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SCALE EFFECT AND OPTIMUM RELATIONS FOR SEA SURFACE PLANING

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From the general dimensional and mechanical similarity theory it follows that a condition of steady motion of a given shape bottom with constant speed on the surface of water is determined by four nondimensional parameters. By considering the various systems of independent parameters which are applied in theory and practice and in special tests, there is determined their mutual relations and their suitability as planing characteristics. In studying the scale effect on the basis of the Prandtl formula for the friction coefficient for a turbulent condition the order of magnitude is given of the error in applying the model data to full scale in the case of a single-step bottom. For a bottom of complicated shape it is shown how from the test data of the hydrodynamic characteristics for one speed with various loads, or one load with various speeds, there may be obtained by simple computation with good approximation the hydrodynamic characteristics for a different speed or for a different load. (These considerations may be of use in solving certain problems on the stability of planing.) This permits extrapolating the curve of resistance against speed for large speeds inaccessible in the tank tests or for other loads which were not tested. The data obtained by computation are in good agreement with the test results. Problems regarding the optimum trim angle or the optimum width in the case of planing of a flat plate are considered from the point of view of the minimum resistance for a given load on the water and planing speed. Formulas and graphs are given for the optimum value of the planing coefficient and the corresponding values of the trim angle and width of the flat plate.

1. GENERAL REMARKS ON THE HYDRODYNAMIC FORCES IN PLANING

First, consider the various systems of parameters by which the hydrodynamic forces in planing with constant speed are determined for a

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bottom of a certain shape. The shape of the bottom is fixed but the scale, determined by the beam at the step, is introduced as a fundamental variable parameter. Furthermore, planing on a single step is chiefly borne in mind.

The force acting on each element of the bottom may be decomposed into the tangential and normal components. The normal component is determined by the pressure of the liquid at the element, while the tangential component depends on the viscosity of the liquid and is determined by the motion of the liquid in the boundary layer. The vector sum of all the elementary forces normal to the bottom is denoted by $\overline{N}$, and the vector sum of all the forces tangent to the bottom by $\overline{F}$, the force $\overline{N}$ being called the normal force, and the force $\overline{F}$ the friction force. For a bottom with cylindrical curvature (only the cylindrical insert is wetted) the force $\overline{N}$ is perpendicular to the generator of the cylinder while the friction force $\overline{F}$ is parallel to the generator (fig. 1).

Denote by M the sum of the moments of all hydrodynamic forces acting on the elements of the bottom relative to the transverse axis passing through the keel at the step. Evidently, the magnitude M is determined by the law of pressure distribution and the frictional forces on the wetted surface of the bottom. For bottoms streamlined in the longitudinal or transverse directions the friction forces in general affect the value of the moment M. In the case of planing of a flat rectangular plate the moment relative to the rear edge does not depend on the friction force. A bottom of given shape may plane with various geometric orientation relative to the water and with various values of the given mechanical magnitudes.

The disturbed steady motion of the fluid for a forward horizontal motion of a bottom of any fixed shape (possibly with several steps) is determined by the following parameters:

b width of the bottom at the first step

l wetted length along the keel

θ trim angle

v speed of motion

In addition to these parameters the motion of the fluid depends also on the values of the physical constants:

g acceleration of gravity

ν coefficient of kinematic viscosity

ρ density of the fluid

Thus for steady planing, all mechanical characteristics are functions of the above seven magnitudes. In particular,

$$N = f(b,\ l,\ \theta,\ v,\ g,\ \rho,\ \nu)$$

$$F = \varphi(b,\ l,\ \theta,\ v,\ g,\ \rho,\ \nu)$$

$$M = \psi(b,\ l,\ \theta,\ v,\ g,\ \rho,\ \nu)$$

To determine these functional relations is the problem of experimental and theoretical investigations.

5 In order to reduce the number of independent variables determining the motion, it is convenient to consider simultaneously the combination of all dynamically similar motions for each condition of motion. For each of these dynamically similar motions all the nondimensional combinations formed from the mechanical magnitudes have the same value. Every combination of motions is characterized by nondimensional magnitudes. The system of similar motions is now determined not by seven independent variables but only by four nondimensional independent parameters. For, since the value of the nondimensional parameters does not depend on the system of measuring units employed for the dimensional magnitudes, it may be assumed in studying the dependence of nondimensional magnitudes on the defining parameters, that the dimensional parameters are measured in a special system of units in which a certain three of the fundamental parameters always have a value equal to unity. For example, take a system of measuring units in which:

$$b' = 1, \quad g' = 1, \text{ and } \quad \rho' = 1$$

7 In this special system of units the magnitudes l', v', and ν' are expressed in terms of the values of the corresponding magnitudes in any system of units by the formulas:

$$l' = \frac{l}{b}; \qquad v' = \frac{v}{\sqrt{gb}}, \text{ and } \qquad \nu' = \frac{\nu}{\sqrt{g}\,b^{3/2}} = \frac{v}{\sqrt{gb}} : \frac{vb}{\nu}$$

Therefore, all the nondimensional characteristics of the planing condition may be considered as functions of the following four nondimensional combinations:

θ trim angle

$\frac{l}{b} = \lambda$ aspect ratio of the wetted surface on the first step

$\frac{v}{\sqrt{gl}} = C_V = F$ velocity coefficient or Froude number

$\frac{vb}{\nu} = R$ Reynolds number

In particular:

$$\frac{N}{\frac{\rho v^2}{2} b^2} = C_B = C_B(\theta, \lambda, C_V, R)$$

$$\frac{F}{\rho \frac{v^2}{2} bl} = C_f = C_f(\theta, \lambda, C_V, R)$$

From these very important considerations it follows that all the hydrodynamic forces acting on the bottom are simply proportional to the density of the fluid. In dealing with tests on a single fluid at a single temperature three of the seven magnitudes determining the steady planing – namely, g, ρ, and ν – are fixed, and therefore the essential magnitudes will be only four: namely, θ, b, l, and v, which are equivalent to the four nondimensional magnitudes θ, λ, C_V, and R.

From this point of view, the introduction of nondimensional parameters appears not to simplify the problem. However, it is more convenient to employ the nondimensional parameters because they have absolute values expressing certain physical effects. If dimensional parameters are employed, their values will depend on the assumed system of measuring units, and for this reason, their numerical values will not in themselves be significant. Moreover, a nondimensional system of parameters is suitable for model tests.

In practice it is often more convenient to use other independent parameters determined by the motion of the body on the water. Instead of the wetted length l and the trim angle θ the following known magnitudes may be taken: namely,

A the load on the water

M_1 the external moment about the transverse axis passing through the rear edge of the step

For steady motion

$$M + M_1 = 0$$

The magnitude M_1 cannot, in general, be preassigned, since it is determined by the weight and the position of the center of gravity[1] and depends on the position of the line of action of the thrust and the magnitude of the thrust. Moreover, for seaplanes the moment M_1 may essentially be determined by the aerodynamic forces, in particular, the forces acting on tail surfaces, which in turn depend on the angle of inclination of the control surface and on the angle of take-off of the seaplane on the water. For this reason, instead of the moment M_1, the trim angle θ is often chosen. In testing models both M_1 and θ may be given. Under full-scale conditions it is generally simpler to measure and assign, with the aid of the control members, the angle θ.

If a nondimensional system of parameters is used, then instead of the aspect ratio λ there may be introduced the parameter C_B or the parameter C_Δ,

$$C_\Delta = \frac{A}{\rho g b^3}$$

The nondimensional coefficient C_Δ is called the load coefficient.

Evidently, in the general case,

$$C_\Delta = C_\Delta(\lambda, \theta, C_V, R)$$

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[1]The moment of the weight in general, depends on the trim angle, but the change in the lever arm of the force for small trim angles is not large, so that in practice this change may always be neglected.

Thus, in replacing λ by C_B or C_Δ the dependence of the magnitudes considered on θ, C_V, and R changes.

There, now, can be written the equations of the forward motion of the bottom with constant horizontal speed. For simplicity, it is necessary to take into account the case where the bottom has a cylindrical fitting strip. By projecting the forces on the horizontal and vertical directions and setting up the equation of moments with respect to the transverse axis passing through the rear edge of the step there is obtained:

$$X - F\cos\theta - N\sin\theta = 0 \tag{1}$$

$$N\cos\theta - F\sin\theta - A = 0 \tag{2}$$

$$M_1 + M = 0 \tag{3}$$

where X is the horizontal thrust, equal to the resistance. For given A, v, θ, and b equation (1) permits determination of the resistance and hence the required thrust.

Equations (1) and (2) may be simplified. Since the angle θ in practice is always small and the force F by comparison with the force N is small,[1] there may be written

$$X = F + A\theta; \qquad N = A$$

whence is obtained:

$$C_B = \frac{N}{\frac{\rho v^2}{2} b^2} = \frac{A}{\frac{\rho v^2}{2} b^2} = \frac{2C_\Delta}{C_V^2}$$

If the external moment is given, the angle θ for a given load is determined from the moment equation (3).

The following three nondimensional systems of parameters determining the state of steady planing have been indicated:

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[1]The angle θ is of the order $\theta < 0.2$ radian $\approx 12^\circ$, the friction force $F < 0.1$ A.

$$\text{I.}\quad \theta,\ \lambda,\ C_V,\ R$$
$$\text{II.}\quad \theta,\ C_B,\ C_V,\ R$$
$$\text{III.}\quad \theta,\ C_\Delta,\ C_V,\ R$$

In certain cases instead of the trim angle θ it is more convenient to assign the nondimensional moment coefficient, connected by a simple relations with the acting external moment. Sometimes, in place of the angle θ there may be given the position of the center of pressure, which is determined by the given distribution and values of the external forces (not hydrodynamic) acting on the planing body. System I is suitable in theoretical investigations, since the wetted length is a fundamental characteristic of the velocity field of the disturbed motion of the fluid. Systems I and II are used mainly in experimental investigations on problems connected with the physical side of the planing phenomenon. System III is widely used in technical applications. The coefficient C_Δ is a characteristic structural parameter which for a given width (given float) is determined by the load on the water. For the model or full-scale test the effect of the coefficient C_Δ is equivalent to the effect of the load on the water, a fundamental characteristic magnitude.

For boats the bottoms of which differ little from a flat shape, systems II and III are evidently connected with each other. In this case:

$$C_B = \frac{2C_\Delta}{{C_V}^2}$$

In a number of problems, systems I and II are more suitable since in this case for the planing condition the dependence of the velocity field of the disturbed motion on the Froude number is not very appreciable, and therefore the number of independent parameters is reduced to three.

The Reynolds number R may also be expressed as:

$$R = \frac{vb}{\nu} = \frac{v}{\sqrt{gb}}\,\frac{\sqrt{g}b^{3/2}}{\nu} = C_V\,\frac{\sqrt{g}}{\nu}\,b^{3/2}$$

In tests with the same fluid at constant temperature and with the same model (b = constant) the Reynolds number is evidently simply proportional to the Froude number. For this reason, in model tests the system of parameters

$$\theta, \; C_\Delta, \text{ and } C_V$$

completely determines the motion and therefore the resistance, hydrodynamic moments, and so forth.

Besides the systems of parameters I, II, and III, the following systems may also be used

$$\theta, \; C_\Delta, \; C_V, \text{ and } B = \frac{\sqrt{g}}{\nu} b^{3/2}$$

or

$$\theta, \; C_B, \; C_V, \text{ and } B$$

The effect of the parameter B is equivalent to the effect of the absolute dimensions of the model. Evidently, this parameter may have an appreciable effect only in the case where the Reynolds number has an appreciable effect, that is, in those cases where the viscosity of the fluid has a considerable value. For the parameter B there may be written:

$$B = \frac{1}{\nu'},$$

where ν' is the coefficient of kinematic viscosity in the special system of units given previously.

The quality of the planing bottom is characterized by the ratio of the resistance to the load on the water:

$$\epsilon = \frac{X}{A}$$

This ratio is denoted as the planing coefficient. The reciprocal magnitude is denoted as the efficiency:

$$K = \frac{1}{\epsilon} = \frac{A}{X}$$

Evidently,

$$\epsilon = \theta + \frac{C_f \lambda}{C_B}$$

Later, the conditions under which the coefficient ϵ has the minimum value which corresponds to the maximum value of the efficiency will be considered.

2. EXPERIMENTAL AND THEORETICAL DATA ON THE DEPENDENCE OF THE LIFT FORCE AND THE FRICTION FORCE ON THE DETERMINING PARAMETERS

Lift Force

As has been seen from the general considerations of the dimensional theory, it follows that for steady planing at small angles of attack

$$\frac{N}{\rho \frac{V^2}{2} b^2} = \frac{A}{\rho \frac{V^2}{2} b^2} = C_B(\lambda, C_V, R, \theta)$$

Now, consider how the coefficient C_B depends on the above variables. The force N is determined by the law of the pressure distribution on the wetted surface. Since the pressures are determined by the external potential flow which may be determined theoretically without taking account of the viscosity, it is evident that the Reynolds number has no effect on the value of $\overline{N}$, and therefore it may be practically assumed that the coefficient C_B does not depend on the Reynolds number R. Planing bodies generally are approximately flat in shape. In planing on the surface of the water the elements of the wetted surface have a small inclination to the horizontal. From theoretical considerations (reference 1) it follows that in this case it may be assumed that the coefficient C_B depends linearly on the trim angle θ. For a bottom with cylindrical insert C_B is simply proportional to θ.

Moreover, it follows from theoretical considerations that for large values of the Froude number C_V the weight of the fluid does not essentially affect the motion of the fluid near the planing bottom (reference 1). The motion of the water by which the pressure on the bottom is determined may be obtained for large values of the Froude number without taking the weight into account. Therefore, for large values of C_V the value of C_B does not depend on C_V. For small values of C_V in the plowing and transition stage the effect of the weight of the liquid is essential. This effect may to a first approximation be taken into account by the addition of hydrostatic forces, which for large values of the Froude number are negligibly small in comparison with the dynamic forces. In the expression for the coefficient C_B the added term determined by the hydrostatic forces is inversely proportional to the square of the Froude number (references 1 and 2). This term very rapidly decreases with increasing C_V.

On the basis of the analogy between planing and the motion of a wing in the fluid there is obtained for the flat plate the following semiempirical formula for the dependence of the coefficient C_B on the fundamental parameters of the motion (references 2, 3).

$$C_B = \frac{0.7\ \pi\lambda\theta}{1 + 1.4\lambda} + \lambda(0.92\lambda - 0.38)\frac{\theta}{{C_V}^2} \tag{1}$$

Figure 2 shows the agreement between this formula and the test results. The second term of the above formula represents the hydrostatic forces. For small $\lambda = \frac{l}{b} < 1$ and large values of C_V only the first term need be used:

$$C_B = \frac{0.7\ \pi\lambda\theta}{1 + 1.4\lambda} \tag{2}$$

Besides the above formula derived on a theoretical basis, there are a number of purely empirical formulas. In 1935 Perring and Johnston (reference 4) representing the lift force in the form

$$A = kbl\ \rho\ v^2\ \theta$$

proposed for the coefficient k the following empirical formula:

$$k = 0.45\lambda^{-0.42} \tag{3}$$

Since $\frac{C_B}{\theta} = 2\ k\ \lambda$, this is equivalent to the following formula:

$$C_B = 0.9\lambda^{0.58}\theta \tag{4}$$

In 1936, B. Sifman (in an unpublished paper) improved somewhat the formula of Perring and Johnston by proposing the formula

$$k = \frac{0.47}{\sqrt{\lambda}}, \qquad \text{whence} \quad C_B = 0.94\sqrt{\lambda}\ \theta \tag{5}$$

This formula is more suitable in computations, since it contains the square root of λ instead of the complicated power 0.58.

In 1938, Sottorf (reference 5) proposed for the coefficient $C_a = 2\ k$ the empirical formula:

$$C_a = \frac{0.85}{\sqrt{\lambda}}, \qquad \text{whence} \quad C_B = 0.85\sqrt{\lambda}\ \theta \tag{6}$$

Thus, it is seen that essentially the same formula was proposed by a number of authors.

Figure 3 shows a comparison of the test results with the theoretical formula

$$k = \frac{0.5\ \pi}{1 + 2\ \lambda} \tag{7}$$

which is obtained from the wing theory on the basis of the analogy between the planing phenomenon and the motion of a wing in an infinite fluid. There is also shown on figure 3 the curve corresponding to the formula:

$$k = \frac{0.35\,\pi}{1 + 1.4\,\lambda} \tag{8}$$

which is obtained from formula (2). This curve is in satisfactory agreement with the given experimental results for small λ. (This figure was published in reference 6.)

On figure 4 are given the curves for the dependence of the coefficient k on λ by formulas (4), (5), (6), and (8).

For very small λ approaching zero, formulas (3), (5), and (6) give large values for the coefficient k approaching infinity. Formula 7 for $\lambda = 0$ gives a finite value for the coefficient k. From theoretical considerations it is evident that the coefficient k should be finite for $\lambda = 0$, and therefore formula (2) for the coefficient C_B for very λ better corresponds with the true conditions. On the other hand, the theoretical considerations on the basis of which formula (2) was obtained correspond better with the true conditions when λ is very small.

If C_Δ, C_V, θ, and R are chosen as the fundamental independent parameters, the expression for C_B is easily obtained from the equation of equilibrium along the vertical. For small trim angles independent of the form of the profile there is obtained:

$$C_B = \frac{N}{\rho \frac{v^2}{2} b^2} = \frac{A}{\rho \frac{v^2}{2} b^2} = \frac{2C_\Delta}{C_V{}^2} \tag{9}$$

From formula (2), for determining λ the relation is obtained:

$$\frac{0.7\,\pi\lambda}{1 + 1.4\,\lambda}\,\theta = \frac{2C_\Delta}{C_V{}^2} \tag{10}$$

whence

$$\lambda = \frac{C_B}{0.7\ \pi\theta - 1.4C_B} \quad \text{or} \quad \lambda = \frac{C_\Delta}{0.35\ \pi\theta\ C_V{}^2 - 1.4C_\Delta} \tag{11}$$

The above formula may serve to determine the wetted length for given trim angles and the coefficient C_B or the coefficients C_Δ and C_V. It is evident that for the planing condition the denominator in formula (11) is positive.

For a flat plate the center of pressure is located at a distance of $0.75\ l$ from the rear edge. (Experimental and theoretical data on the center of pressure are given in references 2 and 3.) For bottoms with cylindrical insert the position of the center of pressure is approximately the same as for the flat plate. On the other hand, the position of the center of pressure is determined by the external moments and the load on the water. For planing bottoms the center of pressure, and therefore also the wetted length, is determined essentially by the center-of-gravity position. In this case, equation (10) may serve to determine the trim angle θ, whence the result is obtained that the trim angle is proportional to the load and decreases with increase in the velocity.

The Friction Force

To determine the friction force

$$F = C_f\ \frac{\rho\ b l v^2}{2}$$

it is necessary to know the functional relation 15

$$C_f = C_f(\lambda,\ C_V,\ R,\ \theta)$$

At the present time, sufficient knowledge is not available on this function. The value of the coefficient C_f is determined by the velocity field of the disturbed motion of the fluid. It may be assumed that in the planing condition the weight of the fluid does not have any effect on the disturbed flow of the fluid outside and within the boundary layer. 16

From this it may be concluded that the friction coefficient does not depend on the Froude number. Thus for the planing condition there results:

$$C_f = C_f(\lambda, \theta, R)$$

In shipbuilding practice there is taken, as a starting value for the friction coefficient, the value of the coefficient for smooth flat plates placed parallel to the towing velocity for the turbulent condition. In this case Prandtl proposed for the friction coefficient the formula:

$$C_f = \frac{0.074}{\left(\frac{vl}{\nu}\right)^{0.2}} = \frac{0.074}{R^{0.2}\,\lambda^{0.2}} \tag{12}$$

where l is the dimension of the plate in the towing direction. Besides this formula a number of others have been proposed. Here also is given the formula of Prandtl-Schlichting (reference 7)

$$C_f = \frac{0.455}{\left(\lg \frac{vl}{\nu}\right)^{2.58}} = \frac{0.455}{(\lg \lambda R)^{2.58}} \tag{13}$$

and the formula for the friction coefficient in the case of an initial laminar layer:

$$C_f = \frac{0.455}{\left(\lg \frac{vl}{\nu}\right)^{2.58}} - \frac{1\ 700}{\frac{vl}{\nu}} \tag{14}$$

The values of the friction coefficient by formulas (12) and (13) differ little from each other in the range $10^5 < \frac{vl}{\nu} < 10^8$ (fig. 5).

On figure 6 are plotted the test results obtained by various authors and the values of the friction coefficient by formula (13). These test results were taken from the work of Schoenherr (reference 8).

In practice it is possible to determine the frictional resistance of ships as the resistance of a flat plate with an area equal to the wetted area of the body. The friction coefficient may be obtained from one of the above formulas and some corrections made on the roughness of the surface, the presence of rivets and projections, and curvature of the surface. At the present time, the effect of the above factors may be roughly estimated from the data of special tests (reference 3). Here it is noted only that the above factors increase the friction force, the amount of increase being at times of the order of 100 percent.

The value of the friction coefficient in planing depends evidently on all three variables. The problem of determining the friction force in planing is complicated by the fact that the pressure distribution over the wetted surface is very irregular and depends on the angle θ and on the aspect ratio λ. At the forward edge of the planing surface the fluid is thrown forward with a velocity greater than the planing velocity, and therefore the friction force at the forward part is directed forward so that the over-all value of the friction force is decreased.

The investigations conducted by the method of the theory of the boundary layer to determine the dependence of the friction coefficient on the trim angle do not provide a reliable formula for the friction coefficient. It is clear only that for otherwise equal conditions the friction coefficient decreases with increase in the angle of attack as was also observed in tests (reference 9). From the above-mentioned tests on flat plates it follows that the planing coefficient of friction in many cases has values near the friction coefficient of flat plates towed in the direction of motion (reference 5).

In view of the absence of reliable data on the dependence of the planing coefficient of friction on the basic parameters of the motion, in what follows use will be made of the Prandtl formula in the theoretical discussion.

3. RESISTANCE AND SCALE EFFECT

For a planing bottom with cylindrical insert the water resistance may be expressed as:

$$W = X = A(\theta + F)$$

or

$$W = \left(\theta + \frac{F}{A}\right)A = \left(\theta + \frac{C_f\ \lambda}{C_B}\right)A = \epsilon\, A \tag{1}$$

The ratio of the magnitude of the angle θ to $\frac{F}{A} = \frac{C_f\ \lambda}{C_B}$ is equal to the ratio of the form drag to the friction drag. In planing at large angles of attack the wetted length is small (λ small). The form drag is large in comparison with the friction drag. At small angles of attack, on the other hand, the wetted length is large (λ large) and there results large friction drag and small form drag.

On the basis of formulas (10) and (12) (sec. 2), for the flat plate there is obtained for the planing coefficient ϵ :

$$\epsilon = \theta + \frac{0.074(1 + 1.4\ \lambda)\ \lambda^{0.8}}{R^{0.2} \times 0.7\ \pi\ \theta} \tag{2}$$

The wetted length $l\left(\lambda = \frac{l}{b}\right)$ is difficult to measure on models and particularly under full-scale conditions. Moreover, for small λ the wetted length fluctuates somewhat, even for quiet steady motion of the model. The indeterminacy of the boundaries of the wetted length is shown by the water droplets and foam-forming spray at the forward edge. For this reason, instead of the aspect ratio λ it is more convenient in practice to make use of the coefficient

$$C_B = \frac{A}{\frac{\rho v^2}{2} - b^2}$$

For a flat plate there results:

$$\lambda = \frac{C_B}{0.7\ \pi\ \theta - 1.4\ C_B}$$

Thus it is found:

$$\epsilon = \theta + \frac{0.074}{C_B^{0.2} R^{0.2} (0.7 \pi \theta - 1.4 C_B)^{0.8}} \tag{3}$$

From formulas (2) and (3) it is seen that the coefficient ϵ (the efficiency) does not depend[1] on the Froude number $C_V = \frac{v}{\sqrt{gb}}$. This is due to the fact that the weight of the water has no appreciable effect on the disturbed motion of the water near the body. Here, no account was taken of the hydrostatic lift force which in planing is negligibly small in comparison with the lift force due to the dynamic reaction of the water. It is evident that for the planing of bottoms of a more complicated two-step shape the weight of the water likewise has no essential effect on the hydrodynamic force. Thus for planing

$$\epsilon = \epsilon \ (C_B, \theta, R) \tag{4}$$

For a flat plate, formula (3) gives the form of this function. For curved bottoms there are neither theoretical nor empirical formulas similar to formula (3).

Since the coefficient ϵ does not depend on the Froude number but depends on the Reynolds number, it is evident that in model tests following the similarity laws the value of ϵ is the same for model and full scale.

The planing states for model and full scale are similar if C_B, θ, and R are the same for both. It is evident that the similarity conditions C_B = constant, θ = constant, and R = constant are necessary and sufficient conditions. In tests with the same fluid the constancy of $R = \frac{vb}{\nu}$ is equivalent to the constancy of the product vb. Hence, on decreasing the dimensions the velocity should increase. If the dimensions of the model are smaller than the full scale, it is necessary for similarity of the motion that the velocity of the model be less than the full-scale velocity. In model tests following the similarity laws of Reynolds, the lift and resistance forces are the same for similar

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[1]The velocity has an effect on the efficiency through the coefficient C_B and through the Reynolds number.

motions. In passing from model to full scale, the observance of the condition R = constant is required only in determining the resistance of the water. In determining the lift force as a function of the fundamental parameters it is not essential that the condition R = constant be observed, because the lift coefficient does not depend on the Reynolds number (see below). If the condition R = constant is not observed, the resistance coefficient ϵ for otherwise equal conditions has different values for model and full scale, and therefore the model resistance computed to full scale does not agree with the full-scale resistance. This difference in the resistance is known as the scale effect.

The scale effect will be estimated in following the Froude laws on the model. For this purpose use will be made of formula (3). In following the Froude laws there is obtained:

$$\theta_1 = \theta_2; \qquad C_{B_1} = C_{B_2}; \qquad C_V = \frac{v_1}{\sqrt{gb_1}} = \frac{v_2}{\sqrt{gb_2}}; \qquad R_1 \neq R_2$$

Evidently

$$R_2 = \frac{v_2 b_2}{\nu} = R_1 \left(\frac{b_2}{b_1}\right)^{3/2} \quad (\text{if } b_2 > b_1, \text{ then } R_2 > R_1)$$

The full-scale Reynolds number is greater than that of the model if the full scale is greater than the model. The change in the resistance coefficient is given by the formula:

$$\epsilon_1 - \epsilon_2 = \frac{0.074}{C_{B_1}^{0.2} R_1^{0.2} (0.7 \pi \theta - 1.4 C_{B_1})^{0.8}}$$

$$\times \left[1 - \left(\frac{b_1}{b_2}\right)^{0.3}\right] = \frac{F_1}{A_1} \left[1 - \left(\frac{b_1}{b_2}\right)^{0.3}\right] \qquad (5)$$

The above formula shows that $\epsilon_1 - \epsilon_2 > 0$ if $b_1 < b_2$, that is, the resistance coefficient decreases in passing from model to full scale. Since

$$R_1 = \frac{v_1 b_1}{\nu} = C_V \frac{\sqrt{g}}{\nu} \sqrt{b_1^3}$$

formula (5) may be given in the form:

$$\epsilon_1 - \epsilon_2 = \left(\frac{\nu}{\sqrt{g}\, C_{V_1}} \right)^{0.2} \frac{\lambda_1^{0.8}}{C_{B_1}} \left[\frac{1}{b_1^{0.3}} - \frac{1}{b_2^{0.3}} \right] \tag{6}$$

The coefficient before the term in brackets does not depend on the scale in model tests with equal Froude number.

For the same ratio $\frac{b_1}{b_2}$ the difference $\epsilon_1 - \epsilon_2$ is smaller the greater the Reynolds number, that is, the greater the velocity or the greater the width of the model. It must be noted, however, that the above decrease in the scale effect with increasing width of the model occurs rather slowly since R_1 enters to the 0.2 power. For a fixed b_1 the magnitude of the scale effect is determined by the factor

$$\xi_1 = 1 - \left(\frac{b_1}{b_2} \right)^{0.3}$$

The full-scale resistance W'_2, obtained by computation from the resistance of the model by the similarity law of Froude, is given by the formula:

$$W^1{}_2 = \epsilon_1 A_2$$

and the actual resistance by the formula

$$W_2 = \epsilon_2 A_2$$

From formula (5) it is evident that

$$\eta_1 = \frac{W'_2 - W_2}{W'_2} = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1} = \frac{\xi_1}{1 + \frac{A_1 \theta}{F_1}} \tag{7}$$

The magnitude η_1 characterizes the scale effect. From formula (7) it follows that η_1 is always less than ξ_1. The value of η_1 is near ξ_1 when the ratio of the form resistance to the frictional resistance is small, $\frac{A_1 \theta}{F_1} \ll 1$. This will be the case in planing at small angles of attack. For planing at large angles of attack the ratio of the form to the frictional resistance is large, $\frac{A_1 \theta}{F_1} \gg 1$ and therefore $\eta_1 \ll \xi_1$. Further on, it will be shown that the minimum resistance occurs when the frictional resistance is near in value to the form resistance; that is, $\frac{A_1 \theta}{F} \approx 1$. Hence for the optimum planing condition

$$\eta_1 \approx \frac{1}{2} \xi_1$$

Making use of the graph of figure 7, the value of the scale effect is readily evaluated. For example, for $\frac{b_1}{b_2} = \frac{1}{8}$ under the optimum planing condition there is obtained

$$100 \eta_1 \approx 24 \text{ percent}$$

In relation to the full-scale resistance the errors in the resistance obtained by computation according to the similarity law of Froude will be still larger: namely,

$$\frac{W'_2 - W_2}{W_2} 100 \approx 33 \text{ percent}$$

Further, also consider the change in ϵ in computing this coefficient from one speed to another for equal C_B and θ. In this case:

$$b_1 = b_2; \qquad C_{B_1} = C_{B_2}; \qquad v_2 \neq v_1 \text{ and } R_2 \neq R_1 \frac{v_2}{v_1}$$

and there is readily obtained:

$$\epsilon_1 - \epsilon_2 = \frac{F_1}{A_1}\left[1 - \left(\frac{v_1}{v_2}\right)^{0.2}\right] = \frac{F_1}{A_1}\xi_2 \tag{8}$$

Thus, for equal C_B and θ if $v_2 > v_1$, then $\epsilon_2 < \epsilon_1$. With increase in the velocity for otherwise equal conditions the resistance coefficient drops. Evidently, the difference $\epsilon_1 - \epsilon_2$ increases with decrease in the ratio v_1/v_2 more slowly than with decrease in the ratio b_1/b_2 in tests following the Froude law.

The resistance for the speed v_2 obtained by the formula $W'_2 = \epsilon_1 A_2$ will be denoted by W'_2; then

$$\eta_2 = \frac{W'_2 - W_2}{W'_2} = \frac{\xi_2}{1 + \frac{A_1\,\theta}{F_1}} \tag{9}$$

for $\frac{v_1}{v_2} = \frac{1}{2}$ for the optimum planing condition there results:

$$\eta_2 \approx \frac{1}{2}\xi_2 \approx 0.07$$

Thus the error in the resistance for the speed $2\,v_1$ obtained from the value of ϵ_1 for the velocity v_1 is of the order of 10 percent.

The results obtained from formulas (6) and (8) are intimately connected with the Prandtl formula for the friction coefficient. It should be noted, however, that the possibility of making use of this formula for the friction coefficient in planing is not yet clarified.

For a model with a complicated profile shape (b is fixed), the dependence of the coefficient ϵ on C_B and θ may be determined experimentally for a certain fixed speed. These data permit determination of the full-scale and model resistances for a given load A and trim angle θ at all planing speeds if the effect of the Reynolds number on the value of the resistance coefficient is neglected. It is sufficient, in fact, to obtain the complete hydrodynamic characteristics only for a single planing speed. With the aid of these tests the dependence of the coefficient ϵ on C_B may be determined for constant values of the trim angle θ. With the aid of this family of curves ϵ can be readily constructed as a function of the speed for given loads and trim angles. The latter may be assumed or chosen as the optimum. This permits constructing the resistance curve for large speeds for which the resistance of the model cannot be measured because of the limited speed of the towing carriage.

Figure 8 gives the test curves from the flat-plate tests of Sottorf on the dependence of ϵ on C_B at constant angles θ. It is seen from these curves that the effect of the speed (Reynolds number $R = \frac{vb}{\nu}$ since $b = 0.3$ m = constant) on the value of ϵ as a function of C_B is small. It may therefore be assumed that in this case the previous remarks are confirmed by test results.

What has been said previously is true only for the planing condition. For the plowing condition at small values of the Froude angle the lift force is determined essentially by the hydrostatic forces. In this case the dynamic forces also depend on the Froude number.

4. THE OPTIMUM PLANING ANGLE OF A FLAT PLATE

The efficiency of planing of a bottom is characterized by the ratio of the resistance to the lift $\epsilon = \frac{W}{A}$. For a given velocity and load on on the water the smaller this ratio the more favorable the bottom. For a flat plate then results:

$$\epsilon = \theta + \frac{C_f \lambda}{C_B} \tag{1}$$

At the planing condition the coefficient ϵ depends on three parameters. These may be taken as the nondimensional magnitudes θ, λ, and R; or θ, C_B, and R. For practical purposes, however, it is convenient to use the trim angle θ and the dimensional magnitudes A – the load, and bv – the product of the speed by the width of the plate. Evidently, this dimensional system of parameters is uniquely expressed in terms of the system of nondimensional parameters.

Now express the planing coefficient ϵ in terms of θ, A, and $u = \rho \pi b^2 v^2$. Then:

$$C_B = \frac{2\,A}{\rho\, b^2\, v^2} = \frac{f\,\pi\,\lambda\,\theta}{1 + 2\,f\,\lambda}, \text{ whence } \qquad \lambda = \frac{2\,A}{f(u\,\theta - 4\,A)} \tag{2}$$

where f is a test coefficient the theoretical value of which is equal to unity. According to test data for $\lambda > 0.2$ this coefficient may be taken equal to 0.7. This coefficient is left undetermined since very small aspect ratios are being dealt with for which no test data are available, and for small aspect ratios the theoretical value $f = 1$ is very probable.

For the friction coefficient the Prandtl formula is used.[1]

$$C_f = \frac{0.074}{\left(\frac{v\,l}{\nu}\right)^{0.2}} = \frac{0.074(\rho\,\pi)^{0.1}\,\nu^{0.2}}{\lambda^{0.2}\,u^{0.1}} \tag{3}$$

With the aid of formulas (2) and (3) there is obtained for the planing coefficient:

$$\epsilon = \theta + E\,A^{-0.2}\,u^{0.9}\,(u\,\theta - 4\,A)^{-0.8} \tag{4}$$

where

$$E = 0.074\,\rho^{0.1}\,\pi^{-0.9}\,f^{-0.8}\left(\frac{\nu}{2}\right)^{0.2} \tag{5}$$

---

[1]All qualitative results obtained below remain valid if for the friction coefficient the more general formula

$$C_f = \frac{m}{\left(\frac{v\,l}{\nu}\right)^{\alpha}}$$

is taken, where m and α are constant $(0 < \alpha < 1)$.

By setting for water $\rho = 1000/g = 102$ kg sec^2/m^4, $\nu = 11 \times 10^{-7}$ m^2/sec (this corresponds to the value of the kinematic viscosity coefficient of water at a temperature of 15° to 16°) and $f = 1$, there is obtained for the value of $\lg_{10} E$:

$$\lg_{10} E = \overline{3},\ 3707$$

From formula (4) for a given load, speed, and width of plate, the value of the trim angle θ^* is readily determined, for which the coefficient ε has the minimum value. For θ^* the equation,

$$\frac{d\varepsilon}{d\theta} = 1 - 0.8\ E\ u^{1.9}\ A^{-0.2}\ (u\ \theta^* - 4\ \overline{A})^{-1.8} = 0$$

results, whence is obtained:

$$\theta^* = 0.012\ \frac{A}{b^2\ v^2} + 0.042 \left(\frac{b\ v}{A} \right)^{1/9} \tag{6}$$

For this value of the trim angle there is obtained:

$$\epsilon^* = 0.012\ \frac{A}{b^2\ v^2} + 0.095 \left(\frac{b\ v}{A} \right)^{1/9} \tag{7}$$

and

$$\lambda^* = 0.148\ \frac{A^{10/9}}{(bv)^{19/9}} \tag{8}$$

Evidently, the numerical values of the coefficients in these formulas depend on the assumed system of units. In formulas (6), (7), and (8) these values correspond to the kilogram, meter, second system of units.

On figure 9 are given the curves for θ^* and ϵ^* as a function of the load A kilograms for bv = 1.8, 2.4, and 2.85. At a width

$b = 0.3$ meter, this corresponds to the speeds $v = 6$, 8, and 9.5 meters per second. On the figure are plotted the test points obtained by Sottorf. The agreement of the theoretical curves with the test results is entirely satisfactory. At a speed $v = 6$ meters per second there is obtained for the width of the plate $b = 0.3$, 0.4, and 0.475 meters. It is evident from the curves that, in these examples, at large loads the planing coefficient rapidly decreases with increase in the width, and therefore more favorable planing conditions are obtained. As is shown by formula (8), with increase in the load A, the value of λ^* increases rather slowly: with increase in the speed or the width, λ^* decreases extremely rapidly.

The position of the center of pressure along the keel is determined by the value of the wetted length and therefore by the value of $\lambda = \frac{l}{b}$. hull For a float, the position of the center of pressure is determined essentially by the position of the center of gravity. Therefore the favorable center-of-gravity position for a given load and width greatly depends on the speed. For a given load and speed the favorable center-of-gravity position strongly depends on the width.

5. ON THE OPTIMUM WIDTH OF A PLANING FLAT PLATE[1]

Formulas (6), (7), and (8) in the preceding section give the most favorable values of θ^*, ϵ^*, and λ^* for a given value of the load A and product bv. From these formulas it is seen that for a given value of the speed the minimum value of the planing coefficient may be affected by a choice of the width b.

Now will be found the value of the product bv for which the minimum value ϵ^* assumes its least value. By differentiating the expression for ϵ^* given by formula (7) of the preceding section there is obtained

$$\frac{d\epsilon}{d(bv)} = -0.024\,\frac{A}{(bv)^{**3}} + 0.011\,\frac{(bv)^{**-8/9}}{A^{1/9}} = 0$$

whence

[1]The results of this section follow from the assumed empirical formulas which are extended to conditions of motion not investigated experimentally.

$$(bv)^{**} = 1.51\ A^{10/19} \tag{1}$$

Substituting the value for the product bv in formulas (6), (7), and (8) (sec. 4), there is obtained:

$$\theta^{**} = 0.05\ A^{-1/19} \tag{2}$$

$$\epsilon^{**} = 0.105\ A^{-1/19} \tag{3}$$

and

$$\lambda^{**} \doteq \frac{1}{16} \tag{4}$$

From (2) and (3) there is obtained

$$\epsilon^{**} = \frac{19}{9}\,\theta^{**} \tag{5}$$

The obtained formulas give the values for the fundamental characteristics under the most favorable planing condition of a flat plate as a function of the load. From these formulas the following conclusions may be derived. At a given speed v and load A there exists an optimum width. The optimum value of the width is proportional to $A^{10/19}$, that is, increases approximately as the square root of A with increasing load. For a given load the optimum width decreases with increasing
25 speed inversely proportional to the speed. The optimum value of $1/\epsilon$ (the minimum value of the planing coefficient) depends only slightly on the load, and increases with increasing load. (These results are, of
26 course, not valid for very small loads.) For a flat plate for loads A = 10 to 1000 kg, ϵ^{**} = 0.09 to 0.077, and therefore the optimum value varies within the limits 11 to 13. The optimum angle θ^{**} depends only slightly on the load and drops with increasing load.

27 At the optimum planing condition the frictional resistance constitutes somewhat more than 50 percent of the total resistance. The optimum planing condition exists for small wetted lengths, since $\lambda^{**} = \frac{1}{16}$. At the optimum condition the center of pressure is very near the rear edge of the plate.

The plates used in the tests of Sottorf were narrower than the optimum. (For the test data of Sottorf, formula (6) gives an optimum width equal to 0.77 meter for a very small load.) This explains why in the tests of Sottorf the minimum value for ϵ decreased continuously with increasing width. 128 129

In the planing condition when the wetted length is very small, $\lambda << 0.2$, it is probable that the actual value gives the effect of the spray falling on the surface of the water at the front edge, thereby disturbing the steady character of the motion of the water. Moreover, for small values of λ the planing is unsteady. Special tests are required to explain the existence and possibility of using optimum conditions in changing the width of the plate. 130 131

If for the coefficient C_B one of the empirical formulas of the form:

$$C_B = c\lambda^{\beta}$$

is used where c is constant — $\beta = \frac{1}{2}$ in the formula of Sifman and Sottorf, and 0.48 in the formula of Perring and Johnston — the optimum width is not obtained; that is, from these formulas it follows that the planing coefficient always decreases with increasing width. This result was obtained as a consequence of the unlimited increase of the coefficient $k = \frac{C_B}{2\lambda\theta}$ which is refuted theoretically, and test data for very small λ are not available. The influence of the scale effect on the magnitude of the optimum width will now be considered. The optimum model and full-scale widths are denoted by b_1 and b_2, respectively. For model and full-scale speeds and loads for which the comparison is made, the relations are:

$$A_2 = m^3 A_1; \qquad v_2 = \sqrt{m}\, v_1$$

These relations correspond to the similarity law of Froude. From formula (1) there is obtained:

$$b_1 = 1.54 \frac{A_1^{10/19}}{v_1}; \qquad b_2 = 1.54 \frac{A_2^{10/19}}{v_2} = b_1 m^{30/19-1/2} = b_1 m^{41/38}$$

hence,

$$b_2 = m^{1.08}\ b_1 \tag{6}$$

and not $b_2 = mb_1$ as follows from the Froude law. Thus, the optimum width determined on the model and computed to full scale by the Froude law will not be the optimum for full scale. This must be understood in the sense that after computing the full-scale width according to Froude, an efficiency may be obtained equal to the optimum on the model but for the full-scale efficiency a higher value may be obtained by a different
32 choice of width. Since the exponent 1.08 differs little from unity, the above conclusion is only of theoretical value. How the planing speed varies with increasing load for a given power will now be determined. If the power is denoted by E, there is obtained:

$$E = W \cdot v = \epsilon^{**} A\ v$$

whence by formula (3) there results:

$$v = \frac{E}{\epsilon^{**} A} = \frac{10E}{A^{18/19}} \tag{7}$$

Hence the planing speed for the optimum condition is approximately inversely proportional to the load. Formula (7) may be also expressed as:

$$v = 10 \frac{E}{A} A^{1/19} = 10\ p\ A^{1/19} \tag{8}$$

where p denotes the power per unit load on the water. From formula (8) it follows that with increasing load for a given power per unit weight of the float, the speed increases with increasing load. Since in
33 formula (8), A enters to the 1/19th power, this increase is very slow.

Translation by S. Reiss,
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for Aeronautics.

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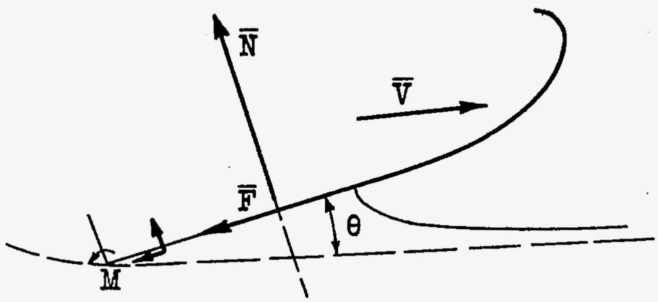


Figure 1.– Schematic representations of hydrodynamic forces acting on a bottom with cylindrical insert in planing on the surface of water.

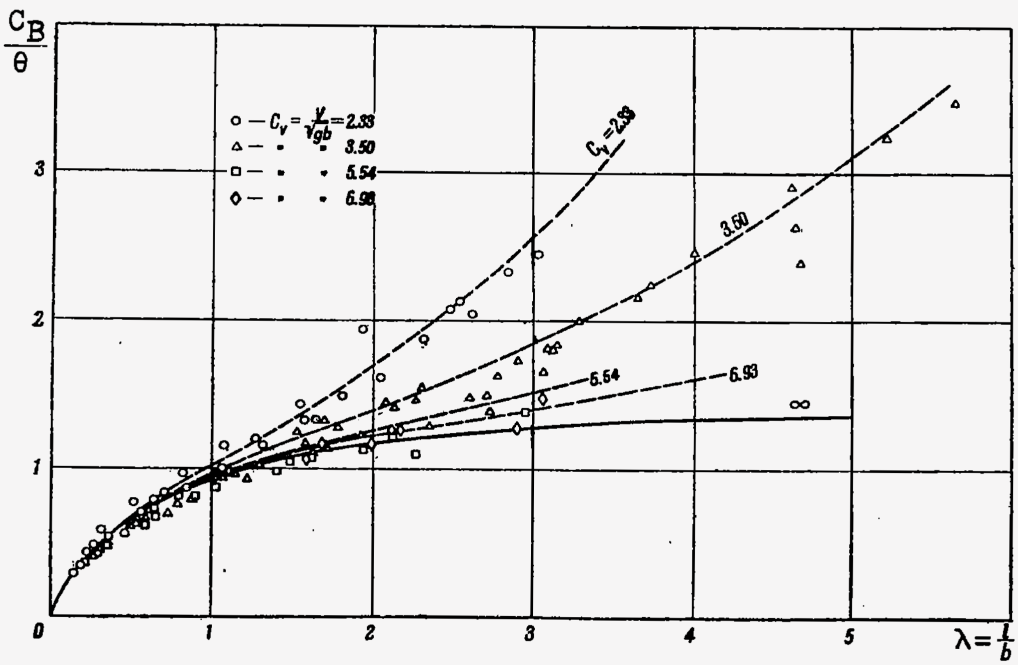


Figure 2.– Comparison of curves for C_B/θ by formula (1) with test results of Sottorf and Sambraus.

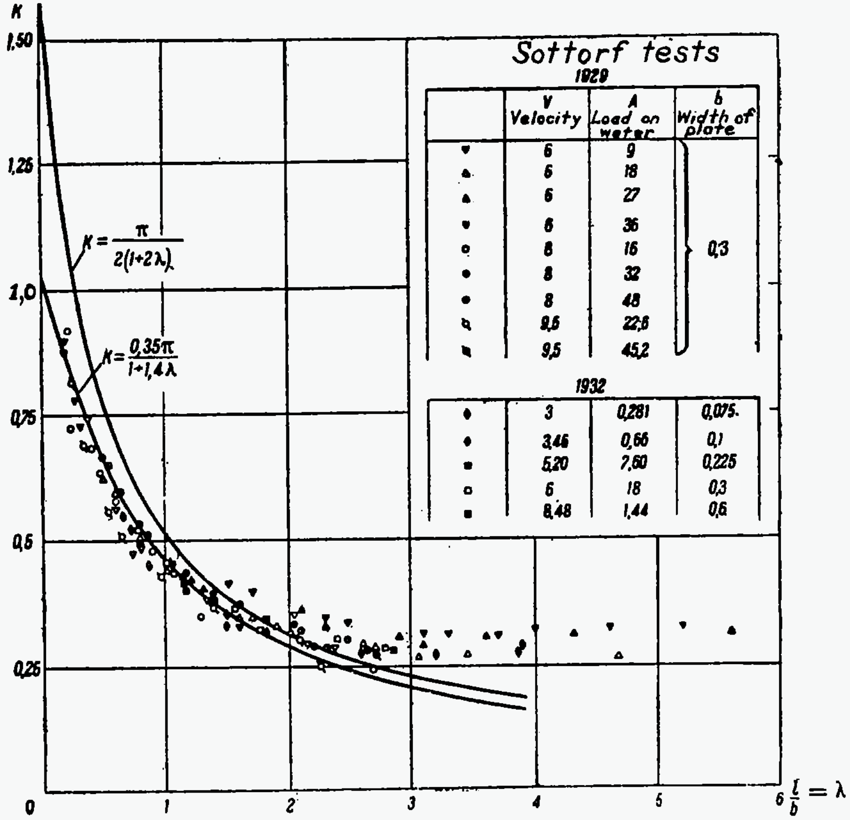


Figure 3.- Comparison of test results with the theoretical curve for $k = 0.5\pi/(1 + 2\lambda)$ and the empirical curve $k = 0.35\pi/(1+1.4\lambda)$ $(A = k\,\rho b l v^2 \theta)$.

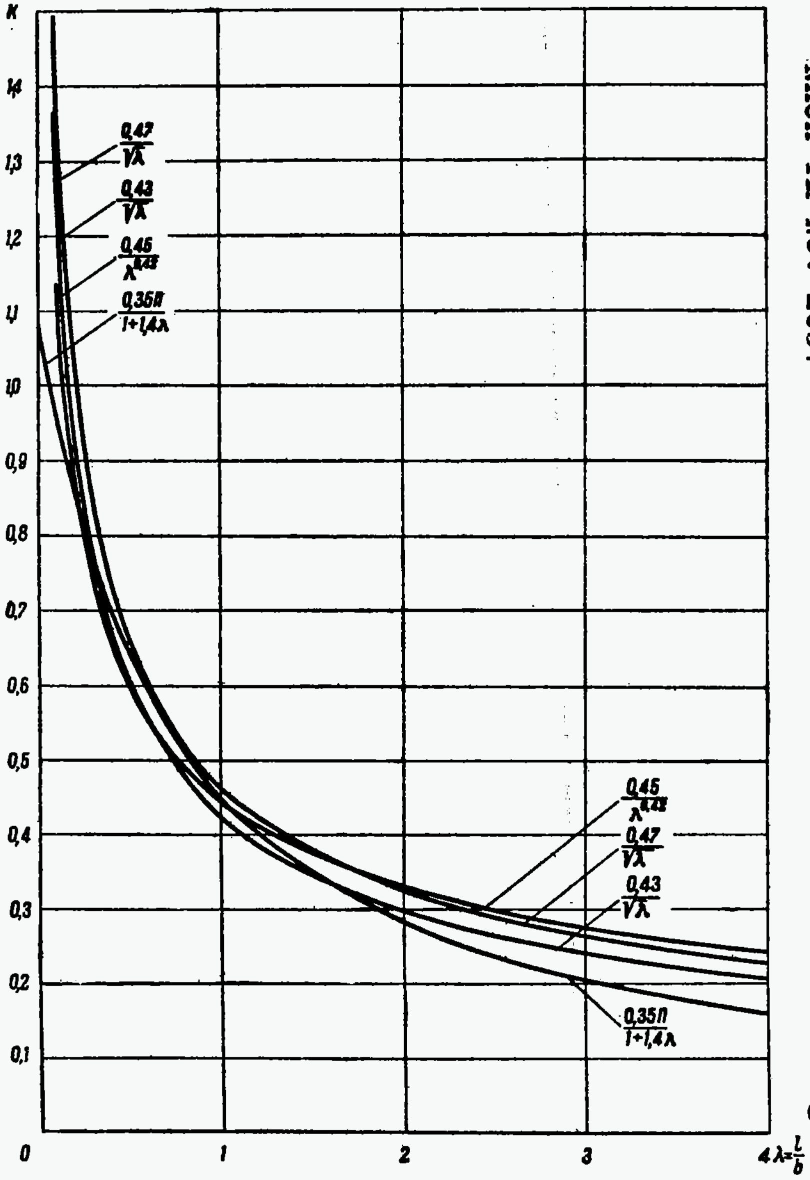


Figure 4.- Comparison of empirical formulas for the coefficient k proposed by various authors.

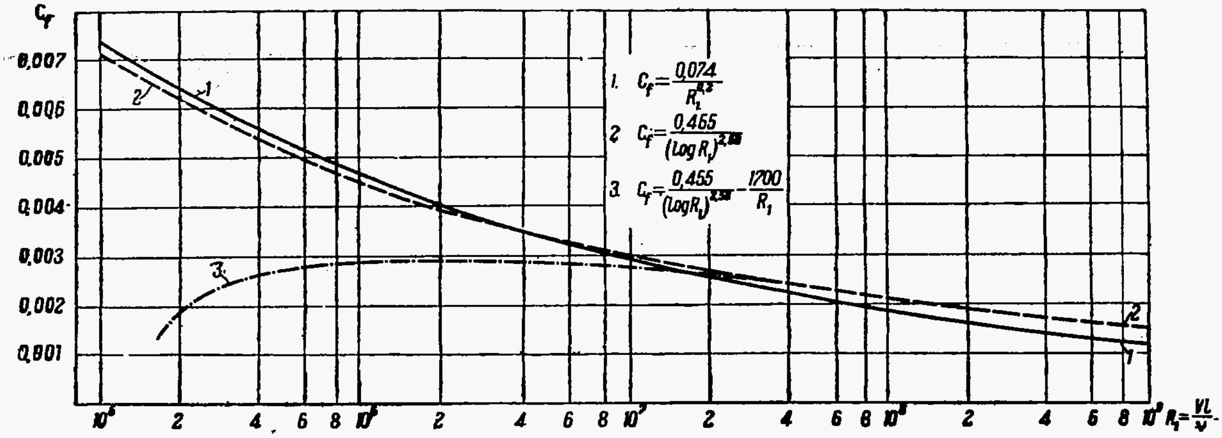


Figure 5.- Curves for the friction coefficient as a function of the Reynolds number $R_1 = vl/\nu$ for towing of flat plates in the direction of their plane according to the formulas of Prandtl and Schlichting.

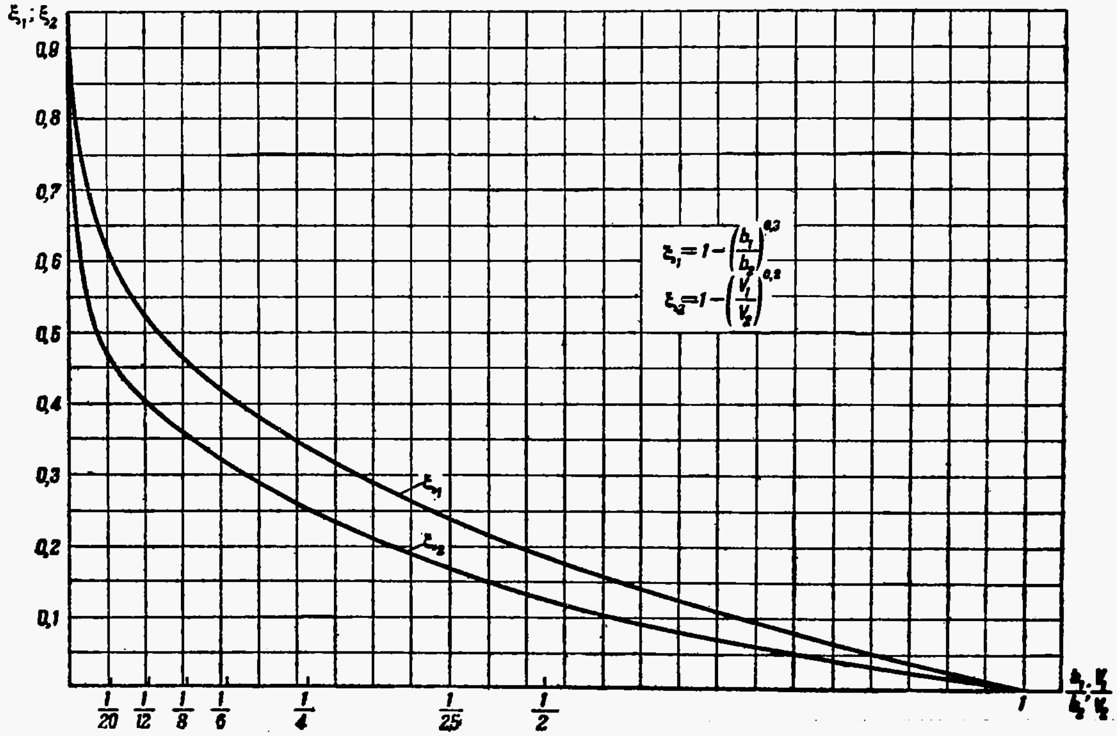
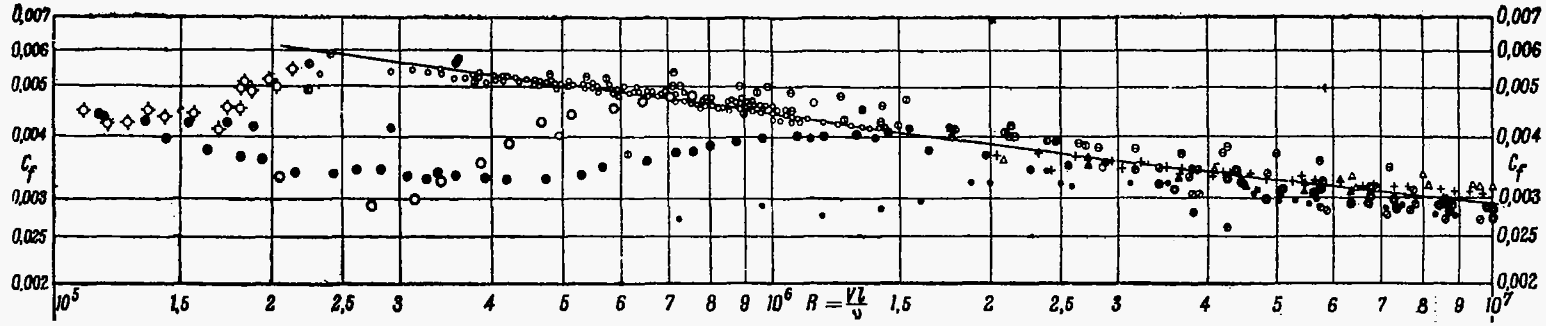


Figure 7.- Coefficient ξ_1 (b_1/b_2) characterising scale effect for computing the resistance according to Froude: coefficient ξ_2 (v_1/v_2) characterising the error in computing the resistance from one velocity to another for the same C_B and θ.



△ -Froude plate 16',25' and 50'(4.88:7.62 and 15.25 m). tested in air.

⊙ -Washington plate 20' and 30'(6.1 and 9.15 m).

• -Gebers Uebigau plates 0.6,1.6,3.6,4.6 and 6.52 m.

⊘ - " Jena plates 1.25,2.50,5,7.5 and 10 m.

+ -Froude national laboratory plates 3',8' and 16' (0.915,2.44 and 4.88 m).

◒ -Washington smooth plate 80'(24.4 m).

◐ - " plate with excrescences 80'(24.4 m).

⦶ -Kempf and Kloss plates 0.5 and 0.75 m.

⊖ -Wieselsberger plate covered with fabric and lacquered, 0.5,1,1.5 and 2 m (tested in air).

⊗ -Zahm plate board 2'(0.61 m) and paper, 16'(4.88 m)

▲ -Gibbons glass plate 9 1/2' (2.9 m) tested in air.

○ -Schoenherr catamaran (a kind of raft) flat, smooth, 3'(0.915 m).

⌖ -Schoenherr catamaran with rough forward edge, 3'(0.915 m).

◉ -Schoenherr catamaran smooth, 6'(1.83 m).

∘ - " " under conditions of more rapid occurrence of turbulence, 6'(1.83 m).

□ -Kempf element of hull.

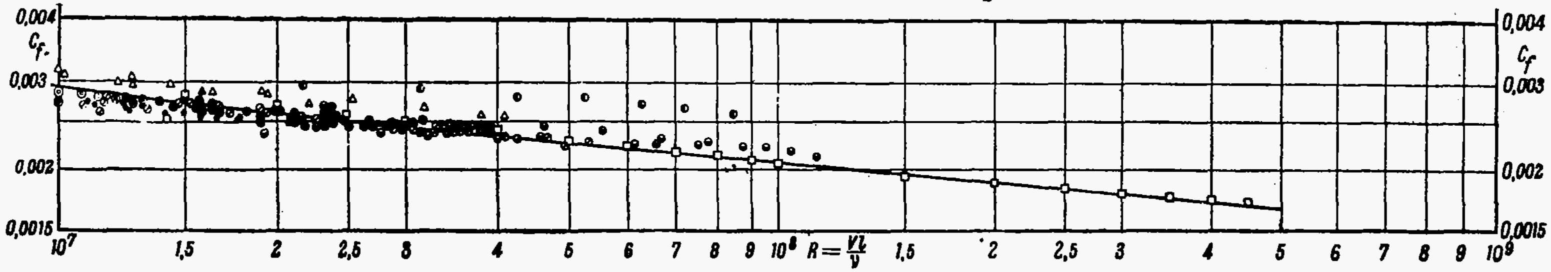


Figure 6.- Comparison of test results for the friction coefficient of flat plates by the formula $C_f = 0.455/(\lg vl/\nu)^{2.58}$.

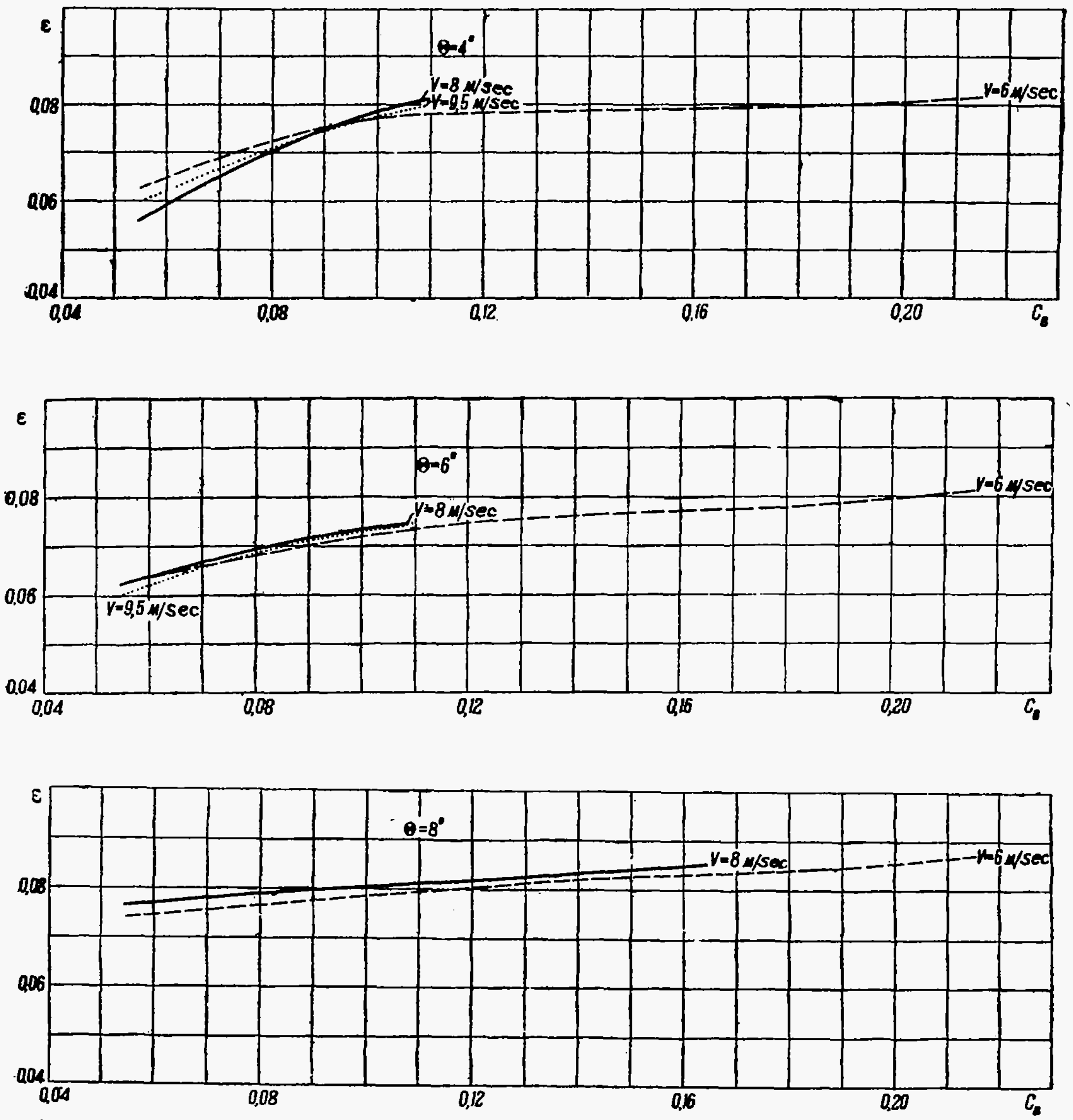


Figure 8.- Dependence of the planing coefficient $\epsilon = W/A$ on C_B for constant θ for various speeds according to the test results of Sottorf.

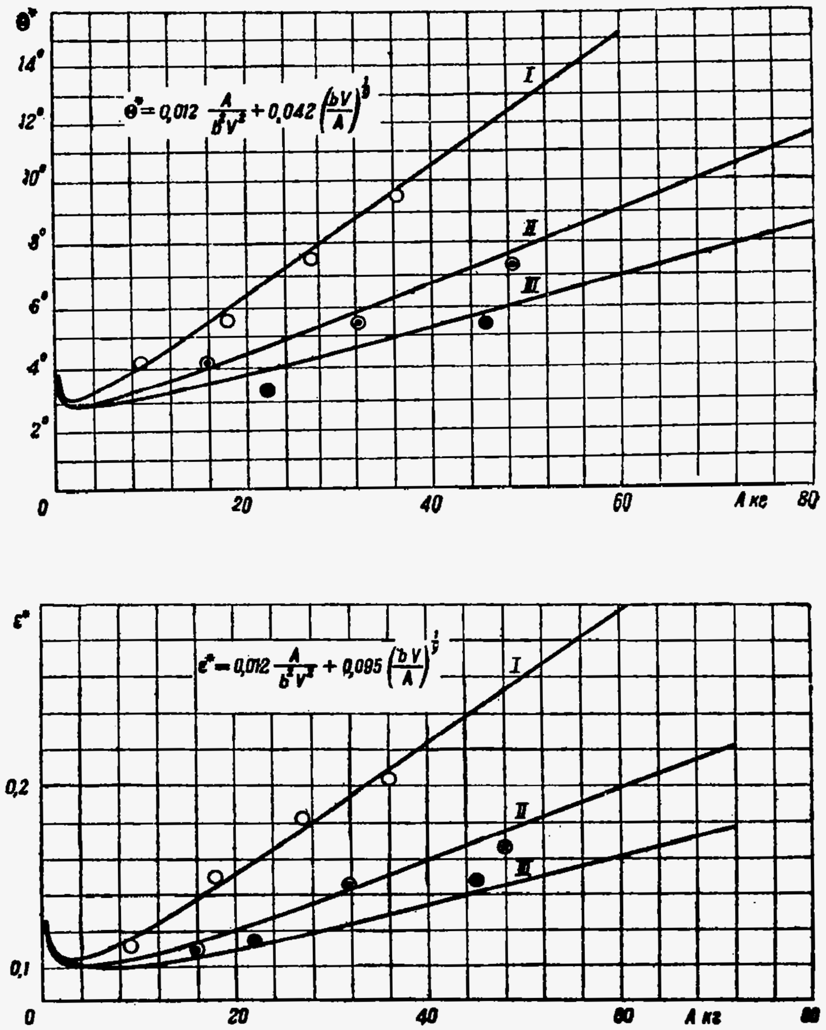


Figure 9.- Curves of optimum values of the trim angles θ* and planing coefficient ε* for constant values of the product bv. Curve I corresponds to bv = 1.8, II – bv = 2.4 and III – bv = 2.85. For b = 0.3, I corresponds to v = 6 m/sec, II – v = 8 m/sec and III – v = 9.5 m/sec. For v = 6 m/sec, I corresponds to b = 0.3 m, II – b = 0.4 m and III – b = 0.475 m.

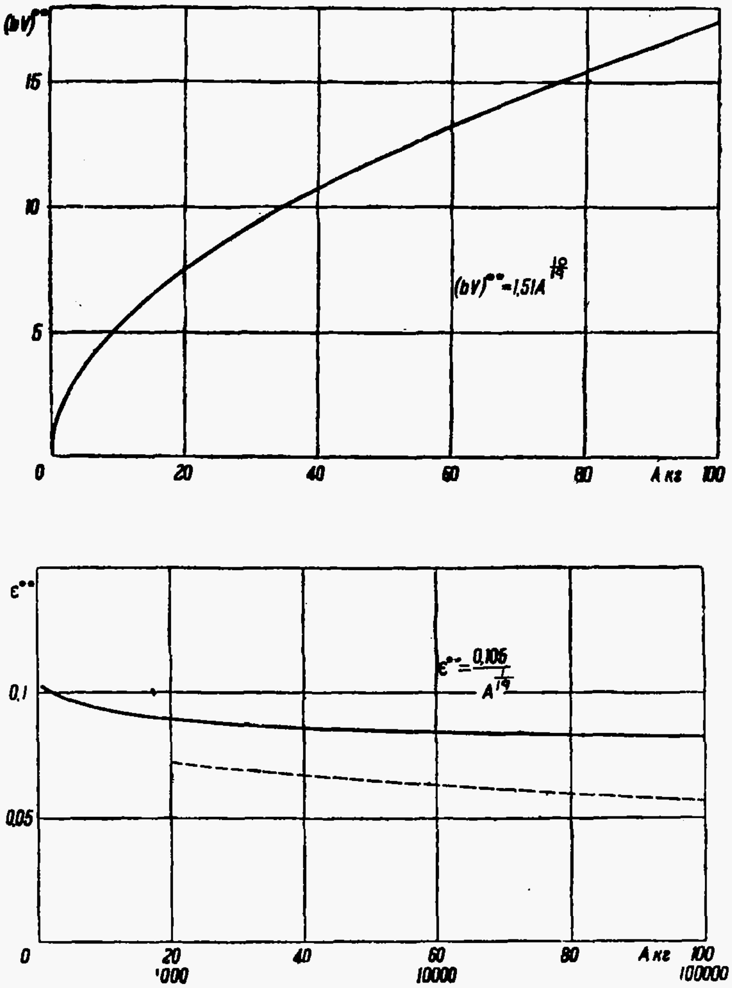


Figure 10.- Curves for optimum values of product bv** and planing coefficient ε** as functions of the load on the water for the optimum value of the trim angle θ** for planing of a flat plate.