the analysis become circular functions having a real period \( 2\omega = 2\pi \) and an imaginary period \( 2\omega' \), which is infinite. From equation (2.23) it is clear that \( \omega' = \infty \) corresponds to \( q = 0 \) and that

\[
\lim_{q \to 0} A(s) = \frac{1}{2} \cot \frac{\gamma}{2} + \frac{1}{2} \cot \frac{\delta}{2} \tag{1}
\]

Let \( \beta = \alpha \), then employing this equation and relation (1.8) it is found that equation (2.12) becomes

\[
\lim_{q \to 0} u(s) e^{\alpha i} = U_1 = T_1 + \frac{a^2}{T_1} \tag{2}
\]

where the origin in the \( U_1 \) plane is referred to the midpoint of the line segment and where

\[
T_1 = e^{\alpha i} (t - ic \coth \alpha) = c \cosh \alpha
\]

For the boundary \( C_2, \) the subscript 1 is replaced by 2. The \( T_1 \) plane corresponds simply to the \( t \) plane translated to a new origin \((0, 1)\) and rotated by an angle \( \gamma' = \gamma + \frac{\delta}{2} \). Equation (2) is nothing more than the well-known transformation leading to Joukowsky airfoils. The line segment in this case \((\beta = \alpha)\) is equal to twice the diameter of the circle in the \( T_1 \) plane, i.e., \( c_1 = 4a \).

\(^{10}\) Strictly speaking, the point at infinity represents another boundary and the flow regions are respectively, triply connected and doubly connected. This fact corresponds to the circumstance that in the flow formula for the biplane case there may be two arbitrary circulations specified, and in the monoplane flow formula, one. The flow formula for a simply connected region without singular points cannot be multiply valued, i.e., cannot possess an arbitrary circulation.
If $\alpha$ instead of $\beta$, is allowed to approach infinity there is obtained

$$\lim_{\alpha \to \infty} u(s) = U_2 = T_2 + \frac{b^2}{T_2}$$

where the origin in the $U_2$ plane is referred to the midpoint of the line segment, and where

$$T_2 = e^{i\pi}(t+i\alpha \coth \beta)$$

$$b = e \csch \beta$$

The chord length $c_2 = 4b$.

In what immediately follows we show how to express $\lambda, \mu$ in the degenerate case in terms of the coordinates $x, y$ of the airfoil section. These results will be of value in the later determination of $\lambda, \mu$ from $x, y$ for the biplane case.

It has been seen that in the degenerate case ($\beta = \infty$) the origin of coordinates may be referred to the midpoint of the chord and

$$U_1 = T_1 + \frac{c^2}{T_1} \quad (2')$$

where the rectangular coordinates in the $U_1$ plane are $(x_1, y_1)$ i.e.,

$$U_1 = x_1 + iy_1$$

$$T_1 = e^{i\pi}(t+i\alpha \coth \alpha)$$

$$a = e \csch \alpha$$

and $\gamma' = \gamma + \delta/2$

Let $T_1$ be written in the form

$$T_1 = a e^{i\pi t + \mu t} \quad (3')$$

Then

$$U_1 = x_1 + iy_1 = 2a \cosh (\psi_1 + i\psi_1) \quad (4')$$

where

$$x_1 = 2a \cos \psi_1 \cos \theta_1$$

$$y_1 = 2a \sinh \psi_1 \sin \theta_1$$

And upon inversion (cf. reference 1)

$$2 \sin^2 \theta_1 = p + \sqrt{p^2 + \frac{y_1^2}{a^2}}$$

$$2 \sinh^2 \psi_1 = -p + \sqrt{p^2 + \frac{y_1^2}{a^2}}$$

where

$$p = 1 - \left( \frac{x_1}{2a} \right)^2 - \left( \frac{y_1}{2a} \right)^2$$

By equation (1.8)

$$t = -c \cot \frac{\delta}{2}$$

where $s = \lambda + i\mu$

Hence,

$$\lambda + i\mu = -2 \cot \frac{t}{c} = -2 \cot^{-1} (T_1 e^{i\pi t} + ic \coth \alpha) \frac{1}{c}$$

or also

$$\lambda + i\mu = -2 \cot^{-1} (l + im) \quad (6)$$

where

$$l = \frac{c}{e} e^{i\pi} \cos (\theta_1 - \gamma')$$

$$m = \frac{c}{e} (e^{i\pi} \sin (\theta_1 - \gamma') + \cosh \alpha)$$

It is known that

$$\cot^{-1} (l + im) = -\frac{i}{2} \log \frac{l + im + i}{l + im - i}$$

$$= -\frac{1}{4} \log \frac{(l^2 + m^2 - 1)^2 + 4l^2}{(l^2 + (m - 1)^2)^2} + \frac{1}{2} \tan^{-1} \frac{2l}{(l^2 + (m - 1)^2)}$$

So that finally equation (6) separates into

$$\lambda = -\tan^{-1} \frac{2l}{(l^2 + (m - 1)^2)}$$

$$\mu = \frac{1}{2} \log \frac{(l^2 + m^2 - 1)^2 + 4l^2}{(l^2 + (m - 1)^2)^2} \quad (7)$$

These relations express $\lambda$ and $\mu$ in the monoplane case in terms of $\theta_1$ and $\psi_1$, and hence by (5) also in terms of $x_1$ and $y_1$.

Similarly, in order to obtain $\lambda$ and $\mu$ for the degenerate case in which $\alpha = \infty$, replace $a$ by $b$ ($= e \csch \beta$) and let

$$T_2 = e^{i\pi}(t+i\alpha \coth \beta)$$

$$= be^{i\pi t + \mu t} \quad (3')$$

Then finally $\lambda$ and $\mu$ are given by equation (7) in which

$$l = \frac{b}{c} e^{i\pi} \cos (\theta_1 - \gamma')$$

$$m = \frac{b}{c} (e^{i\pi} \sin (\theta_1 - \gamma') - \cosh \beta) \quad (0')$$

Inversion of equation (2.12).—It is quite evident that the direct inversion of the elliptic transcendental equation (2.12) (p. 52), if at all possible, would be very laborious. However, an indirect method which employs the results of the degenerate cases and which performs this inversion readily, to any degree of approximation, will now be outlined.

Thus, let a definite biplane arrangement be given (fig. 5(a)). The chords or line segments $c_1$ and $c_2$ may first be chosen in the following convenient manner. Let the chord $c_1$ be defined by the line $F_1 F_1'$, where $F_1$ is the midpoint of the distance between the leading edge and the center of curvature of the leading edge, and $F_1'$ is the midpoint of the distance between the trailing edge and the center of curvature of the trailing edge. It is observed at this point that the only theoretical restriction upon the choice of the chords is that the singular points (the end points of the line segments) be within the contours, or, at most, on the boundaries themselves. The above-mentioned choice is merely of convenience, the object in view being the defining of a smooth ($\lambda, \mu$) relationship. (In reference 1, a similar situation is described in detail.) The above-outlined procedure may be also applied to determine the chord $c_2$ of the lower contour of the biplane cellule.11

11 In the event that the chords so determined are almost, but not quite, parallel it is of some advantage numerically to vary from the foregoing procedure sufficiently to cause the chords to become exactly parallel and to maintain approximately the foot $F$ and $F'$. Small variations from the choice of chords outlined are of minor importance and will not affect the smoothness of the ($\lambda, \mu$) relationship. The desirability of maintaining the chords exactly parallel, if they are reasonably parallel to start with, is due to the circumstance that elliptic functions of the second kind are then avoided.
The chords $c_1$ and $c_2$ having been conveniently chosen, it is possible to determine uniquely by means of the charts outlined previously (fig. 4), or indirectly from the theorems themselves, the values of the constants $\alpha$, $\beta$, $\gamma$, and $\delta$. The quantity $c$ may be regarded throughout as a convenient unit reference length.

Equation (7), it will be recalled, for these values of $\alpha$, $\beta$, $\gamma$, and $\delta$, determines the values of $\lambda$ and $\mu$ (in terms of the rectangular coordinates of the profile sections), which in the degenerate cases correspond to profiles geometrically similar to the upper or lower profiles of the given biplane cellule, the only differences being that the chords are $4a=4c$ csch $\alpha$ and $4b=4c$ csch $\beta$, respectively. When, however, the values of $\lambda$ and $\mu$ thus determined are inserted in equation (2.29) (employing the proper periods $2\omega=2\pi$, $2\omega'=2\pi (\alpha+\beta)$) there is obtained a biplane arrangement which has, necessarily, the required chords and position (i.e., the proper skeleton) and around this skeleton has generated contour shapes $A_1$ and $B_1$, which, in ordinary cases, are almost identical with the given original contours $A_0$ and $B_0$. Thus, the use of the values $(\lambda, \mu)$ of the degenerate cases in the biplane analysis is equivalent to a replacement of the original biplane arrangement by a new arrangement defined by the contours $A_1$ and $B_1$. The differences between $A_0$ and $A_1$, and $B_0$ and $B_1$ are remarkably small in practice. Mathematically the foregoing procedure represents, however, only a first, although important, step in a process of successive approximations which we outline as follows.

Consider only $A_0$ and $A_1$. The contour $A_1$ defines, by means of equation (7), a new degenerate $(\lambda, \mu)$ relation. The differences between this new $(\lambda, \mu)$ relation of $A_1$ and the original degenerate $(\lambda, \mu)$ relation of $A_0$ is a proper criterion of the differences between the contours $A_0$ and $A_1$ themselves since, if these $(\lambda, \mu)$ functions coincide, the contours must coincide. Hence, by a shifting process similar to that commonly employed in methods of successive approximations, the first approximation to the desired $(\lambda, \mu)$ relation of $A_0$ (this first approximation has been here considered to be the degenerate $(\lambda, \mu)$ relation itself of $A_0$) may be corrected by these differences to give a second and better approximation. The process may be repeated $k$ times, if necessary, until the degenerate $(\lambda, \mu)$ relation of $A_k$ coincides with the degenerate $(\lambda, \mu)$ relation of $A_0$, hence $A_k$ coincides with $A_0$, and therefore the actual $(\lambda, \mu)$ relation that defines the contour $A_k$ itself in the biplane case (i.e., by equation (2.29)) is the desired $(\lambda, \mu)$ relation of $A_0$.

**Singular points in the monoplane case.**—The monoplane case may also be useful in obtaining the singular points of the biplane transformation to a good first approximation. From equation (2):

$$U_1=T_1+\frac{z^2}{T_1}$$

the singular points are given by

$$dU_1 \frac{dT_1}{T_1} = 1 - \frac{a^2}{T_1^2}$$

or by

$$T_1 = a e^{\mp i t}$$

Since by equation (3)

$$T_1 = a e^{\mp i t}$$

it is evident that the singular points correspond to $$(\psi_1 , \theta_1) = (0, 0)$$ and $$(\psi_1 , \pi)$$ respectively. On replacing $\psi_1$ and $\theta_1$ in equations (6) and (7) by these values it is seen that the singular points $(\lambda=\lambda_0)$ of the upper chord $(\mu=\alpha)$ are given by

$$-\tan \lambda_0 = \sinh \alpha \cos \gamma'$$

Similarly for the chord of the lower profile $(\mu=-\beta)$ we get

$$\tan \lambda_0 = \sinh \beta \cos \gamma$$

It may also be useful to note that to a first approximation (the approximation being better the greater $\alpha+\beta$) the chords of a biplane cellule are given by $c_1=4c$ csch $\alpha$, $c_2=4c$ csch $\beta$. Hence the chord ratio is approximately

$$\frac{c_1}{c_2} = \frac{\sinh \beta}{\sinh \alpha}$$

It has been shown thus far that the degenerate monoplane case may be of value in the determination of $\mu$ and $\lambda$ for a biplane cellule. The limiting forms of the simultaneous integral equations will now be briefly discussed.

**Forms of the integral equations (3.23) and (3.24) for the monoplane case.**—Consider the simultaneous equations (23) and (24) of section 3, which define the distortion of two contours from two circles. Let $\beta'=-\gamma'=0$, i.e., $R_0=0$, the interior circle of figure 5 (e) degenerates to a point. It is seen then that the Laurent series, equation (3.6), becomes a one-way ascending power series and $a_{+}=A_{+}+iB_{-}=0$. We have also since $\tau = \alpha' + \beta' = 0$ that equations (3.21) and (3.22) give

$$Z(\varphi-\varphi') = 0, Z_1(\varphi-\varphi') = \frac{1}{2} \cot \frac{\varphi-\varphi'}{2}$$

Equations (3.23) and (3.24) then reduce to a single equation

$$g_1(\varphi') = B_0 + \frac{1}{2 \pi} \int_0^{2\pi} f_1(\varphi) \cot \frac{\varphi-\varphi'}{2} d\varphi$$

or also

$$g_1(\varphi') = B_0 + \frac{1}{2 \pi} \int_0^{2\pi} f_1(\varphi) \log |\frac{\varphi-\varphi'}{2}| d\varphi$$

Equation (9), considered as an integral equation, enables the transforming of the contour of a simply connected region into a circle. The process of iteration outlined in the preceding section is directly applicable in this simpler case. (Cf. reference 1.) The constants $A_0$, $B_0$, and $\sigma = \alpha'$ may be determined, as before, from equations (3.32) and (3.33).
In the event that \( \alpha' = \infty \), the outer circle of figure 5(a) becomes infinitely large, i.e., \( R_i = \infty \). Then the Laurent series (3.6) becomes a descending power series and \( a_\infty = A_\infty + iB_\infty = 0 \). With \( f_1 = g_1 = 0 \) and \( \tau = \infty \) we find that equations (3.23) and (3.24) reduce to

\[
g_2(\phi') = B_0 + \frac{1}{2\pi} \int_0^{2\pi} f_2(\phi) \cot \frac{\phi - \phi'}{2} d\phi
\]

(10)

or also

\[
g_2(\phi') = B_0 - \frac{1}{\pi} \int_0^{2\pi} f_2(\phi) \log \sin \frac{\phi - \phi'}{2} d\phi
\]

(10')

The functions \( g_1(\phi) \) and \( g_2(\phi) \), determined by equations (9) and (10) by separate treatment of the two monoplane cases, may when known be employed as a convenient initial approximation (cf. footnote 9) in the more general case.

The well-known flow function for a single circular cylinder may be brought into combination with the results of this section to yield the flow about an arbitrary single airfoil as has been already obtained in reference 3. We proceed at once to the more general case to introduce the flow function for two circular cylinders into the biplane analysis.

5. POTENTIAL FLOW ABOUT THE BIPLANE CONTOURS

Potential flow around two circles.—It has thus far been shown how the two contours of a biplane arrangement, in a \( u \) plane, may be transformed into two concentric circles in the \( z \) plane. It is desirable in what follows to transform the concentric circles \( C_1 \) and \( C_2 \) of the \( z \) plane into the coaxial ones \( K_1' \) and \( K_2' \) in a \( t' \) plane (fig. 9). The circles \( K_1' \) and \( K_2' \) will thus also correspond to the contours of the biplane arrangement and are not to be confused with the contours \( K_1 \) and \( K_2 \) of the \( t \) plane (fig. 6 (d)) which, in general, are not circles. The primes will be retained to denote this difference.

The relation between the \( z \) and \( t' \) planes is (cf. equation (1.4))

\[
t' = \frac{z + c}{z - c}
\]

(1)

Also by the relation

\[
t' = -c \cot \frac{s'}{2}
\]

(2)

the region external to the circles \( K_1' \) and \( K_2' \) is mapped into a rectangular region in the \( s' \) plane bounded by the lines \( l_1' \) and \( l_2' \) (cf. equation (1.8) and fig. 9). For later reference the relation between \( s' \) and \( z \) is also noted here:

\[
z = ce^{-iu'}
\]

(3)

and since \( z = ce^{i\sigma} \) and \( s' = \lambda' + i\mu' \) it is clear that

\[
\begin{align*}
\lambda' &= -\varphi \\
\mu' &= \sigma
\end{align*}
\]

(4)

Hence the boundary lines \( l_1' \) and \( l_2' \) are given respectively by

\[
\begin{align*}
\mu' &= \alpha' = \sigma_1 \\
\mu' &= -\beta' = \sigma_2
\end{align*}
\]

where

\[
\alpha' > 0, \beta' > 0
\]

(It will also be recalled from section 3 (cf. line following equation (3.16)) that \( \alpha_1 - \alpha_2 = \alpha' + \beta' = \tau \).)

In a noteworthy paper (reference 4) Lagally has given the complex flow potential for uniform flow past two circles. His formula makes use of the interme-
The potential flow about arbitrary biplane wing sections is given by:

\[ \Omega(s') = \frac{\Gamma'}{4\pi s'} - \frac{\Gamma}{4\pi} \left[ 2i \log \frac{s'(s' - 2i\beta')}{s'(s' + 2i\beta')} + \left(1 - \frac{4\beta'\eta}{\pi} \right) s' \right] \]

or, in a single expression,

\[ u_\infty + iv_\infty = -V_c e^{-i\alpha}. \]

By derivation of equation (5) with regard to \( s' \), we obtain the complex velocity function in the sl' plane as

\[ W(s') = \left( \frac{\Gamma'}{4\pi} - \frac{\Gamma}{4\pi} \left[ 2i \log \frac{s'(s' - 2i\beta')}{s'(s' + 2i\beta')} + \left(1 - \frac{4\beta'\eta}{\pi} \right) s' \right] \right. \]

\[ \left. -2cV_c \cos \alpha \left[ p(s') + p(s' + 2i\beta') + 2\eta/\pi \right]\right] \]

\[ -2icV_c \sin \alpha \left[ p(s') - p(s' + 2i\beta') \right] \]

This expression gives the velocity components \( U_1 - iV_1 \) at any point of the rectangular region in the sl' plane. It is a real quantity on each of the boundaries, \( \mu' = \alpha' \), \( \mu' = -\beta' \) of the rectangle since these boundaries are streamlines and the normal flow \( V_1 \) vanishes.

Let us evaluate \( W \) for each of the two cases:

1. \( s' = \lambda' + i\alpha' \) corresponding to the boundary \( l' \)
2. \( s' = \lambda' - i\beta' \) corresponding to the boundary \( l_2' \).

In case (1) we obtain

\[ W_1 = \frac{\Gamma'}{4\pi} - \frac{\Gamma}{4\pi} \left( 2i \log \left[ \alpha' \right] + \left( \lambda' + i\alpha' \right) - \left( \lambda' - i\alpha' \right) \right) \]

\[ -2cV_c \cos \alpha \left[ p(\lambda' + i\alpha') + p(\lambda' - i\alpha') + 2\eta/\pi \right] \]

\[ -2icV_c \sin \alpha \left[ p(\lambda' + i\alpha') - p(\lambda' - i\alpha') \right] \]

or

\[ W_1 = \frac{\Gamma'}{4\pi} - \frac{\Gamma}{4\pi} R_1(\lambda', \alpha') \]

\[ -2cV_c \cos \alpha R_2(\lambda', \alpha') \]

where \( R_1 \), \( R_2 \), and \( R_3 \) are real quantities introduced for brevity in notation and defined by the following equations (the primes are dropped for convenience): 13

\[ R_1(\lambda, \alpha) = i \left[ p(\lambda + i\alpha) - p(\lambda - i\alpha) \right] + \frac{2\eta \alpha}{\pi} \]

or

\[ R_1(\lambda, \alpha) = i \left[ p(\lambda + i\alpha) - p(\lambda - i\alpha) \right] + \frac{2\eta \alpha}{\pi} \]

\[ \frac{1}{2} \sinh \alpha \cos \alpha = \cos \lambda - \frac{2}{1 - \frac{\eta}{\alpha} + \sinh m\lambda \cos m\lambda} \]

1 The developments given here for \( R_1, R_2, \) and \( R_3 \) are rapidly convergent when \( \eta = \alpha' + \beta' \) is large, say greater than \( \frac{\pi}{2} \). Other expansions are possible (cf. reference 10, p. 420) and, in some cases, more desirable. For example, when \( \eta \) is small, say less than \( \frac{\pi}{2} \), it is possible by a simple transformation to interchange the real and imaginary periods of the elliptic functions (cf. footnote 4) and obtain more rapidly convergent developments.
\[ R_2(\lambda, \alpha) = p(\lambda + i\alpha) + p(\lambda - i\alpha) + \frac{2n}{\pi} \]  

(11)

\[ R_3(\lambda, \alpha) = iq(\lambda + i\alpha) - p(\lambda - i\alpha) \]  

(12)

where  
\[ q = e^{-\nu} \]

In case (2) there is obtained similarly

\[ W_2 = \frac{\Gamma'}{4\pi} + \frac{\Gamma}{4\pi} R_1(\lambda', \beta') - 2c V_e \cos \alpha R_2(\lambda', \beta') \]

\[ + 2c V_e \sin \alpha R_3(\lambda', \beta') \]  

(13)

In order to specify the circulation \( \Gamma \) and the counter-circulation \( \Gamma' \), we make use of the Kutta-Joukowsky condition for infinite velocities at the sharp trailing edges. Equations (9) and (13) must vanish for the particular values of \( \lambda' \) that correspond to the trailing edges of the upper and lower contours of the biplane combination. Let \( \lambda_1' \) and \( \lambda_2' \) be the values of \( \lambda \) corresponding to the trailing edges of the two contours.  

Then we have

\[ W(\lambda_1', \alpha') = \frac{\Gamma}{4\pi} + \frac{\Gamma}{4\pi} R_1 - 2c V_e \cos \alpha R_2 \]

\[ - 2c V_e \sin \alpha R_3 = 0 \]

\[ W(\lambda_2', \beta') = \frac{\Gamma'}{4\pi} + \frac{\Gamma'}{4\pi} R_1 - 2c V_e \cos \alpha R_2 \]

\[ + 2c V_e \sin \alpha R_3 = 0 \]

where the \( R \)'s with double subscripts are constants defined as follows. (See equations (10)–(12).)

\[ R_1 = R_1(\lambda', \alpha'), \quad R_2 = R_2(\lambda', \alpha'), \quad R_3 = R_3(\lambda', \alpha') \]

\[ R_1 = R_1(\lambda', \beta'), \quad R_2 = R_2(\lambda', \beta'), \quad R_3 = R_3(\lambda', \beta') \]

In order to obtain the angle of zero lift \( \beta_e \), in the plane of the circles, we equate \( \Gamma = 0 \) and solve for the particular value of the angle of attack \( \alpha_e \), which is denoted as \( -\beta_e \) (i.e., for \( a_e = -\alpha_e \), \( \Gamma = 0 \)). Then we have at once (15)

\[ \tan \beta_e = \frac{R_{21} - R_{22}}{R_{21} + R_{22}} \]  

(16)

with this definition of \( \beta_e \), the total circulation may be expressed as

\[ \Gamma = -4\pi cK V_e \sin (\alpha_e + \beta_e) \]  

(17)

where the constant \( K \) is

\[ K = \frac{1}{\cos \beta_e} \frac{R_{21} + R_{22}}{R_{21} + R_{22}} \]  

(18)

In the limiting cases \( \beta' = \infty, \alpha' = \infty \), \( cK \) is equal to \( c \text{csch} \alpha' \) or \( c \text{csch} \beta' \), respectively, which are the radii of \( K_1' \) or \( K_2' \), respectively.)

Similarly, the angle \( \gamma_e \) may be defined as the angle of attack for which counter-circulation \( \Gamma' \) vanishes (i.e., for \( \alpha_e = -\gamma_e \), \( \Gamma' = 0 \)). Then from (15)

\[ \tan \gamma_e = \frac{R_{21} R_{22} + R_{21} R_{22}}{R_{21} R_{22} - R_{21} R_{22}} \]  

(19)

and \( \Gamma' \) may be expressed as

\[ \tan \gamma_e = \frac{R_{21} R_{22} + R_{21} R_{22}}{R_{21} R_{22} - R_{21} R_{22}} \]  

(19)

where the constant \( J \) is

\[ J = \frac{1}{\cos \gamma_e} \frac{R_{11} R_{22} - R_{21} R_{12}}{R_{11} + R_{12}} \]  

(20)

Velocity at the boundary contours of the biplane combination.—Let the complex potential function in the plane containing the biplane contours \( (u \text{ plane}) \) be \( \Omega \) then the complex velocity function is

\[ \frac{\partial \Omega}{\partial u} = v_e - iV_e \]

(21)

where \( v_x \) and \( v_y \) are the velocity components in the direction of the coordinate axes in the \( u \) plane. Introducing the intermediate planes, we have

\[ \frac{\partial \Omega}{\partial u} = \frac{\partial \Omega}{\partial u} \frac{ds}{dz} \frac{dz}{dw} \frac{dw}{du} \]  

(22)

It is first of importance to consider the changes that a velocity at infinity in the \( u \) plane undergoes when transformed to the \( t' \) (or \( s' \) ) planes. (It will be recalled that \( u = \infty \) corresponds to \( t = \infty \), \( s = 0 \), \( w = e, \ z = c, s' = 0, t' = \infty \).) Let \( -V_e e^{i\alpha_e} \) denote the velocity at infinity in the \( u \) plane, and, as before, \( -V_e e^{i\beta_e} \) denotes the velocity at infinity in the \( t' \) plane. (See figs. 9 and 10; the velocity magnitudes are \( V_e \) and \( V_e \) and the angles of attack are \( \alpha_e \) and \( \alpha_e \), respectively.)
Then noting equations (2.10) and (3.29) it is found from (21) that

\[ V_c = rV \]
\[ \alpha_c = \alpha_0 + \gamma + \xi \]

(22)

where \( r \) and \( \xi \) are determined by equation (3.31') \((r \text{ is generally near unity, } \xi \text{ near zero})\), and \( \gamma \) is the angle of stagger of the biplane chords.

The angle of zero lift for the biplane combination is given by

\[ \beta_c = \beta_0 + \gamma + \xi \]

i.e., for \( \alpha_c = -\beta_c \) the lift vanishes.

In order to determine the velocities at each boundary contour of the biplane combination, it is sufficient to obtain the magnitudes of the individual terms in equation (21) at each boundary. For the upper profile we have by equation (9)

\[ \frac{dr}{dz} = W_1 \]

From equation (3)

\[ \frac{ds}{dz} = \frac{1}{s} \]

\[ v_1 = \frac{W_1}{2c} \left( 1 + \frac{dg_1}{d\varphi} \right)^2 + \left( \frac{df_1}{d\varphi} \right)^2 \left( P_1 \cos \gamma + P_1' \sin \gamma \right)^2 + \left( Q_1 \cos \gamma + Q_1' \sin \gamma \right)^2 \]

(23)

Similarly for the velocity at each point of the surface of the lower profile

\[ v_2 = \frac{W_2}{2c} \left( 1 + \frac{dg_2}{d\varphi} \right)^2 + \left( \frac{df_2}{d\varphi} \right)^2 \left( P_2 \cos \gamma + P_2' \sin \gamma \right)^2 + \left( Q_2 \cos \gamma + Q_2' \sin \gamma \right)^2 \]

(24)

The local superstream pressure at points of the boundary surfaces, in terms of the dynamic pressure of the uniform stream, is given by

\[ \frac{p_1}{q} = 1 - \left( \frac{v_1}{V} \right)^2 \]
\[ \frac{p_2}{q} = 1 - \left( \frac{v_2}{V} \right)^2 \]

(25)

where

\[ q = \frac{1}{2} \rho V^2 \]

The general formulas can be somewhat simplified in certain special cases. For example, in the case of biplane contours described about parallel segments as chords the parameter \( \delta = 0 \) and, in addition, if the chords are equal, \( \alpha = \beta \).
the special case of biplane combinations composed of the framework of the line segments themselves, the formulas for the forces and moments will reduce somewhat. These and further special applications are reserved for the future.

As an example of the application of the formulas presented in this paper the pressure distribution for the biplane cellule shown in figure 10 (N. A. C. A. 4412 airfoil section) is developed. The curves representing the $g_1$, $g_2$, $\mu_1$, and $\mu_2$ functions are shown in figure 11. The pressure distribution is given in figure 12 for values of the biplane combination lift coefficient: $C_L=0$, 0.5, 1.0, and 1.5. The pressure distribution for the monoplane case (cf. reference 14) is also presented for comparison. The numerical procedure is outlined under table I.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., June 8, 1935.

REFERENCES
Pressure distribution for the upper wing section of the biplane combination in figure 10.

Pressure distribution for the lower wing section of the biplane combination in figure 10.

Pressure distribution for the monoplane airfoil section (N. A. C. A. 4412).
EXPLANATION OF TABLE I

Table I presents in outline the numerical procedure in obtaining the velocities and pressures at the boundaries of the biplane combination shown in figure 10. The gap/chord = 0.825, stagger/chord = 0.530, chord/chord = 1, and the decalage = 0. The parameters \( \alpha, \beta, \gamma, \) and \( \delta \) are: \( \alpha = \beta = 1.4436, \gamma = -30^\circ, \) and \( \delta = 0. \) The rectangular coordinates \( x, y \) of the profiles are given in columns 1 and 2. (\( \epsilon = \text{unity.} \)) The values of \( \lambda, \mu \) which correspond (i.e., satisfy equation (2.29)) are given in columns 3 and 4 and have been calculated by the method outlined in section 4. The constants \( A_0 = 0.0467, B_0 = -0.0106, \tau = 1.025, \xi = -0^\circ, \alpha' = 1.333, \beta' = 1.303, \tau = 2.636 \) and the angular distortion functions \( g_1 \) and \( g_2 \) (column 5) are determined by the method outlined in section 3. Column 6 presents the angle \( \varphi = \theta - \phi \) where \( \theta = -\lambda. \) The functions \( R_1, R_2, \) and \( R_3 \) given in columns 7, 8, and 9 are determined by equations (5.10)–(5.12). Column 10 gives the quantity \( h = \left( 1 + P^* \right)^* \) for each profile obtained graphically. (See fig. 11.) Column 11 gives the quantity \( k = \left[ (P \cos \gamma + Q' \sin \gamma)^2 + (Q \cos \gamma + Q' \sin \gamma)^2 \right]^{-1} \) for each profile. In determining the next column, we require the singular points of the chord as determined by equation (2.22), \( \lambda_1 = 91^\circ 49', \lambda_2 = 40^\circ 58'. \) Hence \( \varphi_1 = -93^\circ 22', \varphi_2 = -35^\circ 14' \) (cf. footnote 14). The constants occurring in equation (5.14) are then \( R_1 = 0.3444, R_2 = 0.2617, R_3 = 0.3659, R_4 = 0.9431, R_5 = -0.4890, R_6 = 0.7117. \) Equation (5.16) gives then \( \beta_1 = 34^\circ 51' \) and equation (5.17) determines \( K = 1.020. \) The circulation is now \( \Gamma = -4\pi V_c (1.020) \sin (\alpha + 34^\circ 51'). \) Equation (5.18) determines \( \gamma_c = 38^\circ 5' \) and (5.19) gives \( J = -0.0987. \) The countercirculation \( \Gamma' = 8\pi V_c (0.0987) \sin (\alpha + 38^\circ 5'). \) The lift coefficient may be expressed as \( C_L = \frac{1}{q(\text{chord})} \left\{ \frac{1}{2 - q V}\right\} \) where \( q = \frac{1}{2} \sqrt{V}. \) The chord for each profile in the example considered is equal to (2.106), hence \( C_L = -2\pi (0.985) \sin (\alpha + 34^\circ 51'). \) (In the monoplane case the lift coefficient for the N.A.C.A. 4412 airfoil in two-dimensional potential flow is obtained as (cf. reference 14) \( 2\pi (1.114) \sin (\alpha + \text{constant}) \) or the slope of the lift curve in the monoplane case is about 13 percent greater than that in the biplane example treated). Putting \( C_L = 0, 0.5, 1.0, \) and 1.5, respectively, the angles \( \alpha_c \) are determined as \( \alpha_c = -34^\circ 51', -30^\circ 8', -25^\circ 24', \) and \( -20^\circ 35', \) respectively (i.e., by equation (5.22) \( \alpha_c = -4^\circ 48', 0^\circ 5', 4^\circ 33', 9^\circ 22', \) respectively). Columns 12–15 of the table give the values of \( \frac{W}{2cV} \) and \( \frac{W}{2cV} \) for the foregoing four values of \( \alpha_c, \) and are determined by equations (5.9) and (5.12). The velocities at each profile surface \( v_1, v_2, \) and \( v_3 \) are given in terms of the stream velocity \( V \) by the formulas (5.23) and (5.24), \( v_1 = \frac{W_1^1 h_1 k_1}{2cV} \) and \( v_2 = \frac{W_2^2 h_2 k_2}{2cV}. \) The pressure ratios \( \frac{P_1}{V} \) and \( \frac{P_2}{V} \) shown in figure 12 are given by equation (5.25).

The Smithsonian Tables of Hyperbolic Functions were found useful in the numerical work.
### TABLE I. BIPLANE ARRANGEMENT N. A. C. A. 4412

#### A. UPPER PROFILE

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<th>y</th>
<th>λ</th>
<th>μ</th>
<th>φ (=λ−μ)</th>
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<th>R0(y)</th>
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#### B. LOWER PROFILE

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<th>R0(y)</th>
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