REPORT No. 191

ELEMENTS OF THE WING SECTION THEORY AND OF THE WING THEORY

By MAX M. MUNK
National Advisory Committee for Aeronautics

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SUMMARY.

The following paper, prepared for the National Advisory Committee for Aeronautics, contains those results of the theory of wings and of wing sections which are of immediate practical value. They are proven and demonstrated by the use of the simple conceptions of "kinetic energy" and "momentum" only, familiar to every engineer; and not by introducing "isogonal transformations" and "vortices," which latter mathematical methods are not essential to the theory and better are used only in papers intended for mathematicians and special experts.

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I. THE COMPLEX POTENTIAL FUNCTION.

1. I have shown in the paper, reference 1, how each air flow, considered as a whole, possesses as characteristic quantities a kinetic energy and a momentum necessary to create it. Many technically important flows can be created by a distribution of pressure and they then have a "velocity potential" which equals this pressure distribution divided by the density of the fluid with the sign reversed. It is further explained in the paper referred to how the superposition of several "potential flows" gives a potential flow again.

The characteristic differential equation for the velocity potential Φ was shown to be

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \quad \text{(Lagrange's equation)} \qquad (1)$$

where x, y, and z are the coordinates referred to axes mutually at right angles to each other. The velocity components in the directions of these axes are

$$u=\frac{\partial\Phi}{\partial x};\ v=\frac{\partial\Phi}{\partial y};\ w=\frac{\partial\Phi}{\partial z}.$$

I assume in this paper the reader to be familiar with paper reference 1, or with the fundamental things contained therein.

2. The configurations of velocity to be superposed for the investigation of the elementary technical problems of flight are of the most simple type. It will appear that it is sufficient to study two-dimensional flows only, in spite of the fact that all actual problems arise in three-dimensional space. It is therefore a happy circumstance that there is a method for the study of two-dimensional aerodynamic potential flows which is much more convenient for the investigation of any potential flow than the method used in reference 1 for three-dimensional flow.

The method is more convenient on account of the greater simplicity of the problem, there being one coordinate and one component of velocity less than with the three-dimensional flow. But the two-dimensional potential is still a function of two variables and it represents a distribution of velocity equivalent to a pair of functions of two variables. By means of introducing the potential a great simplification of the problem has been accomplished, reducing the number of functions to one. This simplification can now be carried on by also reducing the number of variables to one, leaving only one function of one variable to be considered. This very remarkable reduction is accomplished by the use of complex numbers.

The advantage of having to do with one function of one variable only is so great, and moreover this function in practical cases becomes so much simpler than any of the functions which it represents, that it pays to get acquainted with this method even if the student has never occupied himself with complex numbers before. The matter is simple and can be explained in a few words.

The ordinary or real numbers, x, are considered to be the special case of more general expressions $(x+iy)$ in which y happens to be zero. If y is not zero, such an expression is called a complex number. x is its real part, iy is its imaginary part and consists of the product of y, any real or ordinary number, and the quantity i, which is the solution of

$$i^2=-1;\ i.\ e.,\ i=\sqrt{-1}$$

The complex number $(x+iy)$ can be supposed to represent the point of the plane with the coordinates x and y, and that may be in this paragraph the interpretation of a complex number. So far, the system would be a sort of vector symbolism, which indeed it is. The real part x is the component of a vector in the direction of the real x-axis, and the factor y of the imaginary part iy is the component of the vector in the y-direction. The complex numbers differ, however, from vector analysis by the peculiar fact that it is not necessary to learn any new sort of algebra or analysis for this vector system. On the contrary, all rules of calculation valid for ordinary numbers are also valid for complex numbers without any change whatsoever.

The addition of two or more complex numbers is accomplished by adding the real parts and imaginary parts separately.

$$(x+iy)+(x'+iy')=(x+x')+i(y+y')$$

This amounts to the same process as the superposition of two forces or other vectors. The multiplication is accomplished by multiplying each part of the one factor by each part of the other factor and adding the products obtained. The product of two real factors is real of course. The product of one real factor and one imaginary factor is imaginary, as appears plausible. The product of $i\times i$ is taken as -1, and hence the product of two imaginary parts is real again. Hence the product of two complex numbers is in general a complex number again

$$(x+iy)\ (x'+iy')=(xx'-yy')+i(xy'+x'y).$$

There is now one, as I may say, trick, which the student has to know in order to get the advantage of the use of complex numbers. That is the introduction of polar coordinates. The distance of the point (x,y) from the origin $(0,0)$ is called R and the angle between the positive real axis and the radius vector from the origin to the point is called φ, so that $x=R\cos\varphi$; $y=R\sin\varphi$. Multiply now

$$(R_1\cos\varphi_1+i\,R_1\sin\varphi_1)\ (R_2\cos\varphi_2+i\,R_2\sin\varphi_2).$$

The result is

$$R_1R_2\cos\varphi_1\cos\varphi_2-R_1R_2\sin\varphi_1\sin\varphi_2+i(R_1R_2\cos\varphi_1\sin\varphi_2+R_1R_2\sin\varphi_1\cos\varphi_2)$$

or, otherwise written

$$R_1R_2\,[\cos(\varphi_1+\varphi_2)+i\sin\,(\varphi_1+\varphi_2)]$$

That is: The radius R of the product is the product of the radii R_1 and R_2 of the two factors, the angle φ of the product is the sum of the angles φ_1 and φ_2 of the two factors. Further, as is well known, we may write

$$z = R(\cos \varphi + i \sin \varphi) = R\, e^{\varphi i}$$

where e denotes the base of the natural logarithms.

As a particular case

$$(e^{\varphi i})^n = (\cos \varphi + i \sin \varphi)^n = \cos n\varphi + i \sin n\varphi = e^{ni\varphi}$$

This is Moivre's formula.

I proceed now to explain why these complex numbers can be used for the representation of a two-dimensional potential flow. This follows from the fact that a function of the complex numbers, that is in general a complex number again different at each point of the plane, can be treated exactly like the ordinary real function of one real variable, given by the same mathematical expression. In particular it can be differentiated at each point and has then one definite differential quotient, the same as the ordinary function of one variable of the same form. The process of differentiation of a complex function is indeterminate, in so far as the independent variable $(x+iy)$ can be increased by an element $(dx+idy)$ in very different ways, viz, in different directions. The differential quotient is, as ordinarily, the quotient of the increase of the function divided by the increase of the independent variable. One can speak of a differential quotient at each point only if the value results the same in whatever direction of $(dx+idy)$ the differential quotient is obtained. It has to be the same, in particular, when dx or dy is zero.

The function to be differentiated may be

$$F(x+iy) = R(x, y) + iT(x, y)$$

where both R and T are real functions of x and y. The differentiation gives

$$\frac{\partial F}{\partial x} = \frac{\partial R}{\partial x} + i \frac{\partial T}{\partial x}$$

or again

$$\frac{\partial F}{i\partial y} = -i \frac{\partial R}{\partial y} + \frac{\partial T}{\partial y}$$

These two expressions must give identical results and hence are equal. That is, both the real parts and both the imaginary parts are equal:

$$\frac{\partial R}{\partial y} = -\frac{\partial T}{\partial x}; \quad \frac{\partial R}{\partial y} = -\frac{\partial T}{\partial x}.$$

Differentiating these equations with respect to dx and dy

$$\frac{\partial^2 R}{\partial x \partial y} = \frac{\partial^2 T}{\partial y^2} = -\frac{\partial^2 T}{\partial x^2}, \quad i.\ e., \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

or again

$$\frac{\partial^2 T}{\partial x \partial y} = \frac{\partial^2 R}{\partial y^2} = -\frac{\partial^2 R}{\partial x^2}, \quad i.\ e., \quad \frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} = 0$$

Hence, it appears that the real part as well as the imaginary part of any analytical complex function complies with equation (1) for the potential of an aerodynamic flow, and hence can be such a potential. If the real part is this potential, I shall call the complex function the "potential function" of the flow. It is not practical, however, to split the potential function in order to find the potential and to compute the velocity from the potential. The advantage of having only one variable would then be lost. It is not the potential that is used for the computation of the velocity, but instead of it the potential function directly. It is easy to find the velocity directly from the potential function. Differentiate $F(x+iy) = F(z)$. It is seen that

$$\frac{dF(z)}{dz} = \frac{\partial R}{\partial x} + i \frac{\partial T}{\partial x}$$

But it was shown before that

$$\frac{\partial T}{\partial x} = -\frac{\partial R}{\partial y}$$

Hence

$$\frac{dF(z)}{dz} = \frac{\partial R}{\partial x} - i\frac{\partial R}{\partial y}$$

The velocity has the components $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$. Written as a complex vector, it would be $\frac{\partial R}{\partial x} + i\frac{\partial R}{\partial y}$. It appears therefore:

Any analytical function $F(z)$ can be used for the representation of a potential flow. The potential of this flow is the real part of this potential function, and its differential quotient $\frac{dF}{dz}$, called the "velocity function," represents the velocity at each point "turned upside down." That means that the component of the velocity in the direction of the real axis is given directly by the real part of the velocity function $\frac{dF}{dz}$, and the component of the velocity at right angles to the real axis is equal to the reversed imaginary part of $\frac{dF}{dz}$. The absolute magnitude of the velocity is equal to the absolute magnitude of $\frac{dF}{dz}$.

3. I proceed now to the series of two-dimensional flows which are of chief importance for the solution of the aerodynamic problems in practice. They stand in relation to the straight line. The privileged position of the straight line rests on the fact that both the front view and the cross section of a monoplane are approximately described by a straight line. The different types of flow to be discussed in this section have in common that at the two ends of a straight line, but nowhere else, the velocity may become infinite. At infinity it is zero. This suggests the potential function $\sqrt{z^2-1}$ which has discontinuities at the points ± 1 only, but it does not give the velocity zero at infinity.

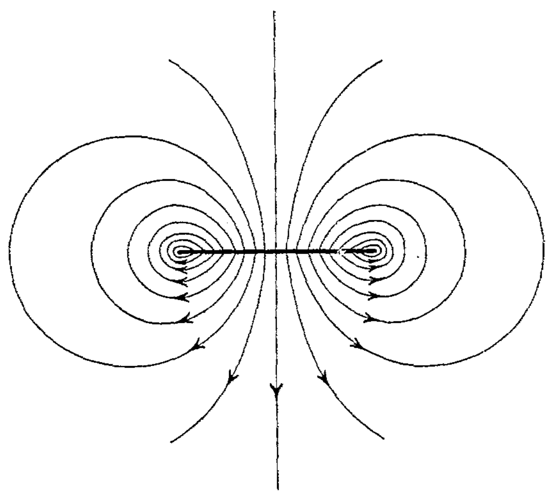

FIG 1.—Transverse flow, produced by a moving straight line.

$$F = \sqrt{z^2-1} - z$$

gives rise to an infinite velocity at the points $z = \pm 1$ which may be regarded as the ends of the straight line, and in addition the velocity becomes zero at infinity. A closer examination shows that indeed the potential function

$$F = i(z - \sqrt{z^2-1}) \qquad (2)$$

represents the flow produced by the straight line extending between the points $z = \pm 1$, moving transversely in the direction of the negative imaginary axis with the velocity 1 in the fluid otherwise at rest. For its velocity function is

$$F' = i - \frac{zi}{\sqrt{z^2-1}}$$

giving for points on the line a transverse velocity -1. This flow may be called "transverse flow." The velocity potential at the points of the line, i. e., for $y=0$ is $\sqrt{1-x^2}$. This gives the kinetic energy of the flow (half the integral of the product of potential, density and normal velocity component, taken around the line (reference 1)).

$$T=\frac{\rho}{2}\cdot 2\int_{-1}^{+1}\sqrt{1-x^2}\,dx=\frac{\rho}{2}\pi \tag{3}$$

giving an apparent mass of the straight line moving transversely equal to the mass of the fluid displaced by a circle over the straight line as diameter. This reminds us of the apparent additional mass of the circle itself, which is the same (ref. 1, sec. 6). It can be proved that the additional mass of any ellipse moving at right angles to a main axis is equal to the mass of the fluid displaced by a circle over this main axis as diameter (reference 6).

The flow around the straight line just discussed can be considered as a special case of a series of more general flows, represented by the potential function

$$F=i(z-\sqrt{z^2-1})^n \tag{4}$$

where n is any positive integer. $n=1$ gives the transverse flow considered before. For n different from 1 the component of the transverse velocity along the straight line is no longer constant, but variable and given by a simple law. Such a flow, therefore, can not be produced by a rigid straight line moving, but by a flexible line, being straight at the beginning and in the process of distorting itself.

It is helpful to introduce as an auxiliary variable the angle δ defined by $z=\cos\delta$. Then the potential function is

$$F=i\ (\cos n\delta-i\sin n\delta)=i\ e^{-in\delta}$$

where δ is, of course, complex. The potential along the line is

$$\Phi=\sin n\delta \tag{5}$$

where now δ is real. The velocity function is

$$F'=\frac{dF}{dz}=\frac{dF}{d\delta}\cdot\frac{d\delta}{dz}=-\frac{n\ e^{-in\delta}}{\sin\delta}=-\frac{n}{\sin\delta}\ (\cos n\delta-i\sin n\delta)$$

giving at points along the line the transverse component

$$u=-\frac{n\sin n\delta}{\sin\delta} \tag{6}$$

and the longitudinal component

$$v=-\frac{n\cos n\delta}{\sin\delta} \tag{7}$$

This becomes infinite at the two ends. The kinetic energy of the flow is

$$T=\frac{1}{2}\,n\rho\ \int_0^{2\pi}\frac{\sin^2 n\delta}{\sin\delta}\sin\delta\ d\delta=n\pi\ \frac{\rho}{2} \tag{8}$$

This impulse is given by the integral

$$\rho\int\Phi\ dx$$

to be taken along both sides of the straight lines, since the velocity potential times ρ represents the impulsive pressure necessary to create the flow. This integral becomes

$$\int_0^{2\pi}\sin n\delta\sin\delta\ d\delta$$

for the nth term. This is zero except for $n=1$.

By the superposition of several or infinitely many of the flows of the series discussed

$$F=i[A_1(z-\sqrt{z^2-1})+A_2(z-\sqrt{z^2-1})^2+\ldots+A_n(z-\sqrt{z^2-1})^n], \tag{9}$$

with arbitrary intensity, infinitely many more complicated flows around the straight line can be described. There is even no potential flow of the described kind around the straight line existing which can not be obtained by such superposition. The kinetic energy of the flow obtained by superposition stands in a very simple relation to the kinetic energy of the single flows which relation by no means is self-evident. It is the sum of them. This follows from the computation of the kinetic energy by integrating the product of the transverse component of velocity and the potential along the line. This kinetic energy is

$$T=\frac{\rho}{2}\int_0^{2\pi} (A_1 \sin \delta+A_2 \sin 2\delta+. .)\ (A_1 \sin \delta+2A_2 \sin 2\ \delta+. .)d\delta \tag{10}$$

But the integral

$$\int_0^{\pi} \sin n\delta \sin m\delta\ d\delta=0\ (n\neq m) \tag{11}$$

is zero if m and n are different integers. For integrating two times partially gives the same integral again, multiplied by $(m/n)^2$. In the same way it can be proved that

$$\int_0^{\pi} \cos n\delta \cos m\delta\ d\delta=0\ (n\neq m) \tag{12}$$

Only the squares in integral (10) contribute to the energy and each of them gives just the kinetic energy of its single term (equation (8)).

It may happen that the distribution of the potential Φ along the line is given, and the flow determined by this distribution is to be expressed as the sum of flows (equation (9)). The condition is, for points on the line, a known function of x is given,

$$\Phi=A_1 \sin \delta+A_2 \sin 2\delta+\ldots+A_n \sin n\delta+\ldots \tag{13}$$

and the coefficients A are to be determined. The right-hand side of equation (13) is called a Fourier's series, and it is proved in the textbooks that the coefficients A can always be determined as to conform to the condition if Φ has reasonable values. At the ends $\delta=0$ or π, hence Φ has to be zero there as then all sines are zero.

Otherwise expressed equation (4) gives enough different types of flow to approximate by means of superposition any reasonable distribution of the potential over a line, with any exactness desired. This being understood, it is easy to show how the coefficients A can be found.

Integrate

$$\int_0^{\pi} (A_1 \sin \delta+A_2 \sin 2\delta\ \ldots) \sin n\delta\ d\delta$$

According to equation (11) all integrals become zero with the exception of

$$A_n \int_0^{\pi} \sin^2\ n\delta\ d\delta=\frac{\pi}{2}.\ A_n$$

Hence

$$A_n=\frac{2}{\pi}\int_0^{\pi} \Phi \sin n\delta\ d\delta \tag{14}$$

These values may be introduced into equation (9), and thus the potential function F is determined.

Another problem of even greater practical importance is to determine the potential functions, equation (4), which superposed give a desired distribution of the transverse component of velocity. The condition is now

$$u=A_1 \frac{\sin \delta}{\sin \delta}+2A_2\ \frac{\sin 2\delta}{\sin \delta}+\ldots \tag{15}$$

That means, now, $u \sin \delta$, a known function, is to be expanded into a Fourier's series

$$u \sin \delta=B_1 \sin \delta+B_2 \sin 2\delta+\ldots+B_n \sin n\delta \tag{16}$$

The *B's* may be determined by an equation like (14), and then the *A's* may be deduced, since

$$A_n = B_n / n \tag{16a}$$

This is always possible if the velocity component is finite along the line. These values may then be introduced in equation (9).

The value of the potential function F as given by series (13) with the values of A_n substituted from (16) may be transformed into a definite integral which sometimes is more convenient for application. Let u_0 be a function of the coordinate z_0, a point on the line joining $z=-1$ and $z=+1$, and let $f(z, z_0)$ be a function to be determined so that

$$F = \int_{-1}^{+1} f(z, z_0) \cdot u_0 \cdot dz_0$$

I have found that this is satisfied by making

$$f = \frac{1}{\pi} [\log (e^{i\delta} - e^{i\delta_0}) - \log (e^{i\delta_0} - e^{-i\delta_0})] \tag{17}$$

and this leads to a physical interpretation of u_0.

Hence, the velocity function

$$F' = \frac{dF}{dz} = \int_{-1}^{+1} \frac{df}{dz} u_0 \, dz_0$$

The potential function and the velocity function are both thought of, then, as being the summation of functions due to "elementary flows." An element gives rise to a potential function $f(z, z_0)\, u_0\, dz_0$ and to the velocity function $\frac{df}{dz} u_0\, dz_0$ (18)

$$f' = \frac{df}{dz} = \frac{1}{\pi} \frac{\sin \delta_0}{\sin \delta} \frac{1}{\cos \delta - \cos \delta_0} = \pm \frac{1}{\pi} \frac{1}{z - z_0} \sqrt{\frac{1 - z_0^2}{1 - z^2}}$$

where the plus sign is to be taken for points on the positive side of the line, and the negative sign for those on the opposite side. In this elementary flow, then, the velocity is parallel to the line at all points of the line excepting the point z_0, being directed away from this point on the positive side of the line and toward it on the other side. For points close to z_0,

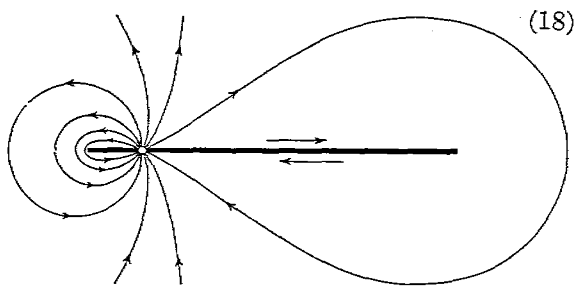

FIG. 2.—Flow around a straight line created by one element of the wing section.

$$f' \, u_0 \, dz_0 = \frac{u_0 \, dz_0}{\pi} \frac{1}{z - z_0}$$

from which the value of the velocity of the flow may be deduced. If a *small* circle is drawn around the point z_0, it is seen that there is a flow out from the point z_0 of amount $u_0\, dz_0$ per second on the positive side and an inflow of an equal amount on the other side; so that this is equivalent to there being a transverse velocity u_0 at points along the element dz_0, positive on one side, negative on the other. The total flow around the line due to $f(z, z_0)\, u_0\, dz_0$ is illustrated in Figure 2.

Substituting the value of f' in F'

$$F' = \pm \int_{-1}^{+1} \frac{1}{\pi} \frac{u_0 \, dz_0}{z - z_0} \sqrt{\frac{1 - z_0^2}{1 - z^2}}$$

Therefore, for any point on the real axis, the transverse velocity is u_0 and the longitudinal velocity

$$v_z = \pm \int_{-1}^{+1} \frac{1}{\pi} \frac{u_0\, dz_0}{z - z_0} \sqrt{\frac{1 - z_0^2}{1 - z^2}}$$

Or, interchanging symbols, writing z for z_0 and *vice versa*

$$v_0 = \mp \int_{-1}^{+1} \frac{1}{\pi} \frac{u dz}{z - z_0} \sqrt{\frac{1 - z^2}{1 - z_0^2}} \tag{19}$$

For a point near the edge on the positive side, write $z_0 = 1 - \epsilon$

$$v_{edge} = -\frac{1}{\pi\sqrt{2\epsilon}} \int_{-1}^{+1} u dz \sqrt{\frac{1+z}{1-z}}$$

or, substituting $\sqrt{2\epsilon} = \sin \delta_{edge}$

$$v_{edge} = -\frac{1}{\pi \sin \delta_{edge}} \int_{-1}^{+1} u dz \sqrt{\frac{1+z}{1-z}} \tag{19a}$$

For the discussion of the elements of the wing theory, in addition to the flows mentioned, there is one flow which needs a discussion of its own. This is given by the potential function

$$F = A_0 \sin^{-1} z \tag{20}$$

The velocity function of this flow is

$$F' = \frac{A_0}{\sqrt{1 - z^2}}$$

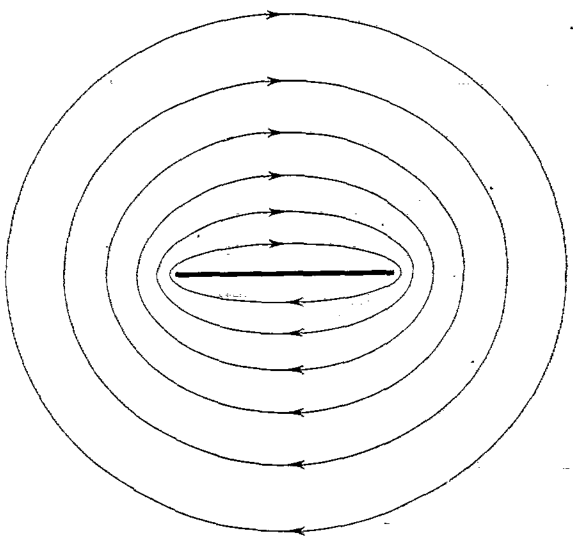

FIG. 3.—Circulation flow around a straight line.

I shall call this flow "circulation flow" as it represents a circulation of the air around the line. The transverse component of the velocity at points along the line is identically zero.

The circulation flow does not quite fit in with the other ones represented by equation (4), because the potential function (20) is a multiple valued one, the values at any one point differing by 2π or multiples thereof. All this indicates that the flow is a potential flow, it is true, but it does not conform to the condition of a potential flow when considered as in equation (4).

This is in accordance with the physical consideration, that it is impossible to produce this flow by an impulsive pressure over the straight line. Such a pressure would not perform any mechanical work, as the transverse components of velocity at points along the line are zero. The kinetic energy of this flow, on the other hand, is infinite, and hence this flow can not even be completely realized. Still it plays the most important part in aerodynamics.

The best way to understand this flow and its physical meaning is to suppose the line to be elongated at one end, out to infinity. On the one side the potential may be considered zero. Then it is constant and will be equal to 2π on the other side. The transverse velocity component is finite. Hence the flow can be produced by a constant impulsive pressure difference along this line extending from the edge $z = 1$ to infinity. This pressure difference makes the fluid circulate around the original straight line, the pressure along the line itself being given by the potential function (20) and not performing any work.

A pressure difference along an infinite line does never actually occur. At least t does not occur simultaneously along the whole line. A very similar thing, however, occurs very often which has the same effect. That is a constant momentum being transferred to the air at right angles to an infinite straight line at one point only, but the point traveling along the line, so that the final effect is the same as if it had occurred simultaneously. This is the fundamental case of an airplane flying along that infinitely long line. During the unit of time it may cover the length V and transfer to the air the momentum L, equal to the lift of the airplane. Then the impulse of the force, per unit of length of the line, is L/V, and hence the potential difference is $\frac{L}{V\rho}$. That makes A_0 in equation (20) $A_0 = \frac{L}{2V\rho\pi}$. If the airplane has traveled long enough, the flow in the neighborhood of the wing, or rather one term of the flow, is described by the circulation flow, provided that the airplane is two-dimensional, that is, has an infinite span.

The velocity at the end of the wing $z = +1$ due to this circulation flow

$$F = A_0 \sin^{-1} z,$$

where

$$A_0 = \frac{L}{2\pi\rho V} \tag{21}$$

is

$$V_{edge} = \left(\frac{A_0}{\sin \delta}\right)_{\delta=0} = \frac{L}{2\pi\rho V\, (\sin \delta)_{\delta=0}} \tag{21a}$$

II. THEORY OF WING SECTION.

4. The investigation of the air flow around wings is of great practical importance in view of the predominance of heavier-than-air craft. It is necessary to divide this problem into two parts, the consideration of the cross section of one or several wings in a two-dimensional flow, and the investigation of the remaining effect. This chapter is devoted to the first question.

All wings in practice have a more or less rounded leading edge, a sharp trailing edge and the section is rather elongated, being as first approximation described by a straight line. The application of the aerodynamic flows around a straight line for the investigation of the flow around a wing section suggests itself. I have shown in section (3) how the potential flow around a straight line is determined, for instance, from the transverse components of velocity along this line. Only one type of flow, the circulation flow, is excepted. This flow does not possess any transverse components at the points of the line and hence can be superposed on a potential flow of any magnitude without interfering with the condition of transverse velocity. I have shown, on the other hand, that it is just this circulation flow not determined so far, which gives rise to the chief quantity, the lift. It is, therefore, necessary to find some additional method for determining the magnitude of the circulation flow.

This magnitude of the circulation flow is physically determined by the fact that the air is viscous, no matter how slightly viscous it is. The additional condition governing the magnitude of the circulation flow can be expressed without any reference to the viscosity and was done so in a very simple way by Kutta. The condition is very plausible, too. Kutta's condition simply states that the air does not flow with infinite velocity at the sharp, rear edge of the wing section. On the contrary, the circulation flow assumes such strength that the air leaves the section exactly at its rear edge flowing there along the section parallel to its mean direction. The wing as it were acts as a device forcing the air to leave the wing flowing in a particular direction.

Consider, for instance, the wing section which consists merely of a straight line of the length 2. The angle of attack may be α. The flow produced by this line moving with the velocity V is then represented by the potential function

$$F = V \sin \alpha \cdot i\, e^{-\delta i}$$

which gives a constant transverse component of velocity along the wing, as shown in equation (6) for $n = 1$. The real axis is parallel to the straight line, its origin is at the center of the line.

The infinite longitudinal velocity at the rear end is

$$-V \sin \alpha \cdot \frac{1}{(\sin \delta)_{\delta=0}}$$

The angle of attack may now be assumed to be small and I change slightly the way of representing the flow, turning the real axis of coordinates into the direction of motion. Instead of referring the flow to the line really representing the wing section I consider the straight line between $z=\pm 1$, which differs only slightly from the wing and is parallel to the motion. The transverse components of the flow relative to this line are approximately equal to the transverse velocity relative to the wing section at the nearest point and therefore constant again and equal to $V \sin \alpha$. Therefore, this way of proceeding leads to the same flow as the more exact way before. It also gives the same infinite velocity at the rear end.

This velocity determines the magnitude of the circulation flow

$$F=A_0 \sin^{-1} z \tag{21}$$

by the condition that the sum of their infinite velocities at the edge is zero.

$$V \sin \alpha \frac{1}{(\sin \delta)_{\delta=0}} - A_0 \frac{1}{(\sin \delta)_{\delta=0}} = 0$$

and hence $A_0 = V \sin \alpha$. The lift is therefore

$$L = 2\pi \ V^2 \rho \sin \alpha.$$

The lift coefficient, defined by $C_L = \frac{L}{S \ V^2 \frac{\rho}{2}}$, since the chord $=2$, is therefore,

$$C_L = 2\pi \sin \alpha, \text{ or approximately } 2\pi\alpha \tag{22}$$

and

$$L = V^2 \frac{\rho}{2} S \ 2\pi\alpha \tag{23}$$

where S denotes the area of the wing.

The representation of the flow just employed is approximately correct and gives the same result as the exact method. This new method now can be generalized so that the lift of any wing section, other than a straight line, can be computed in the same way, too. The section can be replaced with respect to the aerodynamic effect by a mean curve, situated in the middle between the upper and lower curves of the section and having at all points the same mean direction as the portion of the wing section represented by it. The ordinates of this mean wing curve may be ξ, the abscissa x, so that the direction of the curve at each point is $\frac{d\xi}{dx}$. This direction can be considered as the local angle of attack of the wing, identifying the sine and tangent of the angle, with the angle itself. Accordingly it is variable along the section. Since the velocity of the air relative to the wing is approximately equal to the velocity of flight, the component at right angles to the x-axis is $\frac{Vd\xi}{dx}$. As before, the infinite velocity at the rear edge is to be found. It is, according to equation (19a)

$$\frac{-V}{\pi \ (\sin \delta)_{\delta=0}} \int_{-1}^{+1} \frac{d\xi}{dx} \sqrt{\frac{1+x}{1-x}} \, dx \tag{24}$$

At the rear edge $x=1$. The mean apparent angle of attack, that is the angle of attack of the straight line giving the same lift as the wing section, is found by the condition that this infinite value must be the same as that deduced for a straight line; viz, $-\frac{V \sin \alpha}{\sin \delta}$. Hence, replacing $\sin \alpha$ by α

$$\alpha' = -\frac{1}{\pi}\int_{-1}^{+1} \frac{d\xi}{dx}\sqrt{\frac{1+x}{1-x}}\,dx \quad \text{length}=2 \tag{25}$$

$$\alpha' = -\frac{2}{\pi}\int_{-1/2}^{+1/2} \frac{d\xi}{dx}\sqrt{\frac{1+2x}{1-2x}}\,dx \quad \text{length}=1 \tag{25a}$$

This formula holds true for any small angle of attack of the section. The integral can now be transformed into one containing the coordinate ξ rather than the inclination $\frac{d\xi}{dx}$ of the wing curve, provided that the trailing edge is situated at the x-axis, that is, if ξ is zero at the end $x=+1$. This transformation is performed by partial integration, considering $\frac{d\xi}{dx}\,dx$ as a factor to be integrated. It results

$$\alpha' = \frac{1}{\pi}\int_{-1}^{+1} \frac{dx\,\xi}{(1-x)\,\sqrt{1-x^2}} \quad \text{length}=2 \tag{26}$$

$$\alpha' = \frac{4}{\pi}\int_{-1/2}^{+1/2} \frac{dx\,\xi}{(1-2x)\,\sqrt{1-4x^2}} \quad \text{length}=1 \tag{26a}$$

The important formula (26) gives the mean apparent angle of attack directly from the coordinates of the shape of the wing section. The mean height ξ of the section has to be integrated along the chord after having been multiplied by a function of the distance from the leading edge, the same for all wing sections. This integration can always be performed, whether the section be given by an analytical expression, graphically or by a table of the coordinates. In the latter case a numerical integration can be performed by means of Table II, taken from reference 4. The figures in the first column give the distance from the leading edge in per cent of the chord. The second column of figures gives factors for each of these positions. The height ξ of the mean curve of the section over its chord, measured in unit of the chord terms, is to be multiplied by the factors, and all products so obtained are to be added. The sum gives the apparent angle of attack in degrees.

5. The lift of a wing section as computed in section (4) is caused by the circulation flow symmetrical with respect to the straight line representing the wing. Hence the pressure creating this lift is located symmetrically to the wing, its center of pressure is at 50 per cent of the chord, it produces no moment with respect to the middle of the wing. This lift is the entire lift produced by the wing. It is not, however, the entire resultant air force. The remaining aerodynamic flow in general exerts a resultant moment (couple of forces) and this moment removes the center of pressure from its position at 50 per cent.

If the wing section is a straight line of the length 2, its apparent transverse mass is $\pi\rho$, as seen in section (4). The longitudinal mass is zero. Hence, according to reference 1, the resultant moment is

$$M = V^2\,\frac{\rho}{2}\,\pi \sin 2\alpha \quad \text{length}=2 \tag{27}$$

$$M \sim V^2\,\frac{\rho}{2}\,2\,\pi\alpha \quad \text{length}=2 \tag{28}$$

Both the exact and the approximate expressions give the constant center of pressure 25 per cent of the chord from the leading edge, as results by dividing the moment by the lift (23).

The straight sections considered have a constant center of pressure, independent of the angle of attack. The center of pressure does not travel. This is approximately true also for symmetrical sections with equal upper and lower curves, where the center of pressure is also at 25 per cent. If, however, the upper and lower curves are different and hence the mean section curve is no longer a straight line, the potential flow produced at the angle of attack zero of the chord not only gives rise to the circulation flow and thus indirectly to a lift, but also creates a moment of its own. It is simple to compute this moment from the potential flow, which is represented in equation (9) as a superposition of the flows, equation (4).

The longitudinal velocity relative to the line is, according to equation (7),

$$v=-\left(A_1\frac{\cos\delta}{\sin\delta}+A_2\frac{2\cos 2\delta}{\sin\delta}+\cdots+A_n\frac{n\cos n\delta}{\sin\delta}\right)+V$$

As the section is supposed to be only slightly curved, $\frac{d\xi}{dx}$ is always small, so are, therefore, the coefficients A_n when compared to V, so that they may be neglected when added to it. The pressure at each point along the line, according to reference 1, is

$$p=\frac{\rho}{2}v^2$$

The present object is the computation of the resultant moment. When really forming the square of the bracket in the last expression, the term with V^2 indicates a constant pressure and does not give any resultant moment. The squares of the other terms are too small and can be neglected. There remains only the pressure,

$$p=-\rho V\left(A_1\frac{\cos\delta}{\sin\delta}+A_2\frac{2\cos 2\delta}{\sin\delta}\cdots\cdots\right)$$

giving the resultant moment about the origin

$$M=2V\rho\int_0^\pi\left(A_1\frac{\cos\delta}{\sin\delta}+A_2\frac{2\cos 2\delta}{\sin\delta}\cdots\cdots\right)\cos\delta\sin\delta\,d\delta,$$

since the density of lift is twice the density of pressure, the pressure being equal and opposite on both sides of the wing. But according to (12)

$$\int^\pi\cos n\delta\cos m\delta\,d\delta=0 \tag{12}$$

if m and n are different integers. Hence there remains only one term. The resultant moment is

$$M=2\rho VA_1\frac{\pi}{2}$$

A_1 was found according to equation (14) by means of the integral

$$A_1=\frac{2}{\pi}\int_0^\pi V\frac{d\xi}{dx}\sin^2\delta\,d\delta$$

Hence the moment is

$$M=2\rho V^2\int_0^\pi\frac{d\xi}{dx}\sin^2\delta\,d\delta$$

or, expressed by x

$$M=2\rho V^2\int_{-1}^{+1}\frac{d\xi}{dx}\sqrt{1-x^2}\,dx \tag{29}$$

By the same method as used with integral (25) this integral can be transformed into

$$M=2\rho V^2\int_{-1}^{+1}\frac{x\,dx\,\xi}{\sqrt{1-x^2}} \tag{30}$$

It has been shown that for a chord of length 2, the center of pressure has a lever arm $\frac{1}{2}$ and the lift is $V^2 \frac{\rho}{2} \cdot 2\pi\alpha \cdot 2$, giving a moment $V^2\rho\pi\alpha$; so that an angle of attack corresponds to a moment $V^2\rho\pi\alpha$. Consequently the resultant moment is the same as if the angle of attack is increased by the angle

$$\alpha'' = \frac{2}{\pi} \int_{-1}^{+1} \frac{x\, dx\, \xi}{\sqrt{1-x^2}} \text{ length} = 2 \tag{31}$$

$$\alpha'' = \frac{16}{\pi} \int_{-1/2}^{+1/2} \frac{x\, dx\, \xi}{\sqrt{1-4x^2}} \text{ length} = 1 \tag{32}$$

It is readily seen that this angle is zero for sections with section curves equal in front and in rear. Hence such sections have the center of pressure 50 per cent at the angle of attack zero of the mean curve, that is, for the lift (24) produced by the shape of the section only. The additional lift produced at any other angle of attack of the chord and equal to the lift as produced by the straight line at that angle of attack has the center of pressure at 25 per cent. Hence a travel of the center of pressure takes place toward the leading edge when the angle of attack is increased, approaching the point 25 per cent without ever reaching it. The same thing happens for other sections with the usual shape. At the angle of attack zero of the chord the lift produced was seen to be $2\pi V^2\rho\alpha'$, i. e., from (26)

$$L = V^2 \frac{\rho}{2} 4 \int_{-1}^{+1} \frac{dx\, \xi}{(1-x)\sqrt{1-x^2}}$$

and the moment, see equation (30),

$$M = 2\rho V^2 \int_{-1}^{+1} \frac{x\, \xi\, dx}{\sqrt{1-x^2}} \tag{30}$$

giving the center of pressure at the distance from the middle

$$\frac{\displaystyle\int_{-1}^{+1} \frac{x\, \xi\, dx}{\sqrt{1-x^2}}}{\displaystyle\int_{-1}^{+1} \frac{dx\, \xi}{(1-x)\sqrt{1-x^2}}} \text{ length} = 2$$

The lift produced by the angle of attack of the chord, equation (23) as before has the center of pressure 25 per cent. The travel of the center of pressure can easily be obtained from this statement. The moment about the point 25 per cent is independent of the angle of attack.

The center of pressure in ordinary notation at the angle of attack zero is

$$CP = 50\% - 50\ \frac{\displaystyle\int_{-1}^{+1} \frac{x\, \xi\, dx}{\sqrt{1-x^2}}}{\displaystyle\int_{-1}^{+1} \frac{dx\, \xi}{(1-x)\sqrt{1-x^2}}}\ \%$$

The computation of the mean apparent angle of attack with respect to the moment is done in the same way as that of the angle with respect to the lift. Table II, gives the coefficients for numerical integration, by means of two ordinates only, to be used as the other figures in Table II. The final sum is the mean apparent angle in degrees.

6. The problem of two or even more wing sections, combined to a biplane or multiplane and surrounded by a two-dimensional flow can be treated in the same way as the single wing section. The two sections determine by their slope at each point a distribution of transverse velocity components along parallel lines. The distribution determines a potential flow with a resultant moment. According to Kutta's condition of finite velocity near the two rear edges, the potential flow in its turn determines a circulation flow giving rise to a lift and moment. The physical aspect of the question offers nothing new, it is a purely mathematical problem.

This mathematical problem has not yet been solved in this extension. I have attacked the problem within a more narrow scope (reference 4). The method followed by me amounts to the following considerations:

Equation (13) represents different types of flow around one straight line, consisting in a motion of the air in the vicinity of the straight line only. Now the motion of the flows with high order n is more concentrated in the immediate neighborhood of the straight line than the flows of low order n. The transverse velocity components along the line, determining the flow, change their sign $(n-1)$ times along the line. With large n, positive and negative components follow each other in succession very rapidly so that their effect is neutralized even at a moderate distance.

Hence the types of flow of high order n around each of a pair of lines will practically be the same as if each line is single. The flows of high order do not interfere with those of a second line in the vicinity even if the distance of this second line is only moderate. It will chiefly be the types of flow of low order, the circulation flow $n=0$, the transverse flow $n=1$, or it may be the next type $n=2$ which differ distinctly whether the wing is single or in the vicinity of a second wing. Accordingly, I computed only the flows of the order $n=0$ and $n=1$, the circulation flow and the transverse flow for the biplane and used the other flows as found for the single section.

The results are particularly interesting for biplanes with equal and parallel wings without stagger. Their lift is always diminished when compared with the sum of the lifts produced by the two wings when single. The interference is not always the same. If the sum of the angle of attack and the mean apparent angle of attack with respect to the moment is zero, or otherwise expressed, at the angle of attack where the center of pressure is at 50 per cent, it is particularly small. The lift produced at the angle of attack zero is diminished only about half as much as the remaining part of the lift produced by an increase of this angle of attack.

This second part of the lift does not have its point of application exactly at 25 per cent of the chord, although its center of pressure is constant, too. This latter is quite generally valid for any two-dimensional flow. At any angle of attack zero arbitrarily chosen, the configuration of wing sections produces a certain lift acting at a certain center. The increase of the angle of attack produces another lift acting at another fixed point. Hence the moment around this second center of pressure does not depend on the angle of attack; and the center of pressure at any angle of attack can easily be computed if the two centers of pressures and the two parts of the lift are known.

The resultant moment of the unstaggered biplane consisting of portions of equal and parallel straight lines is again proportional to the apparent transverse mass, as the longitudinal mass is zero (reference 1). This mass is of use for the considerations of the next chapter, too. Therefore, I wish to make some remarks concerning its magnitude. If the two straight lines are very close together, the flow around them is the same as around a line of finite thickness and is almost the same as around one straight line. Its apparent mass is the same, too, but in addition there is the mass of the air inclosed in the space between the two lines and practically moving with them. Hence the mass is approximately

$$b\left(b\frac{\pi}{4}+h\right)\rho$$

where b is the length of the lines and h their distance apart, if the distance h of the lines is small. For great distance, on the other hand, the flow around each of the lines is undisturbed, the apparent mass is twice that of the flow around each line if single. It is therefore

$$2b^2\frac{\pi}{4}\rho$$

For intermediate cases the apparent mass has to be computed. Particulars on this computation are given in reference 4. Table 1 gives the ratio of the apparent mass of a pair of lines to that of one single line for different values of $\frac{h}{b}$. This ratio, of course, is always between 1 and 2.

These few remarks on the theory of biplane sections seem to be sufficient in this treatise on the elements of wing theory. The student will find full information on the subject in my paper on biplanes, reference 4. The remarks laid down here, I hope, will assist him in understanding the leading principles of the method there employed.

III. AERODYNAMIC INDUCTION.

7. The last chapter does not give correct information on the aerodynamic wing forces, since the flow in vertical longitudinal planes was supposed to be two-dimensional. The vertical layers of air parallel to the motion were supposed to remain plane and parallel and only the distortion of the two other planes at right angles to it was investigated. This is a very incomplete and arbitrary proceeding, for the vertical longitudinal layers do not remain plane, as little as any other layers remain plane. It is therefore necessary to complete the investigation and to assume now another set of layers, parallel to the lift, to remain plane, thus studying the distortion of the vertical longitudinal layers. Accordingly, I will now assume that all vertical layers of air at right angles to the motion remain plane and parallel, so that the air only moves at right angles to the direction of flight. Hence, I have now to consider two-dimensional transverse vertical flows. This consideration, it will appear, gives sufficient information on the motion of the air at large, whereas the preceding investigation gives information on the conditions of flow in the vicinity of the wing. Both, the longitudinal two-dimensional flow studied before and the two-dimensional flow to be studied presently, possess vertical components of velocity. Both flows and in particular these vertical components are to be superposed, and thus one can determine the final aerodynamic pressures and resultant forces.

The transverse vertical layer of air is at rest originally. The wings, first approaching it, then passing through it and at last leaving it behind them, gradually build up a two-dimensional flow in each layer. The distribution of impulse creating this flow is identical with the distribution of the lift over the longitudinal projection of the wings. It is immaterial for the final effect whether all portions of the wings at every moment have transferred the same fraction of the momentum to a particular layer or not. The final effect and hence the average effect is the same as if they always have. They actually have if all wings are arranged in one transverse plane—that is, if the airplane is not staggered. It may be assumed at present that at each moment each layer has received the same fraction of the impulse from every portion of the wings and it follows then that the shape of the configuration of the two-dimensional flow is always the same and that it is built up gradually by increasing its magnitude while not changing its shape, beginning with the magnitude zero at a great distance in front of the wing and having obtained its final magnitude at a great distance behind the wings.

The potential of the final two-dimensional flow long after the wings have passed through the layer is easy to find, for the impulsive pressure creating it is known along the longitudinal projection of the wings. It is identical with the distribution of the lift over this projection, acting as long as the airplane stays in the layer. This is the unit of time, if the thickness of the layer is equal to the velocity of flight. Hence the potential difference along the longitudinal projection of the wings is equal to the density of the lift along this projection divided by the product of the density of air and the velocity of flight, since the velocity potential is equal to the impulse of the pressure creating the flow, divided by the density. In general the longitudinal projections of the wings can be considered as lines. The density of lift per unit length of these lines is then equal to the difference of pressure on both sides, and hence the density of the lift is proportional to the difference of the potential on both sides. This statement determines completely the final two-dimensional flow in the transverse vertical layer, and nothing remains unknown if the distribution of the lift over the wings is given. The actual determination of the flow is then a purely mathematical process.

For the present purpose, however, not the final transverse flow but the vertical flow at the moment of the passage of the wings is of interest. It is this flow that is to be superposed on the longitudinal flow in order to determine the actual air forces. It has already been mentioned that this flow can be supposed to differ from the final flow in magnitude only. It remains therefore only to find the ratio of momentum already transferred while the wing passes through the layer, to the momentum finally to be imparted.

The fraction ½ seems to me more plausible than any other fraction. The effect of the wing on the layer is the same at equal distances from the layer, whether in front or back of it and this would involve the factor ½. It is not necessary, however, to have recourse to a mere assumption in this question, however plausible it may be. It can be proved that the assumption of ½ is the only one which does not lead to a contradiction with the general principles of mechanics. I proceed at once to demonstrate this.

If the transverse flow in the plane of the wings is found, only the vertical component downward u', called the induced downwash, is used for the computation. This downwash can be positive or negative, but in general is positive. Such downwash in the neighborhood of a portion of wing changes the motion of the air surrounding the wing portion relative to it. The induced downwash is always small when compared with the velocity of flight. Hence, its superposition on the velocity of flight at right angles to it does not materially change the magnitude of the relative motion between the wing and the air in its vicinity. It changes, however, the direction of this relative velocity, which is no longer parallel to the path of the wing but inclined toward the path by the angle whose tangent is u'/V. This has two important consequences.

The flow produced and hence the air force no longer correspond to the angle of attack between the wing and the path of flight but to the angle given by the motion of the wing relative to the surrounding portion of the air. In most cases the angle of attack is decreased and the effective angle of attack, smaller than the geometric angle of attack between path and wing, determines now the flow and the air forces. Hence, the lift in general is smaller than would be expected from the geometric angle of attack. The angle of attack in the preceding chapter on the wing section is not identical with the geometric angle between the chord and the direction of flight but with the effective angle of attack, smaller in general, as there is an induced downwash motion in the vicinity of the wing. Therefore the geometrical angle of attack is decreased by

$$\alpha_i = \frac{u'}{V} \tag{34}$$

That is not all. The lift is not only decreased but its direction is changed, too. It is no longer at right angles to the path of flight, but to the relative motion between wing and adjacent portion of air. It is generally turned backward through an angle equal to the induced angle of attack. The turning backward of the lift by itself does not materially change the magnitude of the lift, as the angle is always small; the vertical component of the lift remains almost the same, but the effective angle of attack has to be decreased. In addition to this the air force has now a component in the direction of the motion. The wing experiences an "induced" drag, in addition to the drag caused by the viscosity of the air, not discussed in this paper, and the induced drag is often much larger than the viscous drag. The density of the induced drag is $dL \frac{u'}{V}$ where dL is the density of lift, as can be directly seen from Figure 4.

$$dD_i = dL \frac{u'}{V} \tag{35}$$

The existence of a drag could have been anticipated, as there must be a source of energy for the creation of the transverse flow under consideration. The final kinetic energy of this flow in a layer of thickness V is

$$\int \frac{1}{2} dL\, u$$

and this energy is to be delivered by the wing per unit of time, as during this unit of time another layer has been put into motion in the way discussed. On the other hand the energy delivered by the wings is the integral over the drag multiplied by the velocity, that is, $\int dL\, u'$. From which follows immediately $u' = \frac{1}{2} u$, and it is thus confirmed that the transverse flow is only half formed when the wings are passing through the vertical layer.

8. The problem is thus solved in general if the shape of the wings and the distribution of lift over the wings is known. Before passing to special wing arrangements and distributions of lift, in particular to the simple monoplane, there is one general problem to be discussed. The longitudinal projection of the wings being given, as well as the entire lift, the induced drag depends on the distribution of the lift over the projection. The drag is desired to be as small as possible. The question arises, What is the distribution of lift giving the smallest induced drag? The importance of this question is at once obvious.

The entire lift and the entire induced drag of the wings are found again as important characteristics of the final transverse flow, discussed in the last section. The resultant lift is equal to the resultant vertical momentum of this flow for the thickness of the layer equal to the velocity V, and the induced drag is equal to the kinetic energy in the same layer divided by V. The problem is therefore to find such a two-dimensional flow produced by impulsive pressure over the longitudinal projection of the wings as possesses a given magnitude of the vertical momentum, and the kinetic energy of which is a minimum.

It is sufficient for elementary questions to consider only arrangements of wing symmetrical with respect to a vertical longitudinal plane, giving moreover horizontal lines in the longitudinal projections. The results are valid for all conditions (reference 1). It is then easy to find the solution. The momentum of several flows superposed on each other is the sum of their single momenta. The flow is of the desired kind if the superposition of any other flow with the resultant vertical momentum zero increases the kinetic energy of the flow.

The velocity of the superposed flow can be assumed to be small, for instance, so that its own kinetic energy, containing the square of the velocity, can be neglected. The impulsive pressure along the projection of the wings necessary to create the superposed flow acts along a path determined by the magnitude of the downwash at the same points. The increase of kinetic energy is

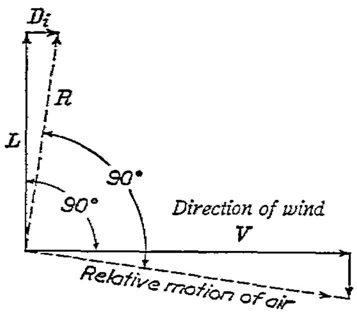

FIG. 4.—Diagram showing the creation of the induced drag.

$$\frac{1}{2} \rho \int u' \Phi \, dx$$

where $\int \Phi \, dx = 0$.

It is readily seen that the first expression can be identically zero for any distribution of the potential Φ restricted by the second condition only if the downwash u' is constant over the entire projection of the wings. Only then a transfer of a portion of lift from one point to another with smaller downwash is impossible, whereas this proceeding in all other cases would lead to a diminution of the induced drag. It is thus demonstrated:

The induced drag is a minimum, if the transverse two-dimensional flow has a constant vertical component of velocity along the entire projection of the wings.

For wings without stagger it follows then that the induced angle of attack u'/V is constant over all wings.

The magnitude of the minimum induced drag of a system of wings is easily found from the apparent mass ρK of their longitudinal projection in the two-dimensional transverse flow. For the vertical momentum equal to the lift is $u V \rho K = L$ where u is the constant downwash of the final flow. This gives

$$u = \frac{L}{V \rho K}$$

The induced drag is equal to the kinetic energy divided by V

$$D_i = u'^2 \frac{\rho}{2} K$$

It follows therefore that the minimum induced drag is

$$D_i = \frac{L^2}{4 V^2 \frac{\rho}{2} K} \tag{36}$$

and the constant or at least average induced angle of attack is

$$\alpha_i = \frac{u'}{V} = \frac{L}{4 V^2 \frac{\rho}{2} K} \tag{37}$$

K is a constant area determined by the longitudinal projection of the wings. It is the area of the air in the two-dimensional flow having a mass equal to the apparent mass of the projection of the wings.

The results (36) and (37) show that the minimum induced drag can be obtained from the consideration that the lift is produced by constantly accelerating a certain mass of air downward from the state of rest. The apparent mass accelerated downward is at best equal to the apparent mass of the longitudinal projection of the airplane in a layer of air passed by the airplane in the unit of time.

In practical applications the actual induced drag can be supposed to be equal to the minimum induced drag, and the average induced angle of attack equal to (37). It is, of course, slightly different, but the difference is not great as can be expected since no function changes its value much in the neighborhood of its minimum.

I proceed now to the application of the general theory of induction to the case of the monoplane without dihedral angle, giving in longitudinal projection a straight line of the length b. Consider first the distribution of lift for the minimum induced drag. It is characterized by the transverse potential flow with constant vertical velocity component along this straight line. This flow has repeatedly occurred in the earlier parts of this paper. For the length 2 of the line, it is the transverse flow given by equation (2) or by equation (4) and $n=1$. The potential function for the length b is

$$F = A_1\, i\left[\frac{2z}{b} - \sqrt{\left(\frac{2z}{b}\right)^2 - 1}\right] \tag{38}$$

giving the constant vertical velocity component along the line

$$u = A_1 \frac{2}{b}$$

The density of the lift per unit length of the span is equal to the potential difference of the final flow on both sides of the line, multiplied by $V\rho$.

$$\frac{dL}{dx} = 2\, V\rho A_1 \sqrt{1 - \left(\frac{2z}{b}\right)^2} = 2\, A_1 V\rho \sin\delta = \frac{4\, L \sin\delta}{b\pi}* \tag{39}$$

where $\cos\delta = \frac{2x}{b}$. Plotted against the span, the density of lift per unit length of the span is represented by half an ellipse, the multiple of $\sin\delta$ being plotted against $\cos\delta$. The lift therefore is said to be elliptically distributed.

$$*L = \int_{-b/2}^{+b/2} \frac{dL}{dx} dx = \int_{-b/2}^{+b/2} 2 V\rho A_1 \sqrt{1 - \left(\frac{}{b}\right)}\, dz = V\rho A_1 \pi \frac{b}{2}$$

Hence

$$A_1 = \frac{2L}{V\rho\pi b}$$

The apparent mass of the line with the length b according to equation (3) is equal to

$$\rho K = b^2 \frac{\pi}{4} \rho$$

Hence, with this distribution of lift, the minimum induced drag is, according to equation (36)

$$D_i = \frac{L^2}{b^2 V^2 \frac{\rho}{2} \pi} \tag{40}$$

and the constant induced angle of attack according to equation (37) is

$$\alpha_i = \frac{L}{b^2 V^2 \frac{\rho}{2} \pi} \tag{41}$$

The density of lift (39) per unit length of the span together with the chord c, different in general along the span, and with equation (22) determines the effective angle of attack at each point, including the apparent mean angle of attack of the section. Namely, from equation (39),

$$\alpha_e = \frac{\text{density of lift}}{2\pi \text{ chord } V^2 \frac{\rho}{2}} = \frac{\frac{dL}{dx}}{2\pi c V^2 \frac{\rho}{2}} = \frac{2L \sin \delta}{b \pi^2 c V^2 \frac{\rho}{2}} \tag{42}$$

The geometric angle of attack is greater by the constant induced angle of attack, and hence

$$\alpha_g = \frac{2L \sin \delta}{b \pi^2 c V^2 \frac{\rho}{2}} + \frac{L}{b^2 \pi V^2 \frac{\rho}{2}} = \alpha_e \left(1 + \frac{\pi c}{2b \sin \delta}\right) \tag{43}$$

Equation (40) indicates the importance of a sufficiently large span in order to obtain a small induced drag.

Any distribution of lift $\frac{dL}{dx}$ over the span other than the elliptical distribution is less simple to investigate, as then the induced downwash is variable. The distribution of lift gives directly the distribution of the potential difference along the two-dimensional wing projection.

$$\Delta\Phi = \frac{\frac{dL}{dx}}{V\rho} \tag{44}$$

The transverse two-dimensional flow can now be obtained by superposition of types of flow given by equation (4) with $z = \frac{2x}{b}$, as now the length of the line is not 2 but b. The condition that the superposition of such flows gives the required potential difference, viz,

$$\frac{1}{2} \Delta\Phi = \frac{\frac{dL}{dx}}{2 V\rho} = A_1 \sin \delta + A_2 \sin 2\delta + \cdots + A_n \sin n\delta + \tag{45}$$

Hence, the distribution of the density of lift, divided by $2 \; V\rho$ is to be expanded into a Fourier's series. The induced angle of attack results then, according to equation (15),

$$\alpha_i = \frac{1}{bV \sin \delta} (A_1 \sin \delta + 2A_2 \sin 2\delta + \cdots + n\, A_n \sin n\delta +, \cdots) \tag{46}$$