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RESEARCH MEMORANDUM

LOCAL ISOTROPY IN TURBULENT SHEAR FLOW

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The Johns Hopkins University

NATIONAL ADVISORY COMMITTEE
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SUMMARY

The mean strain rate in turbulent shear flow must tend to make the structure anisotropic in all parts of the spectrum. It is argued here, however, that, if the spectral energy transfer process destroys orientation, the Kolmogoroff notion of local isotropy can still be justified in spectral regions where the local transfer time is shorter than the characteristic time of the gross shear strain.

INTRODUCTION

Recent measurements on the anisotropy produced by homogeneous strain of the (approximately) isotropic turbulence downstream of a grid (refs. 1 and 2) suggest reexamination of Kolmogoroff's postulate of local isotropy in shear flow (ref. 3). This postulate has been confirmed experimentally in the sense that isotropic-type behavior has been measured for the small "eddies" (refs. 4, 5, 6, and 7). Up to the present time there seems to be no contradictory evidence.

Nevertheless, as pointed out by Uberoi in reference 2, the ubiquitous strain rate of turbulent shear flow must act upon eddies of all sizes, tending to make the turbulence anisotropic at all wave numbers. The largest eddies must be nonisotropic in any case since their dimensions are comparable with the width of the mean shear zone.

Kolmogoroff's idea apparently was that, in the nonlinear transfer of energy from small to large wave numbers, the necessary direction preference of the large structure gets lost. This seems quite plausible, but at least three other effects are involved: (1) The strain-induced tendency to anisotropy at all wave numbers, (2) the general tendency toward isotropy (equipartition?) evident in the absence of strain rate (see refs. 1 and 2) (this must also occur at all wave numbers), and (3) the viscous dissipation to internal energy, especially at very large wave numbers.

It is questioned whether the characteristic times for (a) inertial transfer to higher wave number τ_a, (b) component transfer at the same wave number τ_b, and (c) viscous dissipation at each step of the spectral energy cascade τ_c are sufficiently short to forestall the mean strain-induced anisotropy. In inertial spectral domains if either τ_a or τ_b is short enough, local isotropy remains a plausible notion; in the dominantly viscous domain, τ_c is the critical time. The appropriate characteristic time for comparison is, of course, the inverse of the mean strain rate, or $\left(\frac{1}{2}\frac{\partial\bar{U}}{\partial y}\right)^{-1}$ in boundary-layer-type flows with principal mean velocity field $\bar{U}(x,y)$.

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SYMBOLS

D	any characteristic length
$E(k)$	"three-dimensional" energy spectrum
k	magnitude of wave-number vector
k_c	Kolmogoroff wave number (inverse of Kolmogoroff microscale) characterizing viscous part of spectrum
k_{c_1}	component wave number corresponding to Kolmogoroff wave-number magnitude
q	magnitude of turbulent velocity, $\sqrt{u^2 + v^2 + w^2}$
$q' = \sqrt{\overline{u^2} + \overline{v^2} + \overline{w^2}}$	
R	Reynolds number
R_λ	turbulence Reynolds number, $\frac{1}{\sqrt{3}}\frac{q'\lambda}{\nu}$
U_i	velocity vector
$\bar{U}$	mean velocity in x-direction
u,v,w	turbulent velocity components in x-, y-, and z-directions

V	any characteristic mean velocity component
x,y,z	Cartesian coordinate axes
γ	Heisenberg spectral constant
λ	dissipation scale (Taylor microscale)
ν	kinematic viscosity
$\tau_a(k)$	spectral inertial transfer time
$\tau_b(k)$	spectral time for approach to isotropy in absence of gross strain rate
$\tau_c(k)$	viscous decay time
ϕ	rate of dissipation of turbulent kinetic energy per unit mass

Subscripts:

i,j,k	vector directions

CHARACTERISTIC SPECTRAL TIMES

For a rough estimate of characteristic spectral times a discrete energy cascade in the manner of Onsager (ref. 8) is visualized. The assumption of similarity in successive steps requires a geometric progression; that is, at each step $\Delta k \approx k$. Assume next that at each spectral jump in the cascade process all statistical orientation is lost; that is, the energy arrives at wave number k in isotropic condition, no matter how anisotropic it had become in the previous stage. Possibly by direct dimensional reasoning, Onsager defines the characteristic (inertial) transfer time per stage as

$$\tau_a(k) \equiv \frac{1}{\sqrt{k^3 E(k)}} \tag{1}$$

where k is wave-number magnitude and $E(k)$ is a three-dimensional energy spectrum

$$\frac{1}{2}\left(\overline{u^2} + \overline{v^2} + \overline{w^2}\right) = \frac{1}{2}\,\overline{q^2} = \int_0^\infty E(k)\,dk \tag{2}$$

Here u, v, and w are orthogonal velocity fluctuation components. Physically this form can be deduced from

$$\tau_a \equiv \frac{\text{Kinetic energy per stage}}{\text{Energy transfer rate}} \tag{3}$$

From equation (2) (and $\Delta k \approx k$) the numerator is obviously $kE(k)$. By analogy with the form of the product expressing the rate of transfer of energy from mean shear flow to turbulence, $\overline{u_i u_k} \frac{\partial \bar{U}_i}{\partial x_k}$ in the general turbulent energy equation (see ref. 9), the denominator is expressed as

$$\frac{(\text{Spectral velocity})^3}{\text{Spectral length}} = \frac{(kE)^{3/2}}{1/k} = \sqrt{k^5E^3}$$

Hence, following equation (1),

$$\tau_a(k) = \frac{kE}{\sqrt{k^5E^3}} = \frac{1}{\sqrt{k^3E}}$$

Within each step of the cascade, the characteristic time τ_b for adjustment of an induced energy inequality among the three components is also an inertial phenomenon expressible in terms of pressure-velocity correlations, hence for want of more detailed understanding it is assumed that

$$\tau_b \approx \tau_a \tag{4}$$

The viscous decay time $\tau_c(k)$ of the energy at any stage is physically

$$\tau_c \equiv \frac{\text{Kinetic energy per stage}}{\text{Viscous dissipation rate}} \tag{5}$$

The denominator here, by analogy to the form of the dissipation rate in general $\nu \frac{\partial U_i}{\partial x_j}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$, is

$$\nu \frac{(\text{Spectral velocity})^2}{(\text{Specral length})^2} = \nu \frac{kE}{(1/k)^2} = \nu k^3 E$$

where ν is the kinematic viscosity. Hence

$$\tau_c(k) = \frac{kE}{\nu k^3 E}$$

that is,

$$\tau_c(k) = \frac{1}{\nu k^2} \tag{6}$$

Before returning to the question of local isotropy, it is instructive to write out the two inequalities which identify the inertial and the viscous regions of the spectrum. The former region is temporally characterized by the fact that the energy in any stage jumps to the next before there has been time for appreciable viscous dissipation:

$$\tau_a(k) \ll \tau_c(k) \tag{7}$$

The latter region consists of the cascade stages in which the energy is dissipated before there is time for inertial transfer:

$$\tau_a(k) \gg \tau_c(k) \tag{8}$$

Substituting equations (1) and (6) into these,

$$\left.\begin{aligned} &\text{Inertial region:} \quad \frac{1}{\nu}\sqrt{\frac{E(k)}{k}} \gg 1 \\ &\text{Viscous region:} \quad \frac{1}{\nu}\sqrt{\frac{E(k)}{k}} \ll 1 \end{aligned}\right\} \tag{9}$$

This serves as a check on the τ definitions since $\frac{1}{\nu}\sqrt{\frac{E(k)}{k}}$ is simply the spectral Reynolds number.

INERTIAL REGION

With the assumed model a necessary condition for local isotropy in the purely inertial spectral range is that τ_a be much smaller than the characteristic mean strain time, most simply $\left(\frac{1}{2}\frac{\partial\bar{U}}{\partial y}\right)^{-1}$. In boundary-layer type shear flows (this includes free flows in which Prandtl's boundary-layer approximation is valid) $\frac{1}{2}\frac{\partial\bar{U}}{\partial y}$ is a good approximation to the principal mean strain rate where $\bar{U}(x,y)$ is the mean component of mean velocity and y is the Cartesian coordinate with its highest gradient. Thus, inertial local isotropy can be expected at wave numbers for which

$$\frac{1}{\sqrt{k^3E(k)}} \ll \frac{1}{\frac{1}{2}\frac{\partial\bar{U}}{\partial y}} \tag{10}$$

In the inertial region a spectral behavior roughly like

$$E(k) \approx \phi^{2/3}k^{-5/3} \tag{11}$$

can be expected since this is a formal result when local isotropy is presupposed (see refs. 3 and 8). Here ϕ is the rate of dissipation of turbulent kinetic energy per unit mass. With this spectral form, $\tau_a(k)$ decreases monotonically:

$$\tau_a(k) \approx \phi^{-1/3}k^{-2/3} \tag{12}$$

If this expression were valid out to indefinitely large wave numbers, inertial local isotropy could be found far enough out the spectrum no matter how high the gross strain rate. There would always be a range of k for which

$$\tau_a(k) \ll \frac{1}{\frac{1}{2}\frac{\partial\bar{U}}{\partial y}} \tag{13}$$

Of course, equation (12) cannot apply to indefinitely large wave numbers, so $\tau_a(k)$ will actually increase (eq. (28) or (30)). In any case, the inviscid form, equation (12), converts equation (13) to

$$k^{2/3} \gg \frac{1}{2}\phi^{-1/3}\frac{\partial\bar{U}}{\partial y} \tag{14}$$

In a real fluid, the inertial range can exist only at wave numbers much smaller than the inverse of the Kolmogoroff microscale, that is,

$$k \ll \left(\frac{\phi}{\nu^3}\right)^{1/4} \tag{15}$$

so that equations (14) and (15) make a pair of necessary conditions for inertial local isotropy. They can now be combined into a very crude Reynolds number criterion.

Introduce the following symbol conventions: (a) If $\alpha^{2/3} \gg \beta^{2/3}$ then $\alpha \ggg \beta$; (b) if $\alpha \gg \beta$ and $\beta \gg \gamma$ then $\alpha \ggg\!> \gamma$. For example, suppose that $\alpha \gg \beta$ implies $\alpha = O(20\beta)$: then $\epsilon \ggg\!> \delta$ implies $\epsilon = O\left[(20)^2\delta\right]$.

With this representation, equation (14) can be written as

$$k \ggg \left[\frac{\left(\frac{\partial \bar{U}}{\partial y}\right)^3}{8\phi}\right]^{1/2} \tag{16}$$

which can now be combined with equation (15) as

$$\left(\frac{\phi}{\nu^3}\right)^{1/4} \ggg\!\ggg \left[\frac{\left(\frac{\partial \bar{U}}{\partial y}\right)^3}{8\phi}\right]^{1/2} \tag{17}$$

For a turbulent shear flow in which the total production rate for turbulent energy from mean flow energy is of the same order as the total dissipation rate (they are exactly equal in pipe flow),

$$\phi = O\left(\overline{uv}\,\frac{\partial \bar{U}}{\partial y}\right) \tag{18}$$

But, empirically,

$$\overline{uv} \approx O\left(\frac{1}{10}\,\overline{q^2}\right) \tag{19}$$

according to measurements in a variety of shear flows, so

$$\frac{\partial \bar{U}}{\partial y} = O\left(\frac{10\phi}{\overline{q^2}}\right) \tag{20}$$

Furthermore, approximate ϕ by its isotropic form

$$\phi \approx 5\nu \frac{\overline{q^2}}{\lambda^2} \tag{21}$$

where λ is the Taylor microscale. Equations (20) and (21) convert equation (17) to

$${R_\lambda}^{3/2} >>>>> 16 \tag{22}$$

In turbulent shear flows it seems reasonable to define the turbulence Reynolds number R_λ as

$$R_\lambda \equiv \frac{1}{\sqrt{3}} \frac{q'\lambda}{\nu} \tag{23}$$

where

$$\left(q'\right)^2 \equiv \overline{q^2} = \overline{u^2} + \overline{v^2} + \overline{w^2}$$

VISCOUS REGION

In the predominantly viscous region of the spectrum,

$$k >> \left(\frac{\phi}{\nu^3}\right)^{1/4} \tag{24}$$

and the temporal inequality for possible isotropy is

$$\tau_c(k) = \frac{1}{\nu k^2} << \frac{1}{\frac{1}{2}\frac{\partial \bar{U}}{\partial y}} \tag{25}$$

or

$$k^2 >> \frac{1}{2\nu} \frac{\partial \bar{U}}{\partial y} \tag{26}$$

Since $\tau_c(k)$ decreases monotonically with k, it is clear that there will always be some wave number above which this necessary inequality will be satisfied.

It should be noted that in this spectral range the simple dimensional spectral transfer theory of Heisenberg in reference 10 gives

$$E(k) \propto k^{-7} \tag{27}$$

a result which has had rough experimental confirmation. This leads to an <u>increasing</u> inertial time

$$\tau_a(k) \propto k^2 \tag{28}$$

Although τ_a is of negligible dynamic significance in the viscous region, this suggests a detailed look at the mixed region, where

$$k = O\left[\left(\frac{\phi}{\nu^3}\right)^{1/4}\right]$$

Here both τ_a and τ_c are important and conceivably may violate equation (13).

MIXED REGION

In the spectral region with both inertial transfer and viscous dissipation, $\tau_a(k)$ can be estimated from equation (1) by inserting for $E(k)$ the function obtained by solving Heisenberg's equation (see ref. 10)

$$E(k) = \left(\frac{8\phi}{9\gamma}\right)^{2/3}\left[1 + \frac{8\nu^3}{3\gamma^2\phi}\,k^4\right]^{-4/3} k^{-5/3} \tag{29}$$

Here $\tau_c(k)$, as given by equation (6), is independent of $E(k)$.

The value of Heisenberg's constant γ in equation (29) has been variously estimated from experiment in the range 0.2 to 0.85, with 0.45 an acceptable compromise value. For simplicity, take $\gamma = 4/9$. Assuming the Kolmogoroff wave number is written as

$$k_c \equiv \left(\frac{\phi}{\nu^3}\right)^{1/4}$$

equation (29) turns equation (1) into

$$\tau_a\left(\frac{k}{k_c}\right) \approx 2^{-1/3}\phi^{-1/2}\nu^{1/2}\left[1 + \frac{27}{2}\left(\frac{k}{k_c}\right)^4\right]^{2/3}\left(\frac{k}{k_c}\right)^{-2/3} \tag{30}$$

For values of k/k_c sufficiently small that the second term in the brackets is negligible, this reduces to the inertial estimate, equation (12).

The viscous decay time in terms of k/k_c is

$$\tau_c\left(\frac{k}{k_c}\right) = \phi^{-1/2}\nu^{1/2}\left(\frac{k}{k_c}\right)^{-2} \tag{31}$$

Figure 1 is a dimensionless plot of τ_a and τ_c. The values of τ_a and τ_c are necessarily of the same order in the vicinity of $k \approx k_c$, since this is the region of equal inertial and viscous effects, that is, of Reynolds number about unity.

By equating the derivative of equation (30) to zero, the minimum value of τ_a is found to be

$$\left[\tau_a\left(\frac{k}{k_c}\right)\right]_{min} = \tau_a(0.40) = 1.8\left(\frac{\nu}{\phi}\right)^{1/2} \tag{32}$$

On the whole, a necessary condition for local isotropy is that

$$1.8\left(\frac{\nu}{\phi}\right)^{1/2} \ll \frac{1}{\frac{1}{2}\frac{\partial\bar{U}}{\partial y}} \tag{33}$$

or simply

$$\left(\frac{\phi}{\nu}\right)^{1/2} \gg \frac{\partial\bar{U}}{\partial y} \tag{34}$$

A less certain but more convenient criterion follows from the use of equations (20) and (21):

$$R_\lambda \gg 15 \tag{35}$$

It is worth noting that for $k/k_c = 2$, for example, the inertial condition $\tau_a \ll \frac{2}{\partial\bar{U}/\partial y}$ could be violated while the viscous condition $\tau_c \ll \frac{2}{\partial\bar{U}/\partial y}$ is satisfied. Yet this is a spectral range in which inertial forces are not negligible, and equation (33) does not rule out the possibility of an anisotropic local spectral range for $k \gtrsim k_c$. Of course, for $k \gg k_c$, τ_a is not pertinent and $\tau_c \ll (\tau_a)_{min}$, so equation (33) implies isotropy here.

WAVE NUMBER OF TURBULENT ENERGY PRODUCTION

Following Prandtl's simple kinetic-theory type of model in turbulent shear flow, it is inferred that the production of turbulent energy from mean flow energy is primarily due to the lateral fluctuating motion of "fluid balls" in the presence of a velocity gradient. Therefore the mean production rate must depend at least upon the lateral root-mean-square fluctuation v' and the chief mean velocity gradient $\partial\bar{U}/\partial y$ in a boundary-layer-type flow.

Of course, the turbulent energy equation gives the rate per unit mass as $\overline{uv}\,\frac{\partial\bar{U}}{\partial y}$. Presumably this energy is fed into the turbulence over a wide range of wave numbers, each receiving an allotment proportional to its spectral contribution to $\overline{uv}$.

It is noted, however, that v' and $\partial\bar{U}/\partial y$ are sufficient to reproduce the dimensions of wave number k, so an estimate of the order of magnitude of the energy production wave number k_p is

$$k_p = O\left(\frac{1}{v'}\,\frac{\partial\bar{U}}{\partial y}\right) \tag{36}$$

Clearly local isotropy can be expected only at wave numbers much larger than k_p:

$$k \gg \frac{1}{v'}\,\frac{\partial\bar{U}}{\partial y} \tag{37}$$

For turbulent shear flows whose gross production rate is of the same order as the dissipation rate, equations (20) and (21) can be introduced so that

$$k\lambda \gg \frac{30}{R_\lambda} \tag{38}$$

For inertial local isotropy, $1/k$ must also be much larger than the Kolmogoroff microscale; that is,

$$k \ll \left(\frac{\phi}{\nu^3}\right)^{1/4} \tag{39}$$

which may be written together with equation (37) as

$$\frac{1}{v'}\frac{\partial \bar{U}}{\partial y} \ll k \ll \left(\frac{\phi}{\nu^3}\right)^{1/4} \tag{40}$$

or

$$\frac{1}{v'}\frac{\partial \bar{U}}{\partial y} \lll\ll \left(\frac{\phi}{\nu^3}\right)^{1/4} \tag{41}$$

Using equations (20) and (21) this gives a Reynolds number inequality

$$R_\lambda^{3/2} \ggg\gg 25 \tag{42}$$

which is much like equation (22).

COMPARISON WITH EXPERIMENT

The three semiempirical Reynolds number criteria can be summarized as follows:

(a) Lower bounds on R_λ for the existence of an isotropic inertial subrange are given by equation (22) based on inertial transfer time and Kolmogoroff wave number and by equation (42) based on turbulent production wave number and Kolmogoroff wave number.

(b) The lower bound on R_λ for local isotropy in inertial or mixed range is given by equation (35) based on minimum inertial transfer time assuming a Heisenberg spectrum.

Next these inequalities are compared with the conditions in typical past experiments. In the round-jet experiments which include those of references 5 and 11 $R_\lambda \approx 120$, hence $R_\lambda^{3/2} \approx 1,300$. Since there is no a priori basis for assigning a specific numerical factor to the strong inequality $\alpha \gg \beta$, the experimental results are used. In these jet measurements the shear correlation spectrum reached zero at a one-dimensional wave number of about 45 per centimeter, whereas the Kolmogoroff wave number k_c (the inverse of the Kolmogoroff microscale) is 100 per centimeter.

Since k_c is a three-dimensional wave-number magnitude, comparison is properly made by defining a corresponding one-dimensional parameter k_{c_1}. A plausible, though arbitrary, definition is the first moment of the one-dimensional spectrum that corresponds to a Dirac-function three-dimensional spectrum, that is, a spherical shell of radius k_c:

$$k_{c_1} = k_c \frac{\int_0^1 \left[1 - \left(\frac{k_1}{k_c}\right)^2\right]\left(\frac{k_1}{k_c}\right) d\left(\frac{k_1}{k_c}\right)}{\int_0^1 \left[1 - \left(\frac{k_1}{k_c}\right)^2\right] d\left(\frac{k_1}{k_c}\right)} \tag{43}$$

whence $k_{c_1} = \frac{3}{8} k_c \approx 38$ per centimeter.

Therefore local isotropy is reached just around the Kolmogoroff region where the spectral Reynolds number is of order unity. Hence equation (35) should be just barely satisfied, and it is noted that this is so if $\alpha \gg \beta$ is interpreted to mean $\alpha \geqq 8\beta$.

In this flow there certainly can be no isotropic inertial subrange, and, in fact, neither equation (22) nor (42) is satisfied. The largest value of R_λ attained in a well-defined turbulent flow appears to be that in Laufer's pipe (see ref. 12), $R_\lambda \approx 250$. In this case, $R_\lambda^{3/2} \approx 4{,}000$, and he does find some evidence for an isotropic inertial subrange.

In conclusion, it seems worthwhile to set up a turbulent flow with still higher values of R_λ than those attained by Laufer, perhaps greater than or equal to 500. Since R_λ increases more slowly than R (where $R \equiv VD/\nu$ and V and D are characteristic gross velocity and width)[1] for fixed geometrical boundaries, values of 500 or higher will not be easy to attain with air in a small laboratory.

The Johns Hopkins University,
Baltimore, Md., August 16, 1956.

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[1]Neglecting the slow decrease in turbulence level which sometimes accompanies increasing Reynolds numbers in shear flows, $R_\lambda \propto R^{1/2}$.

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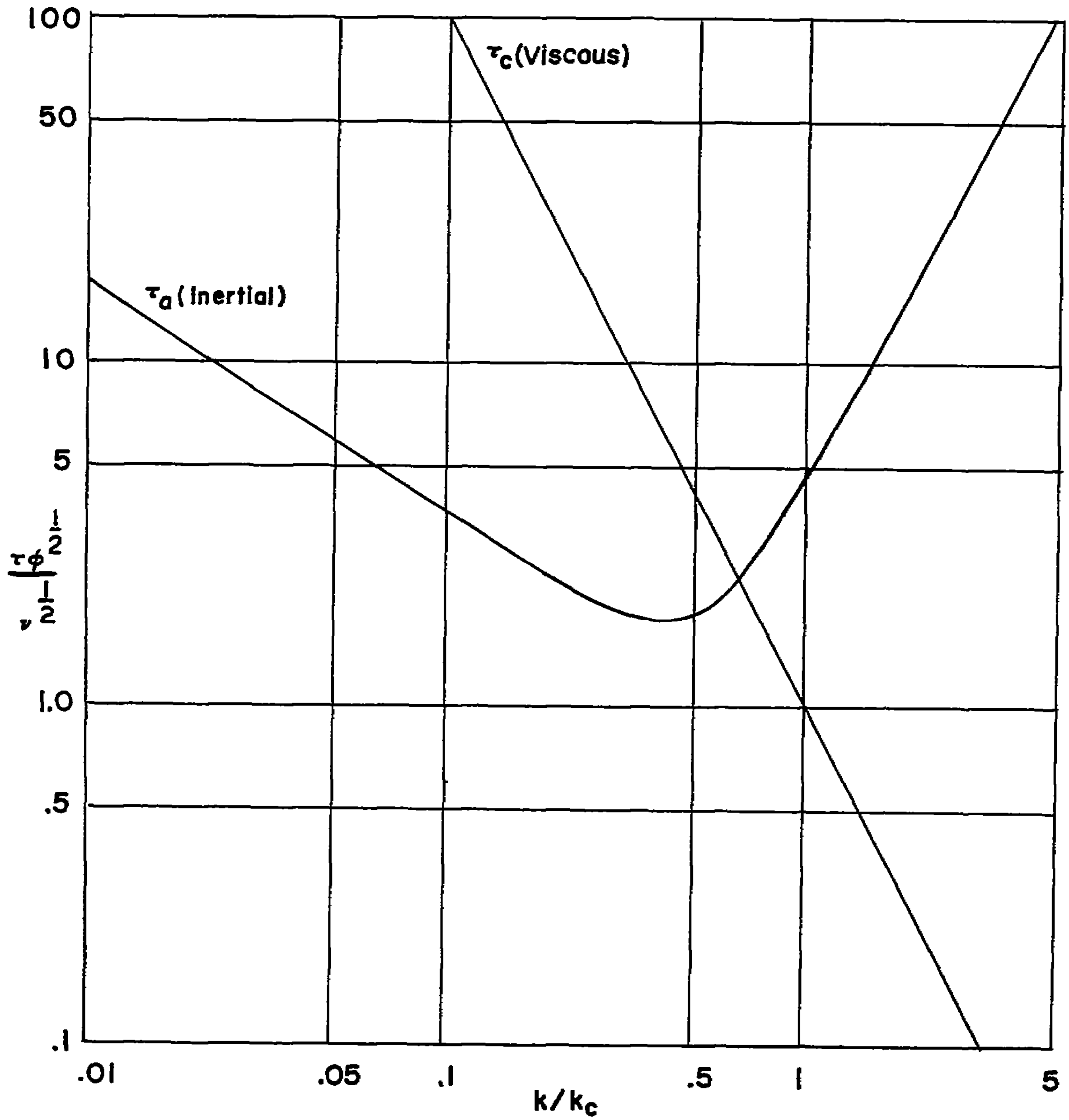

Figure 1.- Characteristic spectral times (assuming a Heisenberg spectrum).