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CHARTS FOR CHECKING THE STABILITY OF PLANE SYSTEMS OF RODS

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CHARTS FOR CHECKING THE STABILITY OF PLANE SYSTEMS OF RODS*

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Charts are presented for checking the stability of plane systems of rods, the use of the charts being illustrated by examples. It is also indicated how best to combine the individual members to form stable structures.

I. INTRODUCTION

1. Subject Matter Treated

In report 36/1 of the DVL (reference 1) charts are constructed and discussed, by the aid of which the stability of various plane trusses with spatially fixed joints may be investigated for stability with respect to buckling in the plane of the truss.** These trusses consist of one member designated as the "buckling member" which is stressed above its "natural buckling load" (the load at which it would buckle when unrestrained by the other members (see reference 2)) and an arbitrary number of "neighboring members" under tension, compression, or without stress, that are jointed at one or both ends of the buckling member and are stiffened against bending (open truss). The group of three members forming a closed triangle (closed truss) is likewise included.

In what follows, two of these charts will be presented in extended form, thus enabling the investigation of the stability with respect to buckling of the plane systems of rods shown on figure 1,a and b, where the individual members may be arbitrarily stressed in tension or compression.

*"Kurventafeln für den Stabilitätsnachweis ebener Stabgruppen." Luftfahrtforschung, vol. 14, no. 2, February 20, 1937, pp. 86-92.

**For investigations on the general case of buckling in any plane, see references 4 and 5.

Besides these general cases, there are treated in the above-mentioned work the particular cases of open trusses where only one member - the middle member of the system represented on figure 1,b - may be stressed above its "natural buckling load." In many cases the treatment of these particular cases alone will be sufficient, since in the choice of a truss it is generally a question of obtaining a properly chosen system of rods in tension, compression, or unstressed, that together with a rod stressed above its "natural buckling load," form a stable truss. (See, for example, reference 5.)

2. Preliminary Remarks and General Notation

The charts presented are directly applicable to members stressed within the proportionality limit. By introducing in place of the modulus of elasticity E, a reduced modulus of elasticity (τE) corresponding to the stress σ, the use of the charts may be extended to include the region above the proportionality limit (references 1, 7, and 8).

As given in reference 1, cited above, for a compression member the "measure of instability" (see, for example, reference 2) is:

$$\alpha = l \sqrt{\frac{S}{EJ}}$$

where l is the length of the member

S, the absolute value of the axial (compressive) force

and EJ, the stiffness in bending

In what follows, the value $\left(\frac{\alpha}{\pi}\right)^2$, which is equal to the ratio of the compressive force to the "natural buckling load," will be employed.* When the rod system under con-

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*Hence, when

$\left(\frac{\alpha}{\pi}\right)^2 < 1$: the compressive force lies below the "natural buckling load" and the member is stable.

$\left(\frac{\alpha}{\pi}\right)^2 = 1$: the compressive force is equal to the natural

(Continued in footnote, page 3.)

sideration reaches the buckling condition, the value $\left(\frac{\alpha}{\pi}\right)^2$ of a member reaches the value $\left(\frac{\alpha_K r}{\pi}\right)^2$, which represents the ratio of the actual force it can bear to its "natural buckling load," and thus represents the effect of the restraint imposed by the presence of the other members of the truss.

Correspondingly, for tension members: $\alpha = l \sqrt{\frac{Z}{EJ}}$ where Z denotes the absolute value of the axial (tensile) force.

There is further introduced the following "member parameter"

$$\varphi_N = \left(\frac{EJ}{l}\right)_N : \left(\frac{EJ}{l}\right)_K$$

where the subscript K refers to one member of the truss that is taken as a reference member, and the subscript N refers to any of the other members. The symbol K was chosen, in the work mentioned above (reference 1) because, in the cases there considered the member stressed beyond its "natural buckling stress" was chosen as reference member (so-called "Knickstab").

II. EXPLANATION OF THE CHARTS

1. Application of the Charts

(See in this connection in section III, the numerical examples worked through.)

a) Buckling of a plane system of rods whose members are all joined at a common point in their plane.- All of the members are pin-ended (fig. 1,a).

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*(Continued from footnote, page 2)

buckling load and the pin-ended member will buckle as a whole.

$\left(\frac{\alpha}{\pi}\right)^2 > 1$: the compressive force lies above the natural buckling load and the member considered by itself (that is, unrestrained by being joined to the other members) is unstable.

In this group one or more members may have $\left(\frac{\alpha}{\pi}\right)^2 > 1$ and would thus buckle if considered by themselves without the restraint imposed by the neighboring members. Of these compression members (overstressed if considered by themselves), one is arbitrarily chosen as a reference member K, while the remaining members that may be under compression, tension, or without stress are denoted in common by N_i. For each of the members N_i a reduced member parameter $\bar{\varphi}_{N_i}$ (referred to $\left(\frac{\alpha_N}{\pi}\right)^2 = 0$) is formed in the following manner. On chart I, we pass perpendicularly from the abscissa $\left(\frac{\alpha_{N_i}}{\pi}\right)^2$ up or down, respectively, to the point of intersection D_{N_i} with the curve whose parameter has the value φ_{N_i}. From D_{N_i} we pass horizontally up to the point of intersection $\bar{D}_{N_i}$ with the axis of ordinates at $\left(\frac{\alpha_N}{\pi}\right)^2 = 0$. The parameter of the curve going through $\bar{D}_{N_i}$ is the reduced "member parameter" $\bar{\varphi}_{N_i}$. (For the overstressed members, when considered by themselves, negative values of $\bar{\varphi}_{N_i}$ are obtained.)

The intersection of the curve whose parameter is equal to $\Sigma\,\bar{\varphi}_{N_i}$ with the axis of ordinates at $\left(\frac{\alpha_N}{\pi}\right)^2 = 0$ gives the ordinate $\left(\frac{\alpha_{K_r,K}}{\pi}\right)^2$ and this corresponds to the compressive force $S_{K_r,K}$ on the reference member K at which the critical buckling condition of the entire system would set in. We then have the criterion:

If the actual value

$$\left(\frac{\alpha_K}{\pi}\right)^2 < \left(\frac{\alpha_{K_r,K}}{\pi}\right)^2, \quad \text{that is,} \quad S_K < S_{K_r,K}$$

the truss is stable; if, on the other hand,

$$\left(\frac{\alpha_K}{\pi}\right)^2 > \left(\frac{\alpha_{K_r,K}}{\pi}\right)^2, \quad \text{that is,} \quad S_K > S_{K_r,K}$$

the system is no longer stable.

If it is desired to avoid using one member as reference member, then the procedure will be as follows:

An arbitrary value $\left(\frac{EJ}{l}\right)_B$ is chosen as reference value. With this value the parameter for the system becomes:

$$\varphi_i = \left(\frac{EJ}{l}\right)_i : \left(\frac{EJ}{l}\right)_B$$

There is then formed, in a manner similar to the above, for each of the members i the reduced member parameter $\bar{\varphi}_i$ (referred to $\left(\frac{\alpha_N}{\pi}\right)^2 = 0$). The system will then be stable when for all the members the sum $\Sigma\,\bar{\varphi}_i > 0$ and unstable when $\Sigma\,\bar{\varphi}_i < 0$.

b) <u>Buckling of an open plane* system of rods consisting of three members joined together and stiffened against bending</u>.- All end points are pin-jointed (fig. 2).

Of this system, any one or two of the three members may be overstressed $\left(\frac{\alpha}{\pi}\right)^2 > 1$. The following notation is used: The center rod is denoted by the subscript M; the outer two rods by the subscripts N_l and N_r, respectively. Further,

$$\varphi_{N_l} = \left(\frac{EJ}{l}\right)_{N_l} : \left(\frac{EJ}{l}\right)_M$$

$$\varphi_{N_r} = \left(\frac{EJ}{l}\right)_{N_r} : \left(\frac{EJ}{l}\right)_M$$

In the case of a given loading condition, for which the stability is investigated, where only one of the compression members is overstressed $\left(\frac{\alpha}{\pi}\right)^2 > 1$, it is first determined whether this member does not form a stable system with one of the other members joined to it, using the

*The buckling of the closed system of three rods (rod triangle) within its plane has been considered in detail in the previous work (reference 1).

method of II,1,a, since in that case further investigation would be unnecessary. If such is not the case the investigation proceeds as follows:

From chart I there are obtained the values:

1. $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)_l^2$ the ordinate of the point on the curve with parameter φ_{N_l} and whose abscissa is $\left(\frac{\alpha_{N_l}}{\pi}\right)^2$ and

2. $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)_r^2$ the ordinate of the point on the curve with parameter φ_{N_r} and whose abscissa is equa to $\left(\frac{\alpha_{N_r}}{\pi}\right)^2$.

If the value $\left(\frac{\alpha_N}{\pi}\right)^2$ of a compression member is greater than 1, then it should be noted whether the value $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2$ thus obtained is in the "compression member region" or "tension member region" of the ordinates $\left(\frac{\alpha_K}{\pi}\right)^2$. There is then obtained from chart 2 the parameter of that curve that goes through the point with coordinates $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)_l^2$ and $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)_r^2$. The parameter thus found is $\left(\frac{\alpha_{K_r,M}}{\pi}\right)^2$, and corresponds to that value of the axial force on the member M for which the critical buckling condition of the bar system will set in. We then have the criterion: If the values of $\left(\frac{\alpha}{\pi}\right)^2$ in the "tension member region" are given negative signs and if

$$\left(\frac{\alpha_M}{\pi}\right)^2 < \left(\frac{\alpha_{K_r,M}}{\pi}\right)^2$$

the system is stable; if, on the other hand,

$$\left(\frac{\alpha_M}{\pi}\right)^2 > \left(\frac{\alpha_{K_r,M}}{\pi}\right)^2$$

then the system is no longer stable.

c) Buckling of a plane system in the form of a member to which an arbitrary number of rods are rigidly joined at each end (fig. 1,b).

With this system each member may be arbitrarily stressed. The following notation is used: The middle member (reference member) is denoted by the subscript M; the members joined to the left end of M are denoted by N_{l_i} and those to the right by N_{r_i}. When the system is investigated for stability at a given loading it is first determined whether the system as a whole can not be subdivided into a series of groups, each of which may be investigated for stability according to the method of II,1a, or b, since in that case no further investigation is necessary. If such is not the case the investigation proceeds as follows:

There is first determined for each of the rods N_{l_i} and N_{r_i} the reduced parameters $\bar{\varphi}_{N_{l_i}}$ and $\bar{\varphi}_{N_{r_i}}$ (referred to $\left(\frac{\alpha_N}{\pi}\right)^2 = 0$). (See II,1,a.)

From chart 1 there are obtained the values:

1. $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)_l^2$, the ordinate of the point at which the curve with parameter $\Sigma\,\bar{\varphi}_{N_{l_i}}$ cuts the axis of ordinates at $\left(\frac{\alpha_N}{\pi}\right)^2 = 0$ and

2. $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)_r^2$ the ordinate of the point at which the curve with parameter $\Sigma\,\bar{\varphi}_{N_{r_i}}$ cuts the axis of ordinates at $\left(\frac{\alpha_N}{\pi}\right)^2 = 0$.

After determining both these values the computation proceeds exactly as in II,1,b.

2. Proper Choice of Rod System

Although the extended charts I and II have been set up for the investigation of general arrangements of rods into trusses, it should nevertheless be observed that for the arrangement represented in figure 1,b, the particular case where the "buckling member" K is the center rod is the most advantageous. Whenever therefore a compression member of a system of rods is overstressed $\left(\left(\frac{\alpha}{\pi}\right)^2 > 1\right)$, it is advantageous to make it the center member, with the other pressure or tension members joined to both its ends. The most favorable arrangement will be obtained when two adjoining members form a triangle with the center member.

This may be made clear by the following example: The rods are considered to be of equal length and equal resistance to bending, one of them being under a compressive stress while the others are unstressed. When there is only one additional member the buckling load of the compression member as compared with the "natural buckling load" $\frac{E J \pi^2}{l^2}$ of the member is increased 1.41 times (fig. 3,a). With two members added, one behind the other (fig. 3,b), the buckling load is increased 1.45 times. With the arrangement of figure 3,c the load is increased 1.59 times; with the arrangement of figure 3,d 1.90 times, and with the three members forming a triangle as in figure 3,e 2.13 times. It is assumed, of course, that these increased buckling loads are still within the region of applicability of Hooke's law.

3. Method of Construction of Charts

a) Preliminary remarks.- With regard to the formulas upon which the charts were based, the following may be said: The symbol $e_{i,j}$ (or $\hat{e}_{i,j}$) denotes the "unit rotation" of the end i of a hinged bar i,j as a result of a concentrated bending moment $M = 1$ applied at the same end i (or the opposite end j) in the same sense as the moment. From the assumption of constant strength in bending over the length of the rod and from considerations of symmetry, it follows that $e_{i,j} = e_{j,i}$ and, according to Maxwell, $\hat{e}_{i,j} = \hat{e}_{j,i}$. Considering the moments at the joints of a buckling rod system as functions of the rotations of the joints and assuming the latter as unknown, the equilibrium conditions for each of these points form a linear homogeneous system of equations for the determination of these unknowns (since before the instability condition sets in there are only axial forces acting on the rods). Rotations of the joints different from zero are only possible when the determinant of these equations in the denominator is equal to zero (buckling conditions, references 2-6).

The above unit rotations in the buckling condition of the individual members appear in the forms

$$\frac{1}{e}, \quad \frac{e}{e^2 - \hat{e}^2}, \quad \frac{\hat{e}}{e^2 - \hat{e}^2}$$

which may be set equal, respectively, to

$$\frac{1}{e} = \left(\frac{EJ}{l}\right) u; \quad \frac{e}{e^2 - \hat{e}^2} = \left(\frac{EJ}{l}\right) v; \quad \frac{\hat{e}}{e^2 - \hat{e}^2} = \left(\frac{EJ}{l}\right) w$$

where u, v, w are the following functions of α:

	For rods under compression	For rods under tension	For unstressed rods
u =	$\dfrac{\alpha^2}{1 - \dfrac{\alpha}{\tan\alpha}}$	$\dfrac{\alpha^2}{\dfrac{\alpha}{\tanh\alpha} - 1}$	3
v =	$\dfrac{\alpha^2\left(1 - \dfrac{\alpha}{\tan\alpha}\right)}{\left(1 - \dfrac{\alpha}{\tan\alpha}\right)^2 - \left(1 - \dfrac{\alpha}{\sin\alpha}\right)^2}$	$\dfrac{\alpha^2\left(\dfrac{\alpha}{\tanh\alpha} - 1\right)}{\left(\dfrac{\alpha}{\tanh\alpha} - 1\right)^2 - \left(\dfrac{\alpha}{\sinh\alpha} - 1\right)^2}$	4
w =	$\dfrac{\alpha^2\left(1 - \dfrac{\alpha}{\sin\alpha}\right)}{\left(1 - \dfrac{\alpha}{\tan\alpha}\right)^2 - \left(1 - \dfrac{\alpha}{\sin\alpha}\right)^2}$	$\dfrac{\alpha^2\left(\dfrac{\alpha}{\sinh\alpha} - 1\right)}{\left(\dfrac{\alpha}{\tanh\alpha} - 1\right)^2 - \left(\dfrac{\alpha}{\sinh\alpha} - 1\right)^2}$	-2

Chart I.- The buckling condition of the system of rods represented in figure 1,a reads (see reference 5, p. 302):

$$\Sigma \frac{1}{e_i} = 0 \quad \text{or} \quad \Sigma\, u_i \left(\frac{EJ}{l}\right)_i = 0$$

For a system that consists of only one overstressed rod $K \left[\left(\frac{\alpha_K}{\pi}\right)^2 > 1\right]$ and one additional rod N, the buckling condition is therefore:

$$u_{K_r} + u_N\, \varphi_N = 0$$

The subscript K_r corresponds to the "measure of instability" $\alpha_{K_r,K}$ of the buckling member K for that axial force $S_{K_r,K}$ for which, with the given value of α_N and given member parameter φ_N of the neighboring member, the critical condition of the system (buckling in the plane of the system) occurs. The relation

$$\underline{u_{K_r} + u_N\, \varphi_N = 0}$$

is plotted on chart I with $\left(\frac{\alpha_N}{\pi}\right)^2$ and $\left(\frac{\alpha_{K_r}}{\pi}\right)^2$ as coordinates, and φ_N as parameter of the family of curves. The ordinate of a point on the curve with parameter φ_N and abscissa $\left(\frac{\alpha_N}{\pi}\right)^2$ therefore represents the "restraint effect" $\left(\frac{\alpha_{K_r,K}}{\pi}\right)^2$ on the compression member K exerted by the member N when the latter possesses the measure of instability α_N and the parameter φ_N.

If the system (fig. 1,a) consists of more than two members that are connected at a common joint, then the buckling condition, with one member K chosen as reference member, is

$$u_{K_r} + \Sigma\, u_{N_i}\, \varphi_{N_i} = 0$$

or, in case no rod is taken as reference member

$$\Sigma\, u_i\, \varphi_i = 0$$

In order to be able to make use of the relation $u_{K_r} + u_N \varphi_N = 0$ on chart I, a single product of the form $u\,\varphi$ must replace the sum. For this purpose the members considered under the summation sign - for example, all the members N_i - are replaced by equivalent unstressed members with $u = u_o$ and parameters $\bar{\varphi}_{N_i}$ (reduced value of φ_{N_i}), so that

$$u_{N_i}\,\varphi_{N_i} = u_o\,\bar{\varphi}_{N_i}\ (= -\,u_{K_r})_i$$

Since the point D_N with abscissa $\left(\frac{\alpha_{N_i}}{\pi}\right)^2$ on the curve φ_{N_i} determines by its ordinate the "restraint effect" $\left(\frac{\alpha_{K_r,K}}{\pi}\right)_i^2$ which is exerted on the buckling member K by the neighboring member N_i the required point $\bar{D}_{N_i}$ on the axis of ordinates through $\left(\frac{\alpha_N}{\pi}\right)^2 = 0$ and having the same ordinate $\left(\frac{\alpha_{K_r,K}}{\pi}\right)_i^2$ corresponds to the case where where the buckling member under consideration would experience the same restraint effect except that the latter is exerted by a neighboring member that is unstressed. The parameter of the curve passing through $\bar{D}_{N_i}$ gives the parameter of this equivalent member and hence the reduced parameter $\bar{\varphi}_{N_i}$ of the neighboring member N_i.

If the neighboring members are themselves overstressed $\left(\left(\frac{\alpha_N}{\pi}\right)^2 > 1\right)$, $\bar{\varphi}_{N_i}$ is naturally negative. For such members the $\bar{\varphi}_N$ values are numbered negatively on the corresponding portion of the axis of ordinates.

Having introduced the equivalent members in the manner described above, the common factor u_o may be put outside the summation sign and the buckling condition then reads:

$$u_{K_r} + u_o\,\Sigma\,\bar{\varphi}_{N_i} = 0 \quad \text{or} \quad u_o\,\Sigma\,\bar{\varphi}_i = 0$$

With one member K chosen as reference member $\Sigma\,\bar{\varphi}_{N_i}$

may be looked upon as the parameter of an unstressed equivalent neighboring member. The problem is then treated in the same manner as that of the two-panel system considered above except that the measure of instability of the neighboring rod is now zero and its parameter is equal to $\Sigma \bar{\varphi}_{N_i}$.

When no member is used as a reference member the system is stable when $\Sigma \bar{\varphi}_i > 0$; since the limit of stability is reached when $\Sigma \bar{\varphi}_i = 0$ and the relation $\Sigma \bar{\varphi}_i = a > 0$ may be used to express the condition that to a rod system near the buckling condition an unstressed member with parameter a is added that introduces an additional restraint.

Corresponding to the equation, $u_{K_r} + u_N \varphi_N = 0$, the curves drawn with the parameter φ and with the corresponding parameter $1/\varphi$ are symmetrical to the dot-dash line through the pole P (inclined 45° to the axes). This, however, is no longer true for the "tension member region" where a portion of the scale was projectively distorted so as to include all the values of $\bar{\varphi}_N$ from -1 to $-\infty$.

c) Chart 2.- The buckling condition for the rod system represented in figure 1,b is:

$$\begin{vmatrix} \left(\dfrac{e}{e^2 - \hat{e}^2}\right)_{K_r,M} + \Sigma \left(\dfrac{1}{e}\right)_{N_{l_i}} & - \left(\dfrac{\hat{e}}{e^2 - \hat{e}^2}\right)_{K_r,M} \\ - \left(\dfrac{\hat{e}}{e^2 - \hat{e}^2}\right)_{K_r,M} & \left(\dfrac{e}{e^2 - \hat{e}^2}\right)_{K_r,M} + \Sigma \left(\dfrac{1}{e}\right)_{N_{r_i}} \end{vmatrix} = 0$$

or

$$\begin{vmatrix} v_{K_r,M} \left(\dfrac{EJ}{l}\right)_M + \Sigma\, u_{N_{l_i}} \left(\dfrac{EJ}{l}\right)_{N_{l_i}} & - w_{K_r,M} \left(\dfrac{EJ}{l}\right)_M \\ - w_{K_r,M} \left(\dfrac{EJ}{l}\right)_M & v_{K_r,M} \left(\dfrac{EJ}{l}\right)_M + \Sigma\, u_{N_{r_i}} \left(\dfrac{EJ}{l}\right)_{N_{r_i}} \end{vmatrix} = 0$$

where the subscript K_r,M corresponds to the instability measure $\alpha_{K_r,M}$ of the center bar M for the axial force for which, with given values of α_N and given member parameters φ_N of all the remaining members, the critical condition of the system (buckling in the plane of the system) sets in.

The above determinant after expansion and division by $\left(\frac{EJ}{l}\right)_M$ becomes:

$$(v^2_{K_r,M} - w^2_{K_r,M}) + v_{K_r,M}\left[\Sigma\, u_{N_{l_i}}\, \varphi_{N_{l_i}} + \Sigma\, u_{N_{r_i}}\, \varphi_{N_{r_i}}\right] + $$
$$+ \Sigma\, u_{N_{l_i}}\, \varphi_{N_{l_i}}\, \Sigma\, u_{N_{r_i}}\, \varphi_{N_{r_i}} = 0$$

(With an open three-bar system the summation signs drop out.)

As under II,3,b there may be determined from chart I for each of the two summation expressions an equivalent expression $u_o\, \Sigma\, \bar{\varphi}_{N_{l_i}}$ and $u_o\, \Sigma\, \bar{\varphi}_{N_{r_i}}$, respectively.

If we write

$$\Sigma\, u_{N_{l_i}}\, \varphi_{N_{l_i}} = u_o\, \Sigma\, \bar{\varphi}_{N_{l_i}} = -\, (u'_{K_r,M})_l$$

and

$$\Sigma\, u_{N_{r_i}}\, \varphi_{N_{r_i}} = u_o\, \Sigma\, \bar{\varphi}_{N_{r_i}} = -\, (u'_{K_r,M})_r$$

then, according to II,3,b, chart I gives the corresponding values of $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2_l$ and $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2_r$, respectively; i.e., the restraining effects which are exerted on the center rod M when the system formed by latter member and those joined to its left end only (or right end, only) would buckle by itself.

Using the general relation $v^2 - w^2 = u\, v$ (see formulas under II,3,a), the buckling condition of the system represented in figure 1,b becomes:

$$u_{K_r,M}\, v_{K_r,M} - v_{K_r,M}\left[(u'_{K_r,M})_l + (u'_{K_r,M})_r\right] + (u'_{K_r,M})_l\, (u'_{K_r,M})_r = 0$$

This relation is plotted on chart II as a family of curves with $\left(\frac{\alpha_{K_r,M}}{\pi}\right)^2$ as parameter and $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2_l$ and $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2_r$ as coordinates.

The parameter of the curve which passes through the point with coordinates $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2_l$ and $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2_r$ thus represents the restraint $\left(\frac{\alpha_{K_r,M}}{\pi}\right)^2$ exerted on member M by all the adjoining members if with the given dimensions and stresses of the adjoining members the center rod M were to experience

1) a restraint effect $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2_l$ due to the members on its left alone, and

2) an effect $\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2_r$ due to the members on its right alone.

III. NUMERICAL EXAMPLES

In order to illustrate the practical application of the table, several examples are computed.

To illustrate II,1,a: Is the truss consisting of the five members with the data given below, stable? (See fig. 4.)

Member	EJ (kg cm²)	l (cm)	S (kg)
1	8×10^5	60	+200
2	7×10^5	75	-2,000
3	2×10^5	45	0
4	7×10^5	75	+2,000
5	8×10^5	60	-3,000

Solution according to the first method of the section: Member 2 is chosen as the "buckling member" K. For the latter

$$\left(\frac{\alpha_K}{\pi}\right)^2 = \left(\frac{l_K}{\pi}\right)^2 \left(\frac{S}{EJ}\right)_K = 1.63$$

For the remaining members:

Member i	$\left(\frac{\alpha_{N_i}}{\pi}\right)^2 = \left(\frac{l_{N_i}}{\pi}\right)^2 \left(\frac{S}{EJ}\right)_{N_i}$	$\varphi_{N_i} = \left(\frac{EJ}{l}\right)_{N_i} : \left(\frac{EJ}{l}\right)_K$	$\bar{\varphi}_{N_i}$ from chart I
1	0.09 tension	1.43	1.50
3	0	.48	.48
4	1.63 tension	1.00	1.81
5	1.37 compression	1.43	-1.23
			$\Sigma\ \bar{\varphi}_{N_i} = 2.56$

To $\left(\frac{\alpha_N}{\pi}\right)^2 = 0$ and $\varphi_N = \Sigma\ \bar{\varphi}_{N_i} = 2.56$ there corresponds

$$\left(\frac{\alpha_{K_r,K}}{\pi}\right)^2 = 1.66 \quad \text{(See chart I.)}$$

Since the actual value

$$\left(\frac{\alpha_K}{\pi}\right)^2 = 1.63 < \left(\frac{\alpha_{K_r,K}}{\pi}\right)^2 = 1.66$$

the system is stable.

Solution according to the second method: No member is preferred as reference member. The reference value chosen is $\left(\frac{EJ}{l}\right)_B = 1 \times 10^4$ kg cm

and we have:

Member i	$\left(\frac{\alpha_i}{\pi}\right)^2 = \left(\frac{l_i}{\pi}\right)^2 \left(\frac{S}{EJ}\right)_i$	$\varphi_i = \left(\frac{EJ}{l}\right)_i : \left(\frac{EJ}{l}\right)_B$	$\bar{\varphi}_i$ from chart I
1	0.09 tension	1.33	+1.39
2	1.63 compression	.93	-2.00
3	0	.44	+.44
4	1.63 tension	.93	+1.69
5	1.37 compression	1.33	-1.01
			$\Sigma\ \bar{\varphi}_i = +0.51$

Since $\Sigma \bar{\varphi}_i > 0$, the system is stable.

To illustrate II,1,b: At which of the loading conditions 1-5 below is the three-bar structure with the following data, stable? (See fig. 2.)

Dimensions of Members

Member	EJ (kg cm²)	l (cm)
N_L	3×10^6	40
M	3.5×10^6	80
N_r	4×10^6	50

Forces on Members

Loading condition	Force S (kg) in N_l	M	N_r
1	-26,000	+7,500	-19,000
2	-22,000	-4,000	+30,000
3	-5,500	-3,000	-20,000
4	-22,000	-8,000	+8,000
5	-7,000	-6,000	-20,000

The parameters for each of the five loading conditions are determined and given in the table below.

Loading condition	$\left(\frac{\alpha_M}{\pi}\right)^2 = \left(\frac{l_M}{\pi}\right)^2 \left(\frac{S}{EJ}\right)_M$	$\left(\frac{\alpha_{N_l}}{\pi}\right)^2 = \left(\frac{l_{N_l}}{\pi}\right)^2 \left(\frac{S}{EJ}\right)_{N_l}$	$\varphi_{N_l} = \left(\frac{EJ}{l}\right)_{N_l} : \left(\frac{EJ}{l}\right)_M$	$\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2_l$ from chart I
1	-1.39*	1.40 compression	1.72	1.35 compression
2	.22	1.19 "	1.72	.50 "
3	.16	.30 "	1.72	1.49 "
4	1.48	1.19 "	1.72	.50 "
5	1.11	.38 "	1.72	1.46 "

*negative, since tension.

Loading condition	$\left(\frac{\alpha_{N_r}}{\pi}\right)^2 = \left(\frac{l_{N_r}}{\pi}\right)^2 \left(\frac{S}{EJ}\right)_{N_r}$	$\varphi_{N_r} = \left(\frac{EJ}{l}\right)_{N_r} : \left(\frac{EJ}{l}\right)_M$	$\left(\frac{\alpha'_{K_r,M}}{\pi}\right)^2_r$ from chart I	$\left(\frac{\alpha_{K_r,M}}{\pi}\right)^2$ from chart II
1	1.21 compression	1.83	0.37 compression	-1.65*
2	1.90 tension		1.74 "	1.06
3	1.27 compression		.08 tension	.21
4	.51 tension		1.64 compression	.99
5	1.27 compression		.04 tension	.24

*negative, since in tension member region.

Under loading conditions 2 and 3, the rod system under consideration is stable since $\left(\frac{\alpha_M}{\pi}\right)^2 < \left(\frac{\alpha_{K_r,M}}{\pi}\right)^2$; under loading conditions 1, 4, and 5 the structure is no longer stable since $\left(\frac{\alpha_M}{\pi}\right)^2 > \left(\frac{\alpha_{K_r,M}}{\pi}\right)^2$.

To illustrate II,1,c: Is the truss with the dimensions and loads given below, stable? (See fig. 5.)

Member	EJ (kg cm²)	l (cm)	S (kg)
M	3×10^5	50	-350
N_{l_1}	3×10^5	50	-1,700
N_{l_2}	2×10^5	40	-500
N_{l_3}	1×10^5	64	+500
N_{r_1}	1×10^5	64	+250
N_{r_2}	3×10^5	50	-1,500

We have:

$$\left(\frac{\alpha_M}{\pi}\right)^2 = \left(\frac{l_M}{\pi}\right)^2 \left(\frac{S}{EJ}\right)_M = 0.30$$

Member	$\left(\frac{\alpha_{N_{l_i}}}{\pi}\right)^2 = \left(\frac{l_{N_{l_i}}}{\pi}\right)^2 \left(\frac{S}{EJ}\right)_{N_{l_i}}$	$\varphi_{N_{l_i}} = \left(\frac{EJ}{l}\right)_{N_{l_i}} : \left(\frac{EJ}{l}\right)_M$	$\bar{\varphi}_{N_{l_i}}$ from chart I
N_{l_1}	1.44 compression	1.00	-1.10
N_{l_2}	.41 "	.83	.58
N_{l_3}	2.07 tension	.26	.49
			$\Sigma\ \bar{\varphi}_{N_{l_i}} = -0.03$

Member	$\left(\frac{\alpha_{N_{r_i}}}{\pi}\right)^2 = \left(\frac{l_{N_{r_i}}}{\pi}\right)^2 \left(\frac{S}{EJ}\right)_{N_{r_i}}$	$\varphi_{N_{r_i}} = \left(\frac{EJ}{l}\right)_{N_{r_i}} : \left(\frac{EJ}{l}\right)_M$	$\bar{\varphi}_{N_{r_i}}$ from chart I
N_{r_1}	1.04 tension	0.26	0.38
N_{r_2}	1.27 compression	1.00	-.54
			$\Sigma\ \bar{\varphi}_{N_{r_i}} = -0.16$

whence we obtain (from chart I):

$$\left(\frac{\alpha'_{K_r,M}}{\pi}\right)_l^2 = 0.98 \text{ compression}$$

$$\left(\frac{\alpha_{K_r,M}}{\pi}\right)_r^2 = 0.90 \text{ tension}$$

and from chart II:

$$\left(\frac{\alpha_{K_r,M}}{\pi}\right)^2 = 0.88$$

The structure is therefore stable, since

$$\left(\frac{\alpha_M}{\pi}\right)^2 = 0.30 < \left(\frac{\alpha_{K_r,M}}{\pi}\right)^2 = 0.88$$

Translation by S. Reiss,
National Advisory Committee for Aeronautics.

IV. REFERENCES

The charts mentioned at the beginning of this paper for the most important particular cases are discussed in

1. Borkmann, Karl: Charts for Checking the Stability of Compression Members in Trusses. T.M. No. 800, N.A.C.A., 1936.

Additional detailed investigation on the stability of trusses and systems of rods as well as further references are found, for example, in

2. Bleich, Fr.: Theorie und Berechnung eiserner Brücken. III. Abschnitt: Die Knicksicherheit der gedrückten Glieder eiserner Brücken. Julius Springer, 1924.

3. Zimmerman, H.: Knickfestigkeit der Stabverbindungen. Ernst & Sohn, 1925.

4. Teichmann, Alfred: Spatial Buckling of Various Types of Airplane Strut Systems. T.M. No. 647, N.A.C.A., 1931.

5. Thalau, K., and Teichmann, A.: Aufgaben aus der Flugzeugstatik. Kapitel IV: Stabilitätsaufgaben. Julius Springer, Berlin, 1933.

6. Kaul, Hans W.: Zur Knicklast einer Zweistabgruppe. Luftfahrtforschung, Bd. 11, Nr. 2, 1934, S. 53-56.

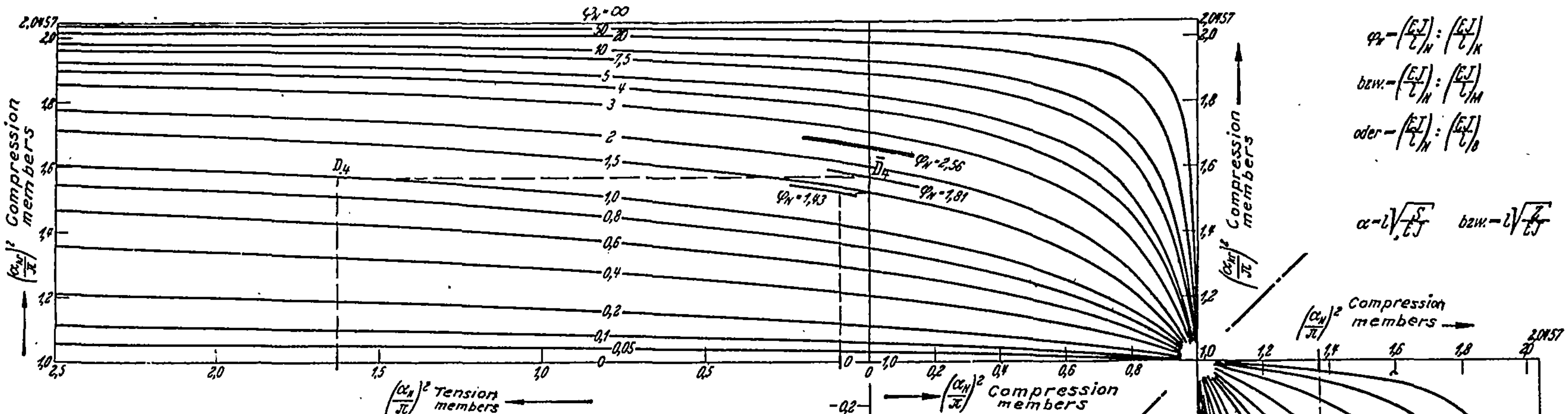
For the application of charts to stresses beyond the elastic range, the references below may be consulted.

7. Borkmann, K.: Zur Berechnung der Stabilität von Stabgruppen bei Beanspruchung jenseits der Proportionalitätsgrenze. Z.F.M., Bd. 24, Nr. 5, 1933, S. 139.

8. Kaul, Hans W.: The Stress Criterion of a Tension Member with Graded Flexural Stiffness (Contribution to the Problem of "Clamping Effect" outside of the Elastic Range). T.M. No. 804, N.A.C.A., 1936.

The reader is also referred to a French paper:

9. Riviére, H.: Calcul des poutres comprimées encastrées élastiquement à leurs extrémités - Flambage des barres de treillis assemblées rigidement. L'Aéronautique, Nr. 200, 1936, pp. 7-12.



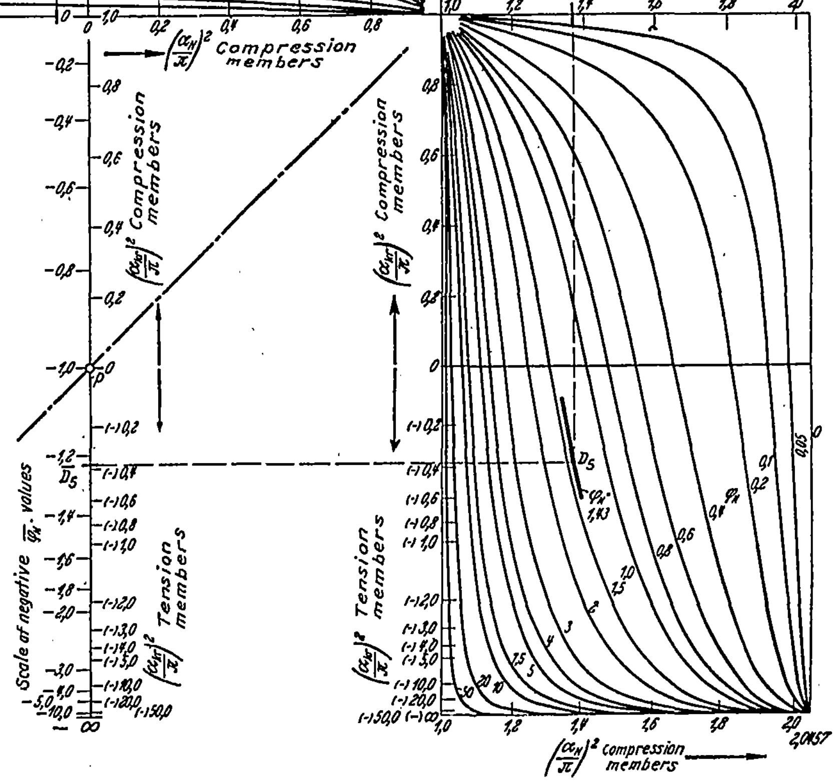


Chart I.- Buckling of a two-panel system of rods in the plane of the system.

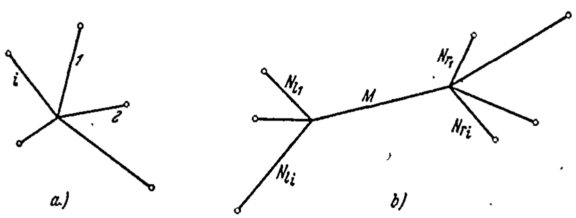


Figure 1.- Rod systems whose stability (with respect to buckling in their planes) may be investigated with the aid of the charts.

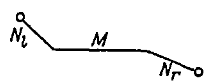


Figure 2.- Three-bar system.

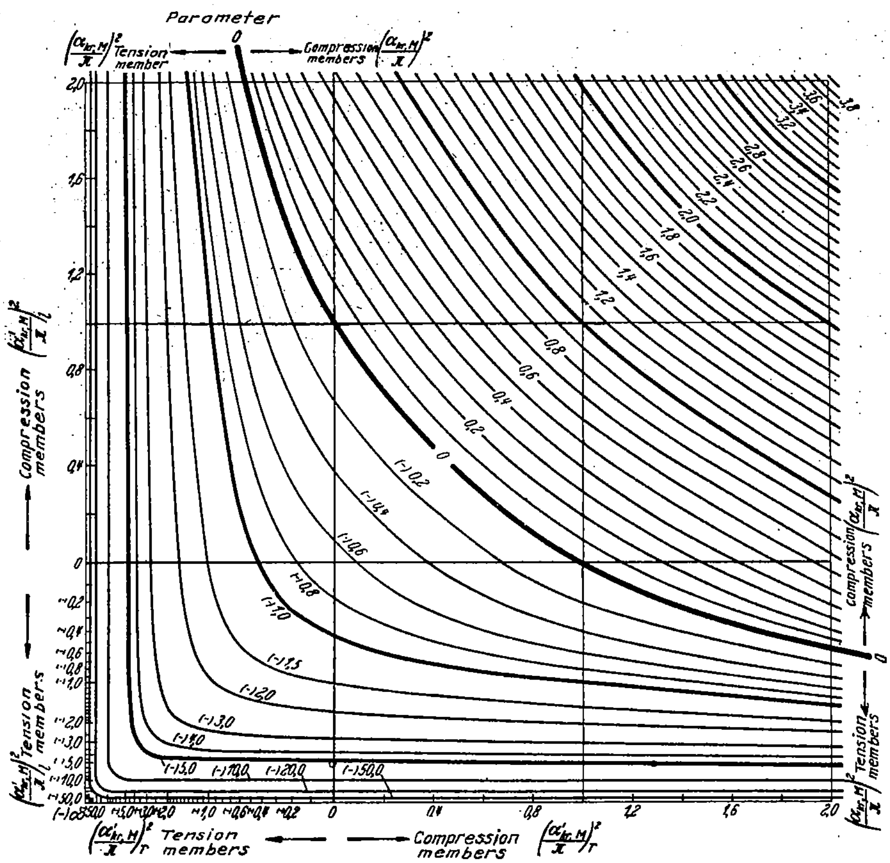


Chart II.- Buckling of a three-panel system in the plane of the system.

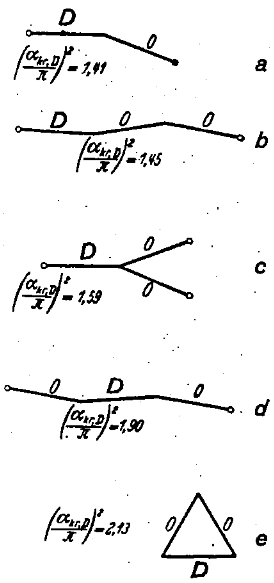


Figure 3.- Comparison of various arrangements of compression member D with unstressed members O.

a, only one member joined to D.

b, two members O joined to one end of D one behind the other.

c, two members O joined to one end of D side by side.

d, two members O joined to D one at each end.

e, two members O and member D forming a triangle.

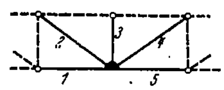


Figure 4.- Section of truss for illustrating numerical example under II 1a).

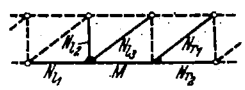


Figure 5.- Section of truss for illustrating numerical example under II 1c).