Coupling Various Methods for Convection-Diffusion Problems with Applications to Flows in Porous Media

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Coupling Various Methods for Convection-Diffusion Problems with Applications to Flows in Porous Media

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1. Introduction

We consider the model convection–diffusion problem: given \( f \in H^{-1}(\Omega) \), \( a = a(x) \) s.p.d. matrix, find \( p \in H^1(\Omega) \) s.t.

\[
-\nabla \cdot a \nabla p + C_p = f(x), \quad x \in \Omega,
\]

where \( C_p = \nabla \cdot (pb) + c_0 p, \quad x \in \Omega, \)

\( p(x) = 0, \quad x \in \partial \Omega. \)

Here \( \Omega \) is a polygonal domain in \( \mathbb{R}^d \), with \( d = 2, 3 \), the coefficients \( c_0(x) \) and \( b = b(x) = (b_1, \ldots, b_d) \) satisfy:

\[
c_0(x) + \frac{1}{2} \nabla \cdot b(x) > \gamma_0 = \text{const} > 0, \quad x \in \Omega.
\]

If \( b(x) \equiv 0 \) then we can have \( \gamma_0 = 0 \) This guarantees the coercivity of the problem and the existence of its solution in \( H^1_0(\Omega) \).
Applications:

(1) heat and mass transfer;
(2) diffusion-reaction processes: $b = 0$;
(3) flow in porous media: $b = 0$, $c_0 = 0$;
(4) transport in porous media: $b$ - given.

Example: bioscreen model.

Partition $\Omega$ into $\Omega_1$ and $\Omega_2$ with an interface $\Gamma$.

(1) In $\Omega_1$ we can use:
(a) Mixed Method, or (b) Finite Volumes,

(2) In $\Omega_2$ we can use one of:
(a) Galerkin FEM; (b) FV; (c) Mixed FEM.
2. Composite Formulations

We shall consider as a model situation when the domain $\Omega$ is split into two subdomains with a common interface $\Gamma$, i.e. $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$. In each subdomain we shall consider a separate formulation of the original problem.

Most of the methods extend to many subdomains. However, we have not studied the dependence of the parameters in the equivalence and a priori estimates on the number of the subdomains.
2.1. Coupling Mixed and Galerkin Methods

In $\Omega_1$ we use mixed formulation, while in $\Omega_2$ we shall use standard Galerkin weak formulation.

In the mixed setting we introduce a new variable - velocity/flux: $u = -a \nabla p$. To distinguish between the problem formulations we write

$$p_1 = p|_{\Omega_1} \text{ and } p_2 = p|_{\Omega_2}.$$ 

Different smoothness requirements on the components $u$, $p_1$ and $p_2$ are imposed, namely

$$u \in H(div, \Omega_1),$$
$$p_1 \in L^2(\Omega_1),$$
$$p_2 \in H^1_0(\Omega_2, \partial \Omega_2 \setminus \Gamma) \equiv H^1_*(\Omega_2).$$
Further, we shall use the standard notations for the Sobolev spaces for $i = 1, 2$: 

$H^1_*(\Omega_i), \quad L^2(\Omega_i), \quad H(\text{div}, \Omega_1), \quad H^{1/2}_0(\Gamma), \quad H^{-1/2}(\Gamma)$.

In the inner product (or corresponding pairing) we shall skip the indexation by subdomain, i.e.

$$(v_i, w_i)_{\Omega_i} \equiv (v_i, w_i) = \int_{\Omega_i} v_i \, w_i \, dx.$$ 

And finally, by $<\cdot, \cdot>$ we shall denote the duality pairing between $H^{1/2}_0(\Gamma)$ and $H^{-1/2}(\Gamma)$ or simply the $L^2$-inner product on functions defined on $\Gamma$, i.e.

$$<f, g> = \int_{\Gamma} f \, g \, ds.$$
The weak composite formulation for the case of **pure diffusion** (i.e. $b(x) \equiv 0$ and $c_0(x) \equiv 0$), which is a basis for the finite element method, is: find

$$
\begin{align*}
\mathbf{u} \in H(div, \Omega_1), \quad & p_1 \in L^2(\Omega_1), \quad p_2 \in H^1_*(\Omega_2) \\
(a^{-1} \mathbf{u}, \mathbf{v}) - (p_1, \nabla \cdot \mathbf{v}) + \langle p_2, \mathbf{v} \cdot \mathbf{n}_1 \rangle \quad & = 0, \\
- (\nabla \cdot \mathbf{u}, \phi) \quad & = (F, \phi), \\
\langle \mathbf{u} \cdot \mathbf{n}_1, \psi \rangle \quad & - (a \nabla p_2, \nabla \psi) \quad = (F, \psi),
\end{align*}
$$

for all

$$
\mathbf{v} \in H(div, \Omega_1), \quad \phi \in L^2(\Omega_1), \quad \psi \in H^1_*(\Omega_2)
$$

**Remark 1.** Note that this weak formulation does not involve Lagrange multipliers.
Theorem 1. The following a priori estimate is valid:

\[ \|u\|_{H(div, \Omega_1)} + \|p_1\|_0, \Omega_1 + \|p_2\|_1, \Omega_2 \leq C \left\{ \|f\|_0, \Omega_1 + \|f\|_{-1}, \Omega_2 \right\}. \]

The proof uses the well-known inf–sup condition

\[ \beta \|p_1\|_0 \leq \sup_{v \in H(div, \Omega_1)} \frac{(p_1, \nabla \cdot v)}{\|v\|_{H(div, \Omega_1)}}. \]
The case of **convection-diffusion problems** is much more complicated.

To describe the weak form of the equation

\[ \nabla \cdot \mathbf{u} + \nabla \cdot (b p_1) + c_0 p_1 = f(x), \ x \in \Omega_1, \]

which is suitable for FE method, we shall need to allow for discontinuous functions \( p_1 \) from

\[ H^1_{\text{loc}}(\Omega_1) = \left\{ v_1 \in L^2(\Omega_1) : \exists \ \text{partition} \ \mathcal{K} \right. \]
\[ \left. \text{s.t.} \ v_1|_K \in H^1(K), \forall K \in \mathcal{K} \right\}. \]

The functions in \( H^1_{\text{loc}}(\Omega_1) \) have traces from both sides of the interfaces of the subdomains \( K \). For \( p_1 \in H^1_{\text{loc}}(\Omega_1) \) we denote these traces by \( p^o_1 \) and \( p^i_1 \), where "o" stands for the outward (with respect to \( K \)) trace and "i" - the interior trace.
For any $K \in \mathcal{K}$ the advection-reaction operator $C_{p_1}$ contributes:

$$C_K(p_1, w_1) = \int_K (\nabla \cdot (bp_1) + a_0(x)p_1) \, w_1 \, dx$$

$$+ \int_{\partial K} (p_1^0 - p_1^i) \, w_1^i \, b \cdot n \, ds,$$

where $\Gamma_- = \{ x \in \Gamma : b(x) \cdot n_1 < 0 \}$ is inflow,

$\Gamma_+ = \{ x \in \Gamma : b(x) \cdot n_1 \geq 0 \}$ is outflow boundary.

We integrate by parts in each $K$ and sum for $K \in \mathcal{K}$. For $p_1, w_1 \in H^1_{\text{loc}}(\Omega_1)$:

$$C(p_1, w_1) =$$

$$\sum \left( - \int_K p_1 \, b \cdot \nabla w_1 \, dx + \int_{\partial K} p_1^0 \, w_1^i \, b \cdot n \, ds \right.$$ 

$$+ \int_K a_0(x) \, p_1 \, w_1 \, dx + \int_{\partial K} p_1^i \, w_1^i \, b \cdot n \, ds \right).$$

Note, that this form is defined for both continuous and discontinuous with respect to $\mathcal{K}$ functions.
If the subdomain $K$ has a side/face on $\Gamma_-$ then the trace $p_1^o$ should be replaced it by its counterpart from $\Omega_2$, namely by $p_2(x)$. Also on $\Gamma_-$ we have $w_1^i = w_1$ and on $\partial \Omega_1^- \setminus \Gamma_-$ we take $p_1^o = 0$. We get the following weak form of the second equation for all $w_1 \in H^1_{loc}(\Omega_1)$:

$$-(\nabla \cdot u, w_1) - a_{11}(p_1, w_1) - a_{12}(p_2, w_1) = -(f, w_1), \forall \ w_1 \in W_1.$$  

Here

$$a_{11}(p_1, w_1) = - \sum_{K \in \mathcal{K}} \int_K p_1 b \cdot \nabla w_1 \, dx + (c_0 p_1, w_1)$$

$$\sum_{K \in \mathcal{K}} \int_{\partial K \setminus \partial \Omega_1^-} \left[ (b \cdot n_1)_- p_1^o + (b \cdot n_1)_+ p_1^i \right] w_1^i \, ds$$

$$a_{12}(p_2, w_1) = \int_{\Gamma_-} p_2(x) w_1(x) b \cdot n_1 \, ds,$$

where for a given function $t(x)$ we have defined $t_- = \min(0, t)$ and $t_+ = \max(0, t)$.  

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Testing the original equation by a function $w_2 \in H_0^1(\Omega_2; \partial \Omega_2 \setminus \Gamma)$, using the zero boundary condition for $w_2$ on $\partial \Omega_2 \setminus \Gamma$, and the fact $u \cdot n_1 = -a \nabla p_1 \cdot n_1 = a \nabla p_2 \cdot n_2$ on $\Gamma$ we get:

$$
< u \cdot n_1, w_2 > - (a \nabla p_2, \nabla w_2) - (\nabla \cdot (b p_2), w_2) - (c_0 p_2, w_2) = -(f, w_2),
$$

for all $w_2 \in H_0^1(\Omega_2; \partial \Omega_2 \setminus \Gamma)$. To account for the "inflow" boundary $\Gamma_+$ we need term $\int_{\Gamma_+} p_1 w_2 b \cdot n_2 \, ds$. We get it by adding it to the equation and subtracting its equal $-\int_{\Gamma_+} p_2 w_2 b \cdot n_1 \, ds$ since on $\Gamma$ we have $p_1 = p_2$.

We get the following form of the last equation:

$$
< u \cdot n_1, w_2 > - a_{21}(p_1, w_2) - a_{22}(p_2, w_2) = -(f, w_2),
$$

for all $w_2 \in H_0^1(\Omega_2)$. 

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Summarizing, we have the following composite system: find

\[ \begin{align*}
    u \in H(\text{div}, \Omega_1), \quad p_1 \in H^1_{\text{loc}}(\Omega_1), \quad p_2 \in H^1_*(\Omega_2) \\
    (a^{-1} u, v) - (p_1, \nabla \cdot v) + \left< p_2, v \cdot n_1 \right> &= 0, \\
    -(\nabla \cdot u, w_1) - a_{11}(p_1, w_1) - a_{12}(p_2, w_1) &= -(f, w_1), \\
    < u \cdot n_1, w_2 > - a_{21}(p_1, w_2) - a_{22}(p_2, w_2) &= -(f, w_2),
\end{align*} \]

for all \( v \in H(\text{div}, \Omega_1), \quad w_1 \in H^1_{\text{loc}}(\Omega_1), \quad w_2 \in H^1_*(\Omega_2) \)

where the bilinear forms are defined as:

\[ a_{11}(p_1, w_1) = \sum_{K \in \mathcal{K}} \int_{\partial K \setminus \partial \Omega_-} \left[ (b \cdot n)_- p_1^o + (b \cdot n)_+ p_1^i \right] w_1^i ds \]
\[ - \sum_{K \in \mathcal{K}} \int_K p_1 b \cdot \nabla w_1 \ dx + (c_0 p_1, w_1) \]
\[ a_{12}(p_2, w_1) = \int_{\Gamma_-} p_2 w_1 b \cdot n_1 \ ds, \]
\[ a_{21}(p_1, w_2) = \int_{\Gamma_+} p_1 w_2 b \cdot n_2 \ ds, \]
\[ a_{22}(p_2, w_2) = (a \nabla p_2, \nabla w_2) + (\nabla \cdot (bp_2), w_2) + (c_0 p_2, w_2) + \int_{\Gamma_-} p_2 w_2 b \cdot n_1 \ ds. \]
The stability of this problem is in much weaker sense. Namely, we have:

**Theorem 2.** The solution of the composite problem exists and satisfies the following a priori estimate:

\[
\|u\|_{L^2(\Omega_1)} + \|p_1\|_{*,\Omega_1} + \|p_2\|_{*,\Omega_2} \leq C\|f\|_{0,\Omega}
\]

where

\[
\|v_1\|_{*,\Omega_1}^2 + \|v_2\|_{*,\Omega_2}^2 =
\]

\[
\frac{1}{2} \sum_{e \in E_0} \int_e [v_1]^2 |b \cdot n_1| \, ds + \gamma_0 \|v_1\|_{0,\Omega_1}^2 + \frac{1}{2} \int_\Gamma (v_1 - v_2)^2 |b \cdot n_1| \, ds +
\]

\[
(\tilde{a} \nabla v_2, \nabla v_2) + \gamma_0 \|v_2\|_{0,\Omega_2}^2
\]
Take stable mixed pairs of spaces: \( V_i = H(d\text{iv}, \Omega_i), W_i = L^2(\Omega_i), i = 1,2, \) and \( \Lambda_2 = H^{1/2}_0(\Gamma). \)

The combined mixed/mixed formulation is: find

\[
\begin{align*}
\frac{\partial}{\partial t}(\mathbf{w}, \mathbf{p}_1) & \in V_1 \times W_1, \quad (\mathbf{u}_2, p_2) \in V_2 \times W_2, \quad \lambda_2 \in \Lambda_2 \\
 a_1(\mathbf{u}_1, \mathbf{v}_1) - (p_1, \nabla \cdot \mathbf{v}_1) + \langle \lambda_2, \mathbf{v}_1 \cdot \mathbf{n}_1 \rangle & = 0, \quad \mathbf{v}_1 \in V_1, \\
 - (\nabla \cdot \mathbf{u}_1, q_1) & = (f, q_1), \quad q_1 \in W_1, \\
 a_2(\mathbf{u}_2, \mathbf{v}_2) - (p_2, \nabla \cdot \mathbf{v}_2) + \langle \lambda_2, \mathbf{v}_2 \cdot \mathbf{n}_2 \rangle & = 0, \quad \mathbf{v}_2 \in V_2, \\
 - (\nabla \cdot \mathbf{u}_2, q_2) & = (f, q_2), \quad q_2 \in W_2, \\
 \langle \mathbf{u}_2 \cdot \mathbf{n}_2, \mu_2 \rangle + \langle \mathbf{u}_1 \cdot \mathbf{n}_1, \mu_2 \rangle & = 0, \quad \mu_2 \in \Lambda_2.
\end{align*}
\]

Here \( \lambda_2 \) plays a role of a Lagrange multiplier and

\[
a_i(\mathbf{u}_i, \mathbf{v}_i) = \int_{\Omega_i} a^{-1}(x) \mathbf{u}_i \cdot \mathbf{v}_i \, dx, \quad i = 1, 2.
\]
2.3. Hybrid Formulation

This formulation is good for coupling two Galerkin FEMs, Galerkin FEM and FVM, and FVM with FVMs. We show it on two subdomains (the case of many subdomains has been studied by Ewing, L. Lin, and Lin).

First we denote by

\[ H^1_*(\Omega_i) = \{ v_i \in H^1(\Omega_i) : v_i|_{\partial \Omega_i \setminus \Gamma} = 0 \}. \]

The space \( X \) and its norm are defined as:

\[ X = H^1_*(\Omega_1) \times H^1_*(\Omega_2), \quad \| v \|_X^2 = \| v \|_{1,\Omega_1}^2 + \| v \|_{1,\Omega_2}^2. \]

The space \( H^1_0(\Omega) \) can be characterized by

\[ H^1_0(\Omega) = \{ v \in X : < v_1 - v_2, \phi >_{\Gamma} = 0, \quad \forall \phi \in H^{-1/2}_{00}(\Gamma) \}. \]

Now we define the bilinear form \( a^* : X \times X \rightarrow \mathbb{R} : \)

\[ a^*(v, w) = \sum_{i=1,2} \int_{\Omega_i} a(x) \nabla v_i \cdot \nabla w_i \, dx. \]
Thus the primal hybrid formulation is: find

\[(p_1, p_2, \psi) \in \left( H^1_*(\Omega_1), H^1_*(\Omega_2), H^{-\frac{1}{2}}(\Gamma) \right) \]

such that

\[
\begin{cases}
    a^*(p, q) + \langle q_1 - q_2, \psi \rangle = (f, p), \quad \forall q \in X, \\
    \langle p_1 - p_2, \phi \rangle = 0, \quad \forall \phi \in H^{-\frac{1}{2}}(\Gamma).
\end{cases}
\]

One has the following result:

**Theorem 3.** The hybrid problem has unique solution \((p, \psi) \in X \times H^{-\frac{1}{2}}(\Gamma)\) and \(p\) is a solution to the original problem. Moreover, \(\psi = a \nabla p_i \cdot n_i, \ i = 1, 2\) and

\[
\|p\|_X + \|\psi\|_{-\frac{1}{2}, \Gamma} \leq C\|f\|_{0, \Omega}.
\]
3. Discretization Methods

We introduce independent triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of the domains $\Omega_1$ and $\Omega_2$ so they do not necessarily match along $\Gamma$ and use different method in each of them. Why blending different methods?
1. **Independent meshing** in the subdomains for practical reasons;
2. **Better or more appropriate** approximation by different methods;
3. **Necessity** of gluing together already existing implementations.

For example, **FVE method** has a large variety of up-winding and stabilizing techniques for convection-diffusion problems, while **mixed FEM** has local conservation properties and superconvergent approximation.
We take \((V, W_1)\) to be a stable pair of mixed finite element spaces: R-T, B-D-M, B-F-D-M, etc. Then the mixed FEM on \(\Omega_1\) will involve \(u_h \in V\) and \(p_{1,h} \in W_1\).

For the problem in \(\Omega_2\) we can apply:

(1) Standard Finite Element Approximation; (Wieners and Wohlmuth’98, L.P.V.’99)

(2) Finite Volume Element Approximation; (L., Pasciak, and Vassilevski’99)

The first two methods have similar formulations:

(1) The FE as Galerkin method with solution and test spaces of conforming linear elements;
(2) The FVE as Petrov-Galerkin method with a solution space of conforming linear FE and a test space of piecewise constants over a partition of the domain into finite volumes.

The mixed/mixed FE approximation requires Lagrange multipliers on the interface $\Gamma$.

All these approximations are stable in an appropriate norm and preconditioning based on Dirichlet-Neumann and Neumann-Dirichlet maps can be applied.
### 3.1. Coupling Mixed FE & Galerkin FEM

The **solution** space \( W_2 \) is the standard conforming space of piece-wise linear functions over \( \mathcal{T}_2 \). Then the approximation for the **pure diffusion** problem reads: find

\[
\begin{align*}
\mathbf{u}_h &\in \mathbf{V}, \quad p_{1,h} \in W_1, \quad \text{and} \quad p_{2,h} \in W_2 \quad \text{s.t.} \quad : \\
(a^{-1}\mathbf{u}_h, \chi) - (p_{1,h}, \nabla \cdot \chi) + <p_{2,h}, \chi \cdot n_1> &= 0, \\
-(\nabla \cdot \mathbf{u}_h, \phi) &= (F, \phi), \\
<\mathbf{u}_h \cdot n_1, \psi> - (a \nabla p_{2,h}, \nabla \psi) &= (F, \psi),
\end{align*}
\]

for all \( \chi \in \mathbf{V}, \ \phi \in W_1, \ \psi \in W_2. \)
This approximation was introduced by
- Wieners and Wohlmuth, 98,
- L., Pasciak, and Vassilevski, 99 proposed and studied several iterative methods based on:

(1) presentation of the system in block form as:

\[
A = \begin{bmatrix}
A_1 & T^T & N^T \\
-T & A_2 & 0 \\
N & 0 & 0
\end{bmatrix}, \quad AU = F, \quad U \in \mathcal{X}
\]

where \( \mathcal{X} = V_h \times W_2 \times W_1 \) and \( A : \mathcal{X} \to \mathcal{X} \) and

(2) using Poincaré-Steklov operators (Neumann-Dirichlet and Dirichlet-Neumann maps for the interface data).
3.2. Mixed FE/FV Approximation

The finite volume approximation will be understood as Petrov-Galerkin method. For this we need also the test space denoted by $W_2^*$ associated with the "triangulation" $\mathcal{T}_2^*$ of $\Omega_2$ into finite (control) volumes.

Then $W_2^*$ is the space of the piece-wise constant functions over the volumes from $\mathcal{T}_2^*$. 

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The balance equation over a volume $V(x) \in T_2^*$ is:

$$- \int_{\partial V} a \nabla v_2 \cdot n ds = \int_V f dx, \quad v_2 \in W_2.$$ 

Introduce the form $a_{2,h}(v, \psi^*)$ for $v \in W_2, \psi^* \subset W_2^*$:

$$a_{2,h}(v, \psi^*) = - \sum_x \psi^*(x) \int_{\partial V(x)} a \nabla v \cdot n ds.$$ 

Then the **MFE/FV approximation** is: find

$$u_h \in V, \quad p_{1,h} \in W_1, \quad p_{2,h} \in W_2 \quad \text{s.t.}$$

$$(a^{-1} u_h, v) - (p_{1,h}, \nabla \cdot v) + \langle I_h p_{2,h}, v \cdot n_1 \rangle = 0,$$

$$-(\nabla \cdot u_h, \phi) = (F, \phi),$$

$$\langle u_h \cdot n_1, \psi^* \rangle - a_{2,h}(p_{2,h}, \psi^*) = (F, \psi^*),$$

for all $v \in V, \phi \in W_1, \psi^* \in W_2^*$.

Here $I^*_h : W_2 \mapsto W_h^*$ is a piece-wise constant interpolation operator.
Remark. To see in what sense the coupling provides continuity of the approximate pressure and the flux at the interface we need to define these quantities on $\Gamma$.

(1) A reasonable approximation of the flux/velocity over each FE in $\Omega_2$ is $-a \nabla p_{2,h}$. Then $p_{2,h}$ satisfies:

$$< a \nabla p_{2,h} \cdot n_1, \psi^* >_{\Gamma} + a_{2,h}(p_{2,h}, \psi^*) = (f, \psi^*)$$

for $\psi^* \in W^*_2$. Thus, the continuity of the flux across $\Gamma$ is imposed in the following weak sense:

$$< (u_h + a \nabla p_{2,h}) \cdot n_1, \psi^* >_{\Gamma} = 0, \quad \psi^* \in W^*_2.$$

(2) The continuity of the pressure is more complicated. Notice that in the mixed side the pressure is discontinuous across the FE interfaces. In general, the degrees of freedom are the medicenters of the FEs. In order to compute the pressure at a FE edge we need Taylor expansion by using the approximations of $p$ and $u$. 

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Another possibility is: find

\[
\begin{align*}
\mathbf{u}_h & \in \mathbf{V}, \; p_{1,h} \in W_1, \; p_{2,h} \in W_2 \quad \text{such that} \\
(a^{-1}\mathbf{u}_h, \mathbf{v}) - (p_{1,h}, \nabla \cdot \mathbf{v}) + \langle p_{2,h}, \mathbf{v} \cdot \mathbf{n}_1 \rangle & = 0, \\
-\langle \nabla \cdot \mathbf{u}_h, \phi \rangle & = (F, \phi), \\
\langle \mathbf{u}_h \cdot \mathbf{n}_1, I_h \psi^* \rangle & - a_{2,h}(p_{2,h}, \psi^*) = (F, \psi^*), \\
\forall \mathbf{v} \in \mathbf{V}, \; \phi \in W_1, \; \psi^* \in W_2^*.
\end{align*}
\]

Here \( I_h : W_2^* \rightarrow W_h \) is a piece-wise linear interpolation operator. This is NOT locally conservative at the cells adjacent to \( \Gamma \).
Theorem 4. The presented above approximations are stable and optimal with respect to the rate of convergence and regularity of the solution. Namely, the following error estimate is valid:

\[
\|u - u_h\|_{H(div, \Omega_1)} + \|p_1 - p_{1,h}\|_{0, \Omega_1} + \\
\quad + \|p_2 - p_{2,h}\|_{1, \Omega_2} \leq \\
C \left\{ h_1 \|u\|_{1, \Omega_1} + h_1 \|p_2\|_{1, \Omega_1} + h_2 \|p_2\|_{2, \Omega_2} \right\}
\]

with a constant \(C\) independent of \(h\).
### 3.3. Mixed & Mixed Discretizations

We take stable mixed pairs of spaces \((V_1, W_1)\) and \((V_2, W_2)\) in \(\Omega_1\) and \(\Omega_2\), respectively. Next, we take \(\Lambda_2 \subset H_0^1(\Gamma)\) a conforming FE space.

Then the combined mixed/mixed FEM is: find

\[
\begin{align*}
(u_1, p_1) \in V_1 \times W_1, & \quad (u_2, p_2) \in V_2 \times W_2, & \quad \lambda_2 \in \Lambda_2 \\
a_1(u_1, v_1) - (p_1, \nabla \cdot v_1) + \lambda_2, v_1 \cdot n_1 = 0, & \quad v_1 \in V_1, & \quad (F, q_1), \quad q_1 \in W_1, \\
- (\nabla \cdot u_1, q_1) = 0, & \quad q_1 \in W_1, & \\
a_2(u_2, v_2) - (p_2, \nabla \cdot v_2) + \lambda_2, v_2 \cdot n_2 = 0, & \quad v_2 \in V_2, & \quad (F, q_2), \quad q_2 \in W_2, \\
(\nabla \cdot u_2, q_2) = 0, & \quad q_2 \in W_2, & \\
\langle u_2 \cdot n_2, \mu_2 \rangle + \langle u_1 \cdot n_1, \mu_2 \rangle = 0, & \quad \mu_2 \in \Lambda_2. &
\end{align*}
\]

Here \(\lambda_2\) plays a role of a Lagrange multiplier and

\[
a_i(u_i, v_i) = \int_{\Omega_i} a^{-1}(x) u_i \cdot v_i dx, \quad i = 1, 2.
\]
3.4. Coupling FV/FV Approximation

We need to formulate the finite element (solution) and finite volume (test) spaces for this setting. Since each domain has its own triangulation we introduce the finite element approximation $X_h$ of $X = H^1_*(\Omega_1) \times H^1_*(\Omega_2)$:

$$X_h = \{ v_i \in C(\Omega_i) \cap H^1_*(\Omega_i) : v_i|_T \in P_1(T), \forall T \in T_i \}.$$

In fact, we have that $X_h \equiv W_1 \times W_2$, where $W_i$ is the conforming finite element spaces in $\Omega_i$, $i = 1, 2$.

Next, we define the FE space for the Lagrange multipliers. We shall choose it in the simplest possible way, as the space of the traces of the functions from $W_1$ modified at the end points, i.e.

$$M_h = \left\{ \phi \in C(\Gamma) : \phi \text{ is linear on } T_1 \text{ and is a constant at end intervals} \right\}.$$
We introduce
\[ a^{FV}(p, q^*) = a_{1,h}(p_1, q_1^*) + a_{2,h}(p_2, q_2^*), \forall p \in V_h, q^* \in X_h^*. \]

The **mortar finite volume element** method is: find \( p_h = (p_{1,h}, p_{2,h}) \in X_h \) and \( \psi_h \in M_h \) s.t.

1. **locally non-conservative along** \( \Gamma \):
   \[
   a^{FV}(p_h, I_h^* q) + \langle q_1 - q_2, \psi_h \rangle = (f, I_h^* q), \quad \forall q \in X_h \\
   \langle p_{1,h} - p_{2,h}, \phi \rangle = 0, \quad \forall \phi \in M_h.
   \]

2. **locally conservative scheme**:
   \[
   a^{FV}(p_h, I_h^* q) + \langle I_h^*(q_1 - q_2), \psi_h \rangle = 0, \quad \forall q \in X_h \\
   \langle I_h^*(p_{1,h} - p_{2,h}), \phi \rangle = 0, \quad \forall \phi \in M_h.
   \]
The following result is a particular case of the many subdomains \((i > 2)\) of Ewing, L., Lin, and Lin, 99:

**Theorem 5.** If \(p_i \in H^{1+\tau_i}(\Omega_i)\), with \(0 < \tau_i \leq 1\), \(i = 1, 2\) then the solution of the mortar FVM satisfies the error estimate:

\[
\|p - p_h\|_{X} + \|\psi - \psi_h\|_{-\frac{1}{2}, \Gamma} \leq C \sum_{i=1,2} (h_i^{\tau_i} \|p_i\|_{1, \Omega_i} + h_i \|f\|_{0, \Omega_i})
\]

with a constant \(C\) independent of \(h\).

Note that this theorem guarantees the convergence of the Lagrange multipliers (which have meaning of the fluxes across \(\Gamma\)) in the negative \(1/2\)-norm.
4. Iterative solution of the composite saddle-point problem

We consider coupling of mixed and Galerkin methods. Other coupled discretizations use the same construction. Let $\mathcal{X}$ denote the product space $V_h \times W_2 \times W_1$ and consider the operator $A : \mathcal{X} \rightarrow \mathcal{X}$ given by

$$A = \begin{bmatrix}
A_1 & T^T & N^T \\
-T & A_2 & 0 \\
N & 0 & 0
\end{bmatrix}.$$

$$(A_1\chi, \theta) = (a^{-1}\chi, \theta) \quad \text{for all } \chi, \theta \in V,$$

$$(N\chi, w_1) = (N^T w_1, \chi) = -(\nabla \cdot \chi, w_1) \quad \text{for all } \chi \in V, w_1 \in W_1,$$

$$(T\chi, w_2) = (T^T w_2, \chi) = \langle w_2, \chi \cdot n_1 \rangle_{\Gamma} \quad \text{for all } \chi \in V, w_2 \in W_2,$$

$$(A_2 v_2, w_2) = a(v_2, w_2) \equiv (a \nabla v_2, \nabla w_2) \quad \text{for all } v_2, w_2 \in W_2.$$
4.1. Direct preconditioning of the saddle-point system

We take the block diagonal operator

\[ D = \begin{bmatrix}
\Lambda & 0 & 0 \\
0 & A_2 & 0 \\
0 & 0 & I 
\end{bmatrix} \]

where \((\Lambda \chi, \theta) = (a^{-1} \chi, \theta) + (\nabla \cdot \chi, \nabla \cdot \theta)\) for all \(\chi, \theta \in V\).

By the stability of the saddle-point system,

\[ c_0 \| U \|_D^2 \leq \| A U \|_D^2 = \sup_{\theta \in X} \frac{(A U, \theta)^2}{(D \theta, \theta)} \leq c_1 \| U \|_D^2, \]

for any \(U \in X\). Here \(\| \cdot \|_D = (D \cdot , \cdot)^{1/2}\) and \((\cdot , \cdot)\) denotes the inner-product in the product space \(X\).


4.2. Iterative solution of the reduced Mixed FE/FVM problem

This applies to all combinations: mixed – standard Galerkin, mixed – finite volume, and mixed – mixed. The technique is based on Dirichlet–Neumann and Neumann–Dirichlet maps.

1. Elimination Procedure: solve two independent problems;
(a) the mixed problem in $\Omega_1$:

$$
(a^{-1}u_h^0, v) - (p_{1,h}^0, \nabla \cdot v) = 0, \quad v \in V,
$$

$$
- (\nabla \cdot u_h^0, q_1) = -(f, q_1), \quad q_1 \in W_1.
$$

On $\Gamma$ we have homogeneous Dirichlet data for $p_{1,h}^0$.

(b) the standard FVM in $\Omega_2$:

$$
a_{2,h}(p_{2,h}^0, q_2) = (f, q_2), \quad q_2 \in W_2^*.
$$

Here on $\Gamma$ we have homogeneous Neumann data for $p_{2,h}^0$.

2. Iteration Procedure: the differences $\hat{u}_h = u_h - u_h^0$ and $\hat{p}_{i,h} = p_i - p_{i,h}^0$ satisfy for $v \in V$, $q_1 \in W_1$ and $q_2^* \in W_2^*$:

$$
(a^{-1}\hat{u}_h, v) - (\hat{p}_{1,h}, \nabla \cdot v) + < I_h^* \hat{p}_{2,h}, v \cdot n_1 > = - < I_h^* p_{2,h}^0, v \cdot n_1 >, 
$$

$$
(\nabla \cdot \hat{u}_h, q_1) = 0.
$$

$$
< \hat{u}_h \cdot n_1, q_2^* > + a_{2,h}(\hat{p}_{2,h}, q_2^*) = - < u_h^0 \cdot n_1, q_2^* >.
$$

(1)
Define the trace space of FE space $W_2$ on $\Gamma$: $\Lambda_2 = W_2|_{\Gamma}$.

We introduce the maps (called often Poincaré-Steklov):

1. the **Dirichlet–Neumann** map $E_1 : \Lambda_2 \mapsto V \cdot n_1|_{\Gamma}$, defined for any $\lambda_2 \in \Lambda_2$ as the normal trace $w_h[\lambda_2] \cdot n_1$ of the mixed $\Gamma \subset$ solution $w_h[\lambda_2]$, i.e. $E_1 \lambda_2 = w_h[\lambda_2] \cdot n_1|_{\Gamma}$, where

\[
(a^{-1} w_h[\lambda_2], v) - (p_{1,h}[\lambda_2], \nabla \cdot v) = - \langle I_h^* \lambda_2, v \cdot n_1 \rangle, \quad v \in V,
\]

\[
(\nabla \cdot w_h[\lambda_2], q_1) = 0, \quad q_1 \in W_1.
\]

2. the **Neumann–Dirichlet** map $S_2 : L^2(\Gamma) \mapsto \Lambda_2$ defined for any $\lambda^* \in L^2(\Gamma)$ as $S_2 \lambda^* \equiv \lambda_2 = p_{2,h}[\lambda^*]|_{\Gamma}$ obtained by the FVE method:

\[
a_{2,h}(p_{2,h}[\lambda^*], I_h^* q_2) = \langle \lambda^*, I_h^* q_2 \rangle, \quad q_2 \in W_2.
\]

In terms of the operators $E_1$ and $S_2$ one can rewrite the coupled system (1) in the form: if $\lambda^* = -u_h^0 \cdot u$, then

\[
\hat{u}_h \cdot n_1 = E_1 \lambda_2,
\]

\[
\lambda_2 = S_2(\lambda^* - \hat{u}_h \cdot n_1) = S_2(\lambda^* - E_1 \lambda_2).
\]

Then, the reduced system reads:

\[
(I + E_1 S_2) \hat{u}_h \cdot n_1 = E_1 S_2 \lambda^*.
\]

(2)
Now we define the extension \( \tilde{w}_1[\lambda^*_1] \) for any \( \lambda^*_1 \in \Lambda^*_1 = V \cdot n_1|_\Gamma \), obtained by solving the problem

\[
\begin{align*}
(a^{-1}\tilde{w}_h[\lambda^*_1], v) - (p_{1,h}[\lambda^*_1], \nabla \cdot v) &= 0, \quad v \in V, \ v \cdot n_1 = 0, \\
- (\nabla \cdot \tilde{w}_h[\lambda^*_1], q_1) &= 0, \quad q_1 \in W_1 \\
\tilde{w}_h[\lambda^*_1] \cdot n_1 &= \lambda^*_1.
\end{align*}
\]

Consider the \( H^{-1/2} \)-equivalent on \( \Lambda^*_1 \) inner product,

\[
< \mu^*_1, \lambda^*_1 >_* = (\tilde{w}_h[\lambda^*_1], \tilde{w}_h[\mu^*_1]).
\] (3)

**Theorem 6.** The reduced problem (2) is well conditioned in this inner product. Namely, the following estimates hold:

\[
< E_1 S_2 \lambda^*_1, \lambda^*_1 >_* = a_{2,h}(p_{2,h}[\lambda^*_1], I_{h}p_{2,h}[\lambda^*_1]) \geq 0
\]

and

\[
< E_1 S_2 \lambda^*_1, \mu^*_1 >_* = a_{2,h}(p_{2,h}[\lambda^*_1], I_{h}p_{2,h}[\mu^*_1]) \\
\leq C\|p_{2,h}[\lambda^*_1]\|_{1,\Omega,1} \|p_{2,h}[\mu^*_1]\|_{1,\Omega,1} \\
\leq C\|\lambda^*_1\|_{-1/2,\Gamma,1} \|\mu^*_1\|_{-1/2,\Gamma,1}.
\]

These estimates guarantee that the preconditioned MINRES in the inner product \( < \cdot, \cdot >_* \) will have optimal convergence rates.
4.3. Iterative solution of the reduced Mixed/Mixed FE problem

1. Elimination Procedure: solve two independent mixed problems;

\[ a_1(u_1^0, v_1) - (p_1^0, \nabla \cdot v_1) = 0, \quad v_1 \in V_1, \]
\[ (\nabla \cdot u_1^0, q_1) = -(f_1, q_1), \quad q_1 \in W_1. \]

\[ a_2(u_2^0, v_2) - (p_2^0, \nabla \cdot v_2) = 0, \quad v_2 \in V_2, \]
\[ -(\nabla \cdot u_2^0, q_2) = -(f_2, q_2), \quad q_2 \in W_2. \]

2. Iteration Procedure: the difference \( \hat{u}_i = u_i - u_i^0 \) and \( \hat{p}_i = p_i - p_i^0 \) and \( \lambda_2 \) satisfy the homogeneous problems,

\[ a_1(\hat{u}_1, v_1) - (\hat{p}_1, \nabla \cdot v_1) + < \lambda_2, v_1 \cdot n_1 > = 0, \quad v_1 \in V_1, \]
\[ -(\nabla \cdot \hat{u}_1, q_1) = 0, \quad q_1 \in W_1. \]  \hspace{1cm} (4)

\[ a_2(\hat{u}_2, v_2) - (\hat{p}_2, \nabla \cdot v_2) + < \lambda_2, v_2 \cdot n_2 > = 0, \quad v_2 \in V_2, \]
\[ -(\nabla \cdot \hat{u}_2, q_2) = 0, \quad q_2 \in W_2, \]
\[ < \hat{u}_2 \cdot n_2, \mu_2 > + < \hat{u}_1 \cdot n_1, \mu_2 > = < \lambda^*, \mu_2 >, \quad \mu_2 \in \Lambda_2. \]  \hspace{1cm} (5)

Here, \( \lambda^* = -u_2^0 \cdot n_2 - u_1^0 \cdot n_1. \)
Introduce:

1. **Dirichlet-Neumann** map \( E_1 : \Lambda_2 \mapsto V_1 \cdot n_1|_\Gamma, \)

\[
E_1 \lambda_2 = w_1[\lambda_2] \cdot n_1|_\Gamma
\]

defined for any \( \lambda_2 \in \Lambda_2 \) as the normal trace of the solution \( w_1[\lambda_2] \) of the mixed problem

\[
a_1(w_1[\lambda_2], v_1) - (p_1[\lambda_2], \nabla \cdot v_1) = -<\lambda_2, v_1 \cdot n_1>,
\]

\[
v_1 \in V_1,
\]

\[
-(\nabla \cdot w_1[\lambda_2], q_1) = 0, \quad q_1 \in W_1:
\]

2. **Neumann-Dirichlet** map \( S_2 : H^{-\frac{1}{2}}(\Gamma) \mapsto \Lambda_2 \)
defined for any \( \lambda^* \in H^{-\frac{1}{2}}(\Gamma) \)

\[
S_2 \lambda^* = \lambda_2[\lambda^*],
\]
as the solution \( \lambda_2[\lambda^*] \) of the hybrid mixed problem,

\[
a_2(w_2, v_2) - (p_2, \nabla \cdot v_2) + <\lambda_2[\lambda^*], v_2 \cdot n_2> = 0, \quad v_2 \in V_2,
\]

\[
-(\nabla \cdot w_2, q_2) = 0, \quad q_2 \in W_2,
\]

\[
< w_2 \cdot n_2, \mu_2 > = <\lambda^*, \mu_2>, \quad \mu_2 \in \Lambda_2.
\]
In terms of the operators $E_1$ and $S_2$ one can rewrite the coupled system (4)-(5) as follows:

$$\hat{u}_1 \cdot n_1 = E_1 \lambda_2,$$

$$\lambda_2 = S_2(\lambda^* - \hat{u}_1 \cdot n_1) = S_2(\lambda^* - E_1 \lambda_2).$$

That is, the reduced system reads:

$$(I + S_2E_1)\lambda_2 = S_2\lambda^*.$$  

(6)

Another reduction is also possible; namely, one has:

$$(I + E_1S_2)\hat{u}_1 \cdot n_1 = E_1S_2\lambda^*.$$  

(7)
Consider now the extension $\tilde{w}_1[\lambda^*_1]$ for any $\lambda^*_1 \in \Lambda^*_1 = V_1 \cdot n_1|_\Gamma$, obtained by solving the problem

\[
\begin{align*}
& a_1(\tilde{w}_1[\lambda^*_1], v_1) - (p_1[\lambda^*_1], \nabla \cdot v_1) = 0, \quad v_1 \in V_1, \\
& - (\nabla \cdot \tilde{w}_1[\lambda^*_1], q_1) = 0, \quad q_1 \in W_1 \\
& \tilde{w}_1[\lambda^*_1] \cdot n_1 = \lambda^*_1.
\end{align*}
\]

Consider the $H^{-1/2}$-equivalent inner product,

$$<\mu^*_1, \lambda^*_1>_* = a_1(\tilde{w}_1[\lambda^*_1], \tilde{w}_1[\mu^*_1]).$$

**Theorem 7.** Then the reduced problem (7) is symmetric and positive definite in the $<\cdot, \cdot>_*$-inner product. And the following spectral equivalence estimates hold:

$$<\mu^*_1, \mu^*_1>_* \leq <\mu^*_1, \mu^*_1>_* + <E_1 S_2 \mu^*_1, \mu^*_1>_* \leq (1 + \text{const}) <\mu^*_1, \mu^*_1>_*, \quad \mu^*_1 \in \Lambda^*_1.$$
Next we define the inner product,

\[ \langle \lambda_2, \mu_2 \rangle^{**} = a_2(\tilde{w}_2[\lambda_2], \tilde{w}_2[\mu_2]) \],

where for a given \( \theta_2 \in \Lambda_2 \), \( \tilde{w}_2[\theta_2] \) is the "mixed harmonic extension" of \( \theta_2 \), i.e. it is the solution of the mixed problem,

\[
\begin{align*}
    a_2(\tilde{w}_2[\theta_2], v_2) - (p_2[\theta_2], \nabla \cdot v_2) &= -\langle \theta_2, v_2 \cdot n_2 \rangle, \\
    (\nabla \cdot \tilde{w}_2[\theta_2], q_2) &= 0, \quad q_2 \in W_2.
\end{align*}
\]

**Theorem 8.** The reduced problem (6) is well-conditioned in this inner product. More specifically, the operator \( I + S_2E_1 \) is a symmetric, uniformly positive definite and uniformly bounded operator in the \( \langle \ldots \rangle^{**} \)-inner product.
5. Numerical Experiments

Test example:

- the domain is $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, where $\Omega_1 = (0,1) \times (0,1)$, $\Gamma = \{(1,y), \ 0 < y < b\}$, $b < 1$ is a given parameter, and $\Omega_2 = (1,1+b) \times (0,b)$;

- the elliptic problem in $\Omega_1$ is $-\nabla \cdot a_1 \nabla p_1 = f_1$, where the coefficient matrix

$$a_1 = \begin{bmatrix} 1 + 10x^2 + y^2 & \frac{1}{2} + x^2 + y^2 \\ \frac{1}{2} + x^2 + y^2 & 1 + x^2 + 10y^2 \end{bmatrix};$$

the exact solution is $p_1(x,y) = (1-x)^2 x (1-y) y$, hence $u = -a_1 \nabla p_1$.

- the elliptic problem in $\Omega_2$ is $-\Delta p_2 = f_2$, where the coefficient matrix is just the identity, i.e. $a_2 = I$, and the exact solution is $p_2(x,y) = 10^5(1+b-x)(x-1)^2 y (b-y)$. 
5.1. Mixed & standard Galerkin tests

We used the following solution methods:

1. the MINRES (minimum residual method) with the block-diagonal preconditioner

\[
\begin{bmatrix}
B_1 & 0 & 0 \\
0 & I & 0 \\
0 & 0 & A_2
\end{bmatrix};
\]

2. CG method applied to the reduced problem

\[(I + S_2 E_1)q_{2,h} = S_2 (u_h^0 \cdot n_1 - E_1 p_{2,h}^0).\]

The stopping criterion: until the relative residual is reduced by a factor of $10^{-6}$.
Error and iteration counts for $b = 0.55$.

<table>
<thead>
<tr>
<th>$h_1 = 1/16$</th>
<th>$h_1 = 1/32$</th>
<th>$h_1 = 1/64$</th>
<th>$h_1 = 1/128$</th>
<th>$\approx$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_2 = b/16$</td>
<td>$h_2 = b/32$</td>
<td>$h_2 = b/64$</td>
<td>$h_2 = b/128$</td>
<td></td>
</tr>
<tr>
<td>$\delta_p$</td>
<td>$3.18e-2$</td>
<td>$7.57e-3$</td>
<td>$1.83e-3$</td>
<td>$4.57e-4$</td>
</tr>
<tr>
<td>$\delta_{u_1}$</td>
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<td>$0.1343$</td>
<td>$3.27e-2$</td>
<td>$7.87e-3$</td>
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<tr>
<td>$\delta_{u_2}$</td>
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<td>$8.87e-2$</td>
<td>$2.21e-2$</td>
<td>$5.51e-3$</td>
</tr>
<tr>
<td>$\delta_{u_{int}}$</td>
<td>$0.3792$</td>
<td>$9.42e-2$</td>
<td>$2.37e-2$</td>
<td>$5.93e-3$</td>
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<tr>
<td>$\delta_{p_{2}}$</td>
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<td>$3.44e-2$</td>
<td>$7.71e-3$</td>
<td>$1.91e-3$</td>
</tr>
<tr>
<td>#</td>
<td>57</td>
<td>71</td>
<td>86</td>
<td>92</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.69</td>
<td>0.74</td>
<td>0.78</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Number of CG iterations and average reduction factors for solving the system

$$(I + S_2 E_1) q_{2,h} = rh s_{2,h}; \ b = 0.55.$$
5.2. Mixed & mixed discretization tests

The same test problem, now discretized by a hybrid mixed method. As described above one reduces the coupled system to (7), i.e.,

\[(I + E_1S_2)\hat{u}_{1,h} \cdot n_1 = E_1S_2\lambda^*,\]

where \(\lambda^* = -u_{2,h}^0 \cdot n_2 - u_{1,h}^0 \cdot n_1.\)

We write the resulting linear system in terms of the normal trace of the discrete solution, i.e., \(u_{1,h} \cdot n_1,\)

\[(I + E_1S_2)u_{1,h} \cdot n_1 = rhs_{1,h}\quad (8)\]
We used the CG method applied to (8) in the inner product $\langle \cdot, \cdot \rangle_*$ so that $I + E_1 S_2$ is symmetric and positive definite in that inner product. The stopping criterion: until relative residual is reduced by a factor $10^{-6}$.

Error and iteration counts for problem (8);

$$b = 0.55.$$

<table>
<thead>
<tr>
<th>$h_1 = 1/16$</th>
<th>$h_1 = 1/32$</th>
<th>$h_1 = 1/64$</th>
<th>$h_1 = 1/128$</th>
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<tr>
<td>$h_2 = b/16$</td>
<td>$h_2 = b/32$</td>
<td>$h_2 = b/64$</td>
<td>$h_2 = b/64$</td>
<td></td>
</tr>
<tr>
<td>$\delta_{-1/2}$</td>
<td>0.1601</td>
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<td>9.89e-3</td>
<td>2.47e-3</td>
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<tr>
<td>$\delta_0$</td>
<td>0.9016</td>
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<td>6.38e-2</td>
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<tr>
<td>$#$</td>
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<td>8</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>0.08</td>
<td>0.11</td>
<td>0.15</td>
<td>0.17</td>
</tr>
</tbody>
</table>
The second test is for the reduced problem (8) with a random $\text{rhs}_{1,h}^*$.

Number of CG iterations and average reduction factors for solving the system

$$(I + E_1S_2)\lambda_{1,h}^* = \text{rhs}_{1,h}^*; \ b = 0.55.$$