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No. 934

APPLICATION OF THE METHODS OF GAS DYNAMICS TO WATER FLOWS WITH FREE SURFACE

PART I. FLOWS WITH NO ENERGY DISSIPATION

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PREFACE

The work here presented was suggested to me by Dr. J. Ackeret, and was carried out at the Institut für Aerodynamik der E.T.H. Problems in the field of supersonic flowe occur with increasing frequency in recent times. It is of interest firet to investigate as to how far the relation' extends between the flow of a liquid on a horisontal bottom with the two-dimensional flow of a compressible gas. Secondly, problems in the field of water flows may be solved directly by the methods of the theory of gas dynamics* which, in recent years, have been highly developed.

The present work was undertaken with two objects in view. In the first place. it is considered as a contribution to the water analogy of gas flows, and secondly, a large portion is devoted to the general theory of the two-dimensional supersonic flows. An attempt has been made to bring the latter Into such shape and detail as to facilitate as much as possible its application by the engineer, who is lees familiar with the subject.

Here, I should like to express my thanks to Dr. Ackeret for his encourcqement nnd aid, and to Dr. de Hnller, Assistant at the Institut für Aerodynnmik, for his friendly support.

*Translator's note: The term "gas dynamics" is defined in the Introduction.

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IBTRODUCTIOB

Let there be considered a gas at rest in space or a portion of space, and let a piston or a movable portion of the boundary set the gas in motion. In the case of an incompressible fluid, the latter will begin to flow simultaneously over the entire space at the Instant the disturbance is applied. With a compressible fluid the case is otherwise. The effect of a disturbance first shows up in a restricted portion of the space only at a definite time interval after the start of the disturbance. If the latter is small, the speed of propagation of its effect is equal to the velocity of sound in the qas. In an ideal gas. it is proportional to the square root of the absolute temperature T and depends only on the latter.

If the velocity of flow in a fluid is small compared to the velocity of sound, the fluid may be treated as incompressible. The relation between velocity c (m/s) and pressure p (kg/m²) at various pofnts of the flow, is in the case of absence of friction, given by the Bernoulli equation. As soon, however, as the velocity differences at various points of the flow attain the order of magnitude of the velocity of sound, the compressibility of the gas may no longer be neglected. Density ρ (mass per unit volume, kg s²/m⁴) and temperature are variable, so that the laws of thermodynamics must be taken Into account. Thethoory of such flow comes under Gas Dynamics (references 1 and 7).

*"Anwendung gasdynamischer Methoden auf Wasserströmungen mit freier Oberfläche." Mitteilungen aus dem Institut für Aerodynamik, No. 7, 1938, Eidgenössische Technische Hochschule, Zürich,

**For Part II, see N.A.C.A. Technical Memorandum No. 935.

Depending on whether the flow velocity is smaller or larger than the velocity of sound, we speak of a subsonic and a supersonic flow, respectively, the two kinds being essentially different in character. They may occur side by side in the same flow since the velocity c and the sound velocity a in general vary from point to point. The quotient velocity of flow per velocity of sound for a definite point of the flow is denoted as the local Mach number $M = c/a$ (reference 4). For $M < 1$ the flow is subsonic: $M > 1$, supersonic. The subsonic flows in the neighborhood of $M = 1$ have as yet been little investigated. We are far better acquainted with the properties of supersonic flows, though chiefly the two-dimensional flows:*

Between the variables, pressure, temperature, and density, there holds the equation of state for an ideal gas

$$p = g R \rho T \tag{1}$$

where R (kg m/kg° = m/o) is a constant that is different for each gas. By the addition of heat, compression, and expansion, all possible states may be attained in the gas. If, however, heat is neither added nor taken away, and in the gas itself no heat arises through friction then, in addition to equation (1), the following adiabatic equations hold between the state variables:

$$p/p_o = (\rho/\rho_o)^k \tag{2a}$$

$$\rho/\rho_o = (T/T_o)^{1/k-1} \tag{2b}$$

$$p/p_o = (T/T_o)^{k/k-1} \tag{2c}$$

where p_o, ρ_o, T_o is any reference state, and k is constant for an ideal gas, being the ratio of the specific heat at constant pressure (c_p) to the specific heat at constant volume (c_v). This case of adiabatic change of state is the one that obtains in an ideal flow (no friction, no addition of heat from the outside, heat conduction and heat radiation in the flow itself negligible). As reference state in a flow there is generally chosen the state at a point of rest.

In order to be able to apply readily the energy equation to thermal processes, there is introduced a further

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*1) Three-dimensional flows: references 6, 8, 20, 26, 29.
2) Two-dimensional flows: references 1 (or 2), (pp. 308-322); 3, 7 (pp. 407-444), 14, 15, 17, 18, 27.
3) Transition region of subsonic and supersonic flows: references 9, 14 (pp. 57-67), 28, 30.

state variable, namel , the heat content i, defined by $i = c_p T$ (in kg m/kg 3*. Let the heat content at a point of reat be i_o. The flow velocity at an arbitrary point (i, P, T, ρ) of the flow is then computed from the energy equation to be

$$c^2 = 2g\,(i_o - i) = 2g\,c_p\,(T_o - T) \tag{3}$$

Transforming with the aid of equations (1) and (2)

$$c^2 = \frac{2k}{k-1}\,\frac{p_o}{\rho_o}\left[1 - \left(\frac{p}{p_o}\right)^{\frac{k-1}{k}}\right] \tag{3a}$$

This equation gives the relation between the pressure and velocity for the compressible adiabatic flow and replaces the Bernoulli equation. To a first approximation, i.e., for small Mach numbers, it goes over Into the Bernoulli equation. Bor the velocity of sound, ma have

$$a^2 = dp/d\rho \text{ (reference 13, p.536)} \tag{4}$$

or, using equation (2a):

$$a^2 = k\,\frac{p}{\rho} = gkR\,T \tag{4a}$$

From (3a) and (4a) there is obtained:

$$M^2 = c^2/a^2 = \frac{2}{k-1}\,\frac{p_o}{\rho_o}\,\frac{\rho}{p}\left[1 - \left(\frac{p}{p_o}\right)^{\frac{k-1}{k}}\right]$$

From the adiabatic 'equation (2a)

$$\frac{p_o}{\rho_o}\,\frac{\rho}{p} = \left(\frac{p_o}{p}\right)^{1-\frac{1}{k}} = \left(\frac{p_o}{p}\right)^{\frac{k-1}{k}}$$

*The heat content i s usually expressed in kcal/kg, Many computations are simplified, however, if the heat is consistently expressed in mkq instead-of kcal. The specific heats c_p and c_v must then be given in mkg/kg° instead of in kcal/kg°. The carrying along of the factor A = 1/427 kgm/kcal is thereby avoided. R is simply $c_p - c_v$, etc. In what follows, this assumption will everywhere be used.

and substituting in the above equation and solving for p_0, we have

$$p_0 = p\left[1 + \frac{k-1}{2} M^2\right]^{\frac{k}{k-1}}$$

Expanding the brackets into a series there is obtained:

$$p_0 = p\left[1 + \frac{k}{k-1}\,\frac{k-1}{2} M^2 + \frac{k}{k-1}\left(\frac{k}{k-1} - 1\right)\frac{1}{1\times 2}\left(\frac{k-1}{2} M^2\right)^2 + \ldots\right]$$

$$p_0 - p = p\left[\frac{k}{k-1}\,\frac{k-1}{2} M^2 + \ldots\right]$$

The common factor $M^2 \frac{k}{2}$ can be taken outside the brackets

$$p_0 - p = p\,\frac{k}{2} M^2\left[1 + \frac{1}{4} M^2 + \frac{1(2-k)}{3!2^2} M^4 + \frac{1(2-k)(3-2k)}{4!2^3} M^6 + \ldots\right]$$ 3

Consider

$$\frac{\rho}{2} c^2 = \frac{\rho}{2}\,\frac{c^2}{a^2}\,a^2$$

Substituting a^2 from equation (4a):

$$\frac{\rho}{2} c^2 = M^2 k \frac{p}{2}$$

We thus have, finally

$$p_0 - p = \frac{\rho}{2} c^2\left[1 + \frac{1}{4} M^2 + \frac{1(2-k)}{3!2^2} M^4 + \ldots\right] \qquad (5)$$

For $M \simeq 0$, the above becomes the Bernoulli equation $\frac{\rho}{2} c^2 = p_0 - p$. A better approximation is $\frac{\rho}{2} c^2 = (p_0 - p)/(1 + \frac{1}{4} M^2)$. The first two coefficients, 1 and 1/4, in the series are independent of k. For k = 1.4, the next two coefficients are 1/40 and 1/1600.

We shall now bring out an important property of the supersonic flows. Let us consider first a parallel flow with constant velocity c. The velocity of sound corresponding to the temperature of the gas also has the same value over the entire flow plane. If a small cylindrical obstacle is situated in such a supersonic flow, the disturbance produced by the obstacle is propagated with respect to the moving gas with the local sound velocity. The waves are circular cylindrical in shape (fig. 1). Let the obstacle be located at point P. If the wave center K_x is at point X, a time interval $t = x/c$, has passed since this wave arose. It then has the radius $r = a\ t = a\ x/c$. At the point P such waves arise continuously. All of them have as their common envelope two straight rays, the Mach rays, which form with the direction of flow the Mach angle α; $\sin \alpha = r/x = a/c$. If the obstacle at P is small, the intensity of the circular waves is small to a higher order. Only along the Mach rays are the circular waves dense enough for the effect of the disturbance to be of the order of magnitude of the latter. The effect of a disturbance at P is propagated only along the Mach rays through P. Now instead of a parallel flow, we shall consider a general supersonic flow. The flow velocity and the sound velocity vary from point to point. For each sufficiently small partial region of flow the same considerations as above are valid, the direction and Mach angle varying only from point to point. The disturbance arising from a small obstacle at P is now propagated along curved lines (fig. 2), those being known as Mach lines. For each flow there are two families of Mach lines. All effects arising from the boundary of the flow are evidenced along those lines of the flow.

It is possible with liquid fluids (water) to produce flows that show a far-reaching analogy to the dimensional flows of a compressible gas (references 5, 11, 13 (p. 537), 21, 22, 23, and 24).

A flow of this kind is obtained if water is allowed to flow over a horizontal bottom under the effect of gravity. The surface of the water is assumed to be free. At the sides it must be bounded by vertical walls or it must flow into water of a definite depth at rest. The fixed vertical walls correspond to the boundaries of the gas flow. A channel with horizontal bottom and rectangular cross section with variable width, the axis of which need not be rectilinear, is an example of this type of boundary. The water flowing into water at rest corresponds to a free gas jet. An open sluice, from which the water flows out, is an example of the second boundary condition. The bottoms of the upstream and downstream water must lie in the same horizontal plane.

The velocities that occur in such flows are very small in comparison with the sound velocity in water (about 1,430 m/s). The latter plays no part at all in the considerations that follow. It is another velocity which is analogous to the velocity of sound in a gas.

In the present work only stationary flows will be investigated. The free upper surface of the water is then a fixed surface in space. The water depth h varies from point to point of the flow. For each depth there exists for long plane waves a wave propagation velocity $\sqrt{gh}$, which depends on the depth alone. On the basis of this wave velocity the water flows described may be divided into two groups which, as in the case of the gases, differ essentially in their properties. If the water velocity is less than $\sqrt{gh}$, the water will be said to "stream"; if greater than $\sqrt{gh}$, the water will be said to "shoot."

PART I. FLOWS WITH NO ENERGY DISSIPATION

Differential Equation of the Water Flow

1. Energy Equation

It will be assumed that the flow of the water is frictionless so that conversion of energy into heat is excluded. The energy equation then simply states that the sum of the potential and kinetic energy of a water particle is constant during its motion.

Let us consider a flow filament (fig. 3) which passes through the point y_0, z_0 of the initial cross section $x = 0$. Along this filament, between the pressure p and the velocity c, there obtains the Bernoulli equation

$$p + \frac{\rho}{2} c^2 + \rho g z = p_1 + \frac{\rho}{2} c_1^2 + \rho g z_1 \qquad (6)$$

On the surface of the water p is constant and equal to the atmospheric pressure p_B. In what follows we may, without error, set this equal to zero since only pressure differences are of physical significance in the case of incompressible flows. The magnitudes denoted with the subscript 1 refer to an arbitrary but fixed point of the flow filament (reference point). The magnitudes without subscript refer to a variable point. If the water flows out from an infinitely wide basin, then the velocity in the basin is $c_o = 0$. Also, the curvature of the free surface is zero. The plane $x = 0$ is assumed to lie in this region. We choose the point x_o, y_o, z_o as reference point. The corresponding water depth is denoted by h_o and is at the same time the maximum depth occurring.

For the above reference point, the Bernoulli equation reads:

$$p + \frac{\rho}{2} c^2 + \rho g z = p_o + \rho g z_o$$

from which

$$c^2 = 2g(z_o - z) + 2(p_o - p)/\rho \tag{7}$$

We now make a simplifying assumption, namely, that the vertical acceleration of the water is negligible compared with the acceleration of gravity. Under this assumption the static pressure at a point of the field of flow depends linearly on the vertical distance under the free surface at that position:

$$p_o = \rho g(h_o - z_o) \tag{8a}$$

and

$$p = \rho g(h - z) \tag{8b}$$

The above substituted in (7) gives, finally,

$$c^2 = 2g(h_o - h) = 2g \Delta h \tag{9}$$

The energy equation (9) holds for the flow filament passing through y_o and z_o at $x = 0$. Since, however, at $x = 0$, all the stream filaments that lie one above the other, have the same h_o and for all of them, $c_o = 0$; and since equation (9) does not contain the coordinate z, the velocity c at x, y, is constant over the entire depth and is given only by the difference in height Δh between the total head and the free level, Δh being, at most, equal to h_o. The maximum attainable velocity therefore is $c_{max} = \sqrt{2g\ h_o}$. The energy equation may thus be written

$$(c/c_{max})^2 = c^2/2g\ h_o = \Delta h/h_o \tag{9a}$$

In a gas the maximum velocity is $c_{max} = \sqrt{2g\ i_o}$.

and equation (3), corresponding to (9a), becomes:

$$(c/c_{max})^2 = c^2/2g\ i_o = \Delta i/i_o = \Delta T/T_o \qquad (10)$$

From these two equations it may be seen that the ratio of the velocity to the maximum velocity for the water and gas flows becomes equally large if

$$(h_o - h)/h_o = (T_o - T)/T_o$$

This is the case for

$$h/h_o = T/T_o$$

With respect to the velocity there exists therefore an analogy between the two flows if the depth ratios h/h_o are compared with the gas-temperature ratios T/T_o. The water depth corresponds to the gas temperature, and conversely.*

2. Equation of Continuity (reference 1?, p. 320)

We shall set up the equation of continuity in differential form. For this purpose we consider at x, y a small fluid prism of edges dx and dy and height h (fig. 4). Let u and v be the horizontal components, and w the vertical component of the velocity c in the direction of the coordinate axes x, y, and z.

Neglecting the vertical acceleration of the water in comparison with the acceleration of gravity, equation (8b) is valid. From it, we have:

$$\frac{\partial p}{\partial x} = \rho\ g\ \frac{\partial h}{\partial x} \quad \text{and} \quad \frac{\partial p}{\partial y} = \rho\ g\ \frac{\partial h}{\partial y}$$

The right sides of the above relations are independent of z, so that the horizontal accelerations for all points along a vertical also are independent of z. The horizontal velocity components u and v are thus constant over the entire depth because they were so in the initial state (of rest).

*It is not a question of setting absolute values of the velocities equal to each other but only, of course, nondimensional magnitudes, as c/c_{max}.

The continuity equation for the stationary flow simply expresses the fact that the quantity of fluid flowing into the prism (fig. 4) per unit time is equal to the outflowing mass. Since the density of the water is constant, the same holds true for the inflowing fluid volume dq_e (m^3/s) and for the outflowing volume dq_a: $dq_e = dq_a$. In the x-direction the volume $u\ h\ dy$ enters per unit time; dq_e becomes $= u\ h\ dy + v\ h\ dx$. The total outflowing volume, except for infinitely small magnitudes of higher order, becomes:

$$dq_a = \left(u + \frac{\partial u}{\partial x}\,dx\right)\left(h + \frac{\partial h}{\partial x}\,dx\right) dy + \left(v + \frac{\partial v}{\partial y}\,dy\right)\left(h + \frac{\partial h}{\partial y}\,dy\right) dx$$

This continuity condition written out and divided by $dx\ dy$ gives the equation of continuity

$$\frac{\partial(h\ u)}{\partial x} + \frac{\partial(h\ v)}{\partial y} = 0 \qquad (11)$$

The continuity equation for a two-dimensional compressible gas flow is

$$\frac{\partial(\rho\ u)}{\partial x} + \frac{\partial(\rho\ v)}{\partial y} = 0 \qquad (12)$$

Comparison of the two equations (11) and (12) shows that, just as the energy equations, the equations of continuity for the two flows have the same form. From these we may derive a further condition for the analogy, that the specific mass ρ of the gas flow corresponds to the water depth h. It may be clearly seen now why the incompressible flow of the water may bear a relationship to the flow of a compressible gas. As a consequence of the compressibility in a two-dimensional gas flow, the gas mass per unit of bottom area is not a constant but varies from point to point of the flow plane. Since the water depth in the flow with free surface varies, the mass per unit bottom area for this flow is also a variable.

From the equation of continuity, we derived the result that the water depth h corresponds to the specific mass ρ. By comparison of the energy equations of the two flows, it followed, however, that the water depth h was simultaneously also the analogous magnitude for the temperature T. This is possible without contradiction only if a very definite assumption is also made as regards the nature of the comparison gas. For the gas flow ρ depends upon T, the relation between the two being the adiabatic equation. (2b)

$$\rho/\rho_0 = (T/T_0)^{1/k-1}$$

Now $\rho/\rho_0 = h/h_0$ and simultaneously $T/T_0 = h/h_0$, and substituting in (2b), we have the equation:

$$h/h_0 = (h/h_0)^{1/k-1}$$

which obviously is satisfied only for

$$\underline{k = 2} \tag{13}$$

Thus we have the result that the flow of the water is comparable with the flow of a gas having a ratio $k = c_p/c_v = 2$. Such gases are not found in nature. There are, however, many phenomena which do not depend strongly on the value of k, so that the analogy has significance also for actual gases.

3. Irrotational Motion

Before introducing the condition of absence of vorticity, we make a slight transformation of the continuity equation (11), taking account of the energy equation (9). The latter solved for h, reads:

$$h = h_0 - c^2/2g$$

Hence

$$\frac{\partial h}{\partial x} = - \frac{1}{2g} \frac{\partial (c^2)}{\partial x}$$

and using the fact that $c^2 = u^2 + v^2$,* this gives

$$\frac{\partial h}{\partial x} = - \frac{1}{g} \left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x}\right) \tag{a}$$

*Since u and v are constant on a vertical, and since from (9), c also is constant, $w = \sqrt{c^2 - (u^2 + v^2)}$ is also constant, and since w vanishes at the bottom, it may be neglected in comparison with the components u and v.

Similarly,

$$\frac{\partial h}{\partial y} = -\frac{1}{g}\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right) \qquad (b)$$

The continuity equation (11) may also be written in the form

$$\frac{\partial u}{\partial x}h + \frac{\partial h}{\partial x}u + \frac{\partial v}{\partial y}h + \frac{\partial h}{\partial y}v = 0$$

Substituting in the above the expressions (a) and (b), there is obtained:

$$\frac{\partial u}{\partial x}h - \frac{u}{g}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right) + \frac{\partial v}{\partial y}h - \frac{v}{g}\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right) = 0$$

The above equation divided by h and rearranged, gives:

$$\frac{\partial u}{\partial x}\left(1 - \frac{u^2}{gh}\right) + \frac{\partial v}{\partial y}\left(1 - \frac{v^2}{gh}\right) - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{uv}{gh} = 0 \qquad (14)$$

We now introduce the condition for absence of vorticity. This will be true if $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$. In this case, there exists a function $\Phi(x,y)$, the velocity potential, of the coordinates x, y such that

$$u = \frac{\partial \Phi}{\partial x} \qquad v = \frac{\partial \Phi}{\partial y}$$

Substituting $\Phi(x, y)$ into equation (14), the latter may be written:*

$$\Phi_{xx}\left(1 - \frac{{\Phi_x}^2}{gh}\right) + \Phi_{yy}\left(1 - \frac{{\Phi_y}^2}{gh}\right) - 2\Phi_{xy}\frac{\Phi_x\,\Phi_y}{gh} = 0 \qquad (15)$$

This is the differential equation for the velocity potential of the ideal free surface water flow over a horizontal bottom. The equation is partial of the second order and

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* Instead of $\frac{\partial \Phi}{\partial x}$, we write in what follows in the usual notation Φ_x; $\frac{\partial^2 \Phi}{\partial x^2} \equiv \Phi_{xx}$; $\frac{\partial^2 \Phi}{\partial x \partial y} \equiv \Phi_{xy}$, etc.

linear in the second derivatives. The coefficients depend on the derivatives of the first order and on these only. It is to be observed that g h is not a constant but, according to the energy equation is

$$gh = gh_o - c^2/2 = g\ h_o - \frac{\Phi_x^2 + \Phi_y^2}{2}$$

The equation corresponding to (15) for the velocity potential of a two-dimensional compressible flow is (reference 1 (or 2), p. 308.

$$\Phi_{xx}\left(1 - \frac{\Phi_x^2}{a^2}\right) + \Phi_{yy}\left(1 - \frac{\Phi_y^2}{a^2}\right) - 2\Phi_{xy}\frac{\Phi_x\Phi_y}{a^2} = 0 \qquad (16)$$

The two equations (15) and (16) become identical if $gh/2gh_o$ is replaced by $a^2/2gi_o$. $\sqrt{g\ h}$ is the basic wave velocity in shallow water, and corresponds to the velocity a in the gas flow.

4. Summary of the Blow Analogy

We shall yet inquire what magnitude in the water flow is analogous to the gas pressure. Writing the equation of state (1) for an arbitrary state and for the state at rest, there is obtained by division:

$$p/p_o = (\rho/\rho_o)\,(T/T_o)$$

Substituting for ρ/ρ_o the corresponding value h/h_o, and for T/T_o also. h/h_o, there is obtained the value corresponding to p/p_o:

$$p/p_o = (h/h_o)^2 \qquad (17)$$

This is also obtained directly from the adiabatic equation (2a) with $\rho/\rho_o = h/h_o$ and $k = 2$.

The pressure p_G on the bottom surface is proportional to the water depth h; with ρ_W as specific mass of the water $p_G = \rho_W g\ h$. This pressure has no analogy in the two-dimensional gas flow. In particular, it is not the magnitude corresponding to the gas pressure since the corresponding magnitude to p is h^2 and not h. The force P of the water flow per unit of length of the vertical wall is, on account of the linear increase of the pressure with distance below the free surface, given by

$$P = \frac{\rho_W g}{2} h^2$$

For P, therefore, we have $P/P_0 = (h/h_0)^2$. Comparison with equation (17) shows that $p/p_0 = P/P_0$. The magnitude of the water flow corresponding to the gas pressure p is thus the force of the water on a unit strip of the side walls. The pressures in the two-dimensional compressible flow are analogous to the forces in the water on the vertical walls.

Prom the differential equation for the velocity Potential, we have derived the fact that the velocity of sound a corresponds to the wave velocity $\sqrt{g\,h}$. Tie differential equation arose through the combination of the energy and continuity equations. Thus the result $a \leftrightarrow \sqrt{gh}$ is not something essentially new but is only a consequence of the results $\rho \leftrightarrow h$, $T \leftrightarrow h$, and k = 2 of these two equations. We have $a^2 = gkRT = g(k - 1)i$, and for k = 2 and $i \leftrightarrow h$, this gives $a^2 \leftrightarrow gh$.

Since the velocity corresponding to a is $\sqrt{gh}$, there corresponds to the subsonic flow $c/a < 1$ the flow with $c/\sqrt{gh} < 1$. The water in this case is said to "stream," while the water flow corresponding to the supersonic flow is Raid to "shoot." The essential difference in character between the supersonic and subsonic flows exists also in the case of water between streaming and shooting flows.

The analogy considered in this section holds for flows with Mach numbers smaller and greater than 1. Essentially, however, only the flow of shooting water will be treated in this work: Application will therefore be made of the extensively developed theory of two-dimensional supersonic flows to the flow of water.

TABLE OF FLOW ANALOGY

	Two-dimensional gas flow	Liquid flow with free surface in gravity field
Nature of the flow medium	Hypothetical gas with $k = c_p/c_v = 2$	Incompressible fluid (e.g., water)
Side boundaries geometrically similar		Side boundary vertical Bottom horizontal
Analogous magnitude	Velocity $c/c_{max}, c/a^*$ Temperature ratio, T/T_0 Density ratio, ρ/ρ_0 Pressure ratio, p/p_0 Pressure on the side boundary walls p/p_0	Velocity $c/c_{max}, c/a^*$ Water depth ratio, h/ho Water depth ratio, h/ho Square of water depth ratio, $(h/h_0)^2$ Force on the vertical walls. $P/P_0 = (h/h_0)^2$
	Sound velocity a Mach number c/a Subsonic flow Supersonic flow Compressive shock (right and slant)	Wave velocity $\sqrt{gh}$. Mach number $c/\sqrt{gh}$ Streaming water Shooting water Hydraulic jump (normal and slant)

MATHEMATICAL BASIS

5. Introduction

For the treatment of fields of flow subjected to the boundary conditions, various mathematical methods, depending on the type of flow considered, are available. The mathematical basis for two-dimensional incompressible flows is the conformal transformation method familiar from the function theory. For the computation of compressible subsonic flows, use is made of the theory of general elliptical differential equations. This theory has not yet been sufficiently developed as a practical method. For the computation of supersonio flows, however, and hence for "shooting" water, there has been perfected the method of characteristics of the theory of hyperbolic partial differential equations by Prandtl, Steichen, and Busemann.

Since the characteristics method is as yet little known and, particularly, since it has not yet been applied to the investigation of flows of "shooting" water, this method in what follows, will be discussed in some detail.

6. Introduction of New Variables

The velocity potential $\Phi(x, y)$ may be geometrically represented by plotting vertically at each point of the flow plane x, y the corresponding value of Φ. We thus obtain a surface in space which we ehall denote as a Φ-surface. The slope of this surface along any direction gives the component of the flow velocity in' this direction.

Let the velocity along a line AS of a shooting flow of water be given in magnitude and direction (fig. 5). This velocity at each point of A3 may be decomposed into component8 ct and c_n, tangential and normal, respectively, to AB. Simultaneously, there will also be given the slopes of the Q-surface corresponding to the flow in the two directions and, finally, the value $\Phi(x, y)$ itself, except for a nonessential constant, will also be dotermined:

$$\Phi = \int_0^s \frac{\partial \Phi}{\partial s} ds + \Phi_A$$

The five magnitudes x, y, Φ (point P) and Φ_x, Φ_y (slope) are denoted as an element of the &surface. An element is simply an infinitesimal piece of the Φ-surface giving the position and elope. The assignment of the velocity along AB is equivalent to the assignment of an elementary strip of the Φ-surface (fiq. 5). The mathematical problem may thus be stated as followe: To find a surface whose curvature and slope satisfy the differential equation (15).

It is possible to put equation (15), by a transformation of variables, into a simpler form (reference 27, p.ö-10).

We consider first a ueual coordinate transformation - a so-called "point transformation." Let x and y be the independent variables; Φ a function of x and y, $\Phi(x, y)$. Then net variables X, Y, χ nag be introduced by defining them through the following equations:

$$\left.\begin{aligned} X &= X(x,y,\Phi(x,y)) \\ Y &= Y(x,y,\Phi) \\ \chi &= \chi(x,y,\Phi) \end{aligned}\right\} \qquad (18)$$

The function χ may be represented by a χ-surface in an X, Y, χ space, taking X and Y as the independent variables. To each point x, y, Φ, there corresponds according to equation (18), an image point X, Y, χ. Conversely, to each image point corresponds its original point since, in general, equations (18) may be solved for x, y, and Φ:

$$\left.\begin{aligned} x &= x(X,Y,\chi) \\ y &= y(X,Y,\chi) \\ \Phi &= \Phi(X,Y,\chi) \end{aligned}\right\} \qquad (19)$$

Let us, for simplicity, consider first a single independent variable x and a function $\Phi = \Phi(x)$. The point transformation in this case is given by the two equations:

$$X = X(x,\Phi(x)) \quad \text{and} \quad \chi = \chi(x,\Phi) \qquad (18a)$$

Solving (18a) for x and Φ, there is obtained:

$$x = x(X,\chi) \quad \text{and} \quad \Phi = \Phi(X,\chi) \qquad (19a)$$

To each pair of values x and Φ (point P), there corresponds according to (18a), a pair of values X and χ (point P*) (fig. 6). An entire curve has another curve as its image and the transformation is uniquely reversible.

We shall now consider a more general transformation. Let an entire element - that is, $x, y, \Phi, \Phi_x, \Phi_y$ be transformed. In place of formulas (18), we now have the more complicated transformation formulas:

$$\left.\begin{aligned} X &= X(x,y,\Phi,\Phi_x,\Phi_y) \\ Y &= Y(x,y,\Phi,\Phi_x,\Phi_y) \\ \chi &= \chi(x,y,\Phi,\Phi_x,\Phi_y) \end{aligned}\right\} \qquad (20)$$

In the case of a single independent variable, an element is given by the triple x, Φ, Φ_x (point and direction). To transform this element the transofrmation formulas would be

$$X = X(x,\Phi,\Phi_x) \quad \text{and} \quad \chi = \chi(x,\Phi,\Phi_x) \tag{20a}$$

From the above we have:

$$dX = X_x\, dx + X_\Phi\, d\Phi + X_{\Phi_x}\, d\Phi_x = (X_x + X_\Phi\, \Phi_x + X_{\Phi_x}\, \Phi_{xx})\, dx$$

and

$$d\chi = (\chi_x + \chi_\Phi\, \Phi_x + \chi_{\Phi_x}\, \Phi_{xx})\, dx$$

so that

$$\chi_X = \frac{d\chi}{dX} = \frac{\chi_x + \chi_\Phi\, \Phi_x + \chi_{\Phi_x}\, \Phi_{xx}}{X_x + X_\Phi\, \Phi_x + X_{\Phi_x}\, \Phi_{xx}} \tag{21}$$

hence, $d\chi/dX$, as (21) shows, in general depends on x, Φ, Φ_x, and Φ_{xx}. If, for example, 8 curve Φ_A (fig. 6) is prescribed, then at each point of the curve these four values are known. From the three formulas (20a) and (21) there are thus determined at each image point P* the values X, χ, and χ_X. Thcrc is thus obtained the curve χ_A as the image of curve Φ_A. Correspondingly, Φ_A may also be drawn If the entire curve χ_A is given. On the other hand, from the element x, Φ, Φ_x, It is not possible to determine anp element X, χ, χ_X from the formulas (20a) and (21), different elements being obtained, depending on how Φ_{xx} is chosen. In one case, however, the transformation is such that the image of an element is again an element, and conversely. This is the case when $d\chi/dX$ in equation (21) becomes Independent of Φ_{xx}, which is true only if

$$\frac{\chi_x + \chi_\Phi\, \Phi_x}{X_x + X_\Phi\, \Phi_x} = \frac{\chi_{\Phi_x}}{X_{\Phi_x}} \tag{22}$$

If the transformation formulas (20a) satisfy the condition (22), then the elements uniquely correspond to one another in the transformation.

An example of the above is the Legendre transformation of x, Φ to X, χ, of which we shall make important use below; for this transformation, the following transformation formulas hold:

$$X = \Phi_x$$

$$\chi = \Phi_x x - \Phi$$

We then have:

$$dX = \Phi_{xx}\, dx$$

$$d\chi = \Phi_x\, dx + \Phi_{xx}\, dx\, x - \Phi_x\, dx = x\, \Phi_{xx}\, dx$$

so that

$$d\chi/dX = x, \quad \text{independent of } \Phi_{xx}$$

The transformation with corresponding elements has in addition, another special property. Let us assume that at point P (fig. 6) two curves Φ_A and Φ_B touch each other. They thus have at point P a oommon element $x_A = x_B$, $\Phi_A = \Phi_B$, and $\Phi_{xA} = \Phi_{xB}$; but $\Phi_{xxA} \neq \Phi_{xxB}$ the curves being assumed in contact' but not osculating. According to the traneformation formulas (20a), we shall also have for this point. $X_A = X_B$ and $\chi_A = \chi_B$. The two image curves χ_A and χ_B then have the point P*, the image of P, also in common. Since,' however, $d\chi/dX$ in general, contains Φ_{xx} according to (21), and this second derivative is different for the curves A and B, the two image curves will intersect in point P* and not touch as the original curves do. Only if $d\chi/dX$ is independent of Φ_{11} will the two Image curves χ_A and χ_B also touch at point P*. This is precisely the case for the transformation with uniquely reciprocal element correspondence.. For this reason such transformations are known as contact traneformatione.

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*1) In correspondence with the concept-point transformation, the term "element transformation" is more logical than contact transformation.

2) The transformation (20a) becomes an element transformation a0 soon as, instead of only the two formulas of (20a), three are used:

$$x = X(x,\Phi,\Phi_x) \quad \chi = \chi(x,\Phi,\Phi_x) \quad \text{and} \quad \chi_X = \chi_X(x,\Phi,\Phi_x) \qquad (20b)$$

There then corresponds to each x, Φ, Φ_x, an X, χ, χ_X, and conversely. It is to be noted, however, that there is a relation between the three variables since $\chi_X = d\chi/dX$. If the left aide of (20b) is independent of Φ_{xx}, the riqht side must be. But this is precisely the contact transformation.

The result found above we shall now apply to two independent variables x, y, and their function Φ. The transformation formula8 are:

$$\left.\begin{aligned} X &= X(x, y, \Phi, \Phi_x, \Phi_y) \\ Y &= Y(x, y, \Phi, \Phi_x, \Phi_y) \\ \chi &= \chi(x, y, \Phi, \Phi_x, \Phi_y) \end{aligned}\right\} \qquad (20)$$

Since X, Y, and χ contain, in addition to x, y, and Φ, also Φ_x and Φ_y, there will in general also occur in

$$\text{and} \quad \left.\begin{aligned} \chi_X &= \partial\chi/\partial X = f_1(x, y, \Phi, \Phi_x, \Phi_y, \Phi_{xx}, \Phi_{xy}, \Phi_{yy}) \\ \chi &= \partial\chi/\partial Y = f_2(x, y, \Phi, \Phi_x, \Phi_y, \Phi_{xx}, \Phi_{xy}, \Phi_{yy}) \end{aligned}\right\} \qquad (23)$$

the second derivatives Φ_{xx}, Φ_{xy}, Φ_{yy}. We shall interpret $\Phi(x,y)$ a0 a eurface (fig. 7). Two surfaces Φ_A and Φ_B, which touch at a point, have x, y, Φ, Φ_x, Φ_y in common at this point. From the transformation equations they will then also have the image point X, Y, χ of the contact point in common. Since, however, χ_X and χ_Y contain the second derivatives of Φ, the tmo transformed surfaces will no longer be in contact at the common point: (XX), and (XX), not being equal - similarly, $(\chi_Y)_A$ and $(\chi_Y)_B$. The transformation again gives a unique correspondence of the elements only If the equations (23) do not contain the magnitudes Φ_{xx}, Φ_{xy} and Φ_{yy}. In this case two surfaces in oontnct at a point, go over after tranefomatlon Into two surfaces which at the image point again have a common tangent plane.

The Legendre contact transformation for two independent variables is

$$\left.\begin{aligned} X &= \Phi_x \\ Y &= \Phi_y \\ \chi &= \Phi_x\, x + \Phi_y\, y - \Phi \end{aligned}\right\} \qquad (24)$$

The eurface $\Phi = \Phi(x,y)$ with the above transformation goes over into a surface $X = \chi(X, Y)$ (fig. 7). We prove first that the above is actually a contact transformation. From equations (24)

$$d\chi = \Phi_x dx + x\, d\Phi_x + \Phi_y dy + y\, d\Phi_y - d\Phi$$

Noting that $d\Phi = \Phi_x dx + \Phi_y dy$, three terms drop out. Substituting for Φ_x and Φ_y, X and Y, respectively, from formulas (24), we have

$$d\chi = x\, dX + y\, dY$$

For the X-surface, the relations must be satisfied:

$$d\chi = \chi_X dX + \chi_Y dY$$

Comparison of the two expressions gives the derivatives of X of the first order:

$$\left.\begin{aligned} \chi_X &= x \\ \chi_Y &= y \end{aligned}\right\} \qquad (24a)$$

These are independent of the derivatives of Φ of the second order. Formulas (24) thus actually express a contact transformation, (24) and (24a) giving the corresponding element $X, Y, \chi, \chi_X, \chi_Y$ when the original element $x, y, \Phi, \Phi_x, \Phi_y$ is given. By simple reversal there is obtained the element correspondence for the reciprocal transformation:

$$\left.\begin{aligned} x &= \chi_X \\ y &= \chi_Y \\ \Phi &= X\chi_X + Y\chi_Y - \chi \end{aligned}\right\} \qquad (25)$$

$$\left.\begin{aligned} \Phi_x &= X \\ \Phi_y &= Y \end{aligned}\right\} \qquad (25a)$$

We wish still to express the derivatives of second order Φ_{xx}, Φ_{xy}, and Φ_{yy} in the new variables $X, Y, \chi, \chi_X, \chi_Y, \chi_{XX}, \chi_{XY}$, and χ_{YY}. There will then be obtained an important result for the applications.

For this purpose we consider x and y as the independent variables. From the first and second of equations (25), there is obtained:

$$dx = \chi_{XX} dX + \chi_{XY} dY$$

$$dy = \chi_{XY} dX + \chi_{YY} dY$$

Solving for dX and dY

$$dX = (\chi_{YY} dx - \chi_{XY} dy)\ 1/N$$

$$dY = (-\chi_{XY} dx + \chi_{XX} dy)\ 1/N$$

where $$N = (\chi_{XX}\,\chi_{YY} - \chi^2_{XY})$$

For the differential of Φ, we have (Φ-surface)

$$d\Phi = \Phi_x dx + \Phi_y dy \tag{26}$$

Substituting in the above (25a), there is obtained:

$$d\Phi = X\ dx + Y\ dy$$

For the second differential, we have:

$$d^2\Phi = dX\ dx + dY\ dy$$

for d^2x and d^2y are equal to zero since x and y are independent variables. In this equation we substitute the previously found expressions for dX and dY, and obtain:

$$d^2\Phi = (\chi_{YY}\ dx^2 - 2\ \chi_{XY}\ dx\ dy + \chi_{XX}\ dy^2)\ 1/N$$

On the other hand, from equation (28):

$$d^2\Phi = \Phi_{xx}\ dx^2 + 2\ \Phi_{xy}\ dx\ dy + \Phi_{yy}\ dy^2$$

Comparison of the coefficients of dx^2, dy^2, and dx dy of the last two equations shows finally that

$$\left.\begin{aligned} \Phi_{xx} &= \chi_{YY}\ 1/N \\ \Phi_{yy} &= \chi_{XX}\ 1/N \\ \Phi_{xy} &= -\chi_{XY}\ 1/N \end{aligned}\right\} \tag{27}$$

These are the required expressions for the derivatives of Φ of the second order..

The coefficients of the differential equation of the flow (15) depend on the derivatives of the velocity potential Φ of the first order. Introducing new variables into that equation (according to the Legendre contact transformation, the coefficients according to (24) will depend on the new independent variables and only on these. The partial derivatives of second order will be replaced, according to equations (27), by the partial derivatives of second order of the new function with the common denominator N. Since the differential equation (15) is linear homogeneous N may be multiplied out. By means of the Legendre contact equation, therefore, (15) becomes linear, homogeneous, of second ardor, and with coefficients that depend on the new independent variables only.

Let us introduce the new variables X, Y. Physically, X and Y are the velocity components u and v. The new variables according to (24) are:

$$\left.\begin{aligned} (X =)\ u &= \Phi_x \\ (Y =)\ v &= \Phi_y \\ \chi &= \Phi_x\, x + \Phi_y\, y - \Phi = u\, x + v\, y - \Phi \end{aligned}\right\} \tag{28}$$

The transformation formulas (25), (25a), and (27) are:

$$\left.\begin{aligned} & x = \chi_u,\ y = \chi_v,\ \Phi = u\, x + v\, y - \chi \\ & \Phi_x = u,\ \Phi_y = v \end{aligned}\right\} \tag{29}$$

$$\Phi_{xx} = \chi_{vv}\ 1/N,\ \Phi_{xy} = -\chi_{uv}\ 1/N,\ \Phi_{yy} = \chi_{uu}\ 1/N \tag{30}$$

The differential equation (15) in the new variables then becomes:

$$\chi_{vv}\left(1 - \frac{u^2}{gh}\right) + \chi_{uu}\left(1 - \frac{v^2}{gh}\right) + 2\chi_{uv}\,\frac{u\,v}{gh} = 0 \tag{31}$$

x and y being the coordinates of the flow. With the Legendre transformation of equation (15) into (31), we passed from the flow over into its "velocity image" - that is, the hodograph (velocity plane) of the flow. At the same time, in place of the velocity potential Φ, which is a function of the position in the flow, we have introduced the "position determining" potential X, which is a function of the velocity in the hodograph.

The assignment of the velocity along a curve AB is equivalent to the assignment of an elementary strip of the Φ-surface. Since the oontaot transformation is an element correspondence, the X-surface must contain the corresponding X-elementary strip.

Bor later use, we ehall introduce in equation (31) in plaoe of the rectangular coordinates u, v, X the cylindrical coordinates c, φ, X (point transformation), figure 8.

The new variables are:

$$c = \sqrt{u^2 + v^2}$$

$$\varphi = (\tan^{-1})(v/u)$$

$$\chi = \chi$$

whence

$$u = c \cos \varphi \quad \text{(a)}$$

$$v = c \sin \varphi \quad \text{(b)}$$

and

$$\frac{\partial c}{\partial u} = \frac{1}{2} \frac{1}{\sqrt{u^2 + v^2}} 2u = \cos \varphi$$

$$\frac{\partial c}{\partial v} = \sin \varphi$$

$$\frac{\partial \varphi}{\partial u} = - \frac{\sin \varphi}{c}$$

$$\frac{\partial \varphi}{\partial v} = \frac{\cos \varphi}{c}$$

We have:

$$\chi = \chi(u,v) = \chi[c,\varphi] = \chi[c(u,v), \varphi(u,v)]$$

so that

$$\left.\begin{aligned}\frac{\partial\chi}{\partial u} &= \frac{\partial\chi}{\partial c}\frac{\partial c}{\partial u} + \frac{\partial\chi}{\partial\varphi}\frac{\partial\varphi}{\partial u} = \frac{\partial\chi}{\partial c}\cos\varphi - \frac{\partial\chi}{\partial\varphi}\frac{\sin\varphi}{c}\\ \frac{\partial\chi}{\partial v} &= \frac{\partial\chi}{\partial c}\frac{\partial c}{\partial v} + \frac{\partial\chi}{\partial\varphi}\frac{\partial\varphi}{\partial v} = \frac{\partial\chi}{\partial c}\sin\varphi + \frac{\partial\chi}{\partial\varphi}\frac{\cos\varphi}{c}\end{aligned}\right\} \qquad (A)$$

Furthermore:

$$\frac{\partial^2\chi}{\partial u^2} = \frac{\partial(\partial\chi/\partial u)}{\partial u} = \frac{\partial(\partial\chi/\partial u)}{\partial c}\cos\varphi - \frac{\partial(\partial\chi/\partial u)}{\partial\varphi}\frac{\sin\varphi}{c}$$

$$\frac{\partial^2\chi}{\partial u\,\partial v} = \frac{\partial(\partial\chi/\partial v)}{\partial u} = \frac{\partial(\partial\chi/\partial v)}{\partial c}\cos\varphi - \frac{\partial(\partial\chi/\partial v)}{\partial\varphi}\frac{\sin\varphi}{c}.$$

$$\frac{\partial^2\chi}{\partial v^2} = \frac{\partial(\partial\chi/\partial v)}{\partial v} = \frac{\partial(\partial\chi/\partial v)}{\partial c}\sin\varphi + \frac{\partial(\partial\chi/\partial v)}{\partial\varphi}\frac{\cos\varphi}{c}$$

Substituting in the above the values of $\partial\chi/\partial u$ and $\partial\chi/\partial v$ from equations (A) there is obtained:

$$\begin{aligned}\chi_{uu} &= \left[\chi_{cc}\cos\varphi - \chi_{c\varphi}\frac{\sin\varphi}{c} + \chi_\varphi\frac{\sin\varphi}{c^2}\right]\cos\varphi\\ &\quad - \left[\chi_{c\varphi}\cos\varphi - \chi_c\sin\varphi - \chi_{\varphi\varphi}\frac{\sin\varphi}{c} - \chi_\varphi\frac{\cos\varphi}{c}\right]\frac{\sin\varphi}{c} =\\ &= \chi_{cc}\cos^2\varphi - \chi_{c\varphi}\frac{2\sin\varphi\cos\varphi}{c} + \chi_{\varphi\varphi}\frac{\sin^2\varphi}{c^2} + \chi_c\frac{\sin^2\varphi}{c} +\\ &\quad + \chi_\varphi\frac{2\sin\varphi\cos\varphi}{c^2}\end{aligned} \qquad (c)$$

and the other two formulas give:

$$\begin{aligned}\chi_{uv} &= \chi_{cc}\sin\varphi\cos\varphi + \chi_{c\varphi}\frac{\cos^2\varphi - \sin^2\varphi}{c} - \chi_{\varphi\varphi}\frac{\sin\varphi\cos\varphi}{c^2} -\\ &\quad - \chi_c\frac{\sin\varphi\cos\varphi}{c} - \chi_\varphi\frac{\cos^2\varphi - \sin^2\varphi}{c^2}\end{aligned} \qquad (d)$$

$$\begin{aligned}\chi_{vv} &= \chi_{cc}\sin^2\varphi + \chi_{c\varphi}\frac{2\sin\varphi\cos\varphi}{c} + \chi_{\varphi\varphi}\frac{\cos^2\varphi}{c^2} + \chi_c\frac{\cos^2\varphi}{c}\\ &\quad - \chi_\varphi\frac{2\sin\varphi\cos\varphi}{c^2}\end{aligned} \qquad (e)$$

The transformation formulae (a) to (e) can now be introduced into equation (31). The latter then reads in polar coordinates:

$$\frac{\partial^2 \chi}{\partial c^2} - \frac{\partial^2 \chi}{\partial \varphi^2} \frac{1}{c^2} \left(\frac{c^2}{gh} - 1\right) - \frac{\partial \chi}{\partial c} \frac{1}{c} \left(\frac{c^2}{gh} - 1\right) = 0 \qquad (31a)$$

7. Characteristics of the Differential Equation
(references 10, p. 153, and 31, p. 282)

The differential equationa (31) and (31a) are a special case of the following general form:

$$A(X,Y)\, Z_{XX} + 2B(X,Y)\, Z_{XY} + C(X,Y)\, Z_{YY} =$$

$$= D_1(X,Y)\, Z_X + E_1(X,Y)\, Z_Y + F_1(X,Y)\, Z \qquad (32)$$

if for the moment we write Z in place of χ, and X and Y for u and v, or c and φ, respectively. The coefficients A to F of differential equations (32) depend on the free variables only. For each pair of variables - i.e., for each point of the hodograph these three magnitudes are given numbers. There is a simple integration method for equation (32) that depends on finding a Taylor series for the solution $Z = Z(X,Y)$.

We seek a solution of (32) that contains a prescribed elementary strip. Let the curve. over which the Z-element strip is given be expressed in parametric form with t as parameter

$$\left.\begin{aligned} X &= X(t) \\ Y &= Y(t) \end{aligned}\right\} \text{(curve AB)}$$

The Z-surface strip (the boundary values of Z) over this curve is then given by

$$Z = F(t) \qquad (33)$$

and $\partial Z/\partial n = Q(t)$ where n is the normal of the curve AD. Along AB:

$$\frac{dZ}{dt} = \frac{\partial Z}{\partial X}\frac{dX}{dt} + \frac{\partial Z}{\partial Y}\frac{dY}{dt} = \frac{\partial Z}{\partial X} X'(t) + \frac{\partial Z}{\partial Y} Y'(t)$$

On the other hand, on account of the prescribed boundary values along the curve AB, we have:

$$\frac{dZ}{dt} = F'(t)$$

so that

$$\frac{\partial Z}{\partial X} X'(t) + \frac{\partial Z}{\partial Y} Y'(t) = F'(t) \qquad (33a)$$

The normal of the curve X(t), Y(t) has the direction cosines

$$\cos(n,X) = -Y'(t)/\sqrt{X'^2(t) + Y'^2(t)}$$

$$\cos(n,Y) = X'(t)/\sqrt{X'^2 + Y'^2}$$

Hence

$$\frac{\partial Z}{\partial n} = \frac{\partial Z}{\partial X}\cos(n,X) + \frac{\partial Z}{\partial Y}\cos(n,Y) = \frac{1}{\sqrt{X'^2 + Y'^2}} \times$$

$$\left(-\frac{\partial Z}{\partial X} Y' + \frac{\partial Z}{\partial Y} X'\right)$$

This expression must be equated to G(t). Thus along AB we also have:

$$-\frac{\partial Z}{\partial X} Y'(t) + \frac{\partial Z}{\partial Y} X'(t) = \sqrt{X'^2 + Y'^2}\, G(t) \qquad (33b)$$

Equations (33a) and (33b) may be solved for $\partial Z/\partial X$ and $\partial Z/\partial Y$, since the denominator determinant of the pair of equations is

$$\begin{vmatrix} X' & Y' \\ -Y' & X' \end{vmatrix} = X'^2 + Y'^2 \neq 0$$

Let the solution be

$$\left.\begin{aligned} \partial Z/\partial X &= p(t) \\ \partial Z/\partial Y &= q(t) \end{aligned}\right\} \qquad (34)$$

Differentiating each of these equations with respect to t, there is obtained:

$$Z_{XX} X'(t) + Z_{XY} Y'(t) = p'(t) \tag{35a}$$

$$Z_{XY} X'(t) + Z_{YY} Y'(t) = q'(t) \tag{35b}$$

For the second derivatives of Z, we have as third condition the differential equation itself:

$$A Z_{XX} + 2B Z_{XY} + C Z_{YY} = D_1 Z_X + E_1 Z_Y + F_1 Z \tag{35c}$$

If the denominator determinant of the system of equations (35)

$$\begin{vmatrix} X' & Y' & 0 \\ 0 & X' & Y' \\ A & 2B & C \end{vmatrix} = C X'^2 - 2B X'Y' + A Y'^2 \tag{36}$$

is not equal to zero, the three equations (35a-c) may be solved for Z_{XX}, Z_{XY}, and Z_{YY}. Let there be obtained for the derivatives of Z of the second order along AB the values:

$$Z_{XX} = R(t); \quad Z_{XY} = S(t); \quad Z_{YY} = T(t) \tag{37}$$

Differentiating (35a) and (35b) with respect to t and equation (35c) partially with respect to X and Y and substituting in the last two equations the values for Z, Z_X... from equations (33), (34) and (37), there is obtained the system of equations:

$$\begin{aligned}
Z_{XXX} X'^2 + 2Z_{XXY} X'Y' + Z_{XYY} Y'^2 &= p''(t) \\
Z_{XXY} X'^2 + 2Z_{XYY} X'Y' + Z_{YYY} Y'^2 &= q''(t) \\
A Z_{XXX} + 2B Z_{XXY} + C Z_{XYY} &= \alpha(t) \\
A Z_{XXY} + 2B Z_{XYY} + C Z_{YYY} &= \beta(t)
\end{aligned}$$

From these equations are obtained the four derivatives of third order of Z along the projection curve of the given elementary strip, since the determinant of the denominator is equal to the square of the determinant (36) and thus not equal to zero if that determinant is different from zero.

Proceeding in this manner there are obtained all of the higher derivatives of Z starting from the boundary values B(t) and Q(t) {equations (33). (34), (37), etc.). It is thus possible to write the solution of $Z = Z(X,Y)$ also for points which do not lie on the curve AB as a Taylor series:

$$Z(X,Y)=Z(X_0,Y_0)+\frac{1}{1!}\left[Z_X(X_0,Y_0)(X-X_0)+Z_Y(X_0,Y_0)(Y-Y_0)\right]+$$
$$+\frac{1}{2!}\left[Z_{XX}(X_0,Y_0)(X-X_0)^2+2Z_{XY}(X_0,Y_0)(X-X_0)(Y-Y_0)+\right.$$
$$\left.+Z_{YY}(X_0,Y_0)(Y-Y_0)^2\right]+\cdots$$

This method of solution falls, however, if the determinant (36) assumes the value, zero, i.e., if

$$C(X,Y)\left(\frac{dX}{dt}\right)^2-2B(X,Y)\frac{dX}{dt}\frac{dY}{dt}+A(X,Y)\left(\frac{dY}{dt}\right)^2=0$$

or

$$C\,dX^2-2B\,dX\,dY+A\,dY^2=0 \tag{38}$$

This equation, decomposed into linear factors, becomes:

$$\left[A\,dY-(B+\sqrt{B^2-A\,C})\,dX\right]\left[A\,dY-(B-\sqrt{B^2-A\,C})\,dX\right]=0$$

The denominator determinant (36) thus vanishes if either

$$A(X,Y)\,dY-\left(B(X,Y)+\sqrt{B^2(X,Y)-A(X,Y)\,C(X,Y)}\right)dX=0 \tag{38a}$$

or

$$A\,dY-(B-\sqrt{B^2-A\,C})\,dX=0 \tag{38b}$$

It is important tn observe that the pair of equations (38a) and (38b) are given by the coefficients of the differential equation (32) alone. They are two ordinary differential equations. The solution of each represents a family of curves $f(X,Y) = k$. These two families of curves are denoted as the characteristics of differential equation (32). If these families of curves, defined by (38a) and (38b) are real, then (32) in this region is denotod as hyperbolic. If the two families coincide, then (32) is parabolic. In regions within which the two sets of characteristics are imaginary, (32) is denoted as an elliptic differential equation.

If, therefore, the curve AB along which the Z-elementarp strip is prescribed as boundary value to (32) is a characteristic, the described method of solution by development of $Z(X,Y)$ into a Taylor series, fails.

As an application we shall now compute the characteristics of the differential equation of the flow. The computation is simplest if we start from the equation in polar coordinates (31a). Comparison of (31a) with (32) shows that for this case the magnitudes A, B, and C assume the following values:

$$A = 1, \quad B = 0, \quad C = -\frac{1}{c^2}\left(\frac{c^2}{gh} - 1\right)$$

and the variables X and Y are now c and φ. The ordinary differential equations of the characteristics (38a) and (38b) then become:

$$d\varphi \mp \sqrt{\frac{1}{c^2}\left(\frac{c^2}{gh} - 1\right)}\, dc = 0 \qquad (39a,b)$$

Substituting in the above the energy equation (9):

$$gh = gh_o - c^2/2$$

there is obtained the differential equationa of the two families of characteristics:

$$\pm\, d\varphi = \frac{1}{c}\sqrt{\frac{c^2 - \frac{2}{3}\, gh_o}{\frac{2}{3}\, gh_o - \frac{c^2}{3}}}\, dc \qquad (40a,b)$$

Before we integrate this equation, we wish yet to Introduce another concept.

The critical velocity a^* (m/e> is given by the condition that the flow velocity is -equal to the wave propagation velocity $a = \sqrt{gh}$, so that the Each number $M = 1$. Thus if $c^2 = g\,h$, $a^* = c = \sqrt{g\,h}$. Let us compute the water depth at the critical positions. From the energy equation

$$c^2 = 2gh_o - 2gh$$

and this should be equal to

$$a^2 = gh$$

that is,

$$2g\,h_o - 2g\,h = g\,h \quad \text{so that} \quad \underline{h^* = \tfrac{2}{3}\,h_o} \tag{41}$$

and hence,

$$c^{*2} = a^{*2} = \tfrac{2}{3}\,gh_o \tag{42}$$

The **critical positions** in a water flow without **energy** dissipation are located where the water depth is two-thirds of the total head. These poeitions in an **accelerated** flow are the **transition points** from **"streaming"** to **"shooting"** water and **conversely**, for decelerated flow.

Substituting (42) into equations (40), the latter after a small transformation, become:

$$\pm\, d\varphi = \frac{1}{(c/a^*)}\sqrt{\frac{(c/a^*)^2 - 1}{1 - (c/a^*)^2/3}}\; d(c/a^*)$$

We shall denote **c/a*** **as** the **velocity** ratio $\bar{c}$, for **which** a* is taken as the refcronco **velocity.** Hence,

$$\pm\, d\varphi = \frac{1}{\bar{c}}\sqrt{\frac{\bar{c}^2 - 1}{1 - \bar{c}^2/3}}\; d\bar{c} \tag{43a,b}$$

The variables in the above equation are already separated, and the equation map be **integrated** by a **simple** quadrature. **We** first introduce a new **integration variable:**

$$z = \bar{c}^2$$

so that we **have:**

$$\int \pm d\varphi = \int \frac{1}{2z}\,\frac{\sqrt{z - 1}}{\sqrt{1 - z/3}}\,dz = \frac{1}{2}\int \frac{z - 1}{z}\,\frac{\sqrt{3}\,dz}{\sqrt{(z - 1)(3 - z)}} =$$

$$= \frac{1}{2}\int \left(1 - \frac{1}{z}\right)\frac{\sqrt{3}\,dz}{\sqrt{-3 + 4z - z^2}}$$

This integral splits up into two parts, J_1 and J_2, of which the first may be directly evaluated:

$$J_1 = \int \frac{\sqrt{3}\,dz}{\sqrt{-3+4z-z^2}} = \sqrt{3}\int \frac{dz}{\sqrt{1-(z-2)^2}} = \sqrt{3}\,(\sin^{-1})\,(z-2)$$

In the second integral

$$J_2 = -\int \frac{\sqrt{3}\,dz}{z\sqrt{-3+4z-z^2}}$$

we make the substitution, $w = 1/z$, so thnt:

$$z = 1/w$$

$$dz = -\frac{1}{a^2}\,dw.$$

We now have:

$$J_2 = +\int \frac{\sqrt{3}\,dw}{\sqrt{-3w^2+4w-1}} = \int \frac{d(3w)}{\sqrt{1-(3w-2)^2}} = (\sin^{-1})\,(3w-2)$$

$$= (\sin^{-1})\,(3/z-2)$$

Denoting

$$f(\bar{c}) \equiv \int \frac{1}{\bar{c}} \sqrt{\frac{\bar{c}^2 - 1}{1 - \bar{c}^2/3}}\,d\bar{c} \qquad (44a)$$

we have finally with J_1 and J_2

$$f(\bar{c}) = \frac{1}{2}\left[\sqrt{3}\,(\sin^{-1})\,(\bar{c}^2-2) + (\sin^{-1})\,(3/\bar{c}^2-2)\right] \qquad (44b)$$

The solutions of (43) are thus:

$$\varphi - \varphi_1 = f(\bar{c}) \qquad (45a)$$

$$-\varphi + \varphi_2 = f(C) \qquad (45b)$$

where φ_1 and φ_2 are the constants of integration - these being the parameters of the two families of characteristics. The latter are shown in figure 9; they are epicycloids, the loci of the points of the circumference of a circle which rolle on another circle (fig. 10). This statement can be confirmed in the following manner.

From the equations (39) (characteristics). and from the energy equation (9), It follows that for h = 0 the magnitude of the velocity becomes a maximum. In the velocity diagram the extremity of $\bar{c}_{max}$ then lies on a circle K_{max} (fig. 9). For all possible velocities that occur, $c(u,v) < c_{max}$ h > 0. For $c^2 > gh$, the radicand of (39) then becomes positive and the root real. Hence, for that region of the hodoqraph in which $c_{max} > c > \sqrt{gh}$ (region II), there are two real families of characteristics. This holds for the shooting water (supersonic flow). For a flow in which $c < \sqrt{gh}$, the root in (39) becomes imaginary and there exist in this region (I) no real characteristics. This is the case for streaming water.

Let the angle ψ be chosen as parameter (fig. 10). Then, on account of the "rolling condition,"

$$\alpha = (r/R)\ \psi$$

From the triangle PSO, there is obtained for β

$$\beta = (\tan^{-1}) \left[\frac{r \sin \psi}{(R+r) - r \cos \psi}\right]$$

From these two equations, we have:

$$\varphi = \alpha - \beta = (r/E)\ \psi - (\tan^{-1}) \left[\frac{r \sin \psi}{(R+r) - r \cos \psi}\right] \qquad (a)$$

From the cosine law for the triangle PTO:

$$\bar{c} = \sqrt{(R+r)^2 + r^2 - 2(R+r)\ r \cos \psi} \qquad (b)$$

Differentiating (a) and (b), there is obtained:

$$d\varphi = \frac{[(R+r)^2 + r^2 - 2(R+r) r \cos \psi]\ r/R - (R+r) r \cos \psi + r^2}{(R+r)^2 + r^2 - 2(R+r) r \cos \psi}\ d\psi \qquad (c)$$

$$d\bar{c} = \frac{r(R+r)\sin\psi}{\sqrt{(R+r)^2 + r^2 - 2(R+r)r\cos\psi}}\, d\psi \qquad \text{(d)}$$

Eliminating in these two equations $\sin\psi$ and $\cos\psi$ with the aid of equation (b), and than dividing (c) by (d), there is obtained:

$$\frac{d\varphi}{d\bar{c}} = \frac{1}{\bar{c}} \frac{\bar{c}^2(R+2r)/R - R(R+2r)}{\sqrt{\cdots + \bar{c}^2(2R^2+4Rr+4r^2) - R^2(R+2r)^2}} \qquad \text{(e)}$$

Dividing numerator and denominator of this fraction by $\sqrt{c^2 - R^2}\,(R+2r)/R$, we have, finally:

$$\frac{d\varphi}{d\bar{c}} = \frac{1}{\bar{c}} \sqrt{\frac{\bar{c}^2 - R^2}{R^2 - [R/(R+2r)]^2\,\bar{c}^2}} \qquad \text{(f)}$$

8s was to be proved. Bor $R = 1$ and $(R+2r)/R = \sqrt{3}$, this is the differential equation (43). The epicycloid drawn in figure 10 is thue a characteristic of the family (458).

The characteristics of shooting water flow are epicycloids between two circles whose radii are in the ratio $\sqrt{3}:1$. They are drawn on chart 2 of the supplement. For 8 gas, the characteristics lie between circles whose radii are In the ratio $\sqrt{(k+1)/(k-1)}$ to 1. They are shown on chart 1 for air ($k = 1.405$).

8. Further Properties of the Characteristics

We have seen that If an elementary strip be given as boundary value over the characteristics of a partial differential equation, the solution method by a series development of the required function fails. Some further properties of the characteristics will now be *discussed.* The physical character of the Bupersonlc flow (shooting water) - which differs essentially from subsonic flow '(streaming aster) - will thereby receive an interesting explanation from the mathematical point of view.

In equation (32):

$$A\, Z_{XX} + 2B\, Z_{XY} + 0\, Z_{YY} = D_1\, Z_X + E_1\, Z_Y + F_1\, Z$$

let new variables be introduced by making use of a point transformation. Let the new variables be:

$$\left.\begin{aligned} \lambda &= \lambda(X,Y) \\ \mu &= \mu(X,Y) \end{aligned}\right\} \qquad (46)$$

where for the moment we do not fix any definite transformation formulas. From (46) we obtain the inverse formulas:

$$X = X(\lambda,\mu)$$

$$Y = Y(\lambda,\mu)$$

The solution of the differential equation (32) $Z = Z(X,Y)$ is thus a function of λ and μ.

$$Z = Z[\lambda,\mu] = Z[\lambda(X,Y),\mu(X,Y)]$$

From tho above, wo havo:

$$\left.\begin{aligned} Z_X &= Z_\lambda \lambda_X + Z_\mu \mu_X \\ Z_Y &= Z_\lambda \lambda_Y + Z_\mu \mu_Y \end{aligned}\right\} \qquad (47a)$$

Differentiating a second time, there are obtained the derivatives of second order of Z in the new variables:

$$Z_{XX} = Z_{\lambda\lambda}(\lambda_X)^2 + 2Z_{\lambda\mu}\lambda_X\mu_X + Z_{\mu\mu}(\mu_X)^2 + Z_\lambda \lambda_{XX} + Z_\mu \mu_{XX}$$

$$Z_{XY} = Z_{\lambda\lambda}\lambda_X\lambda_Y + Z_{\lambda\mu}(\lambda_X\mu_Y+\lambda_Y\mu_X) + Z_{\mu\mu}\mu_X\mu_Y + Z_\lambda\lambda_{XY} + Z_\mu\mu_{XY}$$

$$Z_{YY} = Z_{\lambda\lambda}(\lambda_Y)^2 + 2Z_{\lambda\mu}\lambda_Y\mu_Y + Z_{\mu\mu}(\mu_Y)^2 + Z_\lambda\lambda_{YY} + Z_\mu\mu_{YY}$$

Putting these expressions in differential equation (32), it becomes:

$$Z_{\lambda\lambda}\left[A\lambda_X^2 + 2B\lambda_X\lambda_Y + C\lambda_Y^2\right] + 2Z_{\lambda\mu}\left[A\lambda_X\mu_X + B(\lambda_X\mu_Y + \lambda_Y\mu_X) + C\lambda_Y\mu_Y\right]_1 +$$

$$+ Z_{\mu\mu}\left[A\mu_X^2 + 2B\mu_X\mu_Y + C\mu_Y^2\right]_3 = D_2 Z_\lambda + E_2 Z_\mu + F_2 Z$$

We shall now determine the transformation formulas (46). The differential equation of the characteristics is

$$C\,dX^2 - 2B\,dX\,dY + A\,dY^2 = 0 \qquad (38)$$

If equation (32) is hyperbolic, (38) has tfo real families of curves as solutions. Let these be

$$\left. \begin{aligned} f_1 (X,Y) &= \text{constant} \\ \text{and} \qquad f_2 (X,Y) &= \text{constant} \end{aligned} \right\} \qquad (49)$$

Along each of these curves

$$f_X \, dX + f_Y \, dY = 0$$

This equation together with (38) gives for both f_1 and f_2, the relation:

$$A f_X^2 + 2B f_X f_Y + C f_Y^2 = 0 \qquad (50)$$

An essential simplification is obtained if, for the traneformatlon formulae (46), the following special ones are chosen:

$$\left. \begin{aligned} \lambda &= f_1 (X,Y) \\ \mu &= f_2 (X,Y) \end{aligned} \right\} \qquad (51)$$

[curvilinear coordinatea in the hodographs, fig. 11b). The two coefficients of $Z_{\lambda\lambda}$ and $Z_{\mu\mu}$ by (50) then vanish in the transformed differential equation, the latter receiving the form

$$\frac{\partial Z}{\partial\lambda\partial\mu} = -\left[a(\lambda,\mu) \frac{\partial Z}{\partial\lambda} + b(\lambda,\mu) \frac{\partial Z}{\partial\mu} + c(\lambda,\mu) \; Z\right] \qquad (52)$$

This form is called the normal form of the linear hyperbolic differential equation. It is well suited to numeribal integration by means of the difference method.

As an application, let the characteristics (45a and b) be Introduced as curvilinear coordinates of the position-determining potential x (31a). We then obtain the normal form of the differential equation of flow.

By elimination of h and h_o from the three equations:

(9) $c^2 = 2gh_o - 2gh$, (42) $a^{*2} = 2gh_o/3$, and $a^2 = gh$

there is obtained:

$$c^2 = 3a^{*2} - 2a^2$$

from which, after short computation and substitution of the velocity ratio $\bar{c} = c/a^*$, there is obtained:

$$\frac{c^2}{a^2} = \frac{2\bar{c}^2}{3 - \bar{c}^2} \quad \text{and} \quad \frac{c^2}{a^2} - 1 = 3\,\frac{\bar{c}^2 - 1}{3 - \bar{c}^2}$$

Substituting this expression in (31a) and multiplying the latter by the critical velocity a^* (42), then (31a) may be written in nondimensional form:.

$$\frac{\partial^2 \chi}{\partial \bar{c}^2} - \frac{\partial^2 \chi}{\partial \varphi^2}\,\frac{3(\bar{c}^2 - 1)}{\bar{c}^2(3 - \bar{c}^2)} - \frac{\partial \chi}{\partial \bar{c}}\,\frac{3(\bar{c}^2 - 1)}{\bar{c}(3 - \bar{c}^2)} = 0$$

In the above we now introduce the coordinates λ and μ through the following expressions:

$$\chi_{\bar{c}} = \chi_\lambda \chi_{\bar{c}} + \chi_\mu \mu_{\bar{c}}$$

$$\chi_{\bar{c}\bar{c}} = \chi_{\lambda\lambda}(\chi_{\bar{c}})^2 + 2\chi_{\lambda\mu}\lambda_{\bar{c}}\mu_{\bar{c}} + \chi_{\mu\mu}(\mu_{\bar{c}})^2 + \chi_\lambda \lambda_{\bar{c}\bar{c}} + \chi_\mu \mu_{\bar{c}\bar{c}}$$

$$\chi_{\varphi\varphi} = \chi_{\lambda\lambda}(\lambda_\varphi)^2 + 2\chi_{\lambda\mu}\lambda_\varphi \mu_\varphi + \chi_{\mu\mu}(\mu_\varphi)^2 + \chi_\lambda \lambda_{\varphi\varphi} + \chi_\mu \mu_{\varphi\varphi}$$

After substitution and rearrangement, there is obtained:

$$\frac{\partial^2 \chi}{\partial \lambda^2}\left[\left(\frac{\partial \lambda}{\partial \bar{c}}\right)^2 - \frac{3(\bar{c}^2-1)}{\bar{c}^2(3-\bar{c}^2)}\left(\frac{\partial \lambda}{\partial \varphi}\right)^2\right] + \frac{\partial^2 \chi}{\partial \mu^2}\left[\left(\frac{\partial \mu}{\partial \bar{c}}\right)^2 - \frac{3(\bar{c}^2-1)}{\bar{c}^2(3-\bar{c}^2)}\left(\frac{\partial \mu}{\partial \varphi}\right)^2\right] +$$

$$+ 2\,\frac{\partial^2 \chi}{\partial \lambda \partial \mu}\left[\frac{\partial \lambda}{\partial \bar{c}}\,\frac{\partial \mu}{\partial \bar{c}} - \frac{3(\bar{c}^2-1)}{\bar{c}^2(3-\bar{c}^2)}\,\frac{\partial \lambda}{\partial \varphi}\,\frac{\partial \mu}{\partial \varphi}\right] +$$

$$+ \frac{\partial \chi}{\partial \lambda}\left[\frac{\partial^2 \lambda}{\partial \bar{c}^2} - \frac{3(\bar{c}^2-1)}{\bar{c}^2(3-\bar{c}^2)}\,\frac{\partial^2 \lambda}{\partial \varphi^2} - \frac{3(\bar{c}^2-1)}{\bar{c}(3-\bar{c}^2)}\,\frac{\partial \lambda}{\partial \bar{c}}\right] +$$

$$+ \frac{\partial \chi}{\partial \mu}\left[\frac{\partial^2 \mu}{\partial \bar{c}^2} - \frac{3(\bar{c}^2-1)}{\bar{c}^2(3-\bar{c}^2)}\,\frac{\partial^2 \mu}{\partial \varphi^2} - \frac{3(\bar{c}^2-1)}{\bar{c}(3-\bar{c}^2)}\,\frac{\partial \mu}{\partial \bar{c}}\right] = 0 \qquad \text{(A)}$$

The two sets of characteristics (45a) and (45b) in the implicit form are now

$$f(\bar{c}) + \varphi = \text{constant}$$

$$f(\bar{c}) - \varphi = \text{constant}$$

Substituting in (A) for λ and μ by (51). the two values

$$\lambda = f(\overline{c}) + \varphi \qquad (53a)$$

and

$$\mu = f(\overline{c}) - \varphi \qquad (53b)$$

the coefficients of $\chi_{\lambda\lambda}$ and $\chi_{\mu\mu}$ become zero and, since

$$\lambda_\varphi = 1, \quad \mu_\varphi = - 1, \quad \lambda_{\varphi\varphi} = 0, \quad \mu_{\varphi\varphi} = 0$$

$$\lambda_{\overline{c}} = df(\overline{c})/d\overline{c} \qquad \mu_{\overline{c}} = df(\overline{c})/d\overline{c}$$

$$\lambda_{\overline{c}\overline{c}} = d^2 f(\overline{c})/d\overline{c}^2 \qquad \mu_{\overline{c}\overline{c}} = d^2 f(\overline{c})/d\overline{c}^2$$

(A) becomes:

$$2\frac{\partial^2 \chi}{\partial\lambda\partial\mu}\left[\left(\frac{df}{d\overline{c}}\right)^2 + \frac{3(\overline{c}^2-1)}{\overline{c}^2(3-\overline{c}^2)}\right] + \left[\frac{\partial\chi}{\partial\lambda} + \frac{\partial\chi}{\partial\mu}\right]\left[\frac{d^2 f}{d\overline{c}^2} - \frac{3(\overline{c}^2-1)}{\overline{c}(3-\overline{c}^2)}\frac{df}{d\overline{c}}\right] = 0$$

and the normal form finally reads;

$$\frac{\partial^2 \chi}{\partial\lambda\partial\mu} = -\left(\frac{\partial\chi}{\partial\lambda} + \frac{\partial\chi}{\partial\mu}\right)\frac{1}{2}\,\frac{\dfrac{d^2 f(\overline{c})}{d\overline{c}^2} - \dfrac{3(\overline{c}^2-1)}{\overline{c}(3-\overline{c}^2)}\dfrac{df}{d\overline{c}}}{\left(\dfrac{df}{d\overline{c}}\right)^2 + \dfrac{3(\overline{c}^2-1)}{\overline{c}^2(3-\overline{c}^2)}} = -K\left(\frac{\partial\chi}{\partial\lambda} + \frac{\partial\chi}{\partial\mu}\right) \qquad (53c)$$

where A and μ are defined by (53a) and (53b), and K is obtained by substituting the expression for $f(\overline{c})$ from (44b):

$$K = K(\lambda,\mu) = K(\lambda+\mu) = K(\overline{c}) = \frac{\overline{c}^2(1-\overline{c}^2/2)}{\sqrt{3}\,\sqrt{(3-\overline{c}^2)}\,\sqrt{(\overline{c}^2-1)^3}} \qquad (53d)$$

The numerical values for K are collected in table II.

The lines λ = 'constant and μ = constant are characteristice since we had so chosen the transformation formulae (51). If, after the transformation, λ and μ are plotted as rectangular coordinates (fig. 11c), it appears that the normal form (53c) of the hyperbolic equation has as characteristics, the sets of parallels to the A and μ axes. For equation (52), which is also of the form (32), A = 0, B = $\frac{1}{2}$, C = 0, and the variables X and Y are now A and μ. These substituted in the general equation (38) of the characteristics, give:

$$d\lambda \, d\mu = 0$$

The two solutions of this differential equation are:

$$\lambda = \text{constant}$$

and

$$\mu = \text{constant} \quad \text{(fig. 12)}$$

The solution Z of the differential equations (32) and (52) may be determined if, along a general curve, an element strip is prescribed as boundary value. This curve may not, however, be a characteristic. But if it is made up of two characteristics .of different families, it is surprising that a solution of the differential equation may still be determined. For this purpose, the function Z alone is sufficient as boundary value while no elementary strip may be prescribed since this would be imposing too many conditions.

Let the values $Z = \varphi(\lambda, \mu_0) = \varphi(\lambda)$ and $Z = \psi(\lambda_0, \mu) = \psi(\mu)$ with $\varphi(\lambda_0) = \psi(\mu_0)$ bo given along two segments A_0A_1 and A_0A_2 of tmo characteristics (fig. 12). Along A_0A_2 there is therewith also given $\partial Z/\partial \mu$, but $\partial Z/\partial \mu$ is assumed not to be prescribed; similarly, along A_0A_1. It is to be observed that no elementary strip is prescribed along $A_1A_0A_2$ of Z but only the values of Z itself. By the method of so-called "successive approximation," it is then possible to find A solution Z of the partial differential equation (52) for the entire region $A_1A_0A_2A_3$, which assumes the given values of Z along $A_1A_0A_2$.

As a first approximation, Horn (reference 10)

$$Z_\alpha = \varphi(\lambda) + \psi(\mu) - \varphi(\lambda_0, \mu_0)$$

for all values A and μ of the region $A_1A_0A_2A_3$. On the boundaries A_0A_1 and A_0A_2 Z_α becomes equal to the prescribed values.

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*The proof will not be given here. It is carried out by J. Horn (reference 10), 1913, sec. 30, pp. 164-169. For us it is of importance to know only that the prescribed function $Z(\lambda, \mu)$ satisfies tho boundary values and the hyperbolic differential equation (52).

We now form with the right side of equation (52):

$$Z_\beta(\lambda,\mu) = -\int_{\lambda_0}^{\lambda}\int_{\mu_0}^{\mu}\left(a\frac{\partial Z_\alpha}{\partial\lambda} + b\frac{\partial Z_\alpha}{\partial\mu} + cZ_\alpha\right)d\lambda d\mu$$

where the integration is to be taken over the doubly hatched rectangle. Proceeding in this manner, we form

$$Z_\sigma(\lambda,\mu) = -\int_{\lambda_0}^{\lambda}\int_{\mu_0}^{\mu}\left(a\frac{\partial Z_{\sigma-1}}{\partial\lambda} + b\frac{\partial Z_{\sigma-1}}{\partial\mu} + cZ_{\sigma-1}\right)d\lambda d\mu$$

Setting

$$Z(\lambda,\mu) = Z_\alpha + Z_\beta + Z_\gamma + \ldots\ldots$$

then this sum is the required solution and it converges, as shown by Horn, in the rectangle $A_1A_0A_2A_3$.

There will now be shorn a last property of the characteristics - the most important for tho application to shooting water. At tho same time, in addition to the method of solution of (32) by series development and the method of successive approximation, we shall become acquainted with the method of integration of Riemann.

We denote by W(Z) the most general homogeneous linear differential expression:

$$N(Z) \equiv A\,Z_{XX} + 2B\,Z_{XY} + C\,Z_{YY} + D\,Z_X + E\,Z_Y + F\,Z \qquad (55)$$

where the coefficients A to F depend only on the free variables X and Y. The general linear homogeneous differential equation of the second order is the equation (32):

$$N(Z) = 0 \qquad (56)$$

To the expression N(Z) another one M(W) is made to correspond, having the same coefficients A, B, C, etc. as in (55), where

$$M(W) \equiv (AW)_{XY} + 2(BW)_{XY} + (CW)_{YY} - (DW)_X - (EW)_Y + F\,W \qquad (57)$$

$$= M(W) = A\,W_{XX} + 2B\,W_{XY} + C\,W_{YY} + 2\,W_X(A_X + B_Y - \tfrac{1}{2}D) +$$

$$+ 2\,W_Y(B_Y + C_Y - \tfrac{1}{2}E) + W\,(A_{XX} + 2B_{XY} + C_{YY} - D_X - E_Y + F) \qquad (57a)$$

M(W) is then denoted as the adjoint of N(Z) and the equation

$$M(W) = 0 \tag{58}$$

the adjoint differential equation of N(Z) = 0. Z and W are functions of X and Y: Z = Z(X,Y), W = W(X,Y). M(W) = 0 has the same characteristics as N(Z) = 0, since in (57a) and in (55) the coefficients of the partial derivatives of the second order are the same and since, according to (38), the characteristics depend on these coefficients only.

By addition of the identities:

$$AWZ_{XX} - Z(AW)_{XX} = \frac{\partial}{\partial X}\left[AWZ_X - Z(AW)_X\right]$$

$$BWZ_{XY} - Z(BW)_{XY} = \frac{\partial}{\partial Y}\left[BWZ_X\right] - \frac{\partial}{\partial X}\left[Z(BW)_Y\right]$$

$$BWZ_{XY} - Z(BW)_{XY} = \frac{\partial}{\partial X}\left[BWZ_Y\right] - \frac{\partial}{\partial Y}\left[Z(BW)_X\right]$$

$$CWZ_{YY} - Z(CW)_{YY} = \frac{\partial}{\partial Y}\left[CWZ_Y - Z(CW)_Y\right]$$

$$DWZ_X + Z(DW)_X = \frac{\partial}{\partial X}\left[DZW\right]$$

$$EWZ_Y + Z(EW)_Y = \frac{\partial}{\partial Y}\left[EZW\right]$$

$$FWZ - ZFW = 0$$

there is obtained the identity:

$$W\,N(Z) - Z\,M(W) = \frac{\partial}{\partial X}\left[AWZ_X - Z(AW)_X + BWZ_Y - Z(BW)_Y + DZW\right] + \frac{\partial}{\partial Y}\left[BWZ_X - Z(BW)_X + CWZ_Y - Z(CW)_Y + EZW\right] \tag{59}$$

Denoting for a moment the two expressions in brackets by P and Q, respectively, the above equation reads:

$$W\,N(Z) - Z\,Y(W) = \partial P/\partial X + \partial Q/\partial Y \tag{59a}$$

This equation we shall integrate over the region F of the X,Y plane; Let the boundary of the region of integration, to be more definitely fixed later, be C (fig. 13):

$$\iint_{(F)} \left[W\, N(Z) - Z\, M(W) \right] dX\, dY = \iint_{(F)} (\partial P/\partial X + \partial Q/\partial Y)\, dX\, dY$$

The right side may by integration by parts be converted into a line integral. There is obtained:

$$\iint_{(F)} \left[W\, N(Z) - Z\, M(W) \right] dX\, dY = \oint_{(C)} (P\, dY - Q\, dX) \qquad (60)$$

The generalized Green's theorem (60) will now be applied to the normal form (52) of the hyperbolic differential equation. For this purpose there is to be set in (60) $A = 0$, $B = \frac{1}{2}$, $C = 0$, $D = a$, $E = b$, and $F = c$. In place of X and Y, we have λ and μ. The expressions P and Q then become:

$$\left.\begin{aligned} P &= \tfrac{1}{2}(W\, Z_\mu - Z\, W_\mu) + a\, Z\, W \\ Q &= \tfrac{1}{2}(W\, Z_\lambda - Z\, W_\lambda) + b\, Z\, W \end{aligned}\right\} \qquad (61a)$$

Green's formula (60) now reads:

$$\iint_{(F)} \left[W\, N(Z) - Z\, M(W) \right] d\lambda\, d\mu = \oint_{(C)} (P\, d\mu - Q\, d\lambda) \qquad (61b)$$

With this formula we may now prove the following:

If Z is a function of λ and μ, $Z = Z(\lambda, \mu)$, which satisfies the hyperbolic differential equation (52) and for which, along a' curve from A_1 to B_1 (fig. 14) - which thus, in general, is not a characteristic - an elementary strip* is given; then by these boundary values and the differential equation, the function Z is determined in the characteristic rectangle $A_1 O_1 B_1 O_1'$, which contains the curve $A_1 B_1$ with its end points.

In order to show this we apply the formula (61b) to the region G and its boundary AOBA of figure 14, where

*Along A_1B_1 therefore Z and the slopes $\partial Z/\partial\lambda$ and $\partial Z/\partial\mu$ are given where naturally along A_1B_1, the condition $dZ = Z_\lambda\, d\lambda + Z_\mu\, d\mu$ must be satisfied.

0 is an arbitrary interior point $(\lambda = p, \mu = q)$ of the characteristic rectangle $A_1O_1B_1O_1'$. In integrating along OB, only P dμ contributes anything; Q dλ does not contribute anything, since dλ = 0. Similarly,

$$\int_A^0 (P\,d\mu - Q\,d\lambda) = -\int_A^0 Q\,d\lambda$$

since along AO μ = q = constant, so that dμ = 0. We thua obtain from (61b) applied to the hatched region G

$$\iint_{(G)} \left[W\,N(Z) - Z\,Id(W)\right] d\lambda\,d\mu = \int_0^B P\,d\mu - \int_A^0 Q\,d\lambda + \int_B^A (P\,d\mu - Q\,d\lambda) \qquad (62)$$

Non from (61a), if the first term is integrated by parts*

$$\int_0^B P\,d\mu = \int_0^B \left(\tfrac{1}{2} W \frac{\partial Z}{\partial \mu} - \tfrac{1}{2} Z \frac{\partial W}{\partial \mu} + aZW\right) d\mu =$$

$$= \tfrac{1}{2}(W\,Z)_B - \tfrac{1}{2}(W\,Z)_0 - \int_0^B Z\,(\partial W/\partial\mu - a\,W)\,d\mu \qquad (a)$$

Similarly, by integration by parts of the first term

$$-\int_A^0 Q\,d\lambda = +\int_A^0 \left(-\tfrac{1}{2} W \frac{\partial Z}{\partial \lambda} + \tfrac{1}{2} Z \frac{\partial W}{\partial \lambda} - bZW\right) d\lambda$$

$$= -\tfrac{1}{2}(W\,Z)'' + \tfrac{1}{2}(W\,Z)_A + \int_A^0 Z(\partial W/\partial\lambda - b\,W)\,d\lambda \qquad (b)$$

With expressions (a) and (b), formula (62) becomes:

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*
$$\int_0^B \tfrac{1}{2} W \frac{\partial Z}{\partial \mu}\,d\mu = \tfrac{1}{2}(WZ)\Big|_0^B - \int_0^B \tfrac{1}{2} Z \frac{\partial W}{\partial \mu}\,d\mu$$

$$\iint_{(G)} \left[W\, N(Z) - Z\, M(W)\right] d\lambda\, d\mu = -(W\,Z)_O + \tfrac{1}{2}\left[(W\,Z)_A + (W\,Z)_B\right] +$$

$$+ \int_A^O Z\,(\partial W/\partial\lambda - b\,W)\,d\lambda - \int_O^B Z\,(\partial W/\partial\mu - a\,W)\,d\mu \; +$$

$$+ \int_A^B (Q\,d\lambda - P\,d\mu) \tag{63}$$

We now choose for each point O which is given by the coordinates $\lambda = p$, $\mu = q$, a definite function W of the coordinates λ and μ: $W = W(\lambda,\mu)$. In this function, p and q occur as parametera, the function $W(\lambda,\mu)$ being different for each choice of the point $O(p,q)$. We thus have:

$$W = W(\lambda,\mu) = W(\lambda,\mu;\, p,q)$$

where the function is to have the following properties:

1. At the point O itself (p,q), W is to assume the value 1.

2. The function W is to satisfy over the entire region G (fig. 14) the adjunct differential equation $M(W) = 0$; i.e., be a solution of

$$M(W) = 0 \tag{64}$$

3a) Along the straight line OB ($\lambda = p$ constant, μ variable) the function W is to assume the values:

$$W(p,\mu) = e^{\int_q^\mu a(p,\mu)\,d\mu} \tag{65a}$$

Condition 1 is thereby satisfied since for the point $\lambda = p$, $\mu = q$, $W(p,q) = e^0 = 1$. Differentiating (65a) with respect to μ, there is obtained for the function W along OB the relation

$$\partial W/\partial\mu - a\,W = 0 \tag{66a}$$

3b) Similarly along the straight line AO ($\mu = q$ constant; λ variable) the function is to assume the values:

$$W(\lambda, q) = e^{\int_p^\lambda b(\lambda, q)\, d\lambda} \tag{65b}$$

Here, too, the condition $W(p,q) = 1$ is satisfied. Differentiating (65b) along AO with respect to λ there is obtained along this line the relation:

$$\partial W/\partial\lambda - b\, W = 0 \tag{66b}$$

The function defined by the conditions 1, 2, and 3, is known as Green's function $W(\lambda, \mu; p, q)$ of the differential equation $N(Z) = 0$. It is determined only by the coefficients of this equation. That it exists we know for W, according to condition 2, is a solution of the partial differential equation of the second order ($M(W) = 0$, for which the values of W along the two characteristics AO and OB are prescribed according to requirements 1 and 3, as boundary values. It is thus possible to determine W by the method, for example, of successive approximation.

Substituting now in (63) $N(Z) = 0$, and Green's function W, with its properties (64) and (66a,b), there is obtained:

$$0 = -Z_0 + \tfrac{1}{2}\left[(WZ)_A + (WZ)_B\right] + \int_A^B (Q\, d\lambda - P\, d\mu)$$

so that

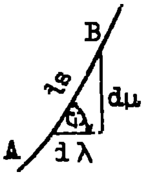
$$Z_0 = Z(p,q) = \tfrac{1}{2}\left[(WZ)_A + (WZ)_B\right] + \int_A^B (Q\, d\lambda - P\, d\mu) \tag{67}$$

Substituting further the expressions (61a) for P and Q, we have:

$$Z_0 = Z(p,q) = \tfrac{1}{2}\left[(WZ)_A + (WZ)_B\right] +$$

$$+ \int_A^B \left(\tfrac{1}{2} WZ_\lambda - \tfrac{1}{2} ZW_\lambda + bZW\right) d\lambda + \left(-\tfrac{1}{2} WZ_\mu + \tfrac{1}{2} ZW_\mu - aZW\right) d\mu =$$

$$= \tfrac{1}{2}\left[(WZ)_A + (WZ)_B\right] + \int_A^B \Big[\tfrac{1}{2} W(\partial Z/\partial\lambda \cos\varphi - \partial Z/\partial\mu \sin\varphi)$$



$$- \tfrac{1}{2} Z (\partial W/\partial\lambda \cos\varphi - \partial W/\partial\mu \sin\varphi)$$

$$+ ZW(b\cos\varphi - a \sin\varphi)\Big]\, ds \tag{67a}$$

We here thus expressed the required solution Z at point O(p,q) by the given boundary values; i.e., by a portion of the elementary strip A_1B_1. The considerations hold for every arbitrary point O which belongs to the characteristic rectangle determined by the points A_1 and B_1. It may be remarked further that Z is already determined at point O by its elementary strip along AB and therefore that the portions AA_1 and BB_1 (fig. 14) of the boundary value strip A_1B_1 have no effect on the value of Z at point O.

By means of the elementary strip A_1B_1 therefore, the solution $Z(\lambda,\mu)$ of the differential equation $N(Z) = 0$ is certainly determined in the largest characteristic rectangle which is fixed by A_1B_1. We wish to show, furthermore, that it is determined only within it, and not outside of it. Let Q be a point without $A_1O_1B_1O_1'$. Z is not determined in Q since, according to formula (67a) Z_Q depends on the elementary strip AR (fig. 14). The portion B_1R of this required elementary strip, however, is not given. Thus the above theorem is proven.

A special case which we still must examine in particular, is that for which the curve A_1B_1 - along which an elementary strip of Z is given - degenerates into the line $A_1O_1'B_1$ (fig. 15), consisting of two characteristics. From the method of successive approximation, we know that Z is then determined in the region $A_1O_1B_1O_1'$ by the assignment of the values of Z alone, along $B_1O_1'A$. This fact will now also be derived from Riemann's method of integration.

We start from the solution

$$Z(p,q) = \frac{1}{2}\left[(WZ)_A + (WZ)_B\right] + \int_{(AO_1'B)}^{B} (Q\,d\lambda - P\,d\mu) \qquad (67)$$

Since along AO_1' $\lambda =$ constant, $d\lambda = 0$, and along $O_1'B$ $\mu =$ constant, $d\mu = 0$, the integral on the right side breaks up into two-part integrals

$$\int_{A-O_1'-B}^{B} (Q\,d\lambda - P\,d\mu) = \int_{A}^{O_1'} - P\,d\mu + \int_{O_1'}^{B} Q\,d\lambda$$

Substituting in the above the expressions P and Q (equations 61a), there is obtained, as before:

$$-\int_{A}^{O_1'} P\,d\mu = +\int_{O_1'}^{A} (\tfrac{1}{2}\,W\,\partial Z/\partial\mu - \tfrac{1}{2}\,Z\,\partial W/\partial\mu + a\,Z\,W)d\mu$$

This time we integrate the second term by parts and obtain:

$$-\int_{A}^{O_1'} P\,d\mu = \tfrac{1}{2}(W\,Z)_{O_1'} - \tfrac{1}{2}(W\,Z)_A + \int_{O_1'}^{A} W(\partial Z/\partial\mu + a\,Z)d\mu \tag{a}$$

Similarly (again the second term integrated by parts):

$$\int_{O_1'}^{B} Q\,d\lambda = \tfrac{1}{2}(W\,Z)_{O_1'} - \tfrac{1}{2}(W\,Z)_B + \int_{O_1'}^{B} W(\partial Z/\partial\lambda + b\,Z)d\lambda \tag{b}$$

Substituting (a) and (b), we have, finally:

$$Z(p,q) = (W\,Z)_{O_1'} + \int_{O_1'}^{A} W(\partial Z/\partial\mu + a\,Z)d\mu + \int_{O_1'}^{B} W(\partial Z/\partial\lambda + b\,Z)d\lambda \tag{68}$$

With the prescribed values of Z as boundary values $\partial Z/\partial\mu$ is also given along $O_1'A$. The integral from O_1' to A may thus be evaluated without the necessity of giving also $\partial Z/\partial\lambda$ and hence an elementary strip. Similarly with the Z values alone, the values $\partial Z/\partial\lambda$ along $O_1'B_1$ and also the second integral in (68) may be evaluated by assigning Z alone. The formula (68) thus represents the solution $Z(p,q)$ in the entire characteristic rectangle $A_1O_1B_1O_1'$.

9. Summary

From the differential equation of. the velocity potential (15) of a compressible flow and from the flow space, we were led by the Legendre contact transformation to the differential equation of the poeitlon-determining potential X (31) in the velocity plane. In aonneation with this partial differential equation of seaond order, we became familiar with the characteristic curves and some of their properties. For 'shooting water and for supersonic flows, these aonsfst of two real families of curves, namely, epiayalolds. The Riemann method of solution showed that the solution of the hyperbolic partial differential equation by the boundary values is always determined within a complete characteristic reatenqle, namely, the smallest rectangle which contains all the boundary values.

THE METHOD OF CHARACTERISTICS

10. Introduction

Important contributions to the solution of the differential equation of two-dimensional supersonic flows have been made by Prendtl, Meyer, Steichen, Ackeret, and Busemann. Whereas the first solution methods are purely computational,. it was pointed out by J. Ackeret that, with' the aid of the characteristics a graphical method may be developed. This has been carried out for flows without energy dissipation by Prandtl end Busemann. For the case of flaws with Impulsive discontinuities, Busemenn has developed - on the basis of the method for nondissipative flows - a graphical method where the characteristics are replaced by the so-called "shock polars" (references 1 (or 2), 7, (pp. 421-440), 14, 15, 17, 18 (pp. 499-509), end 27).

Let the velocity of a two-dimensional supersonic flow or a shooting-water flow be given along a portion of a curve AB (fig. 16). Let the flow be from left to right, 0' a point downstream through which pass the two Mach lines BO' and AO'. The region of the flow bounded by the Yeah lines OA, OB, BO', AO', we shall denote as the Mach quadrilateral, We shell assume that no restriction of the flow (vertical walls) is located in its interior:: that is, neither boundary nor any other object. It may be shown by a simple consideration that under these assumptions the flow, if prescribed along AB, determines the condition in the entire Mach quadrilateral AO'BOA. Outside of this quadrilateral, influences from other points are effective. At point F, for example, another wave GF may arrive and produce a disturbance without producing a change on AB, since. GF is a wave of the same family as BO'.

Since every nondissipative flow is also a possible flow in the opposite direction, the same considerations apply to the upstream region AOB. This statement is not in contradiction of the general fact that in a flow with the above critical velocity, the effects of disturbances make themselves felt only downstream. We ao not state that the condition at 0, for example, is caused by effects on AB, but rather, from the effects on AB, conclude as to the upstream-lying causes.

It is to be observed that the Mach quadrilateral AO'BOA in general has curved sides which, as Mach lines, are determined with the flow itself. In the preceding section, from the integrals of the hyperbolic differential equation, we became familiar with the remarkable fact that boundary values act as determining factors only within restricted regions. To the characteristic quadrilateral, the region of solution of the differential equation, there corresponds in the flow the Mach quadrilateral. The Mach lines are no other than the "characteristics" of the differential equation of the velocity potential. The characteristics in the flow plane are'not given, however, in advance as those in the hodograph, but become known simultaneously with the solution $\Phi(x,y)$. This is due to the fact that the coefficients of that partial differential equation (15) contain not only the free variables but also the first derivatives of the function Φ, that is, Φ_x and Φ_y. This is also the reason why we passed from the flow space to the velocity plane (equations (31), (31a), and (53c)).

11. Physical Basis of the Method of Characteristics

By means of the characteristics in the velocity plane, it is simple to draw the field of flow of two-dimensional supersonic flows and also shooting water if the flow of approach and the side boundaries are given. With a velocity prescribed alone; a line, the flow may be determined in general in the circumscribed Mach quadrilateral.' It is thus a question of Graphical method of solution of the partial differential equation (15) or (31). The *flow is* known if the velocity (u,v) is known at each point (x,y). Hence, It is not necessary to know the velocity potential $\Phi(x,y)$ or the position-determining potential $\chi(u,v)$ themselves. It Is sufficient only to determine χ_u, χ_v 'and Φ_x, Φ_y. (Compare formulas (29): $\chi_u = x$, $\chi_v = y$ and $\Phi_x = u$, $\Phi_y = v$.)

The graphical method Is based on the simultaneous construction of the flow in the velocity field (u,v) and In the field of flow (x,y).

Let us consider first a parallel-flow assumed *to* be bounded on one side. At the position S, the wall receives a small deflection δ (fig. 17). In the case of supersonic flow and shooting water, this leads to a pressure increase.*

If the wall has a convex corner, a flow arises with diverging cross section. In the case of shooting water, this leads to *a* level drop and acceleration.

Since in the boundary of the frictionless flow of figure 17, no finite length occurs as reference length, all streamlines must be similar with respect to the corner S. Water depth and velocity in magnitude and direction therefore have constant values along each stream through the corner.

The flow of figure 17a for large deflection angles. is described in Part II of this report (T.M. No. 935), under Shock Polar Diagram, page 1. This flow is nonstationary. The discontinuities of the different streamlines are equal and all lie on a straight stream ST passing through the corner. For extremely small deflections, the corner leads to only a small disturbance in the flow. Since small disturbances have the Mach lines as the wave front, the disturbance line ST is a Mach line. It forms with the

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*The following considerations hold for water and gas flows. Since, however, for the analogous concepts different terms *are* applied in hydrodynamics and gas dynamics, both would always have to be carried along In this work. This difficulty has been avoided as far *as* possible by using the terms from hydrodynamics. Where terms from gas dynamics, nevertheless, occur the corresponding terms are: Expansion = level drop; compression = level rise; Impulse = jump; expansion wave = depression wave, etc.

parallel flow an angle 'a where $\sin \alpha = a/c = \sqrt{gh}/c$. For somewhat larger deflections the discontinuity lies on a stream ST, whose direction lies between the directions of the two Mach lines of flow I before the deflection, and flow II after the deflection.

The flow corresponding to figure 17b for large deflections and hence, strong acceleration, Is treated more In detail In section 21, Part II of this report (T.M. No. 935), under Level Drop about a Corner. In contrast to level rest, the drop is continuous. It begins again on account of the slmllarlty for all streamlines on a stream ST'. This Is a Mach line of flow I before the level drop. The deflection for all streamlines ends on a stream ST", a Mach line of flow II. For small deflections, It may be assumed as a first approximation also for the level drop that It Is concentrated on a mean stream ST. An important slmpllflcatlon Is thus obtained for the qraphlcal method.

Both the small level drop (In the gas expansion) and small level rise (compression) have the followlnq in common: The velocity receives along a disturbance line a change in magnitude and direction. The direction of the disturbance line is given as the mean direction of the two Mach lines of the conditions before and after the change.* In traversing this line, there Is also a change In the pressure. The pressure drop or gradient - that is, the Increase In pressure per unit length in the direction of the most rapid change - is thus normal to the mean Mach line. According to Newton's law, the acceleration and hence also the vector change In the velocity, has the direction of the force. We thus have the result: The velocity vector $\vec{c}_I$ before the deflection (rise and drop) receives as a result of tho deflection, a vector Increment $\vec{\Delta c}$ which is normal to the Mach line. Since the deflection angle Is also known, $\vec{\Delta c}$ Is determined (fig. 18).

The graphical method consists In building up the entire field of flow out of small individual Mach quadrilaterals, in each of which the velocity is constant and deflections occur from ono quadrilateral to the 'other.

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*Wherever necessary for clearness In what follows, a distinctlon will be made between disturbance line and Mach line. The disturbance lines are those along which the discontlnuitles arise. Disturbance lines of Infinitely small intensity are Mach lines. Both pass over into one another In steady flow.

12. Mach Number and Angle.

It is important that the Mach number M and the angle α ($\sin \alpha = 1/M$) are given by the magnitude of the flow velocity alone, since $\sin a = \sqrt{gh}/c$ and according to the energy equation, the water depth h depends uniquely on the flow velocity (equation (9)). We thus have:

$$\sin^2 \alpha = gh/c^2 = (gh_0 - \tfrac{1}{2} c^2)/c^2$$

Dividing numerator and denominator of the right side by a^{*2} (42)

$$a^{*2} = 2gh_0/3$$

we obtain in the notation of nondimensional velocities $\bar{c} = c/a^*$:

$$1/M^2 = \sin^2 \alpha = (\tfrac{3}{2} - \tfrac{1}{2} \bar{c}^2)/\bar{c}^2 \tag{69}$$

For the graphical method, there is applied the graphical representation of equation (69) (fig. 19), a being plotted as arc, and $\bar{c}$, as radius vector. In rectangular coordinates, $\bar{v} = \bar{c} \sin a$,

$$\bar{v}^2 = \bar{c}^2 \sin^2 \alpha = \frac{3}{2} - \frac{1}{2} \bar{c}^2$$

and

$$\bar{u}^2 = \bar{c}^2 (1 - \sin^2 \alpha) = \frac{3}{2} \bar{c}^2 - \frac{3}{2}$$

Eliminating $\bar{c}$ from those two equations, there is obtained the curve in rectangular coordinates

$$(\bar{u}/\sqrt{3})^2 + \bar{v}^2 = 1 \tag{70}$$

This is an ellipse with major and minor semiaxes $\sqrt{3}$ and 1 (fig. 19). For an ideal gas, it is an ellipse with the semiaxes $\sqrt{(k + 1)/(k - 1)}$ and 1.

13. Characteristics

If any nondimensional velocity $\bar{c}_I$ is given at point P of the flow plane, the direction of the Mach line at the point considered is obtained in the following manner: $\bar{c}_I$ is drawn in the velocity plane (fig. 20). The ellipse is now rotated about 0 until the extremity of $\overline{o}_I$ lies on it (two possible oases).. Then, according to figure 19, the principal axis of the ellipse so rotated gives the direction of the Mach lines in the flow and according to figure 18, the minor axis of tho ellipse gives the direction of the velocity increment Ac. Pour typos of increase are possible, depending on whether the Mach line is a disturbance line of the first or second family, and whother the disturbance is a drop or a rise. In the example shown (fig. 20) no disturbance line of the first family passes through the point P, whereas that of the second family results in a deflection, namely, a level drop. The velocity increment, denoted by a heavy arrow, thus, is the one that comes under consideration for this example. If the disturbance lines of both the first and second families pass through the point P, the apparent difficulty is removed by considering a neighboring streamline. For the latter, the velocity receives tao changes, one following shortly after the other, each of which is uniquely determined.

At each point of the velocity plane there are thus two directions of the velocity Increment. These two directions are given by the minor axis of the ellipse (fig. 21.* There is thus obtained in the circular ring area, between $R = \sqrt{3}$, and $r = 1$, a direction field which determines two families of curves. In figure 21, two representatives of theso two familios are drawn. By the following simple consideration, Busemann shows that we have here the case of the previously found epicycloids.

The direction field is obtained by drawing the small segments a, b, c, d, ... in the direction of the minor axis of the ellipse $(0, \sqrt{3}, 1)$, then rotating the ellipse some-hat, and again drawing the lines. We may now consider a, b, c, . . . as lying, instead of on the ellipse, on the fixed points of the circle chords A_1A_2, B_1B_2, C_1C_2, ... There is then obtained the same direction field as before if these chords are rotated in the circle $(0, \sqrt{3})$ and a, b, c, . . . drawn each time. If all these chords with their points a, b, c, . . . are now arbitrarily drawn in the circle $(0, \sqrt{3})$ (fig. 22), the small segments a, b, c, . . . are still in the direction of the required direction field. By suitable rotation of the chord diagram (fig. 21), we pass a family of ohords through an arbitrarily chosen point A_1, the chord diagram being rotated so that B_1, C_1, D_1, ...

*Figs. 21, 22, and 23 correspond to figs. 40, 41, and 42 of Busemann, 1931, p. 422 (reference 7).

lie successively on A_1 end the segments 8, b, c, ... being drawn. The latter will still be segments in the direction field (fig. 23); The complete field will be obtained by rotating thie diagram about O; 'for example,' A_1 toward A_1', and then again drawing the small segments a, b, c, . . .

Now the points a, b, a, ... divide the chords A_1A_2, B_1B_2, C_1C_2, ... (fig; 21) in the same ratio; the ellipse as effine figure of the circle having this property: The points a, b, c, . . . in figure 23, thus lie on a circle. The directions a, b, c, . . . are normal, respectively, to Ab, Ac, . . .

If the circle with diameter AA_1 is rolled on the' circle about 0 with the radius 1, each of its points describes en epicycloid. The rolling circle at the instant represented, rotates about the point A. All of its points thus also move on normals to the lines joining the corresponding points with A, the direction field of the set of epicycloids being identical with that of the required curves of the possible velocity Increment Ac. These curves are thus the epicycloida described above (figs. 21 and 9).

We have mentioned the same epicycloida before. They are the characteristics of the partial differential equation of the flow. We now see the physical interpretation of the characteristics: During the passing of 8 smell disturbance wave the flow velocity changes along the corresponding characteristic.

14. Graphical Construction of the Flow

The field of flow and the hodograph are drawn simultaneously - in the hodoyraph, the velocities and their changes; in the field of flow, the streamlines. The flow is always assumed from left to right. We may then speak of en upper or a lower boundary. All disturbance lines that start from the upper boundary will be denoted as the upper system of waves, and all thoee from the lower boundary, the lower system.

a) Flow bounded on one side.- The simplest supersonic flow is that bounded on only one side as given by the boundary conditions of figure 24. Let the' approach be parallel end have the *Mach* number $M = 1.5$.' *As* a first step the continuously curved wall is replaced by small straight segments with angle increments of, for example, 2°. In some Cases it may be of advantage to make the angle increments of various amounts.

To the flow of approach (parallel flow), there corresponds, in the velocity plane, a single point P_1 given by the direction of c_1 and the magnitude $\bar{c}_1$*. P_1 is also obtained as the point in the hodograph (fig. 24c) at which the normal to the characteristic forms with the velocity, the Mach angle α_1. At S_1 the flow receives a first discontinuity, a level drop which leads to a deflection by the angle δ. This deflection is of equal magnitude for all streamlines and lies for the entire flow along the disturbance line $S_1 T_1$, whose direction we shall learn from the hodograph. In the latter the velocity $\bar{c}_2$ after the first discontinuity is given by the point P_2 whose radius vector forms the angle δ with that of P_1, and which lies on the characteristic through P_1, corresponding, for $\bar{c}_1 \longrightarrow \bar{c}_2$, to a drop; that is, an increase in velocity. We thus obtain P_2 and $\bar{c}_2$. The disturbance line $S_1 T_1$ in the flow is, as we know, a mean Mach line between the states P_1 and P_2. This direction is now given simply as the normal to the characteristic between P_1 and P_2 in the velocity plane. In the entire region 2, the flow is again a parallel flow with the Velocity c_2 up to the disturbance line $S_2 T_2$. This line and the state after this second disturbance, is determined similarly as for $S_1 T_1$, only now the initial velocity is given in the hodograph by P_2. The velocity after the disturbance is again the velocity OP_3 deflected by δ. The direction of the disturbance line $S_2 T_2$ is the direction of the normal to the characteristic between P_2 and P_3, etc.

With the above construction, the first disturbance thus lies along $S_1 T_1$, the last along $S_{n-1} T_{n-1}$. Actually the beginning and end of the disturbances lie along the dotted lines $S_0 T_0$ and ST, which have the directions of the normals to the characteristic in P_1 and P_n. It is only

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*From equation (69), we have: $\bar{c}^2 = 3 M^2/(M^2 + 2)$

For gases: $\bar{c}^2 = (k + 1) M^2/[(k - 1) M^2 + 2]$

because we must draw the flow discontinuously in finite steps that the actual start of the disturbance and the first disturbance do not accurately coincide. BP decreasing the steps, 'the accuracy may be raised.

Figure 25 shows a flow drawn in this manner with M = 1.5, and for rater (k = 2), the deflection increments being 2°. From this simple example, an important property of shooting water bounded on one aide (supersonic flow) may be recognized, namely, that as' long as no large discontinuous pressure rises (impulses) occur, all the points giving the state in the hodograph lie on a single characteristic; i.e., for such a flow the magnitude of the velocity depends uniquely on its direction and vice versa.

A limiting case of the example considered is the level drop about a corner (fig. 26a-c) (references 14 and 17). This flow is a parallel flow with a Mach number equal to or greater than one. The one-aided rectilinear boundary ends at S. On thd lower side of the boundary the water depth (pressure in the gas) is zero or at least smaller than in the parallel flow of approach. The same results hold as for the flow nf figure 24 except that now the lines S_1T_1, S_2T_2, . . . all pass through the point S. The velocity varies along a streamline in 'such a manner that its end point travels on a characteristic in the velocity plane (fig. 26c). The constant velocity along a stream SP has its end point P' at that position of the corresponding characteristic where the normal to the characteristic is parallel to SP.

b) Interior of a flow bounded on two sides.- Let the velocity c_1 be given in the interior of a flow in a certain region 1 (fig. 27). Let this region be 'bounded on the right aide by an upper (b), and a lower, disturbance line (a). The streamlines α and β, which-may also be considered as walls, are correspondingly assumed to have small defleotiona at A and B. The deflections' δ_α and δ_β are given. The point P_1 in the hodograph is the image point of the region 1 of the flow (fig. 27b). In crossing the disturbance wave a from region 1 to region 2 (drop, since deflection is toward outside) the velocity c_1 receives a change such that the velocity c_2 lies on the characteristic corresponding to the lower disturbance wave system and forma with c_1 the angle δ_α. This gives the point 'Pa. in the hodograph as in a flow bounded on one side and hence also the direction of a as normal. to

$\overline{P_1 P_2}$. The same is true in crossing the disturbance wave b. To this corresponds in the velocity diagram a traveling along the characteristic of the upper system from P_1 toward P_3 (δ_β is given). At a position X the two disturbance waves meet and their effects will "cross." From the point X a disturbance wave of the lower set a' starts out and one from the upper set b'. Crossing a' in the flow means in the hodograph, as in a flow bounded on one side, a change in the velocity from P_3 toward Q_4 (fig. 27b) where Q_4 for the present, is unknown. Similarly the velocity on crossing b' receives a change from P_2 to S_4 where S_4 similarly is for the present, unknown. Now a first condition for Q_4 and S_4 is that the velocity in the region 4q of the flow on passing from from $1 \rightarrow 3 \rightarrow 4$, should have the same direction as the velocity in region 4s on passing $1 \rightarrow 2 \rightarrow 4$. This means in the velocity diagram that the points Q_4 and S_4 must lie on a straight stream through 0 : $OS_4 \parallel OQ_4$. There is, furthermore, to be satisfied, the condition that the water depth (pressure in the gas) in the region 4q must be the same as in 4s. As long as the flow is free from impulse, the water depth is uniquely determined by the velocity. The requirement that the depth ahould be the same in 4q and 4s, means therefore that the velocity OS_4 must have the same magnitude as $OQ_4 : \overline{OS_4} = \overline{OQ_4}$. Both conditions are simultaneously satisfied if S_4 and Q_4 coincide at the point of intersection P_4. The entire region 4 of the flow is thus in the velocity diaqram given by the point P4. We may now draw a' and b'. They start from X in the direction of ths normals to $P_3 P_4$ and Ps P_4, respectively.

Figure 28 shows the intercrossing of two streamlines where now one disturbance is a level rise, the other a drop. The picture would be quite similar if the two disturbances were level rises.

We shall now follow a disturbance line in the interior of a flow in the case where it, encounters several disturbance lines of the other family (fig. 29). The directions of a, b, a', and b' and the points. P_1, Ps. P_2, and P_4 are assumed to be determined by the method given. Then for the regions 3, 4, 5, and 6, we again have P_4 and Ps lying on the characteristics through- P_3. The position of P_5 is determined by the deflection δ_{35} and P_4 is fixed by the characteristics P_2P_4 and P_3P_4. There is now obtained also P_6 and hence the velocity OP_6 in region 6, P_6 being the point of intersection of the two characteristics P_5P_6 and P_4P_6. Similarly, there is finally obtained P_8. The individual portions of the disturbance wave aa' a" a''' are in the directions of the normals at the centers of the portions of the characteristics P_1P_2, P_3P_4, P_5P_6, P_7P_8, respectively.

We thus find the result, namely, that the extremities of all possible velocity vectors before crossing the disturbance wave aa'a" ..., the points P_1, P_3, P_5, ..., all lying on a fixed characteristic through P_1. Similarly, all extremities of the velocities after crossing the disturbance wave a - that is, the points P_2, P_4, P_6 ... lie on the characteristic through P_2. Crossing the disturbance wave aa' a" a''' at any position in the direction of the flow, has the result with respect to the velocity, that there is a transition from the characteristic 1 to the characteristic 2 (both of the same family) each time along a characteristic of the other family. These changes are the heavily drawn portions of figure 29b. Since the two families of characteristics lie symmetrically:

$$\measuredangle P_7OP_8 = \measuredangle P_5OP_6 = \measuredangle P_3OP_4 = \measuredangle P_1OP_2 = \delta_{12}$$

i.e.,

$$\underline{\delta_{12} = \delta_{34} = \delta_{56} = \delta_{78} = \ldots}$$

In figure 30, let the curves denoted by K be circles about O. We then have:

a) $\measuredangle$ AOC = $\measuredangle$ EOF, because each characteristic of the same family arises from the other by rotation about O.

b) $\measuredangle$ AOB = $\measuredangle$ BOE = 1/2 $\measuredangle$ AOE, because AB is symmetrical to EB with axis of symmetry BO.

c) $\measuredangle$ COD = $\measuredangle$ DOF = 1/2 $\measuredangle$ COF, similar to b).

d) $\measuredangle$ COE = $\measuredangle$ COE.

Equation d) subtracted from a) gives

$$\measuredangle AOC + \measuredangle COE = \measuredangle EOF - \measuredangle COE$$

i.e., $\measuredangle AOE = \measuredangle COF$, and hence it follows from b) and c) $\measuredangle BOE = \measuredangle DOF$, as was to be proved.

We thus obtain the moat important result: On crossing a disturbance wave the velocity undergoes a change in magnitude and direction. The change in the velocity direction is the same at all points of the entire disturbance wave Independent of the direction of the velocity before the arrival of the disturbance wave and regardless of whether or not the wave was crossed by disturbances of the other family. This is true on the assumption of flow free from impulse. In section 4 we consider flows with Impulse for which the velocity is not a unique function of the water depth. There it will be found that the deflection angle caused by a disturbance wave may vary along the wave.

c) Fixed wall with 8 flow bounded on two sides.- In figure 31, let SAC be the upper boundary of a flow. Let no disturbance wave from the opposite wall meet the corner S of the wall at first. From the latter, 8 wave s starts out which is identical with that of a disturbance starting from a flow bounded on one side.

We shall now consider the effect of a disturbance wave a which encounters the straight wall SC at point *A*. In region 1, let the velocity be given by the hodograph point P_1 (fig. 31b). On crossing the disturbance wave a from region 1 to region 2, the velocity receives a deflection 8, given by the lower wall. P_2 lying on the characteristic is thereby determined and also the disturbance line a. Since at each point of a flow there are two possible disturbance waves, there can start out from *A* only 8 wave of the uppor family (b). The line b and the velocity in region 3 are determined from the condition that first the velocities c_1 in region 1, and c_3 in region 3, must be parallel, since it was assumed that the wall had no discontinuity at A. In the hodograph this means that P_3 must lie on the straight OP_1. Secondly, b is a disturbance line from the family other than that of a, so that P_3 lies on the characteristic $P_2 P_3$, which passes through P_2. By both of these conditions P_3, the velocity c_3 and also the disturbance line b are determined.

The angle of deflection which the velocity undergoes on crossing the reflected wave is equal and opposite to the angle of deflection by the incident disturbance line. If the incident disturbance is a level rise, then the reflected disturbance is also a rise (fig. 31b). If the disturbance line is a drop, then the reflected line is also a level-drop disturbance (31c).

In case the disturbance line a strikes the wall at the position S where the wall has a discontinuity, no new difficulty arises. It is then only necessary to imagine that the reflected disturbance line b and the newly generated disturbance line s follow shortly upon one another. If b and s are both level-drop waves, each must be drawn separately; if both are level-rise waves, then they are drawn together as a single disturbance starting from S, on the crossing of which the velocity undergoes a deflection equal to the sum of the deflections due to s and b. If, however, one of the disturbance lines is a rise, and the other a drop, then only a single disturbance line starting from S is drawn, along which the deflection angle for the velocity is equal to the difference between the deflection angles for s and b and, depending on the intensities of s and b, may be a rise or a drop line.*

In the third case, where the deflection angles for s and b are opposite, it may also happen that they have the same magnitude. In that case no disturbance at all starts out from that point. This is the case if the wall itself has the same deflection angle as that of the approaching disturbance wave. This fact is made use of where it is desired to produce a parallel flow. In the latter no disturbance waves occur. This condition is obtained by giving the walls in succession discontinuities such that one disturbance wave is "swallowed" when the other strikes it.

d) Free jet.- If a disturbance line strikes a free jet, another type of reflection occurs since the water depth must have a fixed value (fig. 32). Let the point P_1 in the velocity diagram correspond to region 1 ahead of the disturbance wave. The point P_3 which gives the velocity

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*For the third case it is clear that only a single disturbance line starting from S is drawn because the sum of the two disturbances is smaller than that of either individual case. For the first case two, and for the second case only one, disturbance line is drawn in order to approach the true condition for which drops are spread out in the form of a fan (drop about an edge) while rises are concentrated (impulse).

OP_2 of region 2, lies on the characteristic through P_1 belonging to the lower family of disturbance lines and determined by the deflection angle 8,. Since at each point two disturbance waves, at most, pass through, there can etart'out at point A of the flow where the line a strikes the free jet, at most, another disturbance line b of the other family (b).' The disturbance b must be such that the water depth is the same in regions 1 and 3. This means for flow without energy dissipation that the hodograph point P_3 corresponding to region 3, must lie on a circle through P_1 about 0: OP, = OP,. Since, moreover, P_3 lies on the characteristic through P_2 belonging to the upper disturbance line, family P_3 is uniquely determined and hence, also b. On account of the symmetry of the two families of characteristics $\measuredangle P_1OP_2 = \measuredangle P_2OP_3$.

A level-drop wavo is reflected on a freo jot as a level-rise wave, and conversely. It is important to observe that the velocity deflection on crossing tho reflected save is as large as that on crossing the incident. Here again we find that disturbance waves - whether they are crossed by. others or reflected - produce at all points equally large deflection angles of the local velocities.

15. Application: Laval Nozzle

Let a Lava1 nozzle be drawn for mater (k = 2) in which the flow is parallel at the minimum cross section (M = 1) and which is to produce at its exit a parallel flow of Mach number M = 2.

Aside from flows with hydraulic jumps (shocks), all the phenomena have been discussed in detail in the previous sections. There are no difficulties in drawing up the flow with the aid of tho basic elements described above. Instead of drawing Mach lines, however, as normals to the characteristics, the accuracy is considerably Improved by using the ellipse construction described in sections 12 and 13. The normal to the characteristic is then obtained as the direction of the major axis of the ellipse without requiring either the tangent or the normal of the characteristic itsolf (figs. 20 and 33).

A convenient arrangement for the drawing is shown on figure 34. A strip B is glued on the transparent paper A with the ellipse E, the edge of the strip being parallel to the minor axis of the ellipse and rotatable about a needle at point 0 in the origin of the velocity plane. The direction of the major axis is drawn with the triangle F as disturbance wave in the flow.

The Laval nozzle investigated has as its boundary at the approach side of the flow, a cubical parabola PQ with a ehort connecting straight piece QR, in order that at the minimum cross section the flow, for the ahooting-water region to be drawn, ehould be parallel.. There will then be no disturbance waves in ft. To the straight portion there is connected a circular arc RS. The shape of this portion can be chosen at will and the first disturbance waves start out from It. The shape of ST is determined by that assumed for RS since the former must be such that, starting from the channel exit, there are no disturbance waves in the flop.

If the approach flow is parallel, the construction of the flow begins with the first disturbance line from RS, the line being that of a flow bounded on one aide. The construction is then followed As discussed in the preceding paragraphs.

Since we Are constantly passing from the velocity diagram to the flow diagram And in order that corresponding points may be recognized as such, It is necessary to introduce A suitable notstlon. For this purpose the curvilinear coordinates A And μ are convenient (equations (53a) and (53b)). The numbering is shown in figure 34. The number beside each characteristic of the upper family gives the Angle in degrees At which It starts on the unit circle, and similarly, for the coordinates of the characteristics of the lower family. In order that the two families of characteristics may not be confused, the coordinates of the upper family are preceded by a zero.* The coordinates A And μ of the velocity plane Are written in the corresponding field of flow. The numbers thue written have the property (equations (53a, And b)) that $(\lambda - \mu)/2 = \varphi$; that is, their half difference gives the angle of the flow with respect to the horizontal. Their hnlf sum $(\lambda + \mu)/2$ is A number on which the magnitude of the nondimensional velocity and hence also the water-depth ratio h/h_0 uniquely depends, since $\lambda + \mu$ is constant on dir-

*To the curvilinear coordinates $\lambda = 0$, $\mu = 00$, for example, correspond the polar coordinates $\bar{c} = 1$, $\varphi = 0$.

cles about 0. With a definite value $(A + \mu)/2$ is associated the same water-depth ratio h_0/h (gas temperature ratio T/T_0, hence pressure ratio, p/p_0), which corresponds to the level drop about a corner starting from $M = 1$ (fig. 26b) and deflected from the direction of the approach flow by the angle $\omega = (\lambda + \mu)/2$. Corresponding values h/h_0, p/p_0, M, $\bar{c}$, and $w = (A + \mu)/2$ are collected in tables I and II.

In general, the difference of the two coordinate numbers is not required since the direction of the streamlines in each field may be taken directly from the velocity diagram. The streamlines may also be simply and rapidly drawn with the arrangement shown in figure 34, it being only necessary to pass the major axis of the ellipse through the hodograph point given by the coordinate numbers, the triangle then giving the velocity direction in the corresponding field.

The aum of the two coordinates, however, is required If it is desired to draw the lines of constant water depth in the flow. These lines may also be drawn without knowing the coordinate sum if equal deflectlone are chosen for all disturbance lines, namely, as diagonals of the Mach quadrilaterals.

In all problems in which a parallel flow is given as initial flow, we begin, according to the characteristic method, with the first disturbance lines starting from the boundary.

Under suitable assumptions, there may also be prescribed as an initial element, the velocity distribution along a line. The latter must not, however, at any point touch a Yach'line. It must thus be a line which in itself is not a Mach line and which does not intersect the same Mach line twice. Streamlines and their orthogonal trajectories certainly are such lines. The flow may then be computed by the characteristic8 method in the entire Mach quadrilateral described about this line. Thia Mach quadrilateral is only determined on drawing the flow. If the velocity along a line is prescribed as initial element, a further condition is that the position of this line with respect to a side boundary is such that no flow restriction falls *within* the Mach quadrilateral described about the line except when the latter has the form of a streamline.

For the graphical determination of euah flows the line must first be broken up into suitable segments on which the velocity is constant in direction and magnitude. These pieces are then separated by disturbance waves and, starting from these, the flow may be determined with the Mach quadrilateral.

List of Most Frequently Occurring Symbols

- g, acceleration of gravity.
- R, gas constant.
- ν, kinematic viscosity.
- ρ, density.
- p, pressure.
- T, absolute temperature.
- i, heat content.
- c_p, specific heat at constant pressure.
- c_v, specific heat at constant volume.
- $k=c_p/c_v$, adiabatic exponent.
- Φ, velocity potential.
- X, positioning-determining potential.
- x,y,z, rectangular coordinates in the flow space.
- r,ϑ, polar coordinates In the flow plane (x,y).
- λ,μ, curvilinear coordinates in the velocity plane, charncteristlc coordinates.
- X,Y,Z, general variables.
- u,v,w, components of the velocity in the x, y, and z directions.
- c,φ, polar coordinates in the velocity diagram (two-dimensional flow),

c_{max}, maximum velocity.

c, velocity increment.

a, in gas: velocity of sound.
in water: propagation wave velocity $\sqrt{gh}$.

a^*, critical velocity.

$\bar{u}, \bar{v}, \bar{c}, \ldots$, nondimensional velocities (reference velocity a^*; in hydraulic jump a_1^* the critical velocity before the jump).

$M = c/a$, Mach number.

$\alpha = (\sin^{-1})(a/c)$, Mach angle.

h, water depth.

h_0, total head (water depth for c = 0).

h_0', h_0'', total heads after hydraulic jumps.

p_0, T_0, i_0, h_0, subscript 0: stagnation state.

$T^*, h^*, \ldots$, asterisk *: critical state.

u_1, c_1, h_1, M_1, subscript 1: before hydraulic jump.

u_2, c_2, h_2, M_2, subscript 2: after hydraulic jump.

u_{2r}, velocity after right hydraulic jump.

A(X,Y), B, C, coefficients of linear partial differential equation of second order.

a, b, c, coefficients of the differential equation in normal form.

K, coefficient of the differential equation of the flow In normal form.

δ, small deflection angle.

ω, deflection angle of the flow without dissipation (sec. 21, Part II, T.M. No. 935).

β, deflection angle for hydraulic jump (figs. 37 and 38, Part II, T.M. No. 935).

γ, angle of the hydraulic jump wave front (figs. 37 and 38, Part II, T.M. No. 935).

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TABLE I*

Gas, k = 1.405

$\omega = \frac{(\lambda+\mu)}{2}$ (deg.)	$\frac{p}{p_o}$	$\bar{c} = \frac{c}{a^*}$	$M = \frac{c}{a}$	$\omega = \frac{(\lambda+\mu)}{2}$ (deg.)	$\frac{p}{p_o}$	$\bar{c} = \frac{c}{a^*}$	$M = \frac{c}{a}$
0	0.527	1.000	1.000	26	3.130	1.625	1.995
1	.476	1.073	1.090	27	.123	1.640	2.028
2	.449	1.110	1.142	28	,116	1.656	2.065
3	.424	1,141	1.186	29	.109	1.671	2.101
4	6.402	1.172	1.228	30	.103	1.686	2.138
5	.382	1.200	1.265	31	.097	1.700	2.178
6	.363	1.227	1.305	32	.091	1.718	2.215
7	.345	1.253	1.342	33	.086	1.732	2.258
8	,329	1.278	1.376	34	.081	1.748	2.298
9	.313	1.300	1.413	35	.076	1.763	2.338
10	.298	1.322	1.443	36	.071	1.776	2.378
11	.284	1.343	1.474	37	.067	1.791	2.421
12	.270	1.365	1,506	38	.062	1.805	2.460
13	.257	1.387	1.542	39	.058	1.819	2.506
14	.245	1.409	1.575	40	.055	1.832	2.548
15	.233	1.426	1.608	41	.051	1.845	2.592
16	.221	1.447	1.643	42	.048	1.858	2.636
17	.210	1.466	1.680	43	.044	1.872	2.680
18	.200	1.486	1.718	44	.041	1.884	2.730
19	.190	1.503	1.750	45	.039	1.898	2.778
20	.180	1.520	1.780	46	.036	1.910	2.825
21	.171	1.539	1.815	47	,033	1.923	2.875
22	.162	1.556	1.850	48	.031	1.936	2.920
23	.153	1.575	1.885	49	.029	1.948	2.978
24	.145	1.590	1.923	50	,027	1.960	3.028
25	.137	1.608	1.958	129° 19'	0	2.437	∞

*See reference 7, pp. 426-7. For values of K, see reference 1 (or 2), p. 317.

TABLE II

Water, k = 2

$\omega = \frac{\lambda+\mu}{2}$ (deg.)	$\frac{h}{h_0}$	$\bar{c} = \frac{c}{a^*}$	$M = \frac{c}{a}$	K	$\omega = \frac{\lambda+\mu}{2}$ (deg.)	$\frac{h}{h_0}$	$\bar{c} = \frac{c}{a^*}$	$M = \frac{c}{a}$	K
0	2/3	1.000	1.000	∞	26	1.234	1.516	2.56	-0.160
1	.624	1.062	1.098	2.68	27	.223	1.527	2.64	-.177
2	.598	1.101	1.160	2.07	28	.212	1.538	2.73	-.196
3	.576	1.129	1.214	1.40	29	.201	1.549	2.82	-.216
4	.555	1.156	1.267	1.014	30	.190	1.559	2.92	-.234
5	.535	1.182	1.319	.758	31	.180	1.569	3.02	-.252
6	.516	1.207	1.371	.590	32	.170	1.579	3.13	-.271
7	.498	1.229	1.422	.476	33	.160	1.588	3.24	-.291
8	.481	1.249	1.470	.394	34	.151	1.597	3.36	-.313
9	.464	1.269	1.520	.318	35	.141	1.605	3.49	-.336
10	.448	1.288	1.570	.263	36	.132	1.613	3.63	-.36
11	.432	1.306	1.622	.215	37	.123	1.621	3.78	-.38
12	.417	1.323	1.674	.170	38	.115	1.629	3.93	-.40
13	.402	1.340	1.727	.133	39	.107	1.637	4.01	-.43
14	.387	1.356	1.781	.103	40	.099	1.644	4.26	-.46
15	.373	1.372	1.835	.072	41	.092	1.651	4.44	-.49
16	.359	1.387	1.89	.046	42	.085	1.657	4.63	-.52
17	.345	1.402	1.95	.020	43	.078	1.663	4.85	-.54
18	.331	1.416	2.01	-.004	44	.072	1.669	5.08	-.58
19	.318	1.430	2.07	-.028	45	.066	1.675	5.33	-.62
20	.305	1.444	2.13	-.050	46	.060	1.681	5.62	-.66
21	.292	1.457	2.20	-.071	47	.054	1.686	5.95	-.m
22	.280	1.470	2.27	-.089	48	.048	1.681	6.30	-.75
23	.268	1.482	2.34	-.108	49	.043	1.696	6.68	-.81
24	.256	1.494	2.41	-.126	50	.038	1.700	7.11	-.86
25	.245	1.505	2.48	-.143	65° 53'	0	$\sqrt{3}$	∞	-∞

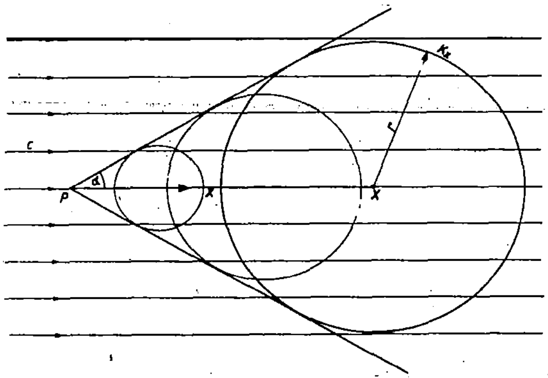


Figure 1.- Mach rays.

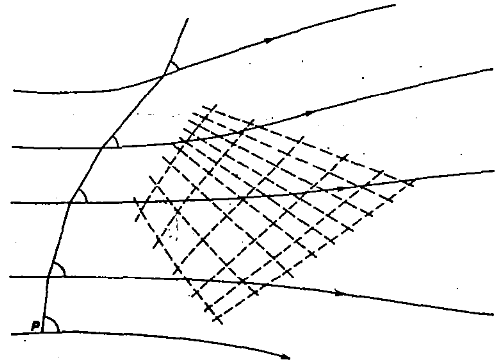


Figure 2.- Mach liner, double family.

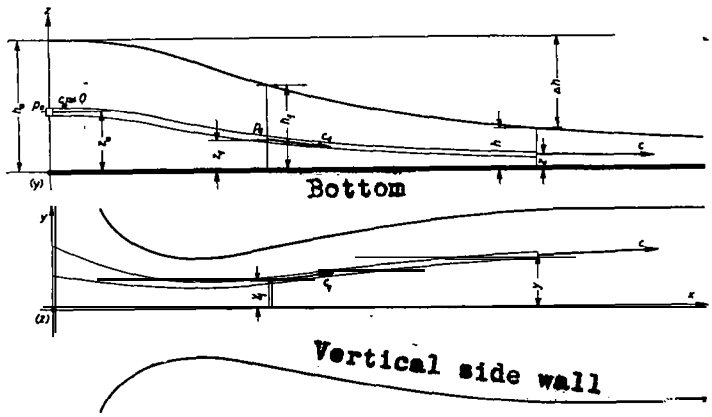


Figure 3.- Notation for energy equation.

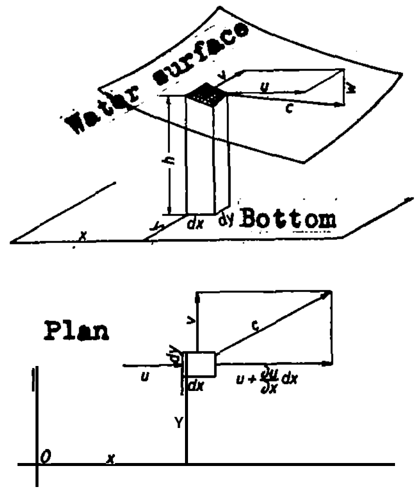


Figure 4.- Sketch for derivation of continuity equation.

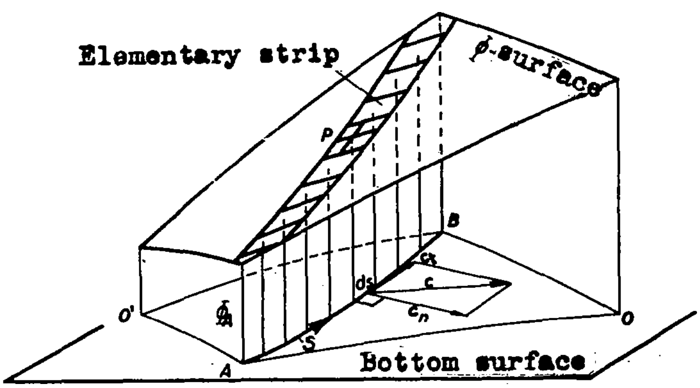


Figure 5.- Φ-surface strip.

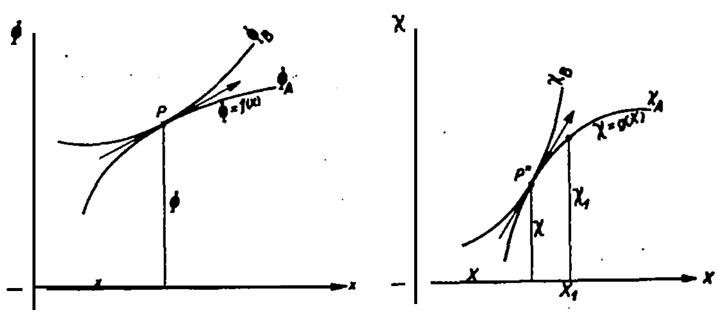


Figure 6.- Contact transformation for one independent variable.

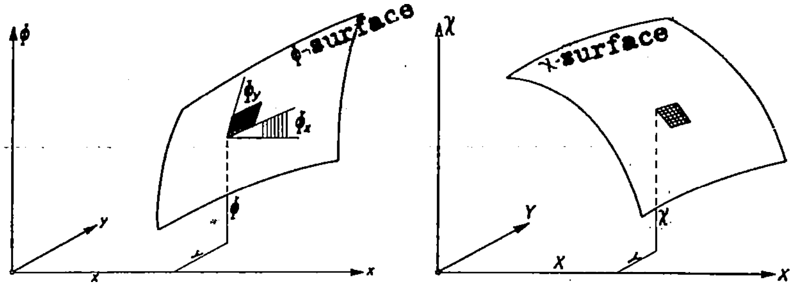


Figure 7.- Element transformation for two independent variables.

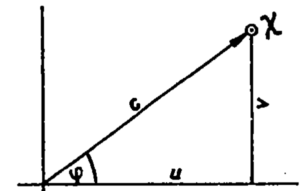


Figure 8.- Polar coordinates in the velocity diagram.

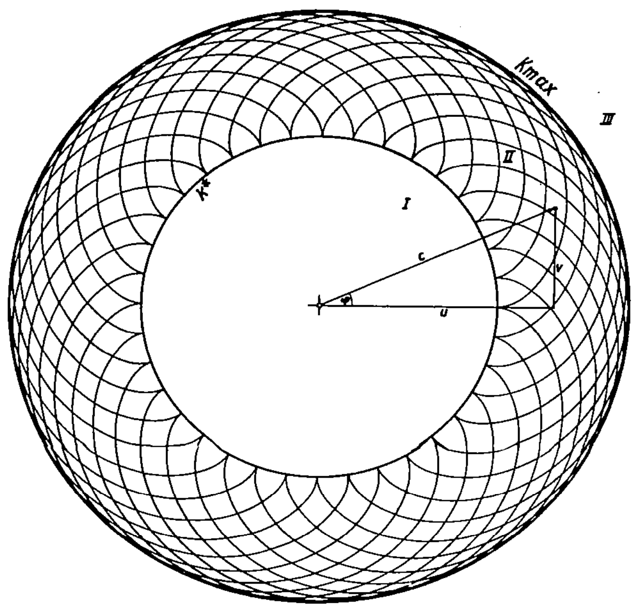


Figure 9.- Characteristics of the flow differential equation.

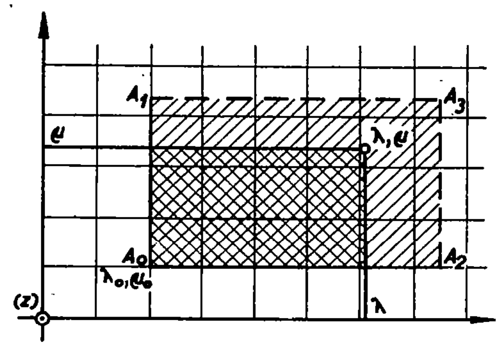


Figure 12.- Characteristics of the normal form. Method of successive approximation.

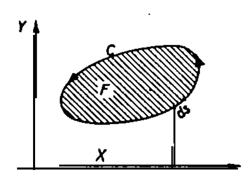


Figure 13.- General region of integration.

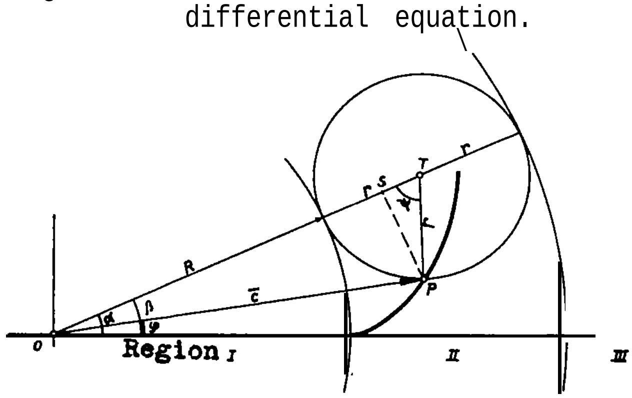


Figure 10.- Construction of the characteristics:

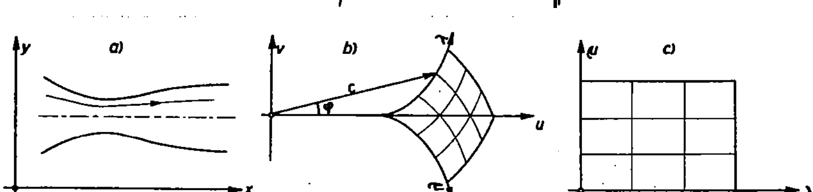


Figure 11.- The various coordinates.

(a) Flow plane. (b) Velocity diagram. (c) Characteristic coordinates

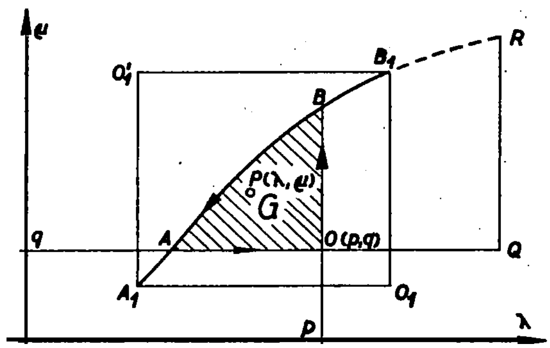
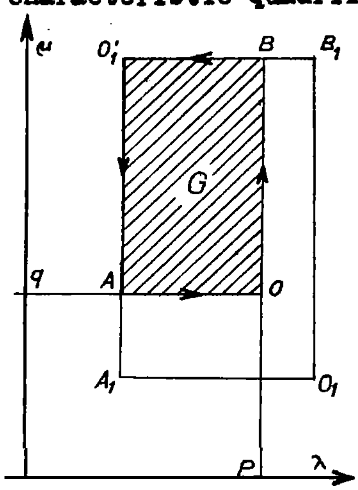


Figure 14.- Region of **integration** for the normal form of the **hyperbolic equation** and **characteristic quadrilateral.**



Pigure 15.e. Notation for application of formula (67) if the boundary value8 **Z** are given along two character; **istics.**

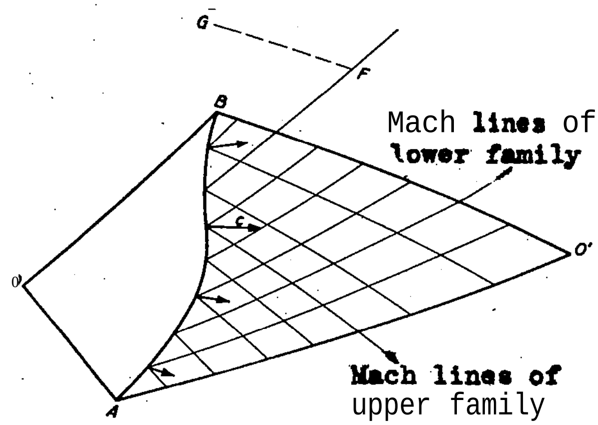
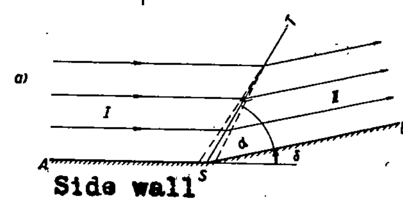
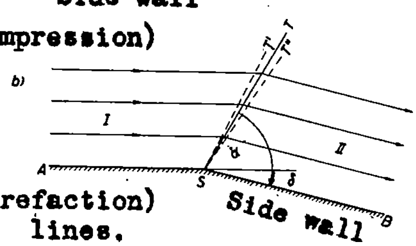


Figure 16.- Mach quadrilateral.



(a) **Rise (compression)**



(b) Sink **(rarefaction)**
-----Mach **lines.**

Figure 17.- Small deflection of a parallel flow,

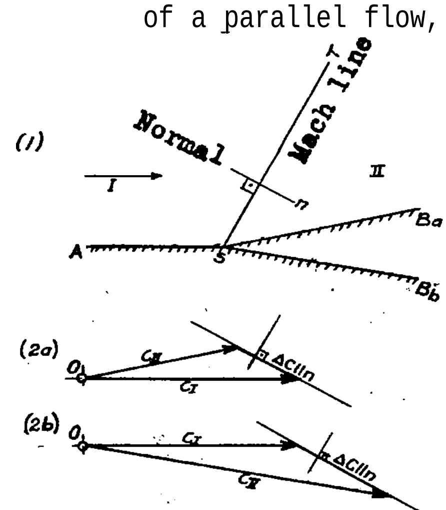


Figure 18.- The change in velocity on' **crossing a** Mach line.

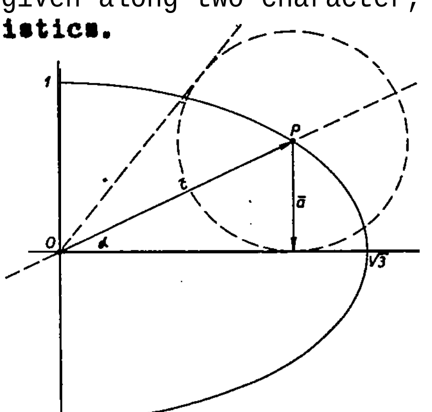
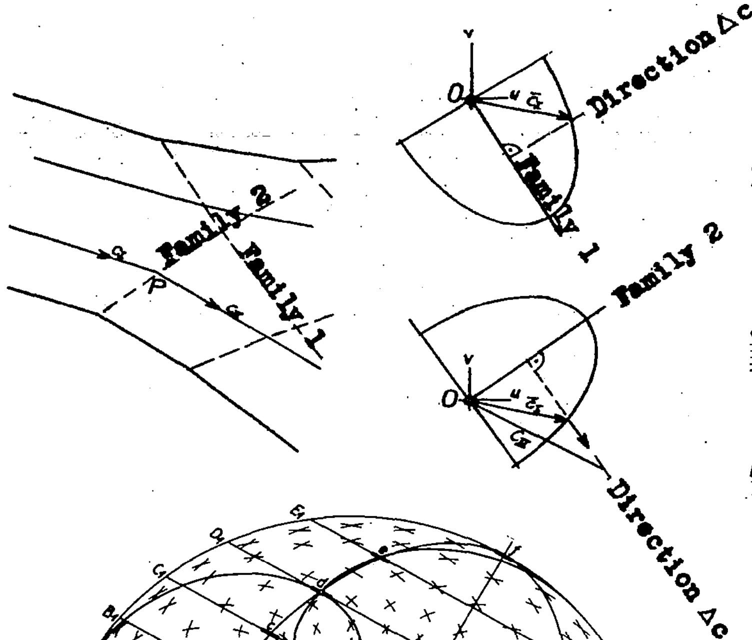


Figure 19.- Relation **between** the **flow velocity** $\bar{c}$ **and the Mach angle** am



(a) Flow: ——— Streamlines; – – – Mach lines.
(b) Velocity diagram.

Figure 20. Employment of the hodograph for the determination of the Mach line in the flow.

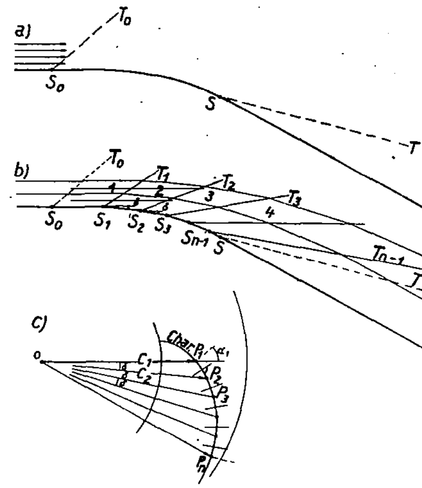


Figure 24.- Flow bounded on one ride.

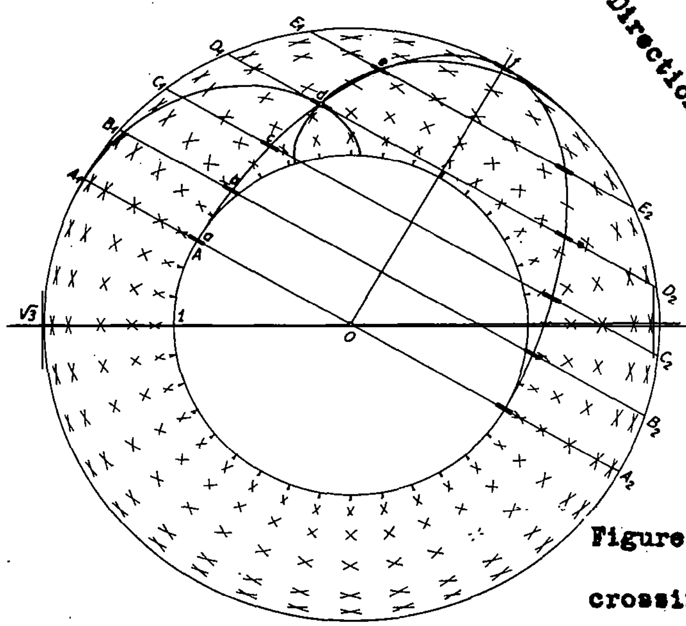


Figure 21.- Field of directions of the velocity change on crossing a disturbance line.

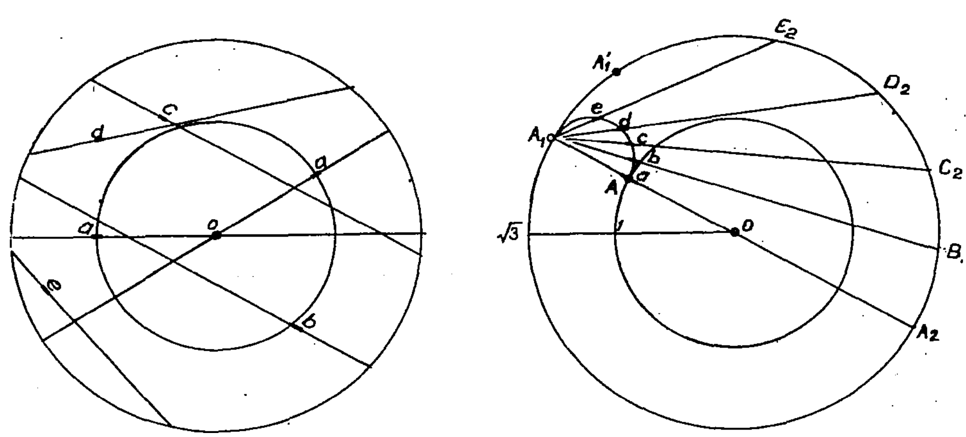


Figure 22 and 23.- Proof that the direction field (fig. 21) belongs to two families of epicycloids.

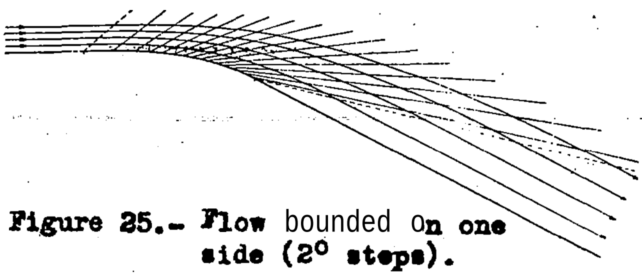


Figure 25.- Flow bounded o**n one side (2° steps).**

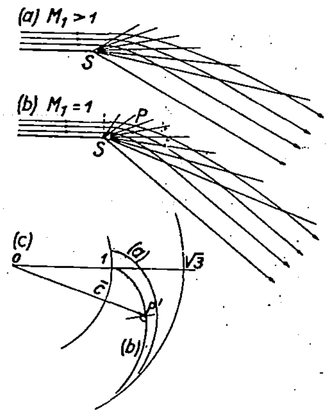
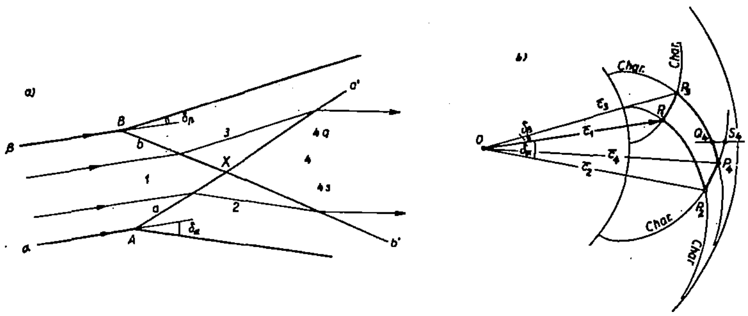
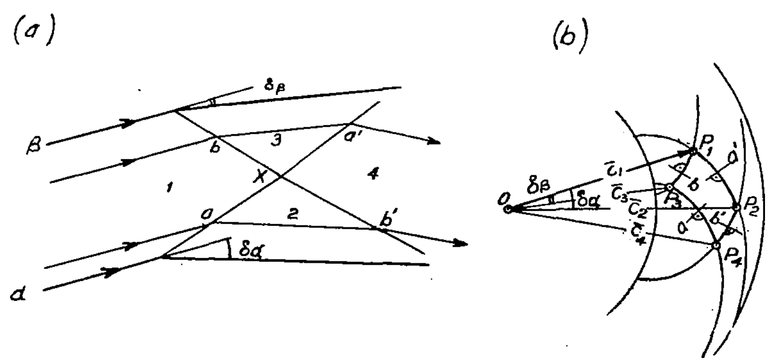


Figure **26.- Sinking at an edge.**
(a) Starting from $M_1>1$.
(b) " " $M_1=1$.
(c) Velocity diagram.



(a) Flow plane. **(b)** Velocity diagram.

Figure **27.-** Interior point of a flow bounded on two sides (the deflection **angles δ** which are of the order of magnitude **of** 1 degree are in thir and the following **figures** drawn exaggerated for clearness).



(a) Flow plane.
(b) Velocity diagram.

Figure **28.-** Interior point of a flow bounded on two **sides.**

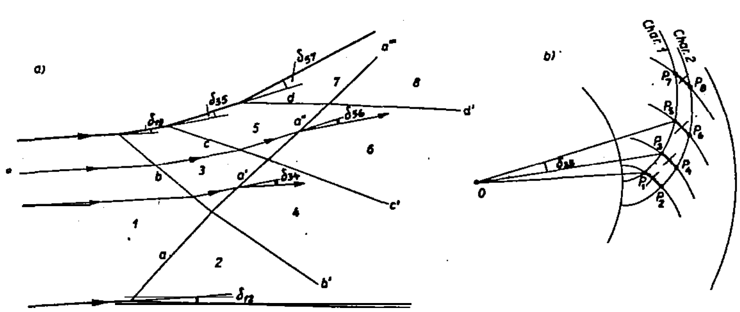


Figure 29. - Conditions along a **disturbance line.**
(a) Flow plane. (b) Velocity **diagram.**

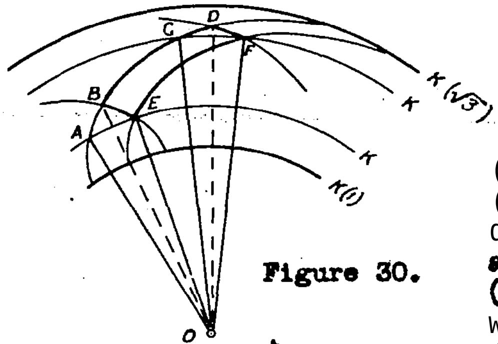
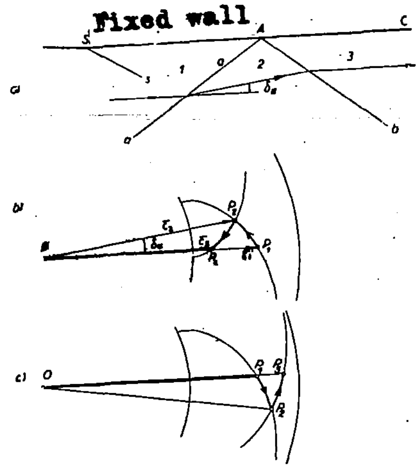


Figure 30.



(a) Flow plane.
(b) Velocity diagram for a level raising (condensation) wave.
(c) Velocity diagram for a level lowering wave.

Figure 31.- Disturbance wave striking a wall.

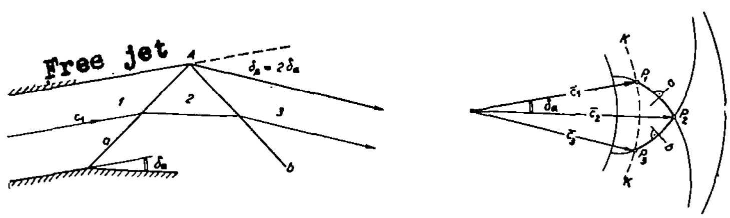


Figure 32.- Disturbance lines striking a free jet boundary.
(a) Flow plane. (b) Velocity diagram.

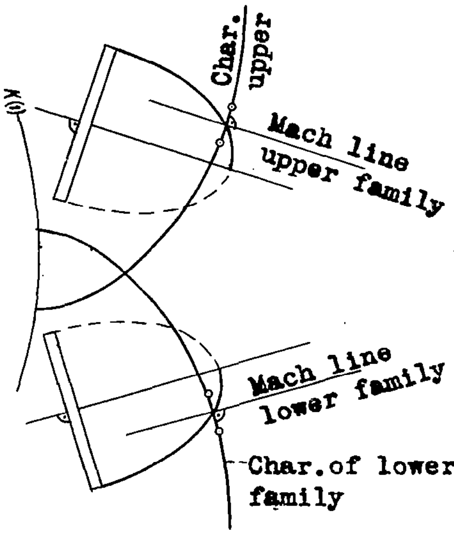


Figure 33.- Sketch showing method of determination of the direction. of the disturbance wave by means of the ellipse.

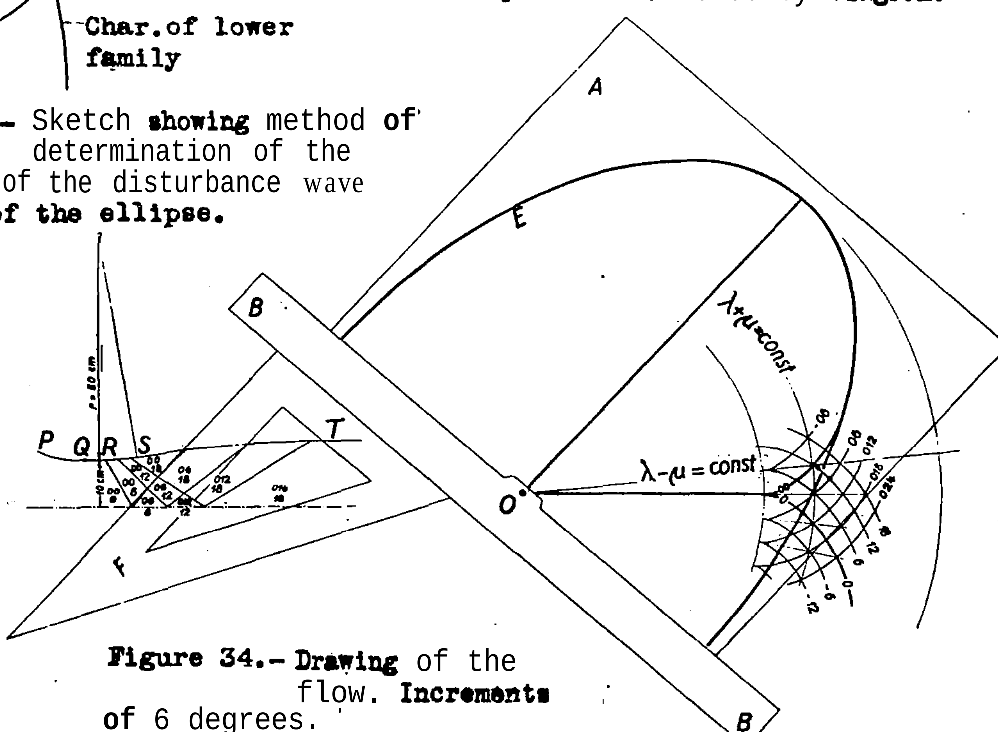


Figure 34.- Drawing of the flow. Increments of 6 degrees.