TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 799

THE TRANSPORT OF VORTICITY THROUGH FLUIDS IN TURBULENT MOTION

(In the light of the Prandtl and Taylor theories)

By C. Ferrari

L'Aerotecnica
Vol. XV, Nos. 11-12, November-December 1935

Washington
July 1936

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 799

THE TRANSPORT OF VORTICITY THROUGH FLUIDS IN TURBULENT MOTION*

(In the light of the Prandtl and Taylor theories)

By C. Ferrari

SUMMARY

The author studies the problem of the transport of vorticity or of momentum in the light of the Taylor and Prandtl theories which he briefly reviews. He shows how the formulas of Prandtl could be brought into agreement with experimental results in those cases where they agree with the principle of statistic similitude of Kàrmàn, and particularly in the problem of the distribution of velocity and temperature in the wake of a heated cylindrical obstacle. He shows that when the formulas are extended to two-dimensional motion with streamlines whose curvature is not zero, they lead to unsatisfactory results and that in this case the formulas differ completely from those derived from the principle of similitude when the latter is applied either to the configuration of disturbed flow or to the distribution functions of the turbulent velocities. He then examines the relations of this problem of transport of motion with the theory of Mattioli, which appears susceptible of some observations. After pointing out the difficulty of obtaining a satisfactory theory of turbulence based on the concept of transport and deduced by the methods of classical mechanics, he indicates the reasons therefor and shows finally how the problem may find a solution by applying the methods of statistical mechanics according to the theory of Gebelein.

1. The essential defining characteristics of the turbulent motion of a fluid is the well-known irregular fluctuations, both in magnitude and direction, of the velocity

---

*"La Teoria della Turbolenza ed il Trasporto della Quantita di Moto e della Vorticita." L'Aerotecnica, vol. XV, nos. 11-12, November-December 1935, pp. 1037-1056.

at each point, corresponding to an energetic mixing of the mass whereby a fluid particle may, in the course of its motion, occupy any position whatever in the field. The fluctuations of the modulus and argument of the velocity vector about their mean value, though both irregular, are not, however, independent. There exists between them a statistical correlation, in consequence of which the mean time value of the product of two velocity components along any two perpendicular directions is different from zero. There result upon each element of surface immersed in the fluid virtual stresses perfectly analogous to the viscous stresses produced by thermal molecular agitation. To characterize this turbulent agitation of the mass Prandtl (reference 1) and Taylor, (reference 2) have independently introduced the concept of mixing length or "Mischungsweg", denoted by l, which is quite analogous to the molecular mean free path λ considered in the kinetic gas theory, and which may therefore be defined as the path normal to the lines of flow which the particles can trace out and still maintain their individuality; that is, without assuming the physical characteristics of the medium in which they are immersed. But whereas Prandtl and his collaborators assume that throughout the path l each fluid particle maintains its momentum, Taylor objects that the instantaneous differences of pressure may cause the velocity of the displaced particle to vary and that therefore it is more logical to assume that it is the vorticity, upon which the instantaneous local pressure distribution has no effect, that is maintained constant.

2. In order to understand these two concepts more clearly, let us limit our considerations to two-dimensional fields of motion in which the flow lines are exactly or approximately straight lines parallel to the X axis, along which the characteristics of the motion may be assumed constant. If u' and v' are the two components of the velocity due to the turbulent agitation, the transport of the momentum across an element dx is equal to

$$\tau_{xy}\, dx = -\rho\, \overline{u'v'}\, dx \tag{1}$$

where $\overline{u'v'}$ denotes the mean time value of the product of u' and v'; the increment of momentum per unit volume communicated to the fluid layer of thickness dy is therefore

$$\frac{d\tau_{xy}}{dy} = -\rho\, \frac{d}{dy}\, \overline{(u'v')} \tag{2}$$

and therefore if the mean motion takes place in the direction of the X axis with a pressure gradient $\partial p/\partial x$, the equation of motion is

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{d}{dy}\overline{(u'v')} \tag{3}$$

On the other hand, considering the motion from the Lagrangian point of view, the fluctuations in velocity appear as a consequence of the transfer of fluid particles from layer to layer, so that if l' is the distance, at a given instant, between the layer from which the particle comes to that which it occupies, and if it is assumed with Prandtl that within the distance l' the velocity remains constant, then

$$u' = -l'\frac{dU}{dy} \tag{4}$$

where U denotes the velocity of the motion. From (3) there results

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{d}{dy}\left\{\overline{l'v'}\;\frac{dU}{dy}\right\} \tag{5}$$

Equation (5) has further been transformed by Prandtl by making the plausible assumption that the correlation between u' and v' and therefore between l' and v' is constant over the whole field, so that it is possible to put

$$\overline{l'v'} = c\sqrt{\overline{l'^2}}\sqrt{\overline{v'^2}} = c\,l\sqrt{\overline{v'^2}}$$

in which c is a constant and l the mean square variation of l', or, according to the definition given above, the mean "mixing length." Considering next two fluid particles which, moving from the layers of height $y + l'$ and $y - l'$, meet at the layer of height y; they will approach each other or move away from each other with a relative velocity $2u'$ and will therefore induce in the fluid a velocity v' which, on account of the continuity of the fluid mass itself, should be of the same order of magnitude as u'; that is, $v' = au'$ and therefore,

$$\sqrt{\overline{v'^2}} = c_1\sqrt{\overline{u'^2}} = c_1 l\frac{dU}{dy} \tag{6}$$

where c_1 is likewise a constant.

By means of (6) equation (5) is transformed into

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{d}{dy}\left\{l^2 \frac{dU}{dy}\left|\frac{dU}{dy}\right|\right\} \tag{7}$$

in which c_1 is included in l.

3. To deduce the increment of momentum communicated per unit volume to the element of thickness dy under the hypothesis that not the momentum but the vorticity remains constant during the transport of the fluid particles, Taylor considers the equation of motion of a perfect fluid which, under the assumption that the mean motion is uniform and parallel to the direction of the X axis, assumes the expression

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = 2v\omega \tag{8}$$

in which

$$\omega = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

is the vorticity; and therefore, in the fields of motion considered above the increment of motion due to the turbulent agitation is, according to Taylor

$$\frac{\partial \tau_{xy}}{\partial y} = -2\overline{v'\omega'} \tag{9}$$

if ω' and v' are the instantaneous values of the oscillations of the vorticity and of the component v' of the velocity. There is then obtained the equation corresponding to (3)

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \overline{2\omega' v'} \tag{3'}$$

in which the correlation between ω' and v' is brought about by the same causes that produce that between u' and v' in the theory of Prandtl. On the other hand, if l' has the meaning defined above, then according to Taylor,

$$\omega' = -\frac{1}{2} l' \frac{d^2U}{dy^2} \tag{4'}$$

and therefore,

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \overline{v'l'}\,\frac{d^2U}{dy^2} \qquad (5')$$

or also, from (6) and the assumption of constant correlation between v' and l'

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = l^2 \left|\frac{dU}{dy}\right| \frac{d^2U}{dy^2} \qquad (7')$$

4. Without entering for the present into the merit or justification of the criteria which led to equations (5) and (5'), these may simply be discussed, as has been indicated by Taylor (reference 3) and Fage (reference 4), on the basis of a comparison of the results to which they lead with those deduced from experiment. Now it is known that in the uniform flow about a flat plate, when l is set equal to cy as required from simple considerations of the homogeneity of the formulas and of dynamic similitude, equation (7) leads immediately to a distribution of the velocity at the surface of contact of the plate itself, which fact has been brilliantly confirmed both by the classical turbulence theory of Kàrmàn (reference 5) and by the experimental investigations of Nikuradse (reference 6). There is, in fact, obtained by double integration of (7)

$$U = a \log y + b \qquad (8)$$

and from the tests of Nikuradse,

$$\frac{U}{\sqrt{\tau_0/\rho}} = 5.5 + 5.75 \log \frac{y\sqrt{\tau_0/\rho}}{\nu}$$

in which τ_0 is the tangential stress at the plate.

Equation (7'), on the contrary, in the case which we are examining, does not give any significant result; in fact, if $\frac{dp}{dx} = 0$, then either l should be equal to zero, which case corresponds to nonturbulent motion, or $\frac{dU}{dy} = 0$ or, finally, $\frac{d^2U}{dy^2} = 0$, and therefore the velocity at the surface of contact of the plate should vary either linearly or parabolically.

Although the example just discussed appears to bear out the theory of Prandtl, a contrary result is obtained

if there is considered the phenomenon of turbulent diffusion in the wake behind a cylindrical obstacle with axis of symmetry in the direction of the X axis. This problem has been studied theoretically and experimentally by Schlichting (reference 7), who has found that the velocity U along the X axis may be expressed by means of the formula

$$\frac{U_0 - U}{U_0} = x^{-1/2} f\left(\frac{y}{\sqrt{x}}\right)$$

where U_0 is the velocity of the undisturbed stream.

The velocity V along y perpendicular to x, by the equation of continuity, is given by

$$\frac{V}{U_0} = -\frac{1}{2} x^{-1} \left(\frac{y}{\sqrt{x}}\right) f\left(\frac{y}{\sqrt{x}}\right)$$

while the mixing length l is proportional to $x^{1/2}$, or $l = ax^{1/2}$. Now if the wake is narrow and therefore y is small compared to x, then according to (7) there is obtained the equation of motion:

$$U_0 \frac{\partial U}{\partial x} = \frac{\partial}{\partial y}\left[\overline{l' v'} \frac{\partial U}{\partial y}\right] = 2a^2 x \frac{\partial U}{\partial y} \frac{\partial^2 U}{\partial y^2} = A\, x \frac{\partial U}{\partial y} \frac{\partial^2 U}{\partial y^2} \qquad (9)$$

while according to (7') there is obtained:

$$U_0 \frac{\partial U}{\partial x} = \overline{l' v'} \frac{\partial^2 U}{\partial y^2} = a^2 x \frac{\partial U}{\partial y} \frac{\partial^2 U}{\partial y^2} = \frac{A}{2}\, x \frac{\partial U}{\partial y} \frac{\partial^2 U}{\partial y^2} \qquad (9')$$

Equations (9) and (9') are formally identical, but the coefficient of turbulent transport resulting from (9) is double that contained in (9'); it follows that the diagrams of velocity deduced from (9) and from (9') are identical, but the results will not be identical if the transport coefficient calculated from (9) or (9') is applied to other problems intimately connected with this one. Let us consider, for example, the temperature in the wake of the same obstacle which is assumed to be heated. The equation of the heat transport is

$$U_0 \frac{\partial T}{\partial x} = \frac{\partial}{\partial y}\left(\overline{l' v'} \frac{\partial T}{\partial y}\right) \qquad (10)$$

and by comparing (10) with (9) it is immediately recognized that the latter is satisfied if we put

$$T = b\,U \tag{11}$$

that is, the law of distribution of temperature in the wake of an obstacle should, according to Prandtl's theory, coincide with that for the velocity. If, however, (11) is applied to (10), the latter is not transformed into (9') and therefore according to the theory of Taylor, the temperature diagram does not coincide with that for the velocity. For the velocity there is obtained

$$\frac{U_o - U}{U_o - U_c} = (1 - \xi^{3/2})^2$$

where U_c is the value of U on the axis of the wake whose width is $2Y$ and ξ is equal to y/γ, for the temperatures

$$\frac{T}{T_c} = 1 - \xi^{3/2}$$

where T_c denotes the temperature on the axis.

Now the tests of A. Fage and of Falkner (reference 8), carried out on two cylindrical heated obstacles of circular section and lenticular section, respectively, have shown an excellent agreement between the velocity and temperature distribution agreeing with the theory of Taylor, and they particularly well bring out the difference between the temperature and velocity diagrams, respectively (fig. 1). In this connection it should, however, be remarked that the similarity between the distribution diagrams of the temperature and velocity affirmed by the theory of Prandtl, has been experimentally confirmed by F. Elias (reference 9) for the flow about a flat plate. It therefore appears that while the theory of transport of momentum is confirmed by experiments in problems of bound turbulence (at the contact of the solid wall), at least for the very simple cases considered above, the theory of transport of vorticity gives better results in the problems of free turbulence. Taylor and Prandtl attribute this singular behavior to the fact that in the case of bound turbulence, as demonstrated by the experiments of Fage and Townend (reference 10), the perturbations which make the particles move from layer to layer are produced

essentially by vortices arranged so that their axis is parallel to the wall of the obstacle and along the undisturbed stream, so that the phenomenon of turbulent agitation is three-dimensional, whereas in free turbulence the transport of the fluid particles is due principally to vortices arranged with their axis normal to the flow line of the mean motion and to the plane of the motion. In the first case migrations of the particles can take place without having the velocity influenced by the differences in local pressure; at any rate, Taylor observes that the transport of vorticity is now given not only by $\overline{2\omega'_z v'}$, but by $2(\overline{\omega'_z v'} - \overline{\omega'_y w'})$, where z is the direction normal to the plane of the mean motion and w' and ω'_y the component of the oscillation velocity and of the rotation, respectively, along the Z and Y axes. From the general laws of vortex motion of Helmholtz (see Lamb, Hydrodynamics, 4th ed., p. 197), Taylor deduces that the transport of vorticity in the case of three-dimensional perturbations is given by $\overline{v' \frac{\partial z}{\partial c}} \omega_z + \overline{l' v' \frac{\partial z}{\partial c}} \frac{d\omega_z}{dy}$, instead of by $\overline{l' v'} \frac{d\omega_z}{dy}$, where c denotes the initial coordinate z of the fluid particle, or the value of z whereby the vorticity is considered equal to that corresponding to the mean motion at the same point. If a fluid element initially parallel to z keeps its orientation constant, then obviously $\frac{\partial z}{\partial c} = 1$, or is always constant, and therefore $\overline{v' \frac{\partial z}{\partial c}} = 0$; but if the instantaneous velocity also has a component along z, $\frac{\partial z}{\partial c} - 1$, which represents the deformation of the element defined above along z itself, it may assume any values whatever and therefore $\overline{v' \frac{\partial z}{\partial c}}$ may be different from zero. Taylor shows that if l is sufficiently small, in the special case of flow about a flat plate, the application of the principle of the transport of vorticity allows an equation to be obtained that is formally identical with (5), derived from the entirely different concept of Prandtl.

5. Taylor's conception is certainly ingenious and may perhaps give useful results in those more complicated problems where the other theorems prove powerless. Never-

theless, it does not appear to the writer that it is necessary to have recourse to a three-dimensional theory of turbulence in order to explain the limits of applicability of formulas (7) and (7'), to give reasons for the erroneous results to which they may lead, and to substitute for these other formulas of greater generality.

It is interesting to compare the results of the preceding theories with those that may be deduced by the correct application of the equations of Navier, which are naturally assumed valid not only for the instantaneous motion but also for the mean motion. Under these conditions, there is obtained from the equation of projection of the momentum in the x direction

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

in which u and v are the instantaneous values of the velocity, or $u = U + u'$, $v = V + v'$; u and v always being connected by the equation of continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. There immediately results in place of (3) for the mean motion

$$-\frac{1}{\rho}\frac{\overline{\partial p}}{\partial x} = \overline{v'\frac{\partial u'}{\partial y}} + \overline{u'\frac{\partial u'}{\partial x}} + U\frac{\partial U}{\partial x} + V\frac{\partial V}{\partial y}$$

But if $\sqrt{\overline{u'^2}} = l\frac{\partial U}{\partial y}$, it is possible to put $u' = -l'\frac{\partial U}{\partial y}$; $v' = a\,l'\frac{\partial U}{\partial y}$ in which <u>a may vary in time but is independent of y</u>,

$$v'\frac{\partial u'}{\partial y} = -\frac{a}{2}\frac{d}{dy}\left[l'^2\frac{\partial U}{\partial u}\left|\frac{\partial U}{\partial y}\right|\right];$$

and therefore, including as before the single constant a in l',

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{2}\frac{\partial}{\partial y}\left[l^2\left(\frac{\partial U}{\partial y}\right)^2\right] + U\frac{\partial U}{\partial x} + \frac{1}{2}\frac{\partial}{dx}\left[l^2\left(\frac{\partial U}{\partial y}\right)^2\right] \qquad (11)$$

If the motion is independent of x, (11) reduces to

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{2}\frac{d}{dy}\left[l^2\left(\frac{dU}{dy}\right)^2\right] \qquad (12)$$

which is formally identical with (7) and therefore, under equal assumptions on the form of l, leads to the same distribution of velocity. In particular, for the flow about a flat plate,

$$\frac{dp}{dx} = 0; \quad l = k\,y$$

and there is therefore obtained

$$\frac{dU}{dy} = \frac{k}{y}; \quad U = k \log y + b$$

On the other hand, the temperature distribution law for the same case is given by

$$U \frac{dT}{dx} = 0 = \frac{d}{dy}\left(\overline{l' v'} \frac{dT}{dy}\right) = \frac{d}{dy}\left(l^2 \frac{dU}{dy} \frac{dT}{dy}\right) \tag{13}$$

which, by putting

$$T = b\,U$$

comes out identical with (12) (since $\frac{\partial p}{\partial x}$ is zero), and therefore assures the similitude of the diagrams of the velocity and temperature distribution in the turbulent flow about a flat heated plate.

In the wake of a heated obstacle, however, at a great distance x downstream, equation (11), with the same degree of approximation by which (9) is written, the term $\frac{\partial}{\partial x}\left[l^2 \left(\frac{\partial U}{\partial y}\right)^2\right]$ being negligible compared with $\frac{\partial}{\partial y}\left[l^2 \left(\frac{\partial U}{\partial y}\right)^2\right]$, gives:

$$U_0 \frac{\partial U}{\partial x} = \frac{1}{2} \frac{\partial}{\partial y}\left[l^2 \left(\frac{\partial U}{\partial y}\right)^2\right] \tag{14}$$

which agrees with (9') resulting from the theory of Taylor, while the temperature distribution is always given by (10).

6. From what we have said above, it therefore turns out that the lack of agreement of the experimental results with those deduced from (7) in the problem of turbulent diffusion in the wake of a cylindrical obstacle, is not a consequence of (4) and (6) and therefore of a possible in-

fluence of the local pressure gradient which, by varying the quantity of motion of the particles, increases the coefficient of heat transport with respect to that of impulse, as deduced by Taylor, but of the fact that the equation of motion is not (9) but (14).

This conclusion does not, however, yet justify us in deducing any principle of general character as regards the possibility of the application of the Taylor and Prandtl concepts to the problems of turbulence. In fact, the fundamental relations (4) and (6) whose validity we have just shown for the cases of the two-dimensional motion considered, and which in the theory of Prandtl define the constancy of the momentum in the transport of turbulence, have been deduced by an entirely distinct procedure by Kàrmàn, and by this method acquire an essentially different significance, which determines also the limits of applicability.

In 1930 Kàrmàn had already determined equations (4) and (6), assuming the condition that the disturbed motion corresponding to the turbulent agitation of the fluid particles is statistically similar at all points of the field, differing from point to point only by the scales of time and of length. This assumption, as Kàrmàn observes, perfectly corresponds to that which is normally made in the kinetic theory of gases and which permits the stresses due to the thermal agitation of the molecules to be simply expressed by means of local derivatives of the general velocity of the motion, and of the mean molecular trajectory, and yields as a consequence the constant correlation between the components of the turbulent oscillation velocity, to which we have already referred above and which has been confirmed by the experiments of Wattendorf (reference 11) and Kuethe at Pasadena, and by Reichardt (reference 12) at Göttingen (fig. 2). In this way, under the assumption that the general motion takes place very approximately along flow lines parallel to the X axis, there result the fundamental relations of Karman,

$$\sqrt{\overline{u'^2}} = l\,\frac{dU}{dy}; \qquad \sqrt{\overline{v'^2}} = cl\,\frac{dU}{dy}$$

which, being formally identical with those established by Prandtl, differ substantially by the concept from which they were derived and by the definition of the length l, which in the theory of Kàrmàn, is given by

$$l = \frac{dU/dy}{\dfrac{d^2U}{dy^2}}$$

The meaning of (4) and (6) has been recently generalized by Dédébant, Schereschewski, and Wehrle (reference 13), who have shown that the Kàrmàn relations may be obtained by applying the law of similitude not to the configuration of the velocity field, but only to the law of distribution of the disturbed velocity; that is, by assuming that the disturbed velocity distribution functions become identical by a suitable change of scale.

It should still be observed that the equation (3') leads to the same laws as those derived from (12) by assuming that the fluctuations of the vorticity at each point are proportional not to $\frac{\partial}{\partial y}\frac{\partial U}{\partial y}$, but to $\frac{\partial}{\partial y}\, l\, \frac{\partial U}{\partial y}$, and therefore not to the derivative of the mean rotation but to the derivative of the displacement.

7. For the more accurate comprehension of the effective meaning of the preceding formulas and of the real possibility that the concepts explained above offer for a satisfactory solution of the problem of turbulence, it is of great aid to consider the motion along flow lines of nonzero curvature. For simplicity of treatment and by analogy with what has been done above, we shall suppose that the mean flow always takes place in a plane and along arcs of concentric curves. We shall denote the velocity tangent to the circle of radius r by v_t and the radial velocity by v_r. The natural extension of the concept of Prandtl (references 14, 15, and 16), leads to the conclusion that the fluid particles in being transferred from layer to layer as a consequence of the turbulent agitation, maintain their velocity moment constant with respect to the center of rotation - that is, $v_t\, r$ = constant = c. A fluid particle, therefore, which arrives at a layer of radius r after transversing a radial distance l' has decreased its own velocity by $\Delta v_t = -\frac{c}{r^2}\, l' = \frac{-v_t}{r}\, l'$, and since it had initially an excess of velocity $l' \frac{dv_t}{dr}$, the increase in velocity that it has with respect to that of the layer to which it arrives, becomes

$$v'_t = l' \left(\frac{dv_t}{dr} + \frac{v_t}{r}\right) = \frac{l'}{r} \frac{d}{dr} (v_t r) \qquad (15)$$

The mean increment in the momentum communicated to the element of radial thickness dr and of length $r\, d\,\varphi$, is therefore,

$$d^2 M = r\, d\varphi\, dr\, \overline{v'_r l'} \;\; \frac{d}{dr} (v_t r)$$

to which there corresponds a virtual tangential stress

$$\tau = \rho \, \frac{\overline{v'_r \, l'}}{r} \, \frac{d}{dr} (v_t \; r) \qquad (16)$$

different from zero on account of the correlation which in general exists between v'_r and v'_t, or between v'_r and l'.

There is thus obtained the equation of motion in the mean direction of flow under the assumption that v_t is constant along a flow line of the mean motion:

$$\frac{1}{\rho} \frac{1}{r} \frac{d}{dr} (\tau \; r^2) = \frac{1}{\rho} \frac{dp}{d\varphi} = \frac{1}{r} \frac{d}{dr} \left[r \; \overline{v'_r l'} \; \frac{d}{dr} (v_t \; r)\right] \qquad (17)$$

From (17) and (16), however, there follow erroneous results or results that have not been confirmed by experiments. Kàrmàn, in fact, observes that from (16) it would be deduced that τ becomes zero at the point at which $v_t r$ assumes its maximum value, and that the sign of τ is determined by that of $\frac{d}{dr} (v_t r)$. Now Wattendorf (reference 17) has recently conducted tests on a channel with circular axis, having an elongated straight rectangular cross section (ratio of the sides 1/18), so that the effect of the secondary flows might be assumed negligible, and from the fall of pressure along the channel, and from the direct measurement of the stress tangential to the walls, deduced the diagram for τ along the radius. Kàrmàn remarks that from this the relations given above between τ and $v_t r$ (fig. 3), are not confirmed. However, this observation of Kàrmàn and Wattendorf does not appear sufficient to the writer to invalidate the assumption of Prandtl, and therefore, indirectly, the hypothesis of constant moment of velocity during the turbulent transport.

In fact, from the equations of Navier, applied to the particular type of flow now considered, there is obtained for the instantaneous motion:

$$-\frac{1}{\rho}\frac{\partial p}{r\partial\varphi} = v_r\frac{\partial v_t}{\partial r} + v_t\left(\frac{v_r}{r} + \frac{1}{r}\frac{\partial v_t}{\partial\varphi}\right)$$

and for the mean motion

$$-\frac{1}{\rho}\frac{\partial p}{\partial\varphi} = \overline{v'_r\left[r\frac{dv'_t}{dr} + v'_t\right]} = \overline{v'_r\frac{d}{dr}(r\,v'_t)} \qquad (18)$$

which, with the assumption of Prandtl, becomes

$$\frac{1}{\rho}\frac{\partial p}{\partial\varphi} = v'_r\frac{d}{dr}\left[l'\frac{d}{dr}(v_t\,r)\right] \qquad (19)$$

Now since for the determination of τ, Wattendorf availed himself of the relation

$$\frac{1}{r}\frac{d}{dr}(r^2\tau) = \frac{\partial p}{\partial\varphi}$$

By comparing (19) with (17), it follow that the virtual stress which gives rise to the pressure gradient is not that which results from (16) but that given by

$$\tau = \frac{1}{r^2}\left[\int r\,\overline{v'_r\frac{d}{dr}\left[l'\frac{d}{dr}v_t r)\right]}\,dr + a\right]$$

A very remarkable observation on the theory of Prandtl is also made by Taylor, who remarks that the theory of the transport of the moment of momentum necessarily leads to expression (16) for the apparent stress, and therefore a rotation of the entire system as a rigid body with angular velocity Ω would change the tangential stresses, increasing them by an amount $2\rho\,\overline{v'_r\,l'}\,\Omega$.

Taylor's theory does not present this incongruence inasmuch as the addition of a constant vorticity to the entire fluid field does not have any effect on the transport of vorticity itself.

8. In this respect, too, however, it seems fitting to show how every difficulty for the determination of the

expression for the fluctuation velocity might be overcome not by assuming a certain transport length, and more or less plausible hypothesis based on the law of motion of the particles during their transfer from layer to layer, but by applying the principle of statistical similitude according to the concept of Kàrmàn mentioned above.

In the coordinates r, φ of the plane, the equation of the transport of the vortices becomes:

$$\frac{\partial}{\partial t} D\psi + v_t \frac{\partial}{\partial \varphi} D\psi + r\ v_r \frac{\partial}{\partial r} D\psi = 0 \tag{20}$$

where ψ denotes the flow function of the field and D the symbol

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

Setting ψ_1 the flow function of the motion corresponding to the turbulent agitation, and denoting by V_t and ω, respectively, the velocity and the mean vorticity at any point, equation (20) may be written:

$$\frac{\partial}{\partial t} D\psi_1 + \left(V_t + \frac{\partial \psi_1}{\partial r}\right)\frac{\partial}{\partial \varphi} D\psi_1 - \frac{\partial \psi_1}{\partial \varphi}\frac{\partial}{\partial r}(D\psi_1 + \omega) = 0 \tag{21}$$

Let us now make the assumption that with respect to a system of axes with origin at any point P of the field and moving with P with the same velocity, the field of flow in the neighborhood of P may be considered as stationary. We can then put, in the neighborhood of P

$$\left.\begin{aligned} V_t &= V_{t_P} + \left(\frac{dV_t}{dr}\right)_P (r - r_P) - \frac{V_{t_P}}{r_P}(r - r_P) + \ldots \\ \omega &= \omega_P + \left(\frac{d\omega}{dr}\right)_P (r - r_P) + \ldots \end{aligned}\right\} \tag{22}$$

since we should evidently put in the second member of the first of equations (22) the disturbing velocity of the fluid element.

There results immediately if, in analogy to what was already done by Kàrmàn for the rectilinear motion, we put

$$r - r_P = l\ r_1 ;\ \psi_1(r,\varphi) = Af(r_1,\varphi)$$

$$\left[-\frac{V_{t_P}}{r_P} + \left(\frac{dV_t}{dr}\right)_P\right](r - r_P)\frac{\partial}{\partial\varphi}D\psi_1 - \frac{\partial\psi_1}{\partial\varphi}\left(\frac{d\omega}{dr}\right)_P +$$

$$+ \frac{\partial\psi_1}{\partial r}\frac{\partial}{\partial\varphi}D\psi_1 - \frac{\partial\psi_1}{\partial\varphi}\frac{\partial}{\partial r}D\psi_1 = 0$$

and

$$\left[\left(\frac{dV}{dr}\right)_P - \frac{V_{f_P}}{r_P}\right] l\ r_1\ \frac{\partial}{\partial\varphi}Df - l^2\left(\frac{d\omega}{dr}\right)_P\frac{\partial f}{\partial\varphi} + \tag{23}$$

$$+ \frac{A}{l}\left[\frac{\partial f}{\partial r_1}\frac{\partial}{\partial\varphi}Df - \frac{\partial f}{\partial\varphi}\frac{\partial}{\partial r_1}Df\right] = 0$$

If the form of the function f which defines the fluctuations of the velocities due to the turbulent motion is to be independent of the particular position of the point P, then it is necessary that

$$\left[\left(\frac{dV_t}{dr}\right)_P - \frac{V_{t_P}}{r_P}\right] l = l^2\left(\frac{d\omega}{dr}\right)_P = \frac{A}{l}$$

or the characteristic length l becomes

$$l = \frac{\frac{dV_t}{dr} - \frac{V_t}{r}}{\frac{d\omega}{dr}} \tag{24}$$

while A is given by

$$A = l^2\left[\left(\frac{dV_t}{dr}\right) - \frac{V_t}{r}\right] \tag{25}$$

and therefore: <u>the amplitude of the turbulence velocity is proportional to</u> $l\left(\frac{dV_t}{dr} - \frac{V_t}{r}\right)$ <u>whereas Prandtl put the amplitude of the fluctuations themselves proportional to</u>

$$l\left(\frac{dV_t}{dr} + \frac{V_t}{r}\right)$$

Since the first expression in a rigid rotation is zero, it has no effect on the amplitudes of the oscillations themselves, and therefore on the expression for the virtual tangential stress.

9. It is interesting to compare (24) and (25) obtained by the principle of similitude of Kàrmàn from the equation of the transport of vortices with those that may be deduced by extending the procedure already mentioned of Dédébant, Schereschewski, and Wehrle to the type of flow we are now examining.

Let us assume that the distribution functions of the perturbed velocities at each point of the field may be made to become idential simply by a suitable change in the scale of the velocity and of the time, and let us suppose, following the method indicated by Dédébant, Schereschewski, and Wehrle, and moreover, the procedure used by Lorentz (reference 18), and by Chapman (reference 19) in the kinetic theory of gases, that the distribution function f is very nearly the same as that of Maxwell, so that indicating the latter function by f_o, ($f_o = a\, e^{-b(u'^2+v'^2)}$), $f = f_o\,(1 + \epsilon)$, in which ϵ is a small quantity of the first order. Assuming as unit velocity at each point of the field the mean quadratic variation σ of the disturbed velocity, the values of f_o are everywhere identical; the values of ϵ should therefore be the same. Now ϵ, which it should be possible to represent by a series in the components of the disturbed velocities u' and v' (which we shall now assume as referred to the mean value σ assumed as the unit), contains the terms which define the nonuniformity of the field; or those of the velocity of the general motion and those which Dédébant calls the "densities." In fact, in order to make the statistics of the disturbed velocities comparable, it is necessary that the time of observation of the velocities themselves vary from point to point in such a manner that the number of the fluctuations examined be everywhere equal. Now in time t the particles observed are proportional to t^2; resulting in a density, if N is the constant number of fluctuations considered proportional to N/t^2. The corresponding nonuniformity is therefore now defined by the variation of t. The distribution function, however, is an invariant; that is, it does not change in form or in value. Since f_o is by itself invariant, ϵ should be the same; but the invariant quantities, functions of u' and v' and of the derivatives along the same axes of t and of the general veloc-

ities of the components U and V are functions of the following elementary invariants:

$$u' \frac{\partial t}{\partial x} + v' \frac{\partial t}{\partial y}$$

corresponding in polar coordinates and by the assumed characteristics of the motion considered to $v'_r \frac{dt}{dr}$, and

$$u'^2 \frac{\partial U}{\partial x} + v'^2 \frac{\partial V}{\partial y} + u'v' \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)$$

corresponding to

$$v'_r v'_t \left(\frac{dV_t}{dr} - \frac{V_t}{r}\right)$$

There results, according to Chapman, the following expression for ϵ:

$$\epsilon = A\, v'_t v'_r \left(\frac{dV_t}{dr} - \frac{V_t}{r}\right) + B\, v'_r \frac{dt}{dr}$$

in which A and B, on account of the homogeneity of the formula, v'_t, v'_r being simply numerical, should be put proportional to t and σ, respectively. Since ϵ should be independent of the coordinates of the point at which the distribution function is determined $\sigma \frac{dt}{dr}$ = constant or σ is proportional to $\frac{1}{dt/dr}$ and $t \left(\frac{dV_t}{dr} - \frac{V_t}{r}\right)$ = constant, or t is proportional to $\frac{1}{\frac{dV_t}{dr} - \frac{V_t}{r}}$. But $\frac{dV_t}{dr} - \frac{V_t}{r}$ is equal to $\gamma_{r\varphi}$ and therefore, the mean quadratic variation of the velocity of fluctuation is

$$\sigma = \frac{{\gamma_{r\varphi}}^2}{\frac{d}{dr}\gamma_{r\varphi}} = l\, \gamma_{r\varphi} \tag{26}$$

where l is now given by

$$l = \frac{\gamma_{r\varphi}}{\frac{d}{dr}\gamma_{r\varphi}} \tag{27}$$

There is thus found the property that the amplitude of the velocity fluctuations due to the turbulent agitation, is proportional to $\gamma_{r\varphi}$ but the characteristic length l does not depend, as before, on the derivative of the rotation, but still on the derivative of the $\gamma_{r\varphi}$. This conclusion immediately makes the results of the preceding investigation more understandable: When the rotation is zero at every point, as in the velocity field which is generated in a viscous fluid about a rotating cylinder of infinite axial length, it is logical to assume that turbulence cannot take place.

Now while (26) and (27) do not tell us anything about this point, (24) and (25) assure us of the impossibility of a motion having the assumed characteristics since $\frac{d\omega}{dr} = 0$ and therefore $l = \infty$.

10. The considerations just developed bring out the difficulty of developing according to the methods of classical mechanics any theory whatever based on the concept of transport, inasmuch as the conclusions to which they lead are intelligible only insofar as the relations, which the introduction of the concept of mean distance l permit to be written down, may be interpreted by means of the principle of similitude of the turbulent oscillations, while a generalization of the relations themselves without this check easily leads to erroneous conclusions. Thus in the case of plane motion with rectilinear streamlines (4) and (6) are correct since dU/dy represents the excess of velocity with respect to the fluid layer at height y of the particle distance l, as well as the fluid displacement between the two contiguous layers, while (15) leads to unintelligible results since $\frac{dv_t}{dr} + \frac{v_t}{r}$ is proportional to the excess of moment of momentum while the fluid displacement is given by $\frac{dv_t}{dr} - \frac{v_t}{r}$. On the other hand, the reasons for such difficulties are easily understood, for although the concept of the mixing length is very intuitive it is not prescribed for a fluid particle which penetrates a certain layer after having traversed more layers, how many of the dynamic and kinematic characteristics that it possessed initially, it conserves after each crossing.

11. With the problem of the transport of momentum and of vorticity is intimately connected the theory - re-

markable in many respects - developed recently by G. D. Mattioli (reference 20). The latter supposes that in turbulent agitation each fluid particle maintains constant not only its momentum, as Prandtl assumes, but also its rotation, as assumed by Taylor. Mattioli, who considers the motion in tubes of circular cross section, derives the following equations:

$$\frac{d}{dr}\left(\epsilon r \frac{dV}{dr}\right) = \frac{1}{\rho}\frac{dp}{dx} r; \quad \frac{d}{dr}\left(\epsilon r \frac{d^2 V}{dr^2}\right) = a\ r \frac{d^2 V}{dr^2} \qquad (28)$$

where V is the general velocity, $\epsilon = l^2 k$, in which l has the meaning given above, while k is a function which has the dimensions of a frequency and is called by Mattioli the "mixing frequency."

Now if Prandtl's theory is applied to this particular type of flow, we have, using the above notation

$$\frac{1}{\rho}\frac{dp}{dx} r = \frac{d}{dr}\left(\overline{l' v'} r \frac{dV}{dr}\right)$$

which is identical with the first of equations (28) putting

$$\overline{l' v'} = l^2 k \qquad (29)$$

On the other hand, from the equation of Navier, which represents the motion in the direction of the tube axis, we have:

$$\frac{1}{\rho}\frac{dp}{dx} = \overline{v'\omega'}$$

where ω' is the instantaneous fluctuation of the vorticity at any point. Therefore, according to the Taylor concept,

$$\omega' = l' \frac{d^2 V}{dr^2}$$

from which

$$\frac{1}{\rho}\frac{dp}{dx} = \overline{v' l'} \frac{d^2 V}{dr^2}, \quad \text{and by (29)} \quad \frac{1}{\rho}\frac{dp}{dx} = l^2 k \frac{d^2 V}{dr^2}$$

or

$$\frac{d}{dr}\left(\epsilon r \frac{d^2 V}{dr^2}\right) = \frac{1}{\rho}\frac{dp}{dx} \qquad (30)$$

Equation (30) does not give satisfactory results, however. Taylor, as explained above, concludes therefrom that the theory of transport of vorticity could not be applied in this case except by considering the turbulent agitation as it is actually, namely, three-dimensional. Mattioli, whose theory is one-dimensional, insofar as the transport of the masses takes place only with radial turbulent velocities, and the continuity of the mass itself is restored by associating with the discontinuous turbulent transport a continuous transport that is always radial, does not at all consider (30) or the connections set up by the equations of Navier, and since he shows that the first member of (30) represents the increment of vorticity communicated to an element of fluid, he deduces that, for the permanence of the motion, there should be applied to the same element a corresponding couple which he puts equal to $ar\frac{d^2V}{dr^2}$.

He maintains (reference 21) that this couple is due to the viscosity of the fluid and therefore dissipates at each point the increment of vorticity which the turbulent agitation produces. This interpretation is open, however, to some reflection even if the fact that the fluid, in the original theory, was assumed to be perfect.

The first member of the second of equations (28), which I shall denote by A, represents the excess of vorticity which is to be dissipated in the time t_o corresponding to the mean period of the oscillations. Now the velocity of dissipation of the vorticity ω in a fluid of kinematic viscosity ν is given by $\frac{d\omega}{dt} = \nu\Delta\omega$, by which the mean velocity $\overline{\frac{d\omega}{dt}}$ in time t_o may be put $\overline{\frac{d\omega}{dt}} = \nu\overline{\Delta\omega}$ and the dissipated vorticity will be

$$\overline{\frac{d\omega}{dt}}\, t_o = \nu\, t_o\, \overline{\Delta\omega} = A \tag{31}$$

In this way it may be understood why in the second member of (28) there does not appear the viscosity, since putting the time t_o necessary for the dissipation of A proportional to $1/\nu$ seems logical, and in any case agrees with what may be deduced in several simple cases (reference 22). Equation (31), however, would make A depend not on the second derivative of V but on the third derivative.

Apart from the difficulty of justifying the second of equations (28) which has already been the subject of some remarks by the author (reference 23), the theory of Mattioli leads to results which agree both with those obtained experimentally and with those deduced from the theory of Prandtl and Kàrmàn; in particular as regards the logarithmic law for the resistance of smooth tubes as indicated by Mattioli himself (reference 24) and the logarithmic distribution law of the velocity at the wall of the obstacle, as may easily be deduced.

11. The problem of the transport of momentum or vorticity in turbulent flow, presents a particularly suggestive aspect and is susceptible of a general solution if the phenomenon is considered as belonging to the domain of statistical physics and is therefore studied with the methods appropriate to the latter. In this connection, it should be observed that the statistical character of the flow is assured, not only by the irregularity of the velocity fluctuations, but by the existence of universal laws of distribution of velocity independent of any initial condition of the flow. Important progress in this sense is represented by the theory recently developed by H. Gebelein (reference 25).

Gebelein assumes that in the fluid motion, the circulation of the momentum, and the energy diffuse in space according to the same law of probability that governs the diffusion of matter in statistical phenomena; that is, in which the motion of any particle from point to point is not determined in an unequivocal manner, with certainty (as in problems of deterministic mechanics), but takes place according to a law of probability, which is a time and space function. Under these conditions, the density of the fluid in a space element, which is proportional to the probability with which molecules of fluid are found in the element considered, received in each elementary volume an increment $\frac{d\rho}{dt} + \rho$ div. V, which by the continuity of motion should be zero, and whose analytical expression is given by the equation of Kolmogoroff:

$$\frac{d\rho}{dt} + \rho \text{ div. } V = \frac{\partial\rho}{\partial t} + \sum_{i=1}^{i=3} \frac{\partial}{\partial y_i}(u_i\rho) - \sum_{\substack{i=1\\k=1}}^{\substack{k=3\\i=3}} \frac{\partial^2}{\partial y_i \partial y_k}(b_{ik}\rho) = 0 \qquad (32)$$

where y_i now represents any of the three coordinate axes (y_1, y_2, y_3), u_i the component in the y_i direction of the mean velocity of the motion, while b_{ik} depends on the mean quadratic variations of the velocity itself. More precisely, let $P(x_1, x_2, x_3, y_1, y_2, y_3; t; t+\Delta)$ be the function which defines the probability with which a particle, which at time t is at the point whose coordinates are x_1, x_2, x_3 will be at the point y_1, y_2, y_3 at time $t+\Delta$, and A_i and B_{ik}, the moments of first and second order, respectively, of P, or the mean values and the quadratic variations, respectively, of the distribution function P, that is

$$A_i = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (y_i - x_i)\, P\, dy_1\, dy_2\, dy_3$$

$$B_{ik} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (y_i - x_i)(y_k - x_k)\, P\, dy_1\, dy_2\, dy_3$$

Then $$u_i = \lim_{\Delta=0} \frac{A_i}{\Delta}; \quad b_{ik} = \lim_{\Delta=0} \frac{B_{ik}}{2\Delta}$$

Thus the equations of motion, according to the statistical mechanics, do not require a knowledge of the form of the function P, but only of the values of the static moments and of the second order moments of P itself, precisely as in the deterministic mechanics, in which the governing equations of motion contain only the static moments of inertia of the mass in motion and do not require a knowledge of the form of the distribution law of the mass.

According to the hypothesis given above by Gebelein, if the mechanical characteristics are diffused by the same law as the density, the increment of the component of the momentum of an element $d\tau$ in the direction y_1,

$$\frac{d\rho u_1}{dt} + \rho u_1 \Sigma \frac{\partial u_i}{\partial y_i}$$

which, on account of the translation equilibrium along the same axis, is equal to $-\partial p/\partial y_1$ is given by

$$\frac{\partial(\rho u_1)}{\partial t} + \Sigma \frac{\partial}{\partial y_i} (u_1 \rho u_i) - \Sigma \frac{\partial^2}{\partial y_i \, \partial y_k} (b_{ik} \rho u_1) = - \frac{\partial p}{d y_1}$$

while analogous equations are obtained for the directions y_2, y_3.

If, instead of the momentum, it is assumed that it is the vorticity that is diffused, there is obtained:

$$\frac{\partial(\rho \omega_1)}{\partial t} + \Sigma \frac{\partial}{\partial y_i} (u_i \rho \omega_1) - \Sigma \frac{\partial^2}{\partial y_i \, \partial y_k} (b_{ik} \rho \omega_1) =$$

$$\Sigma \frac{\partial}{\partial y_i} \left[\rho u_2 \frac{\partial u_i}{\partial y_3} - \rho u_3 \frac{\delta u_i}{\delta y_2} \right]$$

12. The solution of the problem requires a knowledge of the coefficients b_{ik} and this is obtained by Gebelein by using a relation between the mean quadratic variation of the velocity and the characteristics of the mean motion, and the expression for the "characteristic time" T_o (Verweilzeit) which is a measure, so to speak, of the statistic or deterministic character of the phenomenon under consideration, in the sense that the phenomenon itself observed for an interval of time less than T_o appears to be governed by a deterministic law while observed at intervals of time very large compared to T_o, appears as a statistic phenomenon.

Gebelein assumes as a fundamental theorem, that the mean quadratic variation of the perturbed velocity is proportional to the fourth root of the vorticity, corresponding to the mean motion

$$\alpha \equiv \sqrt[4]{\omega} \tag{33}$$

but this proposition is, in fact, not demonstrated, at least not in a convincing manner, and therefore appears essentially as a hypothesis to which Gebelein gives an experimental confirmation based on the tests of Nikuradse, of 1926. Since, however, the determination of the mean quadratic variation in these experiments is not made by direct measurement, it seemed proper to the author to verify the hypothesis of Gebelein by means of the results obtained experimentally with a hot-wire anemometer by Reichardt and by Wattendorf in a very elongated rectangular

channel so as to approximate as far as possible two-dimensional motion. The results which have been calculated using the values indicated above, are shown in figure 4 in which is also indicated by dotted line the theoretical diagram according to the assumption of Gebelein. The approximate character of (33) is deduced by the comparison. On the other hand, if the motion takes place with a variable pressure gradient in the direction of the motion itself between a convergent and a divergent section, (33) does not give any indication of the eventual dissymmetry between the mean quadratic variations of the components u' and v' of V, while it is known, for example, that in the motion within a convergent $\sqrt{\overline{u'^2}}$ decreases, along the axis of the channel, about in proportion to L, with which the mean velocity increases, while $\sqrt{\overline{v'^2}}$ increases in proportion to $L^{1/2}$. In this connection, it is very desirable that systematic tests be conducted to determine the law of variation of the dependence of $\sqrt{\overline{u'^2}}$ and of $\sqrt{\overline{v'^2}}$ on the mean motion, the knowledge of which is essential for any theory of turbulence.

The characteristic time T_0 is determined by Gebelein after an actual analysis of the reasons which could give rise to a statistic phenomenon destroying the causes which tend to produce it in a deterministic manner. The cause is essentially the same as that leading to the production of vortices at the contact of rigid walls which, either on account of the roughness of the walls themselves or the disturbances which may be produced, for example, at the mouth of the tubes, takes place not according to a determinate law but by pure chance. This is what makes the vortices in the boundary layer, in two-dimensional motion, have only in the mean a direction parallel to the walls and normal to the mean motion; to this mean verticose layer is added a layer in which the axes of the verticese are disposed along any direction whatever. A comparison of the results to which the theory leads with those obtained experimentally would indicate that the axes of the perturbed verticese are essentially directed along the mean motion, and this conclusion appears in singular agreement with the hypothesis of Prandtl already referred to, namely, that the transport of particles in bound turbulence takes place by just the vortices having this disposition.

13. The calculation of T_0 has been made by Gebelein for several simple cases (flow about a flat plate, within a convergent and a divergent tube, and within a straight tube); and some of the results obtained by him by relatively simple methods, although at the price of over-simplified assumptions, we shall now point out in what follows.

In the problem of the flow about a flat plate, the equation of Gebelein obtained by assuming that the momentum is diffused according to a law of probability corresponding to that of the diffusion of the mass, considering the fluid as perfect, becomes

$$\frac{1}{\rho}\frac{d\tau}{dy_2} = \frac{d^2}{dy_2}\left(\overline{v'^2}\ T_0\ u_1\right) = 0 \tag{34}$$

while the diffusion of the vorticity leads to the equation

$$\frac{1}{\rho}\frac{d\tau}{dy_2} = \frac{d}{dy_2}\left(\overline{v_2'^2}\ T_0\ \frac{du_1}{dy_2}\right) = 0 \tag{35}$$

By the assumption made on the relation between $\overline{v'^2}$ and $\frac{du_1}{dy_2}$ and with the expression for T_0 calculated by Gebelein, we have for the two cases, respectively:

$$\frac{d^2}{dy^2_2}\left(C_1\ y^2_2\ \frac{du_1}{dy_2}\ u_1\right) = 0 \tag{36}$$

$$\frac{d}{dy_2}\left[C_1\ y^2_2\ \left(\frac{du_1}{dy_2}\right)^2\right] = 0 \tag{37}$$

Equation (36) gives $u\ (y_2) = \sqrt{a + b \log y_2}$, while from (37) is obtained

$$u_1(y_2) = a + b \log y_2 \tag{38}$$

which is the well-known logarithmic formula confirmed by the tests of Nikuradse, mentioned above.

Thus, it is the vorticity and not the momentum which is diffused according to the same law of probability as that corresponding to the diffusion phenomenon of the mass.

This conclusion may, however, be invalidated by the assumptions which are at the base of the calculations; in particular, it may be difficult to admit that the mean quadratic variations of the fluctuation velocities for the flow about a flat plate are proportional to $\sqrt{\frac{du_1}{dy_2}}$ and therefore by (38) to $y^{-1/4}$. This even appears to be contrary to the principle of the constant correlation between u' and v', which would lead, in this problem, and in accordance with the principle of Kàrmàn, to the equation

$$\sqrt{\overline{u'^2}} = \sqrt{\overline{v'^2}} = \text{constant} \tag{39}$$

It may, nevertheless, easily be seen that the conclusion stated above holds true even if equations (39) are admitted. In fact, analogous to what is done in the kinetic theory of gàses, it is possible to assume

$$T_0 = \frac{l}{\sqrt{\overline{v'^2}}}$$

if l has the meaning of "the mean free path" already considered many times above; there is deduced

$$\frac{d}{dy}\left[C\sqrt{\overline{v'^2}}\; l\,\frac{du}{dy}\right] = 0$$

or

$$l\,\frac{du}{dy} = \text{constant}$$

which leads to (38), as already previously derived.

14. The theory of Gebelein, developed according to the methods of statistical mechanics, thus allows the affirmation of the principle of diffusion of the vorticity in turbulent motion. This principle had been confirmed by Taylor, but its application, using the methods of classical mechanics, had not led to a satisfactory solution. The real reason for this fact appears to depend on the circumstance that a theory based on the concept of transport and developed according to classical mechanics, would require a knowledge of the history of the particle during the transport, whereas statistical mechanics, observing

the phenomenon at intervals of time for which every effect of cause is destroyed, renders itself independent of any knowledge of the states assumed by the particle during its deterministic motion.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

REFERENCES

1. Prandtl, L.: Proceedings of Second International Congress of Applied Mechanics, Zurich, 1926.

2. Taylor: Philosophical Transaction A, vol. 215, 1915, p. 1.

3. Taylor, G. I.: The Transport of Vorticity and Heat through Fluids in Turbulent Motion. Proceedings of the Royal Society of London, A, vol. 135, 1932, p. 685.

4. Fage, A.: The Behavior of Fluids in Turbulent Motion. The Jour. of the Roy. Aero. Soc. of London, July 1933, p. 573.

5. Kàrmàn, Th. v.: Proceedings of Third International Congress for Applied Mechanics, Stockholm, vol. I, 1930, p. 85.

6. Nikuradse, J.: Gesetzmässigheiten der turbulenten Strömung in glatten Röhren. Forschungsheft 356, Z.V.D.I., 1932.

7. Schlichting, H.: Über das ebene Windschattenproblem. Ingenieur-Archiv, December 1930, p. 533.

8. Fage, A., and Falkner, V. M.: Note on Experiments of the Temperature and Velocity in the Wake of a Heated Cylindrical Obstacle. Proceedings of the Royal Society of London, vol. 135, 1932, p. 702.

9. Elias, Franz: The Transference of Heat from a Hot Plate to an Air Stream. T.M. No. 614, N.A.C.A., 1931.

10. Fage, A., and Townend, H. C. H.: An Examination of Turbulent Flow with an Ultramicroscope. Proceedings of the Royal Society of London, vol. 135, 1932, p. 656.

11. Wattendorf, F. L., and Kuethe, A. M.: Investigations of Turbulent Flow by Means of the Hot-Wire Anemometer. Physics, vol. V, June 1934, p. 153.

12. Reichardt, G.: Die quadratischen Mittelwerte der Längsschwankungen in der turbulenten Kanalströmung. Z.f.a. M.M., 1933, p. 177.

13. Dédébant, G., Schereschewski, and Wehrle: Sur la similitude statisque dans les mouvements turbulents des fluides. Comptes Rendus de l'Academie des Sciences, January 14, 1935, p. 203.

14. Prandtl, L.: Einfluß stabilisierenden Kräfte auf die Turbulenz. Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete. Aachen, 1929, p. 1.

15. Taylor, G. I.: The Transport of Vorticity and Heat through Fluids in Turbulent Motion. Proceedings of the Royal Society of London, A-1932, vol. 135, p. 696.

16. Kàrmàn, Th. v.: Proceedings of the Fourth International Congress for Applied Mechanics, Cambridge, 1934, p. 81.

17. Wattendorf, F. L.: A Study of the Effect of Curvature on Fully Developed Turbulent Flow. Proceedings of the Royal Society of London, ser. A, vol. 148, no. 865, February 1935, p. 565.

18. Lorentz, H. A.: Theory of Electrons.

19. Chapman, S.: Philosophical Transactions, 1917.

Rocard, Y.: L'hydrodynamique et la theorie cinetique des gas, 1932, p. 40.

20. Mattioli, G. D.: Teoria della turbolenza. Rendiconti della R. Accademia Nazionale dei Lincei, vol. 17, February 1933, p. 217.

21. Mattioli, G. D.: Dissipazione vorticosa nei regimi turbolenti. Rendiconti della R. Accademia Nazionale dei Lincei, 1935, p. 87.

22. Müller, W.: Einführung in die Theorie der zähen Flussigkeiten. Leipzig, 1932, p. 106.

23. Ferrari, C.: Sulla teoria della turbolenza. Rendiconti della R. Accademia Nazionale dei Lincei, 1934, p. 100.

24. Mattioli, G. D.: Sopra l'influenza della viscosità del fluido nei regimi turbolenti. Atti della Reale Accademia delle Scienze di Torino, 1935, p. 220.

25. Gebelein, H.: Turbulenz. Berlin, 1935, (Edit. Springer).

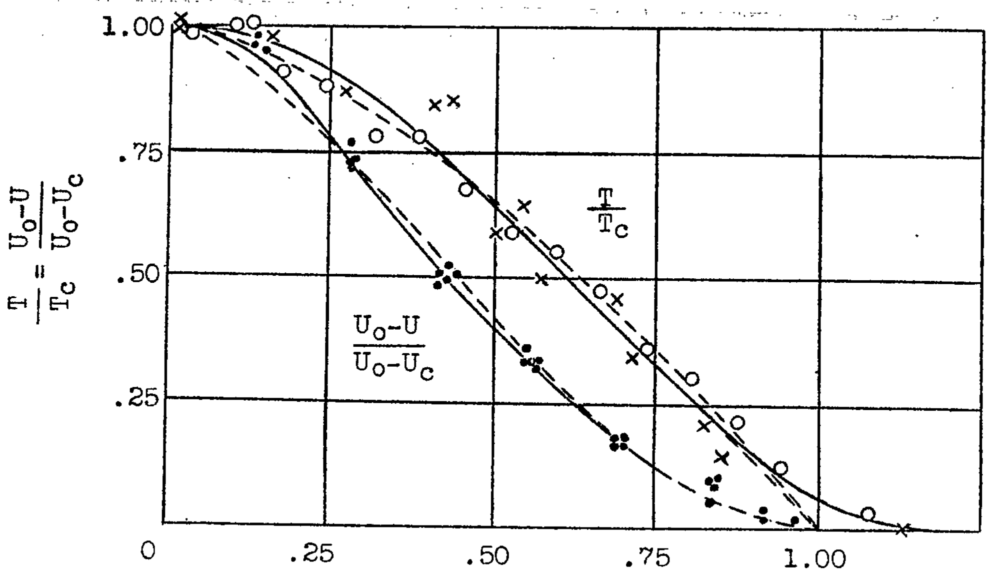

Figure 1.

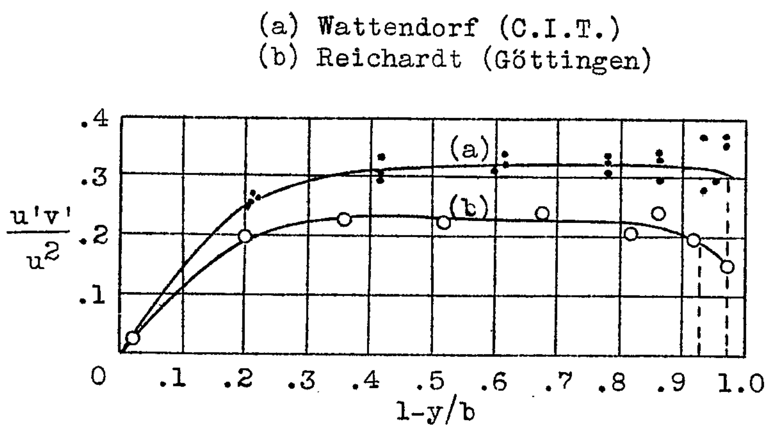

Figure 2.

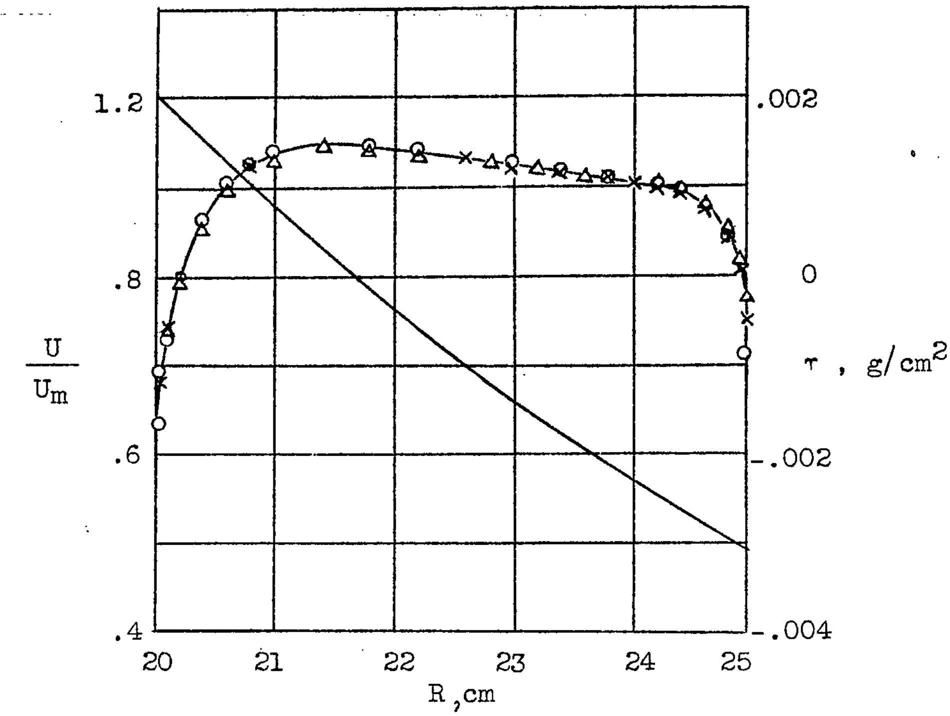

Figure 3.

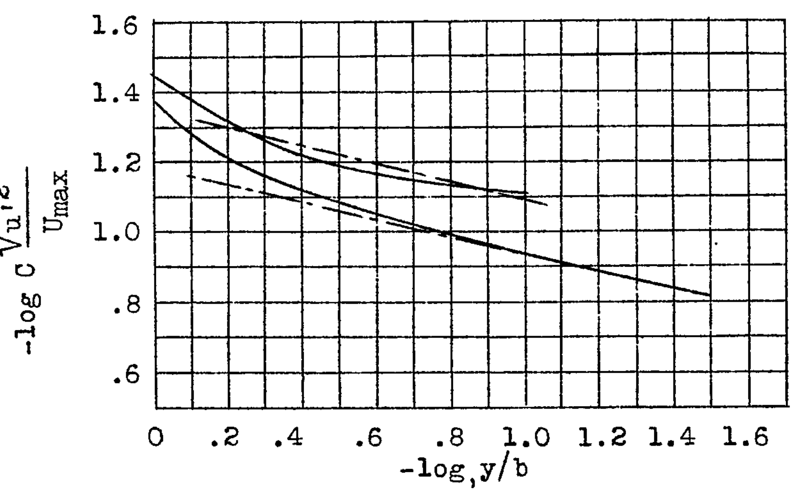

Figure 4.