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CONTRIBUTIONS TO THE THEORY OF INCOMPLETE TENSION BAY*

By E. Schapitz

In a metal panel stressed simultaneously in shear and compression, the stress distribution immediately after buckling is still nonuniform and the second principal stress is not as yet negligible. This condition designated as "incomplete tension bay" is described by the present theory for the general case of combined stresses and elastically flexible stiffeners. The behavior of the buckled plate is marked by two factors, one of which characterizes the distribution of stress, the other the geometrical deformation which occurs with curved metal panels. The variation of the factors with the stresses and deformations must be obtained by experiments.

SUMMARY

The present report offers an approximate theory for the stress and deformation condition after buckling of the skin in reinforced panels and shells loaded in simple shear and compression and under combined stresses. The theory presents a unified scheme for stresses of these types; it is based upon the concept of a nonuniform stress distribution in the metal panel and its marked power of resistance against compressive stresses ("incomplete" tension bay). The stress distribution in the metal skin is designated by the factor ω which disappears with the gradual approach of the condition of "complete" tension bay; that is, the uniform stress distribution. In pure compressive stress a relation exists between factor ω and the "apparent" width. The geometrical distortion of curved metal panels under shear is expressed by the factor ζ_y which also varies with the load and, on approaching the complete tension panel, tends toward its maximum value $\zeta = -\frac{\varphi_1^2}{24}$. At the point of buckling the calculation

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goes over into the theorems of the elasticity theory.

The factors ω and ζ_y are obtainable from stress and deformation measurements on shell bodies. The stress distribution factor ω in flat panel strips can, by virtue of Wagner and Lahde's experiments, be expressed in a simple formula. Further experiments, particularly on curved panels, are under way at the D.V.L.

The condition immediately after buckling is in need of a theoretical analysis by more accurate energy methods, because this field presents great experimental obstacles, and such an analysis would afford a check on several basic assumptions of the calculation method.

I. INTRODUCTION

The stresses and deformations in thin-walled, reinforced panels and shells under shear stress are usually calculated for the condition after buckling of the skin by Wagner's tension bay theory (reference 1). This theory assumes a uniform stress condition in a field bounded by stiffeners while disregarding for the time the compressive stresses taken up by the buckled sheet. The buckling stiffness of the sheet can be allowed for by superposing the stress condition prior to buckling on the tension bay (reference 2).

The experiments of Wagner and Lahde on metal strips (reference 3), the torsion tests of the D.V.L. on stiffened circular cylindrical shells (reference 4), and much practical experience have shown that the condition of Wagner's tension bay is asymptotically reached when the shear stress transmitted by the metal skin reaches around 50 to 100 times the buckling shear stress. But the interim stage immediately after buckling is marked by nonuniform stress distribution and by a considerable influence of the compressive stresses still taken up by the buckled sheet. This condition is hereinafter designated as "incomplete tension bay." The stiffeners in the incomplete tension bay are under smaller load than Wagner's theory would stipulate.

In monocoque airplane designs, buckling of the covering is usually permitted at a load producing stresses between 10 and 50 percent of the allowable stresses.

That is, the covering of the structural part in failing condition is still in the condition of incomplete tensional bay. The need for a far more refined calculation of bodies and wings makes a comprehensive investigation of this condition desirable, although the calculation according to Wagner's assumptions is always on the "safe" side.

The purpose of the present investigation is to develop an approximate theory for the calculation of the stresses and deformations if stiffened panels and shells after buckling of the skin. The theory covers the cases of pure shear, pure compression, and combined stress. The calculation of the stress distribution in the skin buckles and the determination of the form of buckle by energy methods (reference 5) is foregone in favor of an assumed stress distribution law which is adapted to the test results through the introduction of a characteristic factor. The geometrical deformation in curved sheets is treated in the same fashion. These factors can be obtained from tests with metal panels, partial shells, or reinforced circular cylindrical shells.

After a survey of the buckling conditions for flat and curved metal panels, we proceed to a detailed analysis of the fundamental assumptions underlying the approximate theory and to the presentation, according to this theory, of the analysis of the stresses and the deformations in stiffened shells with buckled skin. Lastly, the determination of the stress distribution factor from the experimental results by Wagner and Lahde (reference 3) is given.

The particular problem treated by the approximate theory, is as follows: A cylindrical shell section of uniform (vanishing in the limiting case) curvature (fig. 1) is simultaneously stressed by shearing and normal forces. The applied loads are uniformly distributed over the periphery; the normal forces act only in the direction of the elements of the cylinder.

What, in the case of elastically flexible stiffeners, are the stresses and deformations at given load or the forces necessary to reach predetermined deformations? Special cases are: pure compression, treated heretofore chiefly from the point of view of apparent width (reference 6) and pure shear (references 4, 3, and 2).

II. THE BUCKLING OF METAL PANELS UNDER SIMPLE AND COMBINED STRESSES

The buckling conditions for flat and curved panels and strips as well as for unstiffened circular cylindrical shells under simple stresses, have been compiled in Table of Formulas (p. 22). A discussion of the formulas can be found in Ebner's report on shell bodies (reference 7), while a comprehensive list of references on this subject is given in the report by Ebner and Heck (reference 8).

The conditions of buckling for combined loading due to normal and shearing forces have not been cleared up as yet for all cases. For the flat strip under combined stress, Wagner (reference 9) gives:

$$\frac{\sigma_k}{\sigma_o} = 1 - \left(\frac{\tau_k}{\tau_o}\right)^n \qquad (1)$$

where σ_k and τ_k are the critical stresses for buckling under combined load, and σ_o and τ_o the critical stresses for buckling in simple compression and shear, respectively. He gives $n = 2$ for the exponent n for the long strip, while Chwalla (reference 10) holds this approximation admissible for square plates only, and claims that for greater ratios of a/b (fig. 2), a curve composed of several branches is required. Bridget, Jerome, and Vosseler (reference 11) also give the formula (1) with an exponent $n = 3$ for the critical combined stress of the unstiffened circular cylinders; although the test data, upon which this is based, disclose a wide scatter. Wagner and Ballerstedt (reference 12) found the relation (1) confirmed in their buckling tests with short thin-walled metal cylinders in the range of superposed compressive stresses, while arriving at a linear relationship between $\frac{\sigma_k}{\sigma_o}$ and $\frac{\tau_k}{\tau_o}$ with superposed tension stresses. Concerning the curved panel or curved strip, detailed studies are as yet lacking, although it is likely that formula (1) will still be applicable for determining the critical stresses and that exponent lies between $n = 2$ and $n = 3$. Hereinafter, the relation expressed by formula (1) is designated as "buckling function" (fig. 2). It must always be symmetrical to the σ_k/σ_o axis and have a horizontal tangent for

$\tau = 0$, because a reversal of direction of shear stress does not affect the critical stresses.

For further studies, the <u>critical principal stress</u> σ_{2k}, which at the critical stress (σ_k, τ_k) acts as maximum compressive stress, is of importance. If σ_k is the critical normal stress and τ_k the critical shear stress, the critical principal stress σ_{2k} follows from the general law of the plane stress condition as:

$$\sigma_{2k} = \frac{1}{2} \left(\sigma_k - \left|\sqrt{\sigma_k^2 + 4\ \tau_k^2}\right|\right) \qquad (2)$$

Here the normal stresses, being compressive, are put down negative. For $\sigma_k = -\sigma_o$, σ_{2k} becomes $= -\sigma_o$ and for $\sigma_k = 0$, σ_{2k} becomes $= -\tau_o$. σ_{2k} remains a compressive stress even by superposition of great tensile stresses. If the buckling function is plotted in the $\sigma - \tau$ system of coordinates, then Mohr's circle affords a simple graphical means of determining the critical principal stress σ_{2k} (fig. 3). The radius of Mohr's circle corresponds to the principal shear stress:

$$\tau_{max} = \sqrt{\frac{1}{4}\ \sigma^2 + \tau^2}$$

Buckling is accompanied by a stress rearrangement. In pure compression the compressive stress in the center of the panel between stiffeners remains almost constant, and of the approximate order of magnitude of the buckling stress (reference 5), while with increasing load the stress in the stringers and in the adjoining parts of the panels increases materially. In pure shear the second principal stress σ_2 after buckling is determined by the flexural stiffness of the skin and by the membrane stresses set up at the skin wrinkles as a result of their curvature in the principal tension direction. Here also the sides of the panels are able to carry more than the center. The principal tensile stress σ_1 absorbs the principal share of the shear stress and the stiffeners which carried no load before buckling, receive after buckling considerable stresses which may lead to failure of the structure. (Cf. Schapitz and others.)

III. THEORETICAL CONSIDERATIONS CONCERNING THE INCOMPLETE TENSION BAY

1. Generel Stress and Strain Equations

The calculation of the "incomplete tension bay" formed after buckling, rests on the theories of plane strain and plane stress. Figure 4 shows a section between two stringers and two bulkheads. On curved shell sections (fig. 1) the y-direction is in peripheral direction, and the x-direction is in direction of the elements of the surface and consequently, that of the external normal forces.

According to the theory of plane strain, the angle α' between the principal strain direction and the positive x-axis (ϵ is the maximum positive strain) is:

$$\tan^2 \alpha' = \frac{\epsilon - \epsilon_x}{\epsilon - \epsilon_y} \qquad (3)$$

Assuming for the present, given normal strains, the shearing strain is:

$$\gamma_{xy} = 2 \cot \alpha'(\epsilon - \epsilon_x) = 2 \tan \alpha'(\epsilon - \epsilon_y) = 2\sqrt{(\epsilon - \epsilon_x)(\epsilon - \epsilon_y)} \qquad (4)$$

In pure tension, $\epsilon = \epsilon_x$ and $\tan \alpha' = 0$; in pure compression, $\epsilon = \epsilon_y$ and $\cot \alpha' = 0$. From (4) it follows γ_{xy} disappears in both cases.

In curved metal panels the irregular distortion bound up with the buckling is replaced by a mean contraction of the skin in peripheral direction. This shortening of the periphery due to wrinkling (designated hereafter by ζ_y) is determined as follows: The traces of the central surface of the buckled skin on the longitudinal section planes formed by the radii of curvature and the elements of the cylinder, are of wave form (fig. 5), and permit by sufficiently great shell length the unique determination of one mean skin radius each in every sectional plane. The base line of the cylindrical surface ("substitute surface") formed by these mean radii, passes between the stiffeners usually between the original arc and the chord and is, with great shell length, symmetrical to the "panel center" (fig. 5). If b_1' is its arc length between the stiffeners, and b_1 the original arc length, the wrinkling ζ_y is defined by $\zeta_y = \frac{b_1' - b_1}{b_1}$. If the wrinkling shortens the arc to the chord, then (with $\varphi_1 = b_1/r_H$)

$$\zeta_y = \frac{2r_H \sin\frac{\varphi_1}{2} - r_H \varphi_1}{r_H \varphi_1}$$

The series development of $\sin\frac{\varphi_1}{2}$ leads, when broken off after the term of the third degree, to the form $\zeta_y = -\frac{\varphi_1^2}{24}$.* It is the highest value the wrinkling can assume. In flat metal panels the wrinkling vanishes by virtue of the wave symmetry with respect to the original center plane of the metal skin.

The investigation of the stress condition is based on the consideration of the equilibrium of a skin element (fig. 6). If σ_n is the normal stress in the section perpendicular to the x-direction, σ_r the normal stress in the section perpendicular to the y-direction, and α the angle of principal axis and x-axis, the equilibrium equations read:

$$\sigma_1 = \sigma_n + \tau \tan\alpha \quad \text{and} \quad \sigma_1 = \sigma_r + \tau \cot\alpha$$

From the general laws of the stress condition follows:

$$\sigma_1 + \sigma_2 = \sigma_n + \sigma_r$$

Transformed, these relations give:

$$\left.\begin{aligned} \sigma_1 &= \frac{\tau}{\sin}\,\alpha \cos\alpha + \sigma_2 \\ \sigma_r &= \tau \tan\alpha \qquad + \sigma_2 \\ \sigma_n &= \tau \cot\alpha \qquad + \sigma_2 \end{aligned}\right\} \tag{5}$$

From the fundamental equations of the plane stress condition:

*This method of expressing the wrinkling was first introduced by H. Wagner.

$$\frac{\partial \sigma_r}{\partial y} + \frac{\partial \tau}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \sigma_n}{\partial x} + \frac{\partial \tau}{\partial y} = 0$$

it is seen that with equal shear stress at every point $\frac{\partial \sigma_r}{\partial y} = 0$ and $\frac{\partial \sigma_n}{\partial x} = 0$. So, if the second principal stress σ_2 varies with y, then with constant shear stress, $\tan \alpha$ must also vary with y. If angle α is assumed constant, the stress σ_2 must then also be put down constant. The assumptions for α and σ_2 are treated in the following section.

2. The Fundamental Assumptions for the Incomplete Tension Bay

Like Wagner's ideal tension bay, the incomplete tension bay is an auxiliary concept. The complicated stress and deformation processes immediately after buckling of the sheet are to be analyzed by a method which rests on simple premises and is adaptable to the actual behavior of the panel by introducing suitably chosen index values or factors.

The auxiliary concept of ideal tension bay is first extended to include the "complete" tension bay. The concept of uniform stress and strain condition is preserved at which the stress and strain principal axes are everywhere mutually coincident. But the second principal stress σ_2 is, in general, not neglected. Its magnitude is assumed as independent of the load; we put it down at $\sigma_2 = \sigma_{2k}$. For curved metal panels the wrinkling is assumed to be given by its maximum value of $\zeta_y = -\varphi_1^2/24$ and independent of the load.

The concept of "incomplete" tension bay premises a nonuniform distribution of the normal stress σ_n acting in the sections at right angles to axis x (fig. 1). In the middle of the panel between the stringers (fig. 4), the stress condition is, as in the tension bay theory, premised on an angle of the principal axis α_m and on a second principal stress σ_{2m}; this gives, according to (5), a normal stress σ_{nm} in the middle of the panel. But at the fixation points of the skin this stress is considered as participating with the stringers and the normal stress $\sigma_n = \sigma_{nL}$ is computed from the compressive and bending stresses by the latter. Thus the width of the skin by which it touches the stringer is counted as part of the stringer (fig. 7). In the parts of the bay between center and stringers the normal stress σ_n is to follow the stress distribution law:

$$\sigma_n = \sigma_{nm} - (\sigma_{nm} - \sigma_{nL}) \sin^{1/\omega}\left(\frac{\pi y}{b_1}\right) \tag{6}$$

At panel center $y = 0$, $\sigma_n = \sigma_{nm}$; for $\omega = 0.5$ the distribution curve is a simple sinusoidal wave. For simple compressive stress this distribution law is suggested by Lahde and Wagner's measurements (reference 6). The exponent ω is the factor for the stress distribution in the incomplete tension bay.

As the load increases the stress distribution approaches that of the complete tension bay; according to figure 7, this process corresponds to a decrease of ω toward zero. The principal stress σ_{2m} in panel center is assumed as $\sigma_{2m} = \sigma_{2k}$, so that, according to (5):

$$\sigma_{nm} = \tau \cot \alpha_m + \sigma_{2k}$$

In curved panels the wrinkling ζ_y is considered dependent on the degree to which the buckling load is exceeded. The wrinkling ζ_y is the index value for the geometrical deformation of curved tension panels.

Now, the calculation is materially simplified if the strain in the incomplete tension bay is assumed uniform and the principal axes of stress and strain as being coincident. But to do so is to assume the direction angles α of the stress principal axes as everywhere the same. As already stated under III, 1, the possibility of a scheme compatible with the fundamental equations of the plane stress condition with equal shear stress τ at every point, is predicated on the assumption of equal principal axes angles α and equal stresses at every point. That is to say, after introducing the assumption of a homogeneous state of strain and coinciding principal axes of stress and strain, the nonuniformly distributed stresses must be replaced by mean values.

The mean normal stress value σ_n over the ordinate between $y = 0$ and $y = b_1/2$ is:

$$\bar{\sigma}_n = \sigma_{nm} - (\sigma_{nm} - \sigma_{nL}) \frac{2}{b_1} \int_0^{b_1/2} \sin^{1/\omega} \left(\frac{\pi y}{b_1}\right) dy$$

With $\chi = \pi y/b_1$, we have:

$$\frac{2}{b_1} \int_0^{b_1/2} \sin^{1/\omega} \left(\frac{\pi y}{b_1}\right) dy = \frac{2}{\pi} \int_0^{\pi/2} \sin^{1/\omega} \chi \, d\chi = J_{(\omega)}$$

For values of $1/\omega = m$, which are whole numbers and divisible by 2, the integral $J_{(\omega)}$ follows from the relation (reference 13):

$$J_{(\omega)} = \frac{2}{\pi} \int_0^{\pi/2} \sin^m \chi \, d\chi = \frac{2}{2^m} \frac{m!}{\left(\frac{m}{2}!\right)^2} \qquad (7a)$$

If m is not divisible by 2, the formula

$$J_{(\omega)} = \frac{2}{\pi} \int_0^{\pi/2} \sin^m \chi \, d\chi = \frac{2^m \left(\frac{m-1}{2}!\right)^2}{\pi \, m!} \qquad (7b)$$

should be employed. For small values of ω the calculation may be limited to the whole numbered values of m and the rest interpolated. For higher values (say, for $1/\omega$ between 1 and 2), Gauss' Π-function must be resorted to. Figure 8 gives the curve of the integral $J_{(\omega)}$ for $\omega = 0$ to $\omega = 0.5$.

Accordingly, the mean normal stress value σ_n is:

$$\bar{\sigma}_n = \sigma_{nm} - (\sigma_{nm} - \sigma_{nL}) J_{(\omega)} \qquad (8a)$$

Introducing a mean principal axis angle α, equation (5) results in: $\bar{\sigma}_n = \tau \cot \alpha + \bar{\sigma}_2$ and assuming $\sigma_{2m} = \sigma_{2k}$,

$$\bar{\sigma}_n = (\tau \cot \alpha_m + \sigma_{2k}) (1 - J_{(\omega)}) + \sigma_{nL} J_{(\omega)}$$

To simplify further calculations, we write $\alpha_m \sim \alpha$. The error introduced can be assessed and if necessary, eliminated by successive approximation. With $\alpha_m \sim \alpha$:

$$\left.\begin{aligned} \overline{\sigma}_n &= (\tau \cot \alpha + \sigma_{2k}) \left(1 - J_{(\omega)}\right) + \sigma_{nL} J_{(\omega)} \\ \text{and} \quad \overline{\sigma}_s &= \sigma_{2k} \left(1 - J_{(\omega)}\right) + (\sigma_{nL} - \tau \cot \alpha) J_{(\omega)} \end{aligned}\right\} \tag{8b}$$

The calculation of angle α and of stress σ_{nL} requires an analysis of the stress and strain condition of the whole shell.

3. Calculation of the Stresses and Strains

<u>a) Stiffener stresses</u>.- If s is the skin thickness, F_L the cross-sectional area of one stringer (inclusive of supporting skin strip), P the external normal load for one panel, and σ_x the stringer stress, the mean normal stress σ is defined by:

$$\sigma = \frac{P}{F_L + b_1 s} \quad \text{(positive if tensile stress)}$$

The mean stress σ_x in the stringer follows from the equilibrium of the internal and external forces in a bay as:

$$\sigma_x F_L + b_1 s \overline{\sigma}_n = \sigma (F_L + b_1 s)$$

which, with the abbreviation $\delta = b_1 s/F_L$, becomes:

$$\sigma_x = \sigma (1 + \delta) - \delta \overline{\sigma}_n \tag{9}$$

The normal stress σ_{nL} in the skin across the stringers is composed of the bending and the normal stress $\sigma_{nL} = \sigma_B + \sigma_x$. Then (9) substituted in (8a) for $\overline{\sigma}_n$, followed by solution for σ_x gives:

$$\sigma_x = \frac{\sigma(1 + \delta)}{1 + \delta J_{(\omega)}} - \delta \sigma_{nm} \frac{\left(1 - J_{(\omega)}\right)}{1 + \delta J_{(\omega)}} - \frac{\delta J_{(\omega)}}{1 + \delta J_{(\omega)}} \sigma_B$$

which, with the abbreviation $R = \dfrac{1 - J_{(\omega)}}{1 + \delta J_{(\omega)}}$ finally gives:

$$\sigma_x = \sigma\,(1 + \delta\,R) - \delta\,\sigma_{nm}\,R - \sigma_B\,\frac{\delta}{\delta + 1}\,(1 - R) \qquad (9a)$$

The stringer stresses in consequence depend on the bending moments (acting in the radial plane). This relationship was confirmed in torsion tests with reinforced circular cylindrical shells (references 4, and others). With equal bulkhead spacing a, the mean value of the bending moment between the bulkheads becomes zero, and the mean value of the normal stress σ_x becomes:

$$\overline{\sigma}_x = \sigma\,(1 + \delta\,R) - \delta\,R\,(\tau\,\cot\alpha + \sigma_{2k}) \qquad (9b)$$

In the longitudinal section perpendicular to the bulkheads (cross-sectional area F_y, stress σ_y, spacing a) the sum of the internal forces must be zero, whence

$$\sigma_y = -\,\frac{s\,a}{F_y}\,\overline{\sigma}_r \qquad (10)$$

For brevity, we introduce: $\gamma = \frac{s\,a}{F_y}$.

<u>b) The mean principal stresses $\overline{\sigma}_1$ and $\overline{\sigma}_2$ and the theorems for components of strain.</u>- The mean value of the normal stress σ_n in the whole bay follows from (9) and (9b) as:

$$\overline{\sigma}_r = \frac{\sigma\,(1 + \delta) - \overline{\sigma}_x}{\delta} = \sigma\,(1 - R) + R\,(\tau\,\cot\alpha + \sigma_{2k}) \qquad (11)$$

The factor R contains the quantity $\delta = b_1\,\frac{s}{F_L}$, which is a function of the dimensions of the structure. It is therefore not suitable to characterize in general the stress distribution in a metal panel stressed beyond the buckling limit.

From (5) follows:

$$\left.\begin{aligned}
&\overline{\sigma} = \overline{\overline{\sigma}}_n + \tau\,\tan\alpha = \sigma\,(1 - R) \\
&\qquad\qquad + R\,(\tau\,\cot\alpha + \sigma_{2k}) + \tau\,\tan\alpha \\
&\text{further} \\
&\overline{\sigma}_r = \overline{\overline{\sigma}}_1 - \tau\,\cot\alpha = (1 - R)(\sigma - \tau\,\cot\alpha) + \tau\,\tan\alpha + R\,\sigma_{2k} \\
&\text{and} \\
&\overline{\sigma}_2 = \overline{\overline{\sigma}}_n + \overline{\sigma}_r - \overline{\sigma}_1 = \overline{\overline{\sigma}}_n - \tau\,\cot\alpha = (1 - R)(\sigma - \tau\,\cot\alpha) + R\,\sigma_{2k}
\end{aligned}\right\} \qquad (12)$$

As in the foregoing formulas, the angles α are still unknown, the assumption of equiaxiality of stress and strain condition must be introduced and the angle $\alpha = \alpha'$ computed according to (3). This requires the components of strain ϵ, ϵ_x, and ϵ_y. If $\nu = 1/m$ denotes Poisson's ratio, we have:

$$\epsilon = \frac{1}{E} (\bar{\sigma}_1 - \nu\ \bar{\sigma}_2) \tag{13a}$$

By virtue of the homogeneity of the state of strain, the strain ϵ_x is equal to the stringer strain

$$\epsilon_x = \frac{\sigma_x}{E} \tag{13b}$$

The strain ϵ_y comprises the bulkhead strain due to the stress $\sigma_x = -\gamma\ \sigma_r$ and a share allowing for the mean deflection of the stringers. In curved panels this is supplemented by the wrinkling ζ_y. Under the assumption of equal bulkhead spacings a, the mean deflection under the longitudinal load p is: $f_m = p\ a^4/720\ EJ$. In the case of curved panel with inside reinforcement, we must put $p = \sigma_r\ s\ \varphi$ (φ = angle at center between two stiffeners) and $J = J_L$ (moment of inertia of stiffener section about axis of center of gravity parallel to axis y); the share of the strain ϵ_y amounts to $-\frac{f_m}{r_H}$, because the stiffener deflects in the radial plane. On the other hand, if the stiffener is at the edge, then $p = \sigma_r\ s$ and $J = J_L'$ (moment of inertia about axis perpendicular to shell tangent); the share of the strain is f_m/b. With the abbreviations $\kappa_i = \frac{s\ \varphi\ a^4}{720\ r_H\ J_L}$ and $\kappa_r = \frac{s\ \varphi\ a^4}{720\ b\ J_L'}$ ϵ_y is:

$$\epsilon_y = -\ (\gamma + \kappa_i)\ \frac{\sigma_r}{E} + \zeta_y$$

if the bay lies inside and

$$\epsilon_y = -\ (\gamma + \kappa_r + \tfrac{1}{2}\ \kappa_i)\ \frac{\sigma_r}{E} + \zeta_y$$

if on the edge.

κ_i and $\kappa_r + \frac{\kappa_i}{2}$ are, for short, written as κ.

In the case of pure tensile stress, $\epsilon = \epsilon_x$ and in pure compressive stress, $\epsilon = \epsilon_y$ as stated under III, 1. In both cases the ring stress disappears and the initial stresses between bulkheads and skin are left out of consideration. In pure tension (because $\tan \alpha = 0$) we have $\sigma_2 = \sigma_r = 0$ and $\epsilon = \sigma_1/E$. The component $\tau \cot \alpha \longrightarrow \sigma + \frac{R}{1 - R} \sigma_{2k}$, when $\tau \longrightarrow 0$ and $\alpha \longrightarrow 0$, as follows from $\sigma_r = 0$. Assuming further $R = 0$ (that is, $J_{(\omega)} = 1$) then $\bar{\sigma}_1 = \bar{\sigma}_n = \sigma_x = \sigma$ and consequently, $\epsilon = \epsilon_x$. The assumption for R is necessary because the shell does not buckle under pure tension. In pure compression $\tau \cot \alpha = 0$ and consequently, $\sigma_1 = \sigma_r$. For $\tau = 0$ and $\alpha = \pi/2$, it follows from $\sigma_1 = \sigma_r = 0$, that $\tau \tan \alpha \longrightarrow - [(1 - R) \sigma + R \sigma_{2k}]$. In order that $\epsilon - \epsilon_y = 0$ in pure compression, we must have $\zeta_y = - \nu \sigma_2/E$. However, these assumptions cover only the conditions in curved panels; in straight panels the considerations are different (reference 5).

<u>c) The formulas for the angle of principal axis and the shearing strain.</u>- In order to formulate (3) for $\tan^2 \alpha$, we insert E/σ_{2k} in the fraction $\frac{\epsilon - \epsilon_x}{\epsilon - \epsilon_y}$ and introduce the abbreviations $\vartheta = \tau/\sigma_{2k}$ and $\eta = \sigma/\sigma_{2k}$. Then we put $\tan^2 \alpha = Z/N$, wherein

$$Z = (\vartheta \cot \alpha - \eta) (R (1+\delta-\nu) + \nu) + \vartheta \tan \alpha + R (1+\delta-\nu)$$

and

$$N = \vartheta \cot \alpha [R - (1-R) (\gamma+\kappa-\nu)] + \vartheta \tan \alpha (1+\gamma+\kappa) + \eta (1-R) (1-\nu+\gamma+\kappa) + R (1-\nu+\gamma+\kappa) + E \zeta_y/\sigma_{2k} \tag{14}$$

According to (4)

$$\gamma_{xy} = 2 \sqrt{Z N \sigma_{2k}/E} \tag{15}$$

The previously treated case of pure shear is obtained when $\eta = 0$. This formula differs from the one given previously (reference 4, and others) by the consideration of the transverse strain due to stress σ_a and by the modified formula for the transverse stress σ_r. It should also be borne in mind that the ϑ values become negative because $\sigma_{2k} = -\tau_0$.

If $J_{(\omega)}$ and ζ_y are known for the loading conditions ϑ and η, we determine $R = \frac{1 - J_{(\omega)}}{1 + \delta\, J_{(\omega)}}$ and compute angle α from (14) by trial. The shearing strain γ_{xy} is computed from (15) with the defined values of Z and N, and the stiffener stresses determining the shell strength from (9) to (12).

At the buckling point in pure shear, we have $\eta = 0$ and $\vartheta = -1$. It is readily seen that (14) must be satisfied with $\tan \alpha = 1$. We have $Z = N = -(1 + \nu)$. Then (15) gives:

$$\gamma_{xy} = + \frac{2\,(1 + \nu)\,\tau_0}{E} = \frac{\tau_0}{G}$$

and so establishes the connection with the conventional elasticity equation.

The calculation of incomplete tension bays is accordingly possible when the factors ζ_y and ω are experimentally obtained in their relation to the loading condition.

4. The Index Values of the Incomplete Tension Bay

The factor ω concerns, as stated under III, 2, the distribution of the normal stresses σ_n over the width of the bay. It is therefore important for flat as well as for curved metal bays and is of significance in all the conditions of loading for a buckled skin which have been treated here. In pure shear and in combined stress $\omega \longrightarrow 0$, $J_{(\omega)} \longrightarrow 0$, and $R \longrightarrow 1$ with the approach to the complete tension bay by increasing load. R assumes an indeterminate form at the buckling point. In the case of a pure compressive load, the assumption that the second principal stress σ_{2m} at panel center retains the critical value σ_{2k} independent of further loading, does not exactly hold true; this stress rather increases (reference

5) and $J_{(\omega)}$, and consequently ω, are under the assumption of constant principal stress σ_{2m} greater than corresponds to the actual stress distribution. A relation exists between the apparent width b_m and the factor R. For

$$P = \sigma\ F_L\ (1 + \delta) = \bar{\sigma}_x\ (F_L + b_m\ s) = \bar{\sigma}_x\ F_L + \bar{\sigma}_n\ \delta\ F_L$$

and after insertion of (9b):

$$\left[\sigma\ (1 + \delta\ R) - \delta\ R\ \sigma_{2k}\right]\ \left[1 + \delta\ \frac{b_m}{b_1}\right] = \sigma\ (1 + \delta)$$

which with the abbreviation $\eta = \frac{\sigma}{\sigma_{2k}}$ and solution for R gives:

$$R = \frac{\eta}{\eta - 1}\ \frac{1 - \frac{b_m}{b_1}}{1 + \delta\ \frac{b_m}{b_1}} \tag{16a}$$

The solution for b_m/b_1 gives:

$$\frac{b_m}{b_1} = \frac{\eta - R\ (\eta - 1)}{\delta\ R\ (\eta - 1) + \eta} \tag{16b}$$

Thus for $\eta = 1$ $(\sigma = \sigma_{2k})$ the bay is fully supporting and the corresponding value R becomes indeterminate and must be extrapolated from test data or else arrived at from theoretical considerations. Given the apparent width from test data or theoretical formulas, the relation (16a) allows in the case of pure compression a determination of R and consequently of $J_{(\omega)}$ and ω.

The factor ζ_y concerns only curved metal panels and its practical importance is limited to the case of pure or superposed shear stress. The magnitude of ζ_y approaches with increasing load the limiting value $-\varphi_1^2/24$ asymptotically; the law of this increase must be decided by experiment. It is probable that in pure and superposed shear the wrinkling is solely dependent on the magnitude of the shearing strains present.

For pure and superposed shear the factors ζ_y and ω are obtained from shear-compression tests on metal panels in which the strains γ_{xy} and the stresses $\bar{\sigma}_x$ corresponding to the applied stresses σ and τ must be measured. From (9b) follows:

$$R = \frac{\bar{\sigma}_x - \sigma}{\delta\,[\sigma - \tau \tan\alpha - \sigma_{2k}]} = \frac{\dfrac{\bar{\sigma}_x}{\sigma_{2k}} - \eta}{\delta\,(\eta - \vartheta \cot\alpha - 1)} \qquad (17a)$$

On the other hand, from (4) follows:

$$Z = E\,(\epsilon - \epsilon_x)/\sigma_{2k} = \frac{1}{2}\,\frac{E\,\gamma_{xy}}{\sigma_{2k}} \tan\alpha$$

and, after solution for R (cf. (14) and (19b)):

$$R = \frac{\vartheta(\tan\alpha + \nu\cot\alpha) + \eta(1-\nu) - (\bar{\sigma}_x + \frac{1}{2}\,E\,\gamma_{xy}\tan\alpha)/\sigma_{2k}}{(1-\nu)\,(\eta - \vartheta\cot\alpha - 1)} \qquad (17b)$$

Equating (17a) and (17b) gives a quadratic equation for $\tan\alpha$:

$$a_1 \tan^2\alpha + b_1 \tan\alpha + c_1 = 0 \qquad (18)$$

with the abbreviations:

$$a_1 = (\vartheta - \tfrac{1}{2}\,E\,\gamma_{xy}/\sigma_{2k})\,\delta$$

$$b_1 = \eta\,(1 - \nu)\,(1 + \delta) - \bar{\sigma}_x\,(1 - \nu + \delta)/\sigma_{2k}$$

$$c_1 = \delta\,\vartheta\,\nu$$

The solution gives angle α from the measured values of $\bar{\sigma}_x$ and γ_{xy}, after which R is computed from (17a) and $J_{(\omega)}$ from $J_{(\omega)} = \frac{1 - R}{1 + \delta R}$. The wrinkling ζ_y is obtained from

$$N = E\,(\epsilon - \epsilon_y)/\sigma_{2k} = \tfrac{1}{2}\,E\,\gamma_{xy}\cot\alpha/\sigma_{2k}$$

with consideration of (14) as:

$$\begin{aligned}\zeta_y = \tfrac{1}{2}\,\gamma_{xy}\cot\alpha - [&\vartheta\cot\alpha\,(R-(1-R)\,(\gamma+\kappa-\nu)\\ &+ \vartheta\tan\alpha\,(1+\gamma+\kappa) + \eta\,(1-R)\,(1-\nu+\gamma+\kappa)\\ &+ R\,(1-\nu+\gamma+\kappa)]\,\sigma_{2k}/E\end{aligned} \qquad (19)$$

The differences of stress in (17a) and those of strain in (19) in the immediate vicinity of the buckling point leave some doubt on account of the smallness of the stresses. In this range it seems desirable to supplement the tests by a theoretical analysis by means of energy methods; this would also bring some clearness concerning the most practical assumption for the principal stress σ_{2m} in panel center.

5. Determination of Stress Distribution Factor ω

from the Test Data by Wagner and Lahde

The shear-compression tests by Wagner and Lahde (reference 3) with buckled panels under various stress conditions constitute the only experiments till now which afford some insight into the relation between factor ω and the load. These tests were made on panels with very strong stringers, so that approximately $\delta \sim 0$ and $\sigma \sim \sigma_x$, while the elongation ϵ_y perpendicular to the direction of compression was largely restrained ($\gamma = \kappa = 0$; $\zeta_y = 0$). Nevertheless, their findings are also applicable to more general cases.

For constant load ratios $\frac{\sigma}{\tau} = \lambda$, the tests for sheets of different thickness disclosed a linear decrease of the quotient $\frac{\tau}{E\gamma_{xy}}$ with decreasing square root of the reciprocal stress ratio $\frac{1}{\xi} = \frac{\tau_k}{\tau} = \frac{\sigma_k}{\sigma}$; that is,

$$\frac{\tau}{E\gamma_{xy}} = G/E - \left[\left(\frac{\tau}{E\gamma_{xy}}\right)_z - G/E\right]\left[\sqrt{\frac{1}{\xi}} - 1\right]$$

where $\left(\frac{\tau}{E\gamma_{xy}}\right)_z$ is the extreme value of this quotient for $\frac{1}{\xi} = 0$. But $\frac{\sigma_{2k}}{\tau} \doteq 0$ for $\frac{1}{\xi} = 0$; that is, the case of the ideal tension bay. There we find for $\epsilon_y = 0$: $\tan^2 \alpha_z = 1 - \frac{\sigma_x}{\tau} \sin \alpha_z \cos \alpha_z$ and $\left(\frac{\tau}{E\gamma_{xy}}\right)_z = \frac{\tau}{2E \tan \alpha_z \epsilon} = \cos^2 \alpha_z/2$.

To insure continuity with the elastic condition in the evaluation of the tests at the buckling point, it is necessary to take the transverse elongation due to the crushing (ϵ_x) into consideration at the buckling point. Upon reaching the complete tension bay this strain is neglected. For this reason, we put $\epsilon_y = - \nu \, \epsilon_x \sqrt{1/\xi}$. With this value, formulas (4) and (12) give for R:

$$R = \frac{\eta[1-\nu\ (1-\sqrt{1/\xi})] + \vartheta(\tan\alpha+\nu\cot\alpha) - E\gamma_{xy}\cot\alpha/2\sigma_{2k}}{(1-\nu)\ (\eta-\vartheta\cot\alpha-1)} \tag{17c}$$

Equating (17b) and (17c) gives for $\sigma_x \sim \sigma$:

$$\cot 2\alpha = \lambda \left(1 + \nu \sqrt{\frac{1}{\xi}}\right) \frac{\tau}{E\gamma_{xy}} \tag{19}$$

with $\lambda = \frac{\sigma}{\tau} \sim \frac{\sigma_x}{\tau}$. As $R \sim 1 - J_{(\omega)}$, transformation of (17b) gives:

$$J_{(\omega)} = \frac{\lambda - (\tan\alpha + \cot\alpha) - (1-\nu)\ \sigma_{2k}/\tau + \frac{1}{2}\ E\gamma_{xy}\tan\alpha/\tau}{(1-\nu)\ (\lambda - \cot\alpha - \sigma_{2k}/\tau)} \tag{20}$$

Making use of the linear relationship between $\frac{\tau}{E\gamma_{xy}}$ and $\sqrt{\frac{1}{\xi}}$, we compute from (19) the angle α for all deformation conditions as well as the factor $J_{(\omega)}$ from (20), not forgetting that $\frac{\sigma_{2k}}{\tau} = (\lambda - \sqrt{\lambda^2 + 4})/2\xi$, according to equation (2).

Figure 9 shows the factors $J_{(\omega)}$ plotted against $\sqrt{\frac{1}{\xi}}$ for load ratios: $\lambda = 0$, $\lambda = 0.6$, $\lambda = 1.2$ and $\lambda = \infty$. The curve of $\lambda = \infty$ is computed according to the formula:

$$\frac{b_m}{b_1} = \sqrt[3]{\frac{\sigma_k}{\sigma_x}} \sim \sqrt[3]{\frac{1}{\xi}}.$$

established by Marguerre (reference 5) for the apparent width with pure compressive stress. The curves for all loading conditions could, as is seen, be satisfactorily approximated by one single curve.

The stress distribution factor ω for all λ's is sufficiently exactly obtainable from the formula:

$$\omega = \left(\frac{1}{\xi}\right)^{2/3} \tag{21}$$

If the values of $1/\xi$ are small, the formula is in good agreement with the test data; by greater values it remains on the safe side.

Translation by J. Vanier,
National Advisory Committee
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REFERENCES

1. Wagner, Herbert: Flat Sheet Metal Girders with Very Thin Metal Web.

Part I: General Theories and Assumptions. T.M. No. 604, N.A.C.A., 1931.

Part II: Sheet Metal Girders with Spars Resistant to Bending - Oblique Uprights - Stiffness. T.M. No. 605, N.A.C.A., 1931.

Part III: Sheet Metal Girders with Spars Resistant to Bending - the Stress in Uprights - Diagonal Tension Fields. T.M. No. 606, N.A.C.A., 1931.

2. Wagner, H., and Ballerstedt, W.: Tension Fields in Originally Curved, Thin Sheets During Shearing Stresses. T.M. No. 774, N.A.C.A., 1935.

3. Lahde, R., and Wagner, H.: Tests for the Determination of the Stress Condition in Tension Fields. T.M. No. 809, N.A.C.A., 1936.

4. Schapitz, E.: Über die Drillung dünnwandiger, versteifter Kreiszylinderschalen. Lilienthal-Gesellschaft für Luftfahrtforschung, Jahrbuch 1936.

5. Marguerre, K.: Über die mittragende Breite der gedrückten Platte. Luftfahrtforschung, Bd. 14 (1937), dieses Heft, S. 121.

6. Lahde, R., and Wagner, H.: Experimental Studies of the Effective Width of Buckled Sheets. T.M. No. 814, N.A.C.A., 1936.

7. Ebner, H.: Theorie und Versuche zur Festigkeit von Schalenrümpfen. Luftfahrtforschung, Bd. 14 (1937), dieses Heft, S. 93.

8. Ebner, H., and Heck, O. S.: Methods and Formulas for Calculating the Strength of Plate and Shell Constructions as Used in Airplane Design. T.M. No. 785, N.A.C.A., 1936.

9. Wagner, H.: Structures of Thin Sheet Metal; Their Design and Construction. T.M. No. 490, N.A.C.A., 1928.

10. Chwalla, F.: Die Bemessung der waagrecht ausgesteiften Stegbleche vollwandiger Träger. Vorbericht des zweiten Kongresses 1936 der Internationalen Vereinigung für Brückenbau und Hochbau.

11. Bridget, F. J., Jerome, C. C., and Vosseler, A. B.: Some New Experiments on Buckling of Thin-Wall Constructions. Trans. A.S.M.E., vol. 56, no. 8, 1934, pp. 569-578.

12. Wagner, H., and Ballerstedt, W.: Über die Festigkeit dünner unversteifter Kreiszylinderschalen unter Schub- und Längskräften. Luftfahrtforschung, Bd. 13, Heft 9 (1936), S. 309-312.

13. Jahnke-Emde, S.: Funktionentafeln. 2. Auflage, S. 95.

Table of Formulas for the Critical Buckling Stress of Plates and Shells Under Simple Stresses

s, wall thickness; r, radius of curvature; $\nu = \frac{1}{m}$, Poisson's ratio

Stress: Compression

Flat plates and strips

Critical compressive stress $\sigma_{kr} = k \frac{E}{1-\nu^2}\left(\frac{s}{b}\right)^2$

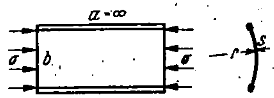
All sides supported

a/b	0,4	0,6	1,0	1,4	1,8	2,4	3,0	∞
k	6,92	4,23	3,29	3,68	3,32	3,40	3,29	3,29

All sides clamped

a/b	1	2	3	∞
k	7,7	6,7	6,4	6,0

Curved plates and strips

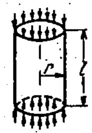


Redshaw's formula

$$\sigma_{kr} = 1/6 \frac{E}{1-\nu^2}\left\{\sqrt{12(1-\nu^2)\left(\frac{s}{r}\right)^2 + \left(\frac{\pi s}{b}\right)^4} + \left(\frac{\pi s}{b}\right)^2\right\}$$

Edges counted as supported

Unstiffened circular cylindrical shells



Critical compressive stress

$$\sigma_{kr} = \frac{0,4}{\sqrt{3}} \cdot \frac{E}{1-\nu^2} \cdot \frac{s}{r}$$

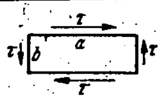
for

$$l \geq 6\sqrt{r \cdot s}$$

Effect of initial buckling, etc. allowed for

Stress: Shear

Flat plates and strips



Critical shear stress $= k \frac{E}{1-\nu^2}\left(\frac{s}{b}\right)^2$

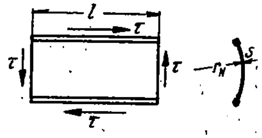
All edges supported

a/b	1,0	1,2	1,4	1,6	1,8	2,0	2,5	3,0	∞
k	7,75	6,58	6,00	5,76	5,59	5,43	5,18	5,02	4,4

All edges clamped

a/b	1	2	∞
k	12,7	9,5	7,4

Curved plates and strips



Critical shear stress

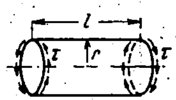
$$\tau_k = \sqrt{\tau_R^2 + \left(\frac{\tau_p}{2}\right)^2} + \frac{\tau_p}{2}$$

with $\tau_p = 4,85\, E\, (s/b)^2$

$$\tau_R = 0,1\, E\, s/r + 5,0\, E\left(\frac{s}{l}\right)^2.$$

Edges counted as supported

Unstiffened circular cylindrical shells



for $l/r < 5$:

$$\tau_k = 0,1\, E\, s/r + 5,0\, E\left(\frac{s}{l}\right)^2$$

(Wagner-Ballerstedt formula)

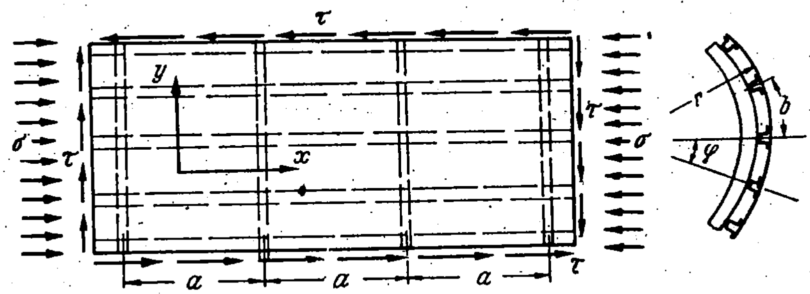


Figure 1.- Shell section diagram.

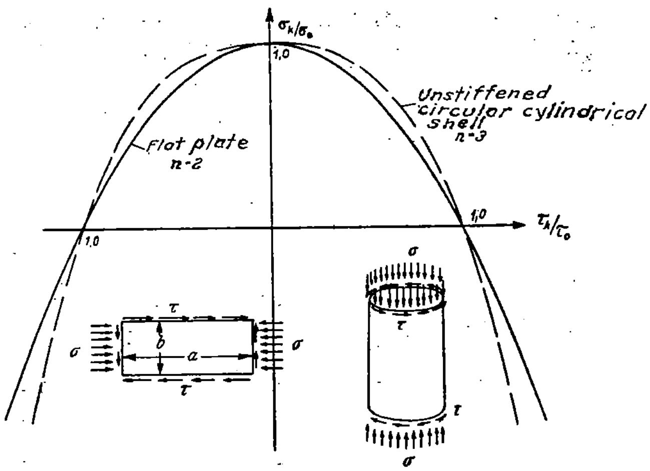


Figure 2.- Buckling formation of plates and shells under combined normal and shearing stress.

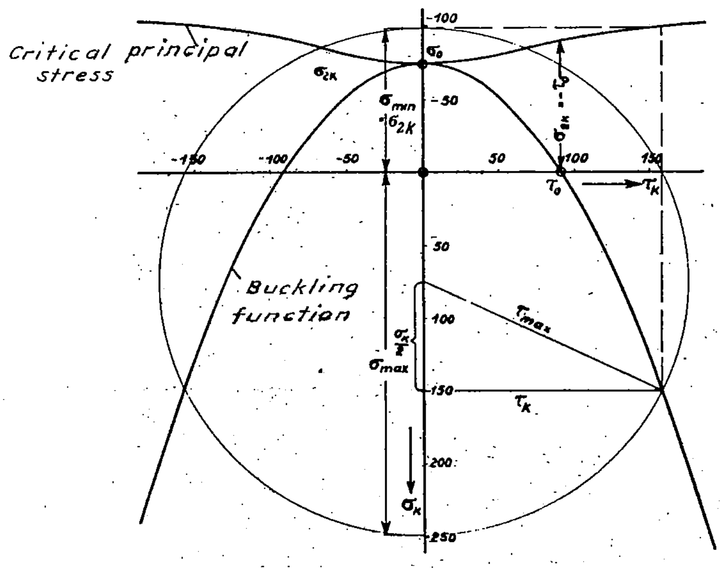


Figure 3.- Determination of critical principal stress σ_{2k}

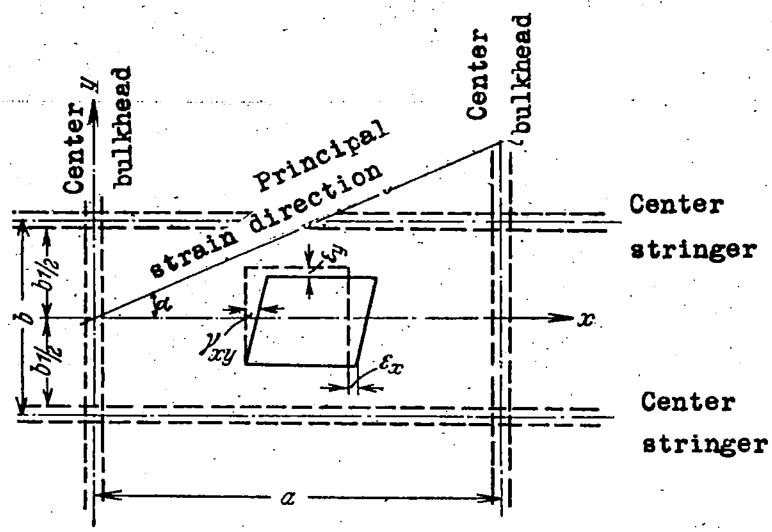


Figure 4.- Coordinate system and components of strain.

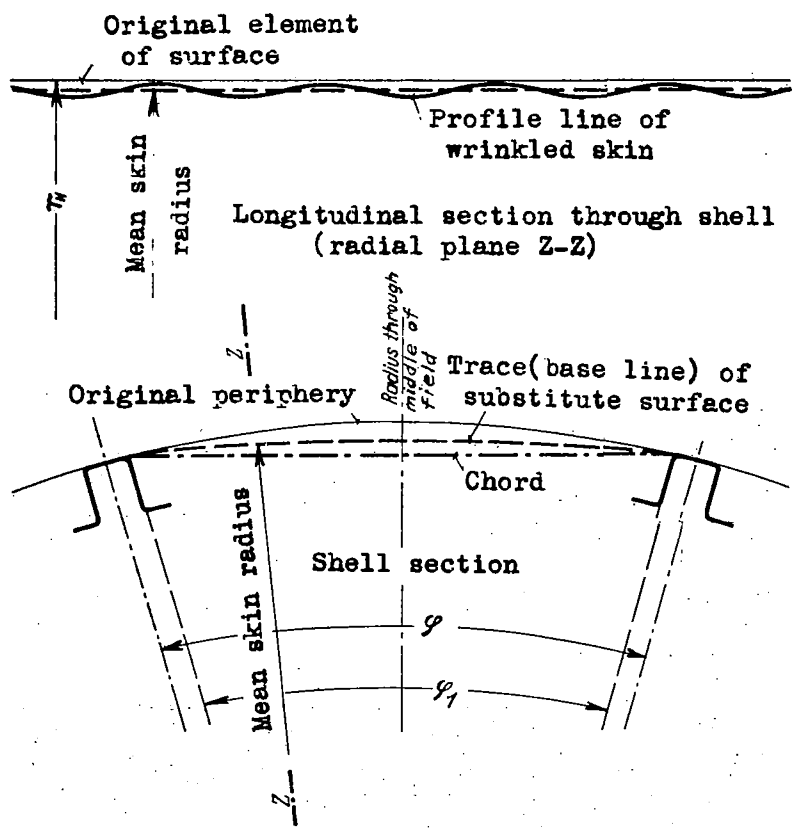


Figure 5.- Determination of wrinkling ζ_y.

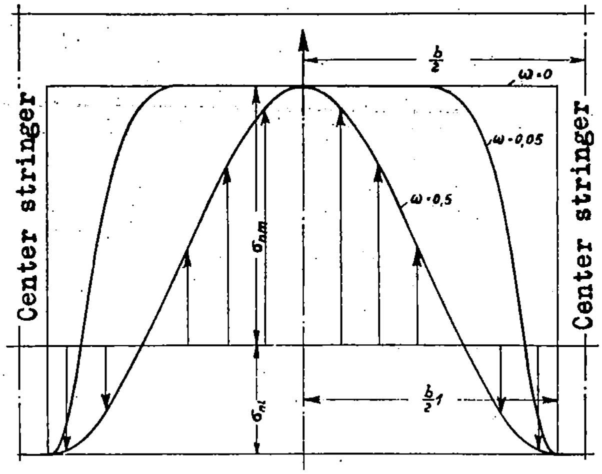


Figure 7.- Distribution of the normal stress σ_n over the width of panel.

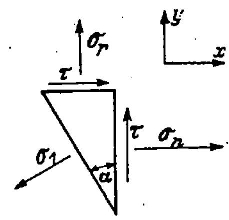


Figure 6.- Equilibrium of skin element.

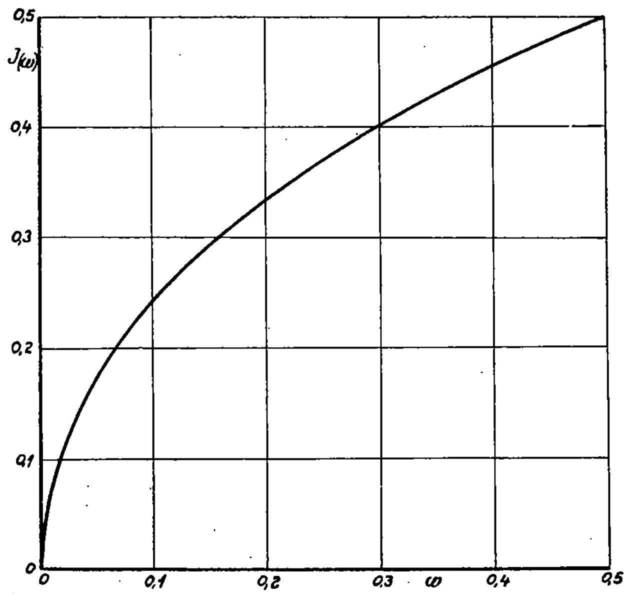


Figure 8.- Curve of integral

$J(\omega)\ 2/\pi \int_0^{\frac{\pi}{2}} \sin \frac{1}{\omega} \cdot \chi^d \chi \cdot$

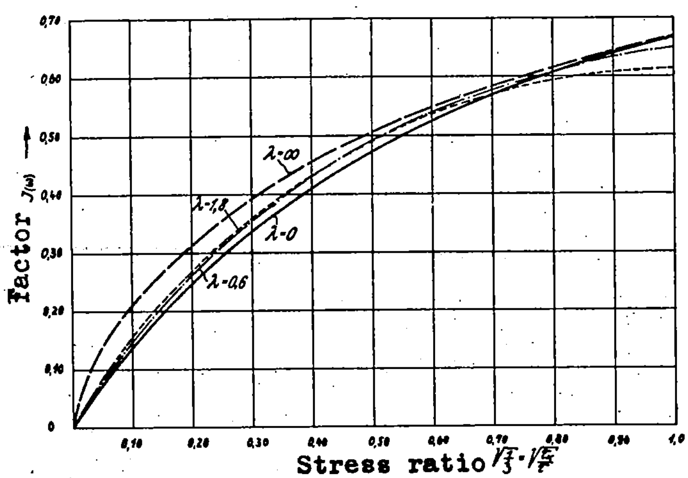


Figure 9.- Curves of $J(\omega)$ plotted against stress ratio $\sqrt{\frac{1}{\varepsilon}}$ for flat metal strip(Wagner and Lahde test)