Nonlinear Electromagnetic Gyrokinetic Equations for Rotating Axisymmetric Plasmas

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Abstract

The influence of sheared equilibrium flows on the confinement properties of tokamak plasmas is a topic of much current interest. A proper theoretical foundation for the systematic kinetic analysis of this important problem has been provided here by presenting the derivation of a set of nonlinear electromagnetic gyrokinetic equations applicable to low frequency microinstabilities in a rotating axisymmetric plasma. The subsonic rotation velocity considered is in the direction of symmetry with the angular rotation frequency being a function of the equilibrium magnetic flux surface. In accordance with experimental observations, the rotation profile is chosen to scale with the ion temperature. The results obtained represent the shear flow generalization of the earlier analysis by Frieman and Chen¹ where such flows were

¹E. A. Frieman and Liu Chen, Phys. Fluids **25**, 502 (1982).

not taken into account. In order to make it readily applicable to gyrokinetic particle simulations, this set of equations is cast in a phase-space-conserving continuity equation form.

I. Introduction

A large number of the microinstabilities believed to be detrimental to energy confinement in fusion plasmas, fall in the category of low frequency ($\omega \ll \Omega_i$, with Ω_i being the ion gyrofrequency), long parallel wavelength ($k_{\parallel}\rho_i \ll 1$, with ρ_i being the ion gyroradius) and short perpendicular wavelength ($k_{\perp}\rho_i \sim 1$) fluctuations. The governing set of equations for these important instabilities were first developed by Rutherford and Frieman¹ and, independently, by Taylor and Hastie.² Later, Catto³ introduced the very convenient gyrokinetic variables, and Catto *et al.*⁴ as well as Antonsen and Lane⁵ presented the electromagnetic generalization of the linear gyrokinetic equations. The first nonlinear extension of this formalism was derived by Frieman and Chen.⁶

In all of the studies just noted, the distribution function is separated into equilibrium and perturbed parts. The equation for the perturbed distribution function is then given in terms of the equilibrium distribution function and the perturbed fields. However, gyrokinetic particle simulation methods^{7,8} often require an equation for the total distribution in terms of the perturbed fields which are in turn obtained by Maxwell's equations. This formalism, which is nonlinear by definition, was first developed by Lee⁷ for electrostatic perturbations in sheared slab geometry. Dubin *et al.*⁹ then derived an appropriate energy-conserving form of this system of

equations using perturbative Hamiltonian methods. The extension of this formalism to general geometry was subsequently carried out by Hahm.¹⁰ Electromagnetic effects in sheared slab geometry were introduced by Hahm *et al.*¹¹ and the extension of this analysis to general geometry was addressed by Brizard.¹²

An important physics feature absent in the gyrokinetic formalisms developed earlier is the possible presence of large equilibrium flows in the plasma. Toroidal flows comparable in magnitude to the ion thermal speed and scaling with the ion temperature have in fact been observed in many fusion devices,^{13,14} and the relevance of such flows to tokamak energy confinement has been discussed by several authors.¹⁵⁻¹⁷ It is thus important to properly address the issue of toroidal rotation in a gyrokinetic framework. Even though the existing gyrokinetic formalism can be easily modified to treat large *parallel* flows properly, the fact that in realistic toroidal geometry; e.g., finite aspect ratio tokamak, the parallel direction has a large *poloidal* component, invalidates simplistic assumptions regarding the equilibrium. The only realistic equilibrium flow on the time scale of interest has been shown¹⁸⁻²⁰ to be in the toroidal direction provided that the angular rotation velocity is a function of the poloidal magnetic flux. This is due to the fact that a differential rotation on a flux surface can twist the field lines and change the field strength.²¹

In this paper, a set of nonlinear gyrokinetic equations that properly treat equilibrium toroidal flows is derived. The analysis is carried out by separating the distribution function into equilibrium and perturbed parts and in that respect can be seen as the toroidal flow generalization of references 4 and 6.

The remainder of this paper is organized as follows. In Sec. II the symmetry properties of the magnetic geometry and the toroidal flow considered are discussed.

Sec. III describes the familiar gyrokinetic ordering in the presence of toroidal rotation, and the coordinate transformation to the moving frame is presented in Sec. IV. The gyrokinetic variables ε, μ as constants of motion are described in Sec. V followed by the calculation of the perturbed distribution function in Sec. VI. In Sec. VII the gyrokinetic equation is transformed to $c_{||}, \mu$ space and cast in the continuity equation form suitable for gyrokinetic particle simulation. Finally, a summary and discussion of the paper is given in Sec. VIII.

II. Geometry and Sheared Flow

The magnetic field for general axisymmetric toroidal geometry is given by:

$$\mathbf{B} = I(\psi)\nabla\zeta + \nabla\psi \times \nabla\zeta \quad (1)$$

where ψ is the poloidal flux and ζ is the toroidal angle variable. This corresponds to

$$\nabla\psi = -R^2\mathbf{B} \times \nabla\zeta = R\hat{\zeta} \times \mathbf{B} \quad (2)$$

where

$$|\nabla\psi| = RB_p, \quad |\nabla\zeta| = \frac{1}{R}, \quad \nabla\zeta \cdot \nabla\psi = 0, \quad \hat{\zeta} = R\nabla\zeta \quad (3)$$

with

$$\hat{\mathbf{n}} \cdot \nabla\psi = 0, \quad \mathbf{B}_p = \nabla\psi \times \nabla\zeta = B_p \hat{\psi} \times \hat{\zeta}, \quad \text{and} \quad \hat{\psi} \equiv \frac{\nabla\psi}{RB_p}. \quad (4)$$

The equilibrium flow considered is in the toroidal direction with the angular frequency ω_R being a function of the equilibrium magnetic flux surface. Specifically,

$$\mathbf{V} \equiv \omega_R(\psi)R\hat{\zeta} \quad (5)$$

where R is the major radius. As pointed out in a number of earlier studies,^{18,20} this form describes the only sustainable plasma equilibrium flow with a magnitude approaching the ion thermal speed. From Eqs. (1) and (5) the following useful relations are readily obtained:

$$\begin{aligned} \mathbf{V} \cdot \nabla \mathbf{V} \cdot \mathbf{V} &= 0, & \mathbf{V} \cdot \nabla Q &= 0, & \mathbf{V} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} &= -\hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{V}, \\ \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} &= 0, & \hat{\mathbf{n}} \cdot \nabla \mathbf{V} &= \mathbf{V} \cdot \nabla \hat{\mathbf{n}}, & \text{and} & \quad \nabla \cdot \mathbf{V} = 0 \end{aligned} \quad (6)$$

where Q is any scalar equilibrium quantity.

III. Ordering

To facilitate the systematic solution of the gyrokinetic equation, first recall the familiar gyrokinetic ordering parameter $\lambda \equiv (\rho_i/L)$. Here L is the shortest equilibrium scale length in the plasma, typically the minor radius in a toroidal geometry, and $\rho_i \equiv v_{ti}/\Omega_i$ is the Larmor radius of the majority ion species in the plasma where $\Omega_i \equiv qB/mc$ is the Larmor frequency. For any perturbed quantity, δQ , the appropriate ordering is

$$\frac{|\hat{\mathbf{n}} \cdot \nabla \delta Q|}{|\nabla_{\perp} \delta Q|} \sim \frac{|(\partial/\partial t + \mathbf{V} \cdot \nabla) \delta Q|}{|\Omega_i \delta Q|} \sim \mathcal{O}(\lambda). \quad (7)$$

Note that this ordering for $(\partial/\partial t + \mathbf{V} \cdot \nabla)$ assures that time variations “in the rotating frame” are much slower than the gyromotion. However, this does not necessarily imply that $\partial/\partial t$ and $\mathbf{V} \cdot \nabla$ individually have to be small with respect to the gyrofrequency. Later in the analysis it will be obvious that these two derivatives always

appear together and can be viewed as

$$\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \sim \frac{\partial}{\partial t_R} \quad (8)$$

where t_R would correspond to the time variable in the rotating frame. Although it may seem that a transformation to the rotating frame in the form $\mathbf{x}_{\text{rot}} = \mathbf{x}_{\text{lab}} - \mathbf{V}t$ and $\mathbf{v}_{\text{rot}} = \mathbf{v}_{\text{lab}} - \mathbf{V}$ would be an appropriate choice, we point out that this would unnecessarily complicate the analysis since $\mathbf{V}$ is a function of $\mathbf{x}$. Even in the simpler case of rigid rotation; i.e., $\omega_R = \text{const.}$, this would be the case since the rotating frame is not inertial. The appropriate coordinate transformation chosen is presented in the next section.

IV. Coordinate transformation

The familiar Vlasov equation in the lab frame is given by:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m} \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad (9)$$

where f is the total distribution function. Following Catto *et al.*,²⁰ we transform to a coordinate frame that is shifted by $\mathbf{V}$ in velocity space but unchanged in configuration space. Specifically, the transformation $\{t, \mathbf{x}, \mathbf{v}\} \rightarrow \{t', \mathbf{x}', \mathbf{c}\}$ is given by:

$$t' = t, \quad \mathbf{c} = \mathbf{v} - \mathbf{V}, \quad \text{and} \quad \mathbf{x}' = \mathbf{x} \quad (10)$$

with the derivatives being expressed as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial \mathbf{x}} = -\nabla \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{c}} + \frac{\partial}{\partial \mathbf{x}'}, \quad \text{and} \quad \frac{\partial}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{c}}. \quad (11)$$

Since the distinction between the primed and unprimed variable is only relevant for the partial derivatives, the superscript can be dropped from t' and $\mathbf{x}'$. The Vlasov equation in the “rotating frame” becomes:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{c} + \mathbf{V}) \cdot \frac{\partial f}{\partial \mathbf{x}} - (\mathbf{c} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \frac{\partial f}{\partial \mathbf{c}} + \frac{q}{m} \left(\mathbf{E}' + \frac{\mathbf{c} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f}{\partial \mathbf{c}} = 0 \quad (12)$$

where $\mathbf{E}' \equiv \mathbf{E} + (\mathbf{V} \times \mathbf{B})/c$ is the electric field observed in the rotating coordinate frame. The requisite electrostatic potential Φ_{-1} for the lowest order force balance is then given by:

$$\nabla \Phi_{-1} = \frac{\mathbf{V} \times \mathbf{B}}{c},$$

and the subscript (-1) refers to the ordering of this potential with respect to temperature; i.e.,

$$\frac{|q\Phi_{-1}|}{T} \sim \mathcal{O}(\lambda^{-1}).$$

Note that in the expression for Φ_{-1} it is usually not necessary to include the pressure gradient term $-\nabla P_i$ since it would formally be $\mathcal{O}(1)$ in the core region of the tokamak.

V. Gyrokinetic Variables

In order to facilitate the analysis, it is desirable to make a gyrokinetic change of variables from $\{t, \mathbf{x}, \mathbf{c}\}$ to $\{t, \varepsilon, \mu, \mathbf{R}, \alpha\}$. Here, ε corresponds to particle energy in the moving frame, μ is the magnetic moment, $\mathbf{R}$ is the guiding center variable, and α is the gyrophase angle. We define $B \equiv |\mathbf{B}|$, $\hat{\mathbf{n}} \equiv \mathbf{B}/B$, $\Omega \equiv qB/mc$, $c_{\parallel} \equiv \hat{\mathbf{n}} \cdot \mathbf{c}$, $\mathbf{c}_{\perp} \equiv (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}) \cdot \mathbf{c}$, and

$$\mathbf{c} \equiv c_{\parallel} \hat{\mathbf{n}} + \mathbf{c}_{\perp} (\hat{\mathbf{e}}_1 \cos \alpha + \hat{\mathbf{e}}_2 \sin \alpha) \quad (13)$$

where q , and m are particle charge and particle mass respectively. Throughout the text, $\mathbf{c}$, $c_{\parallel}$, and $c_{\perp}$ will be used to refer to the velocity variable, and c will be reserved to refer to the speed of light to avoid confusion. The unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are arbitrary and satisfy $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 \equiv \hat{\mathbf{n}}$, and $\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 \equiv 0$.

A. Energy

For the energy ε to be a constant of motion it is of course required that

$$\frac{d\varepsilon}{dt} = 0. \quad (14)$$

Seeking a solution order by order, we can take $\varepsilon = \varepsilon_0 + \varepsilon_1 + \dots$ where $\varepsilon_1/\varepsilon_0 \sim \mathcal{O}(\lambda)$.

Using the total time derivative in the form

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\mathbf{c} + \mathbf{V}) \cdot \frac{\partial}{\partial \mathbf{x}} - (\mathbf{c} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{c}} + \frac{q}{m} \left(\mathbf{E}' + \frac{\mathbf{c} \times \mathbf{B}}{c} \right) \cdot \frac{\partial}{\partial \mathbf{c}}, \quad (15)$$

and defining the lowest order energy as

$$\varepsilon_0 \equiv \frac{c_{\parallel}^2}{2} + \frac{c_{\perp}^2}{2} - \frac{V^2}{2} + \frac{q}{m} \Phi_0, \quad (16)$$

Eq. (14) is satisfied to lowest order; i.e.,

$$\frac{d\varepsilon_0}{dt} \sim \lambda \frac{\varepsilon_0}{\Omega_i}.$$

In Eq. (16), the first term is the kinetic energy in the rotating frame, the second term is the centrifugal potential energy and the last term is the electrostatic potential energy. The lowest order electrostatic potential in the lab frame, Φ_{-1} , does not enter this expression since it was transformed away to satisfy the force balance. In previous work²⁰ it was shown that Φ_0 is a poloidally varying potential set up by the

electrons to achieve quasineutrality by compensating for the ions that are pushed outward by the large centrifugal force they experience due to their mass. As a crude approximation this potential will scale as:

$$|q\Phi_0| \sim \frac{M_i V^2}{2} \sim T_i \quad (17)$$

provided that the rotation velocity can be comparable to the ion thermal speed. The drift velocity due to Φ_0 will then be small compared to the thermal speed, i.e.;

$$\frac{|c \nabla \Phi_0 \times \hat{n} / B|}{|v_{th_i}|} \sim \mathcal{O}(\lambda). \quad (18)$$

This is also in line with the assumptions regarding the equilibrium electrostatic potential Φ_0 in earlier derivations.³⁻⁶

To next order, $\dot{\epsilon} = 0$ yields:

$$-(\mathbf{c} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \mathbf{V} - (\mathbf{c} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \mathbf{c} - \Omega \frac{\partial \epsilon_1}{\partial \alpha} = 0 \quad (19)$$

where use has been made of the relation

$$\frac{q}{m} \frac{\mathbf{c} \times \mathbf{B}}{c} \frac{\partial}{\partial \mathbf{c}} = -\Omega \frac{\partial}{\partial \alpha}.$$

The first order energy ϵ_1 can now be readily evaluated from Eq. (19) by integrating over α . This yields:

$$\begin{aligned} \epsilon_1 = & -\frac{1}{\Omega} \left[(c_{\parallel} \hat{n} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \mathbf{c}_{\perp} \times \hat{n} + \mathbf{c}_{\perp} \times \hat{n} \cdot \nabla \mathbf{V} \cdot (c_{\parallel} \hat{n} + \mathbf{V}) \right. \\ & \left. + \frac{1}{4} (\mathbf{c}_{\perp} \times \hat{n} \cdot \nabla \mathbf{V} \cdot \mathbf{c}_{\perp} + \mathbf{c}_{\perp} \cdot \nabla \mathbf{V} \cdot \mathbf{c}_{\perp} \times \hat{n}) \right] + \langle \epsilon_1 \rangle_{\alpha} \end{aligned} \quad (20)$$

Although $\langle \epsilon_1 \rangle_{\alpha}$, the gyrophase independent part of ϵ_1 , is formally of the same order as the gyrophase dependent part, it is clear that as in previous calculations^{4,6} it is not necessary to explicitly evaluate this term when obtaining the lowest order perturbed distribution function.

B. Magnetic Moment

Similar to the energy variable, one can show that the magnetic moment variable μ_0 defined as

$$\mu_0 \equiv \frac{1}{B} \frac{c_\perp^2}{2}. \quad (21)$$

satisfies $\dot{\mu} = 0$ to lowest order. For μ_1 , the $\mathcal{O}(\lambda)$ correction to the magnetic moment, the constraint $\dot{\mu} = 0$ correct to $\mathcal{O}(\lambda^2)$ then yields:

$$\begin{aligned} & -\frac{1}{B} \frac{c_\perp^2}{2} (\mathbf{c} + \mathbf{V}) \cdot \nabla \ln B - \frac{c_\parallel}{B} (\mathbf{c} + \mathbf{V}) \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c}_\perp \\ & - \frac{1}{B} (\mathbf{c} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \mathbf{c}_\perp - \frac{q}{m} \frac{1}{B} \nabla \Phi_0 \cdot \mathbf{c}_\perp - \Omega \frac{\partial \mu_1}{\partial \alpha} = 0. \end{aligned} \quad (22)$$

It is straightforward now to obtain μ_1 through integration of Eq. (22) in α , the gyrophase angle:

$$\begin{aligned} B\mu_1 &= B \langle \mu_1 \rangle_\alpha - \left[\mathbf{c}_\perp \cdot \mathbf{c}_D + \frac{1}{4\Omega} (\mathbf{c}_\perp \cdot \nabla \mathbf{V} \cdot \mathbf{c}_\perp \times \hat{\mathbf{n}} + \mathbf{c}_\perp \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{c}_\perp) \right. \\ & \left. + \frac{c_\parallel}{4\Omega} (\mathbf{c}_\perp \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c}_\perp \times \hat{\mathbf{n}} + \mathbf{c}_\perp \times \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c}_\perp) \right] \end{aligned} \quad (23)$$

where

$$\mathbf{c}_D = \frac{\hat{\mathbf{n}}}{\Omega} \times \left[\frac{q}{m} \nabla \Phi_0 + \frac{c_\perp^2}{2} \nabla \ln B + (c_\parallel \hat{\mathbf{n}} + \mathbf{V}) \cdot (\nabla \mathbf{V} + c_\parallel \nabla \hat{\mathbf{n}}) \right]. \quad (24)$$

In the expression for $\mathbf{c}_D$ the first term is the $\mathbf{E} \times \mathbf{B}$ drift, and the second term is the ∇B drift. Using the properties of $\mathbf{V}$ one can show that $\mathbf{V} \cdot \nabla \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \nabla \mathbf{V}$, and the third term can accordingly be written as:

$$(c_\parallel \hat{\mathbf{n}} + \mathbf{V}) \cdot (\nabla \mathbf{V} + c_\parallel \nabla \hat{\mathbf{n}}) = c_\parallel^2 \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} + 2c_\parallel \hat{\mathbf{n}} \cdot \nabla \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V}. \quad (25)$$

Note that the first term in Eq. (25) is the familiar curvature drift, the second term is the Coriolis drift, and the last term is the centrifugal drift.

C. Guiding Center Coordinate

The “guiding center” coordinate $\mathbf{R}$ is defined as:

$$\mathbf{R} \equiv \mathbf{x} + \frac{\mathbf{c} \times \hat{\mathbf{n}}}{\Omega}. \quad (26)$$

Using the expression for the total time derivative given by Eq. (15) leads to:

$$\dot{\mathbf{R}} = (\mathbf{c} + \mathbf{V}) \cdot \frac{\partial \mathbf{R}}{\partial \mathbf{x}} - (\mathbf{c} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \frac{\partial \mathbf{R}}{\partial \mathbf{c}} + \frac{q}{m} \left(-\nabla \Phi_0 + \frac{\mathbf{c} \times \mathbf{B}}{c} \right) \cdot \frac{\partial \mathbf{R}}{\partial \mathbf{c}}. \quad (27)$$

Carrying out the gyrophase averaging over α then yields:

$$\langle \dot{\mathbf{R}} \rangle_\alpha = c_\parallel \hat{\mathbf{n}} + \mathbf{V} + \mathbf{c}_D \quad (28)$$

where $\mathbf{V}$ and $c_\parallel \hat{\mathbf{n}}$ are evaluated at the guiding center $\mathbf{R}$.

The gyrophase angle α is the fastest varying coordinate of the particle. To lowest order, the total time derivative of α along the unperturbed particle orbit is given by:

$$\frac{d\alpha}{dt} = -\Omega + \mathcal{O}(\lambda). \quad (29)$$

Although it is straightforward to calculate higher order corrections to $\dot{\alpha}$, it is unnecessary to do so (as will be evident in the course of the present analysis).

D. Equilibrium

Denoting the distribution function at equilibrium by $F \equiv f_{\text{eq}}$ and using the properties of the gyrokinetic variables introduced in the preceding section, the Vlasov operator acting on this function can be expressed as:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \dot{\epsilon} \frac{\partial F}{\partial \epsilon} + \dot{\mu} \frac{\partial F}{\partial \mu} + \dot{\alpha} \frac{\partial F}{\partial \alpha} + \dot{\mathbf{R}} \cdot \frac{\partial F}{\partial \mathbf{R}}. \quad (30)$$

Since $\dot{\varepsilon}$ and $\dot{\mu}$ are required to vanish to all orders and since $\partial/\partial t$ operating on equilibrium quantities is also zero, the significant terms involve $\dot{\mathbf{R}}$ and $\dot{\alpha}$. Note that to lowest order

$$-\Omega \frac{\partial F}{\partial \alpha} = 0 \quad (31)$$

which can be satisfied if F is independent of the gyrophase angle. Using this property of F , the gyrophase average Eq. (30) yields:

$$\langle \dot{\mathbf{R}} \rangle_\alpha \cdot \frac{\partial F}{\partial \mathbf{R}} = 0. \quad (32)$$

Since $\mathbf{V} \cdot \nabla$ acting on equilibrium quantities is zero, the largest term in Eq. (32) is

$$c_{\parallel} \hat{\mathbf{n}} \cdot \frac{\partial F}{\partial \mathbf{R}}$$

which is $\mathcal{O}(\lambda)$. Thus,

$$F = F(\varepsilon, \mu, \mathbf{R}_{\perp}), \quad (33)$$

will satisfy $\dot{F} = 0$ to $\mathcal{O}(\lambda^2)$. This is sufficient to calculate the perturbed distribution function to lowest order.

VI. Perturbed Distribution Function

We define the perturbed distribution function δf via $f = F + \delta f$, with F being the equilibrium distribution function whose properties are given by Eq. (33), and δf is ordered as

$$\frac{\delta f}{F} \sim \mathcal{O}(\lambda).$$

The Vlasov equation for perturbed quantities is:

$$\frac{d\delta f}{dt} = \frac{q}{m} \left(\nabla \Phi + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c} (\mathbf{c} + \mathbf{V}) \times (\nabla \times \mathbf{A}) \right) \cdot \frac{\partial}{\partial \mathbf{c}} (F + \delta f) \quad (34)$$

where Φ and $\mathbf{A}$ are the perturbed electrostatic and vector potentials respectively. We adopt the familiar ordering $|\Phi| \sim |v_t \mathbf{A}/c|$ and $|q\Phi/T| \sim \mathcal{O}(\lambda)$. Using the fact that the equilibrium distribution function is independent of the gyrophase angle α , the velocity derivative on F in Eq. (34) can be expressed as:

$$\frac{\partial F}{\partial \mathbf{c}} \equiv \frac{\partial \mu}{\partial \mathbf{c}} \frac{\partial F}{\partial \mu} + \frac{\partial \varepsilon}{\partial \mathbf{c}} \frac{\partial F}{\partial \varepsilon} + \frac{\partial \mathbf{R}}{\partial \mathbf{c}} \cdot \frac{\partial F}{\partial \mathbf{R}} \quad (35)$$

Then, defining δh as the non-adiabatic portion of δf via

$$\delta f = \delta h + \frac{q}{m} \left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} \right) \frac{\partial F}{\partial \varepsilon} + \frac{q}{m} \frac{1}{B} \frac{\partial F}{\partial \mu} \left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} - \frac{c_{\parallel} A_{\parallel}}{c} \right) + \frac{\mathbf{A} \times \hat{\mathbf{n}}}{B} \cdot \frac{\partial F}{\partial \mathbf{R}}, \quad (36)$$

Eq. (34) becomes:

$$\begin{aligned} & \frac{d\delta h}{dt} + \frac{d}{dt} \left[\frac{q}{m} \left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} \right) \frac{\partial F}{\partial \varepsilon} \right] + \frac{d}{dt} \left[\frac{q}{m} \frac{1}{B} \frac{\partial F}{\partial \mu} \left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} - \frac{c_{\parallel} A_{\parallel}}{c} \right) \right] \\ & + \frac{d}{dt} \left[\frac{\mathbf{A} \times \hat{\mathbf{n}}}{B} \cdot \frac{\partial F}{\partial \mathbf{R}} \right] = \\ & \frac{q}{m} \left(\nabla \Phi + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c} (\mathbf{c} + \mathbf{V}) \times (\nabla \times \mathbf{A}) \right) \cdot \left[\frac{\partial \mu}{\partial \mathbf{c}} \frac{\partial F}{\partial \mu} + \frac{\partial \varepsilon}{\partial \mathbf{c}} \frac{\partial F}{\partial \varepsilon} + \frac{\partial \mathbf{R}}{\partial \mathbf{c}} \cdot \frac{\partial F}{\partial \mathbf{R}} \right] + R_{\text{nl}}. \end{aligned} \quad (37)$$

Here R_{nl} is the nonlinear term given by:

$$R_{\text{nl}} = -\frac{q}{m} \left(\nabla \Phi + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c} (\mathbf{c} + \mathbf{V}) \times (\nabla \times \mathbf{A}) \right) \cdot \frac{\partial \delta f}{\partial \mathbf{c}}. \quad (38)$$

Using the equilibrium condition

$$\frac{dF}{dt} = \mathcal{O}(\lambda^2) \quad \text{with} \quad \dot{\varepsilon} = 0 \quad \text{and} \quad \dot{\mu} = 0,$$

yields

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \varepsilon} \right) \sim \frac{d}{dt} \left(\frac{\partial F}{\partial \mu} \right) \sim \frac{d}{dt} \left(\frac{\partial F}{\partial \mathbf{R}} \right) \sim \mathcal{O}(\lambda^2).$$

Hence, for the terms in Eq. (38) we obtain:

$$\frac{d}{dt} \left[\left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} \right) \frac{\partial F}{\partial \varepsilon} \right] = \frac{\partial F}{\partial \varepsilon} \left(\frac{\partial}{\partial t} + (\mathbf{c} + \mathbf{V}) \cdot \nabla \right) \left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} \right), \quad (39)$$

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{B} \frac{\partial F}{\partial \mu} \left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} - \frac{c_{\parallel} A_{\parallel}}{c} \right) \right] = \\ \frac{1}{B} \frac{\partial F}{\partial \mu} \left[\left(\frac{\partial}{\partial t} + (\mathbf{c} + \mathbf{V}) \cdot \nabla \right) \left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} - \frac{c_{\parallel} A_{\parallel}}{c} \right) \right. \\ \left. - \left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} - \frac{c_{\parallel} A_{\parallel}}{c} \right) \mathbf{c} \cdot \nabla \ln B + \frac{q}{m} \frac{A_{\parallel}}{c} \hat{\mathbf{n}} \cdot \nabla \Phi_0 + \frac{A_{\parallel}}{c} (\mathbf{c} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} \right], \end{aligned} \quad (40)$$

and

$$\frac{d}{dt} \left[\frac{\mathbf{A} \times \hat{\mathbf{n}}}{B} \cdot \nabla_R F \right] = \frac{1}{B} \left[\frac{\partial \mathbf{A}}{\partial t} \times \hat{\mathbf{n}} + (\mathbf{c} + \mathbf{V}) \cdot \nabla (\mathbf{A} \times \hat{\mathbf{n}}) - \mathbf{A} \times \hat{\mathbf{n}} \mathbf{c} \cdot \nabla \ln B \right] \cdot \frac{\partial F}{\partial \mathbf{R}}. \quad (41)$$

Eq. (38) can then be expressed as:

$$\frac{d\delta h}{dt} + \frac{q}{m} \frac{\partial F}{\partial \varepsilon} G_{\varepsilon} + \frac{q}{m} \frac{\partial F}{\partial \mu} G_{\mu} + \mathbf{G}_R \cdot \frac{\partial F}{\partial \mathbf{R}} + R_{\text{nl}} = 0 \quad (42)$$

where G_{ε} , G_{μ} , and G_R are:

$$\begin{aligned} G_{\varepsilon} &= \frac{d}{dt} \left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} \right) - \left(\nabla \Phi + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c} (\mathbf{c} + \mathbf{V}) \times (\nabla \times \mathbf{A}) \right) \cdot \frac{\partial \varepsilon}{\partial \mathbf{c}} \\ G_{\mu} &= \frac{d}{dt} \left[\frac{1}{B} \left(\Phi - \frac{\mathbf{V} \cdot \mathbf{A}}{c} - \frac{A_{\parallel} c_{\parallel}}{c} \right) \right] - \left[\nabla \Phi + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c} (\mathbf{c} \times \mathbf{V}) \times (\nabla \times \mathbf{A}) \right] \cdot \frac{\partial \mu}{\partial \mathbf{c}} \\ \mathbf{G}_R &= \frac{d}{dt} \left[\frac{\mathbf{A} \times \hat{\mathbf{n}}}{B} \right] - \frac{q}{m} \left[\nabla \Phi + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c} (\mathbf{c} + \mathbf{V}) \times (\nabla \times \mathbf{A}) \right] \times \frac{\hat{\mathbf{n}}}{\Omega} \end{aligned} \quad (43)$$

It is important to observe that these coefficients are $\mathcal{O}(\lambda)$ along with the non-linear term R_{nl} . Correct to $\mathcal{O}(\lambda)$, G_{ε} is then given by

$$G_{\varepsilon}^1 = \left[\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right] \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{\mathbf{A} \cdot \mathbf{c}}{c} \right] - \mathbf{c} \cdot \nabla \mathbf{V} \cdot \mathbf{A}$$

$$- \left(\nabla_{\perp} \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{\mathbf{A} \cdot \mathbf{c}}{c} \right] + \frac{1}{c} \mathbf{c}_{\perp} \cdot \nabla \mathbf{A} \right) \cdot \frac{\partial \varepsilon_1}{\partial \mathbf{c}}. \quad (44)$$

Similarly for G_{μ} we obtain:

$$\begin{aligned} BG_{\mu}^1 = & \left(\frac{\partial}{\partial t} + c_{\parallel} \hat{\mathbf{n}} \cdot \nabla + \mathbf{V} \cdot \nabla \right) \left(\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{A_{\parallel} \cdot \mathbf{c}}{c} \right) \\ & - \left(\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{A_{\parallel} c_{\parallel}}{c} \right) \mathbf{c} \cdot \nabla \ln B \\ & - \frac{c_{\parallel}}{c} \mathbf{c} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{A} - \frac{A_{\parallel}}{c} \mathbf{c} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c} - \frac{\mathbf{c} \cdot \nabla \mathbf{V} \cdot \mathbf{A}}{c} + \frac{A_{\parallel}}{c} \mathbf{V} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} \\ & - \frac{A_{\parallel}}{c} \mathbf{V} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c} + \frac{A_{\parallel}}{c} \mathbf{c} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} + \frac{q}{m} \frac{A_{\parallel}}{c} \hat{\mathbf{n}} \cdot \nabla \Phi_0 \\ & - \left[\nabla_{\perp} \left(\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{\mathbf{A} \cdot \mathbf{c}}{c} \right) + \frac{1}{c} \mathbf{c}_{\perp} \cdot \nabla \mathbf{A} \right] \cdot \frac{\partial \mu_1}{\partial \mathbf{c}}. \end{aligned} \quad (45)$$

Finally, $\mathbf{G}_R$ to $\mathcal{O}(\lambda^3)$ can be expressed as:

$$\mathbf{G}_R^1 = -\frac{c}{B} \left[\nabla_{\perp} \left(\Phi - \frac{\mathbf{A} \cdot \mathbf{c}}{c} - \frac{\mathbf{A} \cdot \mathbf{V}}{c} \right) \times \hat{\mathbf{n}} \right]. \quad (46)$$

With regard to the first term in Eq. (38), the total time derivative of δh along unperturbed orbits is

$$\frac{d\delta h}{dt} = \frac{\partial \delta h}{\partial t} + \dot{\mathbf{R}} \cdot \frac{\partial \delta h}{\partial \mathbf{R}} + \dot{\alpha} \frac{\partial \delta h}{\partial \alpha} \quad (47)$$

where $\dot{\varepsilon} = \dot{\mu} = 0$ is used. To lowest order $\dot{\alpha} \sim -\Omega$ such that

$$-\Omega \frac{\partial \delta h}{\partial \alpha} = 0 \quad (48)$$

must be satisfied. This can only be true if δh is gyrophase independent. Hence, by taking the gyrophase average of Eq. (42) we obtain a simplified equation for δh in terms of the equilibrium distribution and the perturbed fields. Correct to $\mathcal{O}(\lambda^2)$

Eq. (38) is written as:

$$\left[\frac{\partial}{\partial t} + \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}} \right] \delta h + \frac{q}{m} \frac{\partial F}{\partial E} G_E^1 + \frac{q}{m} \frac{\partial F}{\partial \mu} G_\mu^1 + \mathbf{G}_R \cdot \frac{\partial F}{\partial \mathbf{R}} + R_{nl} = 0. \quad (49)$$

The gyrophase average of this equation then yields:

$$\left[\frac{\partial}{\partial t} + \langle \dot{\mathbf{R}} \rangle_\alpha \cdot \nabla_R \right] \delta h + \frac{q}{m} \frac{\partial F}{\partial E} \langle G_E^1 \rangle_\alpha + \frac{q}{m} \frac{\partial F}{\partial \mu} \langle G_\mu^1 \rangle_\alpha + \langle \mathbf{G}_R \rangle_\alpha \cdot \nabla_R F + \langle R_{nl} \rangle_\alpha = 0. \quad (50)$$

Using the gyrophase averages of the various terms in Eq. (50), which are derived in App. A, Eq. (50) becomes:

$$\begin{aligned} \left[\frac{\partial}{\partial t} + (c_\parallel \hat{\mathbf{n}} + \mathbf{V} + \mathbf{c}_D) \cdot \frac{\partial}{\partial \mathbf{R}} \right] \delta h = & -\frac{q}{m} \frac{\partial F}{\partial E} \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{R}} \right) \langle \Psi \rangle_\alpha \\ & - \frac{q}{m} \left[\frac{\partial F}{\partial E} + \frac{1}{B} \frac{\partial F}{\partial \mu} \right] \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \left\langle \frac{A_\parallel \mathbf{c}_\perp}{c} - \frac{\mathbf{A} c_\parallel}{c} \right\rangle_\alpha \\ & + \frac{q}{m\Omega} \frac{\partial F}{\partial E} \hat{\mathbf{n}} \times \frac{\partial \langle \Psi \rangle_\alpha}{\partial \mathbf{R}} \cdot \{ (c_\parallel \hat{\mathbf{n}} + \mathbf{V}) \cdot \nabla \mathbf{V} + \nabla \mathbf{V} \cdot (c_\parallel \hat{\mathbf{n}} + \mathbf{V}) \} - \frac{q}{m\Omega} \hat{\mathbf{n}} \times \frac{\partial \langle \Psi \rangle_\alpha}{\partial \mathbf{R}} \cdot \frac{\partial F}{\partial \mathbf{R}} \\ & + \frac{\partial F}{\partial \mu} \left\{ \left(\frac{\partial}{\partial t} + (c_\parallel \hat{\mathbf{n}} + \mathbf{V} + \mathbf{c}_D) \cdot \frac{\partial}{\partial \mathbf{R}} \right) \frac{\langle \Psi \rangle_\alpha}{B} \right\} - \langle R_{nl} \rangle_\alpha \end{aligned} \quad (51)$$

where

$$\Psi \equiv \Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{\mathbf{A} \cdot \mathbf{c}}{c} \quad (52)$$

with its gyrophase average given by:

$$\langle \Psi \rangle_\alpha = J_0 \left(\tilde{\Phi} - \frac{\tilde{\mathbf{A}} \cdot \mathbf{V}}{c} - \frac{\tilde{\mathbf{A}}_\parallel c_\parallel}{c} \right) + J_1 \frac{c_\perp}{k_\perp} \frac{\tilde{B}_\parallel}{c}. \quad (53)$$

Recognizing the total derivative in Eq. (51), it is convenient to define:

$$\delta h \equiv -\frac{q}{mB} \frac{\partial F}{\partial \mu} \langle \Psi \rangle_\alpha + \delta g. \quad (54)$$

Note that the gyrophase average of the nonlinear term yields:

$$\langle R_{nl} \rangle_\alpha = \frac{q}{m\Omega} \hat{\mathbf{n}} \times \frac{\partial \langle \Psi \rangle_\alpha}{\partial \mathbf{R}} \cdot \frac{\partial \delta g}{\partial \mathbf{R}}. \quad (55)$$

Finally, collecting the results from the steps just described produces the gyrokinetic equation in the form:

$$\begin{aligned} \left[\frac{\partial}{\partial t} + (c_\parallel \hat{\mathbf{n}} + \mathbf{V} + \mathbf{c}_D) \cdot \frac{\partial}{\partial \mathbf{R}} \right] \delta g = & -\frac{q}{m} \frac{\partial F}{\partial E} \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{R}} \right) \langle \Psi \rangle_\alpha \quad (56) \\ & -\frac{q}{m} \left[\frac{\partial F}{\partial E} + \frac{1}{B} \frac{\partial F}{\partial \mu} \right] \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \left\langle \frac{A_\parallel c_\perp}{c} - \frac{\mathbf{A} c_\parallel}{c} \right\rangle_\alpha \\ & + \frac{q}{m\Omega} \frac{\partial F}{\partial E} \hat{\mathbf{n}} \times \frac{\partial \langle \Psi \rangle_\alpha}{\partial \mathbf{R}} \cdot ((c_\parallel \hat{\mathbf{n}} + \mathbf{V}) \cdot \nabla \mathbf{V} + \nabla \mathbf{V} \cdot (c_\parallel \hat{\mathbf{n}} + \mathbf{V})) \\ & - \frac{q}{m\Omega} \hat{\mathbf{n}} \times \frac{\partial \langle \Psi \rangle_\alpha}{\partial \mathbf{R}} \cdot \frac{\partial F}{\partial \mathbf{R}} - \frac{q}{m\Omega} \hat{\mathbf{n}} \times \frac{\partial \langle \Psi \rangle_\alpha}{\partial \mathbf{R}} \cdot \frac{\partial \delta g}{\partial \mathbf{R}}. \end{aligned}$$

Here, the term on the left hand side is the linear propagator. The first term on the right is the adiabatic heating term, the second term is associated with the perturbations to the energy and magnetic moment, the third term comes from the free energy due to sheared flows, the fourth term is the familiar linear drive due to radial equilibrium gradients, and the last term is the usual $\mathbf{E} \times \mathbf{B}$ and magnetic flutter nonlinearity in covariant form.

VII. Transformation to $(c_\parallel, \mu)$ Coordinates

The equation derived in the preceding section [i.e., Eq. (56)] is not readily adaptable to most present day gyrokinetic particle simulations. In these simulations the prescription for the distribution function usually involve the coordinates $(v_\parallel, \mu, \mathbf{R})$ in contrast to $(E, \mu, \mathbf{R})$. In our formalism this would correspond to the $(c_\parallel, \mu, \mathbf{R})$

coordinates. In particular, the transformation is given by:

$$t' = t, \quad c_{\parallel} = \sqrt{2(\varepsilon - \mu B - \frac{q}{m}\Phi_0 + \frac{V^2}{2})},$$

$$\mu' = \mu, \quad \mathbf{R}' = \mathbf{R}, \quad \text{and} \quad \alpha' = \alpha,$$

and the partial derivatives in terms of the new variables are expressed as:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \alpha'},$$

$$\frac{\partial}{\partial \mathbf{R}} = \frac{\partial}{\partial \mathbf{R}'} - \left(\mu \nabla B + \frac{q}{m} \nabla \Phi_0 - \nabla \mathbf{V} \cdot \mathbf{V} \right) \frac{1}{c_{\parallel}} \frac{\partial}{\partial c_{\parallel}},$$

$$\frac{\partial}{\partial \varepsilon} = \frac{1}{c_{\parallel}} \frac{\partial}{\partial c_{\parallel}}, \quad \text{and} \quad \frac{\partial}{\partial \mu} = \frac{\partial}{\partial \mu'} - \frac{B}{c_{\parallel}} \frac{\partial}{\partial c_{\parallel}}. \quad (57)$$

Note that the superscript ' from these variables can again be dropped once the partial derivatives are performed. Defining

$$\delta \bar{f} \equiv \delta g + \frac{q}{m} \frac{1}{c_{\parallel}} \frac{\partial F}{\partial c_{\parallel}} \langle \Psi \rangle_{\alpha},$$

the relation between $\delta \bar{f}$ and δf then becomes:

$$\delta f = \frac{q}{m} \frac{1}{c_{\parallel}} \frac{\partial F}{\partial c_{\parallel}} \left(\frac{A_{\parallel} c_{\parallel}}{c} - \left\langle \frac{A_{\parallel} c_{\parallel}}{c} \right\rangle_{\alpha} \right) + \frac{q}{m} \frac{1}{B} \frac{\partial F}{\partial \mu} \left[\frac{\mathbf{A} \cdot \mathbf{c}_{\perp}}{c} + (\Psi - \langle \Psi \rangle_{\alpha}) \right] + \delta \bar{f}. \quad (58)$$

Hence, Eq. (56) can now be written as:

$$\frac{\partial \delta \bar{f}}{\partial t} + (\dot{\mathbf{R}}^{(0)} + \dot{\mathbf{R}}^{(1)}) \cdot \frac{\partial \delta \bar{f}}{\partial \mathbf{R}} + \dot{c}_{\parallel}^{(0)} \frac{\partial \delta \bar{f}}{\partial c_{\parallel}} = -\dot{\mathbf{R}}^{(1)} \cdot \frac{\partial F}{\partial \mathbf{R}} - \dot{c}_{\parallel}^{(1)} \frac{\partial F}{\partial c_{\parallel}} - \dot{\mu}^{(1)} \frac{\partial F}{\partial \mu} \quad (59)$$

where

$$\begin{aligned}
\dot{\mathbf{R}}^{(0)} &= c_{\parallel} \hat{\mathbf{n}} + \mathbf{V} + \mathbf{c}_D \\
\dot{\mathbf{R}}^{(1)} &= \frac{q}{m} \frac{\hat{\mathbf{n}}}{\Omega} \times \nabla \left\langle \Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{\mathbf{A} \cdot \mathbf{c}}{c} \right\rangle_{\alpha} \\
\dot{c}_{\parallel}^{(0)} &= -\hat{\mathbf{n}} \cdot \left(\mu \nabla B + \frac{q}{m} \nabla \Phi_0 - \nabla \mathbf{V} \cdot \mathbf{V} \right) \\
\dot{c}_{\parallel}^{(1)} &= -\frac{q}{m} \left(\hat{\mathbf{n}} + c_{\parallel} \nabla \times \hat{\mathbf{n}} + \nabla \times \mathbf{V} \right) \cdot \nabla \left\langle \Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{\mathbf{A} \cdot \mathbf{c}}{c} \right\rangle_{\alpha} \\
&\quad - \frac{q}{m} \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \left\langle \frac{A_{\parallel}}{c} \right\rangle_{\alpha} \\
&\quad - \frac{q}{m} \langle \delta \mathbf{B}_{\perp} \rangle_{\alpha} \cdot \left(\mu \nabla B + \frac{q}{m} \nabla \Phi_0 + (c_{\parallel} \hat{\mathbf{n}} + \mathbf{V}) \cdot \nabla \mathbf{V} + \nabla \mathbf{V} \cdot \hat{\mathbf{n}} c_{\parallel} \right) \\
\dot{\mu}^{(1)} &= -\frac{q}{mB} \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \left\langle \frac{A_{\parallel} c_{\perp}}{c} - \frac{c_{\parallel} \mathbf{A}}{c} \right\rangle_{\alpha} \tag{60}
\end{aligned}$$

with the expression for $\mathbf{c}_D$ being given in Eq. (24). The superscripts (0) and (1) in this equation denote the time derivatives with respect to equilibrium and perturbed quantities respectively, rather than the gyrokinetic ordering; i.e., all terms in Eq. (60) are correct to $\mathcal{O}(\lambda^2)$. Note that since

$$\frac{\partial}{\partial \mu} (B \dot{\mu}^{(1)}) + \frac{\partial}{\partial c_{\parallel}} (B (\dot{c}_{\parallel}^{(0)} + \dot{c}_{\parallel}^{(1)})) + \nabla \cdot (\dot{\mathbf{R}}^{(0)} + \dot{\mathbf{R}}^{(1)}) \sim \mathcal{O}(\lambda^2), \tag{61}$$

this form of the gyrokinetic equation can be cast in the desired phase-space-conserving continuity equation form. Specifically, using $\mathbf{z} \equiv \{c_{\parallel}, \mu, \mathbf{R}\}$, and $\bar{F} \equiv \delta \bar{f} + F$, Eq. (59) can be expressed as:

$$\frac{\partial}{\partial t} (B \delta \bar{f}) + \frac{\partial}{\partial \mathbf{z}} \cdot (\dot{\mathbf{z}} B \delta \bar{f}) \sim \mathcal{O}(\lambda^2). \tag{62}$$

The exact form of the phase-space volume conservation equation can be derived using symplectic perturbation theory.²² However, for purposes of gyrokinetic particle simulation, Eq. (62) is sufficient.²³ The Poisson equation and Ampère’s law then complete the requisite system of equations to determine the perturbed fields Φ and $\mathbf{A}$. The Poisson equation and Ampère’s law can be evaluated in either gyrokinetic¹¹ or laboratory coordinates depending on the gyrokinetic simulation algorithm used.²⁴

VIII. Conclusion and Discussion

In this paper, the important dynamics associated with subsonic sheared flow have been systematically incorporated in a derivation of the nonlinear electromagnetic gyrokinetic equation for general axisymmetric magnetic field configurations. The gyrokinetic variables are appropriately modified in the presence of such flows. For example, the lowest order unperturbed particle energy ε_0 in the moving frame can no longer be taken to be a constant of motion. Calculating the correction terms to ε and μ , the nonlinear gyrokinetic equation describing the evolution of the nonadiabatic portion of the perturbed distribution function is derived. This is presented in Eq. (56) with the physical significance of each term explained. In order to make these new results readily applicable to gyrokinetic particle simulations, the relevant set of equations have also been cast in a phase-space-conserving continuity equation form and given in Eqs. (59) and (60). Summarily, the present paper has provided the formalism needed to carry out a proper kinetic analysis of the influence of sheared equilibrium flows on the confinement properties of axisymmetric toroidal plasmas.

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Appendix A. Gyrophase Average of Quantities in Eq. (50)

The gyrophase average is defined as:

$$\langle \dots \rangle_\alpha \equiv \frac{1}{2\pi} \int_0^{2\pi} d\alpha \dots \quad (\text{A1})$$

where $\mathbf{R}$, ε , and μ are kept constant. For perturbed quantities Φ and $\mathbf{A}$, the familiar⁴ eikonal approximation has the form:

$$\Phi(\mathbf{x}, t) = \hat{\Phi}(\mathbf{x}, t) \exp(iS(\mathbf{x})). \quad (\text{A2})$$

Here, the eikonal $S(\mathbf{x})$ captures the fast perpendicular variation in Φ , and $\hat{\Phi}$ is a slow varying function of $\mathbf{x}$. Further requiring $\hat{\mathbf{n}} \cdot \nabla S = 0$ corresponds to $\nabla \times \mathbf{k}_\perp = 0$ since $\nabla S = \mathbf{k}_\perp$ for purposes of gyrophase averaging. Since $\mathbf{R}$ is held constant during gyroaveraging, $\mathbf{x}$ can be expanded around $\mathbf{R}$ to obtain:

$$\Phi(\mathbf{x}, t) \simeq \hat{\Phi}(\mathbf{R}, t) \exp(iS(\mathbf{R}) + i\boldsymbol{\rho} \cdot \nabla S).$$

where $\boldsymbol{\rho} = \mathbf{c}_\perp \times \hat{\mathbf{n}} / \Omega$. The gyrophase independent portion of perturbed quantities can be expressed as:

$$\tilde{\Phi}(\mathbf{R}, t) \equiv \hat{\Phi}(\mathbf{R}, t) \exp(iS(\mathbf{R})), \text{ and } \tilde{\mathbf{A}}(\mathbf{R}, t) \equiv \hat{\mathbf{A}}(\mathbf{R}, t) \exp(iS(\mathbf{R})). \quad (\text{A3})$$

Using $\nabla S = \mathbf{k}_\perp$ the gyrophase averages of various useful quantities can be calculated; i.e.:

$$\langle \exp(i\mathbf{k}_\perp \cdot \boldsymbol{\rho}) \rangle_\alpha = J_0$$

$$\langle \mathbf{k}_\perp \cdot \mathbf{c}_\perp \exp(i\mathbf{k}_\perp \cdot \boldsymbol{\rho}) \rangle_\alpha = 0$$

$$\langle \mathbf{c}_\perp \exp(i\mathbf{k}_\perp \cdot \boldsymbol{\rho}) \rangle_\alpha = \frac{ic_\perp}{k_\perp} J_1 \mathbf{k}_\perp \times \hat{\mathbf{n}}$$

$$\langle \mathbf{k}_\perp \cdot \mathbf{c}_\perp \mathbf{c}_\perp \exp(i\mathbf{k}_\perp \cdot \boldsymbol{\rho}) \rangle_\alpha = \frac{c_\perp \Omega}{k_\perp} J_1 \mathbf{k}_\perp$$

$$\langle \mathbf{c}_\perp \mathbf{c}_\perp \exp(i\mathbf{k}_\perp \cdot \boldsymbol{\rho}) \rangle_\alpha = \frac{c_\perp \Omega}{k_\perp^3} J_1 \mathbf{k}_\perp \mathbf{k}_\perp + \frac{c_\perp^2}{k_\perp^2} J_1' \mathbf{k}_\perp \times \hat{\mathbf{n}} \mathbf{k}_\perp \times \hat{\mathbf{n}}$$

$$\langle \mathbf{k}_\perp \cdot \mathbf{c}_\perp \mathbf{c}_\perp \mathbf{c}_\perp \exp(i\mathbf{k}_\perp \cdot \boldsymbol{\rho}) \rangle_\alpha = -\frac{ic_\perp^2 \Omega}{k_\perp^2} \left[J_1' - \frac{\Omega}{k_\perp c_\perp} J_1 \right] (\mathbf{k}_\perp \mathbf{k}_\perp \times \hat{\mathbf{n}} + \mathbf{k}_\perp \times \hat{\mathbf{n}} \mathbf{k}_\perp) \quad (\text{A4})$$

where the argument of the Bessel function is $(k_\perp c_\perp / \Omega)$, and J_1' denotes the first derivative of J_1 with respect to the argument.

It is important to treat the gyrophase average of terms like $\mathbf{V} \cdot \nabla \Phi$ and $\mathbf{V} \cdot \nabla \mathbf{A}$ properly. Since $\mathbf{V} \cdot \nabla$ acting on perturbed quantities is formally of $\mathcal{O}(1)$, ($|k_\perp \mathbf{V} \sim \Omega|$), we have $\boldsymbol{\rho} \cdot \nabla \mathbf{V} \cdot \nabla \Phi \sim \mathcal{O}(\lambda)$. Hence, the fact that gyrophase averages are performed holding $(\mathbf{R}, \varepsilon, \mu)$ constant, requires us to expand $\mathbf{V}$ around the guiding center coordinate $\mathbf{R}$, such that

$$\mathbf{V}(\mathbf{x}) = \mathbf{V}(\mathbf{R}) - \frac{\mathbf{c}_\perp \times \hat{\mathbf{n}}}{\Omega} \cdot \nabla \mathbf{V}.$$

Furthermore, when the Coulomb gauge $(\nabla \cdot \mathbf{A})$ is employed, the parallel and perpendicular components of the vector potential scale as:

$$|k_\parallel A_\parallel| \sim |\mathbf{k}_\perp \cdot \mathbf{A}|. \quad (\text{A5})$$

Hence, using the ordering $k_\parallel / k_\perp \sim \mathcal{O}(\lambda)$, it is apparent that the component of the vector potential along $\mathbf{k}_\perp$ is $\mathcal{O}(\lambda)$ compared to the parallel component $A_\parallel$. Without

loss of any generality, we can then take:

$$\tilde{\mathbf{A}} = \tilde{A}_{\parallel} \hat{\mathbf{n}} + i\tilde{B}_{\parallel} \frac{\mathbf{k}_{\perp} \times \hat{\mathbf{n}}}{k_{\perp}^2} \quad (\text{A6})$$

where $\tilde{B}_{\parallel}$ is the magnetosonic perturbation and $\tilde{A}_{\parallel}$ is associated with the shear Alfvén waves.

Appendix B. Calculation of $\langle G_{\varepsilon}^1 \rangle_{\alpha}$

Using the expression for G_{ε}^1 from Eq. (44) and the velocity derivative of the first order correction of ε ,

$$\begin{aligned} \frac{\partial \varepsilon_1}{\partial \mathbf{c}} = & -\frac{1}{\Omega} \left\{ \hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{c}_{\perp} \times \hat{\mathbf{n}} + \mathbf{c}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}}) \right. \\ & + (\hat{\mathbf{n}} \times \nabla \mathbf{V} \cdot (\mathbf{c}_{\parallel} \hat{\mathbf{n}} + \mathbf{V}) - (\mathbf{c}_{\parallel} \hat{\mathbf{n}} + \mathbf{V}) \cdot \nabla \mathbf{V} \times \hat{\mathbf{n}}) \\ & + \frac{1}{4} (\hat{\mathbf{n}} \times \nabla \mathbf{V} \cdot \mathbf{c}_{\perp} - \mathbf{c}_{\perp} \cdot \nabla \mathbf{V} \times \hat{\mathbf{n}} + \mathbf{c}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} + \nabla \mathbf{V} \cdot \mathbf{c}_{\perp} \times \hat{\mathbf{n}}) \\ & \left. - \frac{\hat{\mathbf{n}}}{4} (\mathbf{c}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} + \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{c}_{\perp} \times \hat{\mathbf{n}}) \right\}, \quad (\text{B1}) \end{aligned}$$

the expression for $\langle G_{\varepsilon}^1 \rangle_{\alpha}$ can be written as:

$$\begin{aligned} \langle G_{\varepsilon}^1 \rangle_{\alpha} = & \left\langle \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{A_{\parallel} c_{\parallel}}{c} \right] \right\rangle_{\alpha} - \left\langle \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \frac{\mathbf{A} \cdot \mathbf{c}_{\perp}}{c} \right\rangle_{\alpha} \\ & - \left\langle \frac{\mathbf{c} \cdot \nabla \mathbf{A} \cdot \mathbf{V}}{c} \right\rangle_{\alpha} - \left\langle \nabla_{\perp} \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{A_{\parallel} c_{\parallel}}{c} \right] \cdot \frac{\partial \varepsilon_1}{\partial \mathbf{c}} \right\rangle_{\alpha} \\ & + \left\langle \frac{\nabla_{\perp} \mathbf{A} \cdot \mathbf{c}_{\perp}}{c} \cdot \frac{\partial \varepsilon_1}{\partial \mathbf{c}} \right\rangle_{\alpha} - \left\langle \frac{\mathbf{c}_{\perp} \cdot \nabla \mathbf{A}}{c} \cdot \frac{\partial \varepsilon_1}{\partial \mathbf{c}} \right\rangle_{\alpha} \quad (\text{B2}) \end{aligned}$$

Each of these terms can in turn be expressed as:

$$\begin{aligned}
& \left\langle \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{A_{\parallel} c_{\parallel}}{c} \right] \right\rangle_{\alpha} = \\
& \left(J_0 \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) - J_1 \frac{c_{\perp}}{k_{\perp} \Omega} \mathbf{k}_{\perp} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_{\perp} \right) \left[\tilde{\Phi} - \frac{\tilde{\mathbf{A}} \cdot \mathbf{V}}{c} - \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \right], \quad (\text{B3}) \\
& \left\langle \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \frac{\mathbf{A} \cdot \mathbf{c}_{\perp}}{c} \right\rangle_{\alpha} = \left(-J_1 \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) + J'_1 \frac{c_{\perp}}{k_{\perp} \Omega} \mathbf{k}_{\perp} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_{\perp} \right) \frac{c_{\perp} \tilde{B}_{\parallel}}{k_{\perp} c}, \\
& \left\langle \frac{\mathbf{c} \cdot \nabla \mathbf{V} \cdot \mathbf{A}}{c} \right\rangle_{\alpha} = J_0 \frac{c_{\parallel} \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \tilde{\mathbf{A}}}{c} + J_1 \frac{ic_{\perp}}{k_{\perp}} \frac{\mathbf{k}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \tilde{\mathbf{A}}}{c}, \\
& \left\langle \nabla_{\perp} \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{A_{\parallel} c_{\parallel}}{c} \right] \cdot \frac{\partial \varepsilon_1}{\partial \mathbf{c}} \right\rangle_{\alpha} = -J_1 \frac{c_{\perp}}{k_{\perp} \Omega} \left[\tilde{\Phi} - \frac{\tilde{\mathbf{A}} \cdot \mathbf{V}}{c} - \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \right] \mathbf{k}_{\perp} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_{\perp} \\
& - J_0 \frac{i}{\Omega} J_0 \left[\tilde{\Phi} - \frac{\mathbf{V} \cdot \tilde{\mathbf{A}}}{c} - \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \right] \left(\mathbf{k}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot (c_{\parallel} \hat{\mathbf{n}} + \mathbf{V}) + (c_{\parallel} \hat{\mathbf{n}} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \mathbf{k}_{\perp} \times \hat{\mathbf{n}} \right), \\
& \left\langle \frac{\nabla_{\perp} \mathbf{A} \cdot \mathbf{c}_{\perp}}{c} \cdot \frac{\partial \varepsilon_1}{\partial \mathbf{c}} \right\rangle_{\alpha} = -J'_1 \frac{c_{\perp}^2}{\Omega k_{\perp}^2} \frac{\tilde{B}_{\parallel}}{c} \mathbf{k}_{\perp} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_{\perp} \\
& + J_1 \frac{ic_{\perp}}{k_{\perp} \Omega} \frac{\tilde{B}_{\parallel}}{c} \left(\mathbf{k}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot (c_{\parallel} \hat{\mathbf{n}} + \mathbf{V}) + (c_{\parallel} \hat{\mathbf{n}} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \mathbf{k}_{\perp} \times \hat{\mathbf{n}} \right),
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \frac{\mathbf{c}_{\perp} \cdot \nabla \mathbf{A}}{c} \cdot \frac{\partial \varepsilon_1}{\partial \mathbf{c}} \right\rangle_{\alpha} &= -J_1 \frac{ic_{\perp}}{k_{\perp}} \frac{\tilde{A}_{\parallel}}{c} (\hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_{\perp} \times \hat{\mathbf{n}} + \mathbf{k}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}}) \\
&- J_1 \frac{c_{\perp}}{k_{\perp}^3} \frac{\tilde{B}_{\parallel}}{c} \mathbf{k}_{\perp} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_{\perp}.
\end{aligned} \quad (\text{B4})$$

Finally, adding all the terms together leads to the result:

$$\langle G_{\varepsilon}^1 \rangle_{\alpha} = \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \left[J_0 \left(\tilde{\Phi} - \frac{\tilde{\mathbf{A}} \cdot \mathbf{V}}{c} - \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \right) + J_1 \frac{c_{\perp}}{k_{\perp}} \frac{\tilde{B}_{\parallel}}{c} \right]$$

$$\begin{aligned}
& + \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}} \left[J_1 \frac{ic_\perp}{k_\perp} \frac{\tilde{A}_\parallel}{c} - J_0 \frac{ic_\parallel}{k_\perp^2} \frac{\tilde{B}_\parallel}{c} \right] \\
& + \frac{i}{\Omega} \left[J_0 \left(\tilde{\Phi} - \frac{\tilde{\mathbf{A}} \cdot \mathbf{V}}{c} - \frac{\tilde{A}_\parallel c_\parallel}{c} \right) + J_1 \frac{c_\perp}{k_\perp} \frac{\tilde{B}_\parallel}{c} \right] \\
& \times \left(\mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot (c_\parallel \hat{\mathbf{n}} + \mathbf{V}) + (c_\parallel \hat{\mathbf{n}} + \mathbf{V}) \cdot \nabla \mathbf{V} \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}} \right). \tag{B5}
\end{aligned}$$

where we have made use of the fact that

$$\nabla \cdot \mathbf{V} = \mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}} + \mathbf{k}_\perp \cdot \nabla \mathbf{V} \cdot \mathbf{k}_\perp = 0$$

since $\hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} = 0$. However, since it is often not desirable to deal with this result in Fourier space, we can use the transformation, $i\mathbf{k}_\perp \rightarrow \nabla_\perp$, to obtain:

$$\begin{aligned}
\langle G_\epsilon^1 \rangle_\alpha &= \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{R}} \right) \left\langle \Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{\mathbf{A} \cdot \mathbf{c}}{c} \right\rangle_\alpha + \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \left\langle \frac{A_\parallel c_\perp}{c} - \frac{\mathbf{A} c_\parallel}{c} \right\rangle_\alpha \\
&+ \frac{\hat{\mathbf{n}}}{\Omega} \times \left((c_\parallel \hat{\mathbf{n}} + \mathbf{V}) \cdot \nabla \mathbf{V} + \nabla \mathbf{V} \cdot (c_\parallel \hat{\mathbf{n}} + \mathbf{V}) \right) \cdot \nabla \left\langle \Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{\mathbf{A} \cdot \mathbf{c}}{c} \right\rangle_\alpha. \tag{B6}
\end{aligned}$$

Appendix C. Calculation of $\langle G_\mu^1 \rangle_\alpha$

Similar to the calculation of $\langle G_\epsilon^1 \rangle_\alpha$, we can calculate $\langle G_\mu^1 \rangle_\alpha$. Equation (46) can be written as:

$$\begin{aligned}
G_\mu^1 &= \left(\frac{\partial}{\partial t} + c_\parallel \hat{\mathbf{n}} \cdot \nabla + \mathbf{V} \cdot \nabla \right) \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} \right] - \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} \right] \mathbf{c} \cdot \nabla \ln B \\
&- \nabla_\perp \left(\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} \right) \cdot \frac{\partial \mu_1}{\partial \mathbf{c}} \\
&- \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \frac{A_\parallel c_\parallel}{c} - \frac{c_\parallel^2 \hat{\mathbf{n}} \cdot \nabla A_\parallel}{c} + \frac{A_\parallel c_\parallel}{c} \mathbf{c} \cdot \nabla \ln B - \frac{A_\parallel}{c} \mathbf{c} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c} \\
&- \frac{A_\parallel}{c} \mathbf{c} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} + \frac{A_\parallel}{c} \mathbf{V} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} - \frac{A_\parallel}{c} \mathbf{V} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c} + \frac{A_\parallel}{c} \mathbf{c} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}}
\end{aligned}$$

$$\begin{aligned}
& + \frac{q}{m} \frac{A_{\parallel}}{c} \hat{\mathbf{n}} \cdot \nabla \Phi_0 + \nabla_{\perp} \left(\frac{A_{\parallel} c_{\parallel}}{c} \right) \cdot \frac{\partial \mu_1}{\partial \mathbf{c}} - \frac{\mathbf{c}_{\perp} \cdot \nabla A_{\parallel}}{c} \hat{\mathbf{n}} \cdot \frac{\partial \mu_1}{\partial \mathbf{c}} - \frac{A_{\parallel} c_{\parallel}}{c} \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c}_{\perp} \\
& - \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \left(\frac{\mathbf{A}_{\perp} \cdot \mathbf{c}_{\perp}}{c} \right) - \mathbf{c} \cdot \nabla \mathbf{V} \cdot \frac{\mathbf{A}_{\perp}}{c} - \frac{c_{\parallel} \hat{\mathbf{n}} \cdot \nabla \mathbf{A}_{\perp} \cdot \mathbf{c}_{\perp}}{c} - c_{\parallel} \mathbf{c}_{\perp} \cdot \nabla \hat{\mathbf{n}} \cdot \frac{\mathbf{A}_{\perp}}{c} \\
& + \nabla_{\perp} \left(\frac{\mathbf{A}_{\perp} \cdot \mathbf{c}_{\perp}}{c} \right) \cdot \frac{\partial \mu_1}{\partial \mathbf{c}} - \frac{\mathbf{c}_{\perp} \cdot \nabla \mathbf{A}_{\perp}}{c} \cdot \frac{\partial \mu_1}{\partial \mathbf{c}}. \tag{C1}
\end{aligned}$$

To carry out the gyrophase average of this equation requires the expression for $\partial \mu_1 / \partial \mathbf{c}$, which can be calculated from Eq. (23):

$$\begin{aligned}
B \frac{\partial \mu_1}{\partial \mathbf{c}} &= -\mathbf{c}_D - \mathbf{c}_{\perp} \frac{\mathbf{c}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \ln B}{\Omega} - \frac{\hat{\mathbf{n}}}{\Omega} \mathbf{c}_{\perp} \times \hat{\mathbf{n}} \cdot (2c_{\parallel} \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} + \hat{\mathbf{n}} \cdot \nabla \mathbf{V} + \mathbf{V} \cdot \nabla \hat{\mathbf{n}}) \\
& - \frac{1}{4\Omega} (\mathbf{I} - \hat{\mathbf{n}} \hat{\mathbf{n}}) \cdot (\hat{\mathbf{n}} \times (\nabla \mathbf{V} + c_{\parallel} \nabla \hat{\mathbf{n}}) - (\nabla \mathbf{V} + c_{\parallel} \nabla \hat{\mathbf{n}}) \times \hat{\mathbf{n}}) \cdot \mathbf{c}_{\perp} \\
& - \frac{1}{4\Omega} \mathbf{c}_{\perp} \cdot (\hat{\mathbf{n}} \times (\nabla \mathbf{V} + c_{\parallel} \nabla \hat{\mathbf{n}}) - (\nabla \mathbf{V} + c_{\parallel} \nabla \hat{\mathbf{n}}) \times \hat{\mathbf{n}}) \cdot (\mathbf{I} - \hat{\mathbf{n}} \hat{\mathbf{n}}) \\
& - \frac{\hat{\mathbf{n}}}{4\Omega} \mathbf{c}_{\perp} \cdot (\hat{\mathbf{n}} \times \nabla \hat{\mathbf{n}} - \nabla \hat{\mathbf{n}} \times \hat{\mathbf{n}}) \cdot \mathbf{c}_{\perp}. \tag{C2}
\end{aligned}$$

where $\mathbf{c}_D$ is given in Eq. (24).

Starting with the terms involving $(\Phi - \mathbf{V} \cdot \mathbf{A}/c)$ we have:

$$\begin{aligned}
\left\langle \left(\frac{\partial}{\partial t} + c_{\parallel} \hat{\mathbf{n}} \cdot \nabla + \mathbf{V} \cdot \nabla \right) \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} \right] \right\rangle_{\alpha} &= \\
\left(J_0 \left(\frac{\partial}{\partial t} + c_{\parallel} \hat{\mathbf{n}} \cdot \nabla + \mathbf{V} \cdot \nabla \right) - J_1 \frac{c_{\perp}}{\Omega k_{\perp}} \mathbf{k}_{\perp} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_{\perp} \right) \left[\tilde{\Phi} - \frac{\tilde{\mathbf{A}} \cdot \mathbf{V}}{c} \right], \tag{C3}
\end{aligned}$$

$$\left\langle - \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} \right] \mathbf{c} \cdot \nabla \ln B \right\rangle_{\alpha} = - (J_0 c_{\parallel} \hat{\mathbf{n}} + J_1 \frac{i c_{\perp}}{k_{\perp}} \mathbf{k}_{\perp} \times \hat{\mathbf{n}}) \cdot \nabla \ln B \left[\Phi - \frac{\tilde{\mathbf{A}} \cdot \mathbf{V}}{c} \right], \tag{C4}$$

and

$$\left\langle - \nabla_{\perp} \left[\Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} \right] \cdot \frac{\partial \mu_1}{\partial \mathbf{c}} \right\rangle_{\alpha} = \left(i J_0 \mathbf{k}_{\perp} \cdot \mathbf{c}_D + \frac{i c_{\perp}}{k_{\perp}} J_1 \mathbf{k}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \ln B \right.$$

$$-J_1 \frac{c_\perp}{2\Omega k_\perp} (\mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot (\nabla \mathbf{V} + c_\parallel \nabla \hat{\mathbf{n}}) \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}} - \mathbf{k}_\perp \cdot (\nabla \mathbf{V} + c_\parallel \nabla \hat{\mathbf{n}}) \cdot \mathbf{k}_\perp) \left[\tilde{\Phi} - \frac{\tilde{\mathbf{A}} \cdot \mathbf{V}}{c} \right],$$

After some cancellations, the terms with $(\Phi - \mathbf{A} \cdot \mathbf{V}/c)$ can be expressed as:

$$B \left(\frac{\partial}{\partial t} + (c_\parallel \hat{\mathbf{n}} + \mathbf{V} + \mathbf{c}_D) \cdot \frac{\partial}{\partial \mathbf{R}} \right) \left(\frac{J_0}{B} \left[\tilde{\Phi} - \frac{\tilde{\mathbf{A}} \cdot \mathbf{V}}{c} \right] \right) \quad (\text{C5})$$

Here we have used the following identities⁴:

$$\mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}} + \mathbf{k}_\perp \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_\perp = k_\perp^2 (\nabla \cdot \hat{\mathbf{n}} - \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) = -k_\perp^2 \hat{\mathbf{n}} \cdot \nabla \ln B, \quad (\text{C6})$$

$$\nabla \times \mathbf{k}_\perp = \nabla \times \nabla S = 0, \quad (\text{C7})$$

$$\mathbf{k}_\perp \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_\perp = -\mathbf{k}_\perp \cdot \nabla \mathbf{k}_\perp \cdot \hat{\mathbf{n}} = -\hat{\mathbf{n}} \cdot \nabla \mathbf{k}_\perp \cdot \mathbf{k}_\perp = -k_\perp \hat{\mathbf{n}} \cdot \nabla k_\perp, \quad (\text{C8})$$

and

$$\left(-J_0 + \frac{k_\perp c_\perp}{2\Omega} J_1 \right) \hat{\mathbf{n}} \cdot \nabla \ln B - \frac{c_\perp}{\Omega} J_1 \hat{\mathbf{n}} \cdot \nabla k_\perp = B \hat{\mathbf{n}} \cdot \nabla \left(\frac{J_0 (k_\perp c_\perp / \Omega)}{B} \right). \quad (\text{C9})$$

Proceeding with the calculation of the terms with $A_\parallel$ we have:

$$\left\langle - \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \frac{A_\parallel c_\parallel}{c} \right\rangle_\alpha = -J_0 \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \frac{\tilde{A}_\parallel c_\parallel}{c} + J_1 \frac{c_\perp}{\Omega k_\perp} \frac{\tilde{A}_\parallel c_\parallel}{c} \mathbf{k}_\perp \cdot \nabla \mathbf{V} \cdot \mathbf{k}_\perp, \quad (\text{C10})$$

$$\left\langle - \frac{c_\parallel^2 \hat{\mathbf{n}} \cdot \nabla A_\parallel}{c} \right\rangle_\alpha = -J_0 \frac{c_\parallel^2 \hat{\mathbf{n}} \cdot \nabla A_\parallel}{c}, \quad (\text{C11})$$

$$\left\langle \frac{A_\parallel c_\parallel}{c} \mathbf{c} \cdot \nabla \ln B \right\rangle_\alpha = J_0 c_\parallel^2 \frac{\tilde{A}_\parallel}{c} \hat{\mathbf{n}} \cdot \nabla \ln B + J_1 \frac{i c_\perp}{k_\perp} \frac{\tilde{A}_\parallel c_\parallel}{c} \mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot \nabla \ln B, \quad (\text{C12})$$

$$\begin{aligned} \left\langle - \frac{A_\parallel}{c} \mathbf{c} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c} \right\rangle_\alpha &= -J_1 \frac{i c_\perp}{k_\perp} \frac{\tilde{A}_\parallel c_\parallel}{c} c \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}} - J_1 \frac{c_\perp \Omega}{k_\perp^3} \frac{\tilde{A}_\parallel}{c} \mathbf{k}_\perp \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_\perp \\ &\quad - J_1' \mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}}, \end{aligned}$$

$$\left\langle -\frac{A_{\parallel}}{c} \mathbf{c} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} \right\rangle_{\alpha} = -J_1 \frac{ic_{\perp}}{k_{\perp}} \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \mathbf{k}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}}, \quad (\text{C13})$$

$$\left\langle \frac{A_{\parallel}}{c} \left(\mathbf{V} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} + \frac{q}{m} \hat{\mathbf{n}} \cdot \nabla \Phi_0 \right) \right\rangle_{\alpha} = J_0 \frac{\tilde{A}_{\parallel}}{c} \left(\mathbf{V} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} + \frac{q}{m} \hat{\mathbf{n}} \cdot \nabla \Phi_0 \right), \quad (\text{C14})$$

$$\left\langle -\frac{A_{\parallel}}{c} \mathbf{V} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c} \right\rangle_{\alpha} = -J_1 \frac{ic_{\perp}}{k_{\perp}} \frac{\tilde{A}_{\parallel}}{c} \mathbf{V} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_{\perp} \times \hat{\mathbf{n}}, \quad (\text{C15})$$

$$\left\langle \frac{A_{\parallel}}{c} \mathbf{c} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}} \right\rangle_{\alpha} = J_1 \frac{ic_{\perp}}{k_{\perp}} \frac{\tilde{A}_{\parallel}}{c} \mathbf{k}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \hat{\mathbf{n}}, \quad (\text{C16})$$

$$\begin{aligned} \left\langle \frac{\nabla_{\perp} A_{\parallel} c_{\parallel}}{c} \cdot \frac{\partial \mu_1}{\partial \mathbf{c}} \right\rangle_{\alpha} &= -iJ_0 \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \mathbf{k}_{\perp} \cdot \mathbf{c}_D - J_1 \frac{ic_{\perp}}{k_{\perp}} \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \mathbf{k}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \ln B \\ &+ J_1 \frac{c_{\perp}}{2\Omega k_{\perp}} \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \left(\mathbf{k}_{\perp} \times \hat{\mathbf{n}} \cdot (\nabla \mathbf{V} + c_{\parallel} \nabla \hat{\mathbf{n}}) \cdot \mathbf{k}_{\perp} \times \hat{\mathbf{n}} - \mathbf{k}_{\perp} \cdot (\nabla \mathbf{V} + c_{\parallel} \nabla \hat{\mathbf{n}}) \cdot \mathbf{k}_{\perp} \times \hat{\mathbf{n}} \right), \end{aligned}$$

$$\begin{aligned} \left\langle -\frac{\mathbf{c}_{\perp} \cdot \nabla A_{\parallel}}{c} \hat{\mathbf{n}} \cdot \frac{\partial \mu_1}{\partial \mathbf{c}} \right\rangle_{\alpha} &= J_1 \frac{ic_{\perp}}{k_{\perp}} \frac{\tilde{A}_{\parallel}}{c} \left(2c_{\parallel} \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} + \hat{\mathbf{n}} \cdot \nabla \mathbf{V} + \mathbf{V} \cdot \nabla \hat{\mathbf{n}} \right) \cdot \mathbf{k}_{\perp} \times \hat{\mathbf{n}} \\ &\frac{c_{\perp}^2}{2k_{\perp}^2} \frac{\tilde{A}_{\parallel}}{c} \left(J_1' - \frac{\Omega}{c_{\perp} k_{\perp}} J_1 \right) \left(\mathbf{k}_{\perp} \times \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_{\perp} \times \hat{\mathbf{n}} - \mathbf{k}_{\perp} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_{\perp} \right), \end{aligned}$$

and

$$\left\langle -\frac{A_{\parallel} c_{\parallel}}{c} \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{c}_{\perp} \right\rangle_{\alpha} = -J_1 \frac{ic_{\perp}}{k_{\perp}} \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_{\perp} \times \hat{\mathbf{n}}. \quad (\text{C17})$$

Using these identities, we can write the terms involving $A_{\parallel}$ as:

$$B \left(\frac{\partial}{\partial t} + (c_{\parallel} \hat{\mathbf{n}} + \mathbf{V} + \mathbf{c}_D) \cdot \frac{\partial}{\partial \mathbf{R}} \right) \left[\frac{J_0}{B} \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \right] + J_1 \frac{ic_{\perp}}{k_{\perp}} \frac{\tilde{A}_{\parallel} c_{\parallel}}{c} \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_{\perp} \times \hat{\mathbf{n}} \quad (\text{C18})$$

where

$$\hat{\mathbf{n}} \cdot \nabla c_{\parallel} = -\frac{c_{\perp}^2}{2c_{\parallel} B} \hat{\mathbf{n}} \cdot \nabla \ln B - \frac{q}{mc_{\parallel}} \hat{\mathbf{n}} \cdot \nabla \Phi_0 + \frac{q}{mc_{\parallel}} \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{V}, \quad (\text{C19})$$

and

$$J'_1 = J_0 - \frac{\Omega}{k_\perp c_\perp} J_1 \quad (\text{C20})$$

has been used.

Finally, we can calculate the terms with $\mathbf{A}_\perp$. The gyrophase average of each of the terms in Eq. (C1) are:

$$\left\langle - \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \frac{\mathbf{A}_\perp \cdot \mathbf{c}_\perp}{c} \right\rangle_\alpha = J_1 \frac{c_\perp}{k_\perp} \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \frac{\tilde{B}_\parallel}{c} + J'_1 \frac{c_\perp^2}{k_\perp^2 \Omega} \frac{\tilde{B}_\parallel}{c} \mathbf{k}_\perp \cdot \nabla \mathbf{V} \cdot \mathbf{k}_\perp, \quad (\text{C21})$$

$$\langle -\mathbf{c} \cdot \nabla \mathbf{V} \cdot \mathbf{A}_\perp \rangle_\alpha = -J_0 \frac{c_\parallel}{k_\perp^2} \frac{\tilde{B}_\parallel}{c} \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}} + J_1 \frac{c_\perp}{k_\perp^3} \frac{\tilde{B}_\parallel}{c} \mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}}, \quad (\text{C22})$$

$$\left\langle - \frac{c_\parallel \hat{\mathbf{n}} \cdot \nabla \mathbf{A}_\perp \cdot \mathbf{c}_\perp}{c} \right\rangle_\alpha = -J_1 \frac{i c_\perp c_\parallel}{k_\perp} \hat{\mathbf{n}} \cdot \nabla \mathbf{A}_\perp \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}}, \quad (\text{C23})$$

$$\left\langle - \frac{c_\parallel \mathbf{c}_\perp \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{A}_\perp}{c} \right\rangle_\alpha = J_1 \frac{c_\parallel c_\perp}{k_\perp^3} \frac{\tilde{B}_\parallel}{c} \mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}} \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}}, \quad (\text{C24})$$

$$\begin{aligned} \left\langle \frac{\nabla_\perp \mathbf{A}_\perp \cdot \mathbf{c}_\perp}{c} \cdot \frac{\partial \mu_1}{\partial \mathbf{c}} \right\rangle_\alpha &= J_1 \frac{i c_\perp}{k_\perp} \frac{\tilde{B}_\parallel}{c} \mathbf{k}_\perp \cdot \mathbf{c}_D - \frac{i c_\perp^2}{k_\perp^2} \frac{\tilde{B}_\parallel}{c} \left(J'_1 - \frac{\Omega}{k_\perp c_\perp} J_1 \right) \mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot \nabla \ln B \\ &+ J'_1 \frac{c_\perp^2}{2\Omega k_\perp^2} \frac{\tilde{B}_\parallel}{c} \left(\mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot (\nabla \mathbf{V} + c_\parallel \nabla \hat{\mathbf{n}}) \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}} - \mathbf{k}_\perp \cdot (\nabla \mathbf{V} + c_\parallel \nabla \hat{\mathbf{n}}) \cdot \mathbf{k}_\perp \right), \end{aligned}$$

and

$$\begin{aligned} \left\langle - \frac{\mathbf{c}_\perp \cdot \nabla_\perp \mathbf{A}_\perp}{c} \cdot \frac{\partial \mu_1}{\partial \mathbf{c}} \right\rangle_\alpha &= \frac{i c_\perp^2}{k_\perp^2} \frac{\tilde{B}_\parallel}{c} \left(J'_1 - \frac{\Omega}{k_\perp c_\perp} J_1 \right) \mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot \nabla \ln B \\ &- J_1 \frac{c_\perp}{2k_\perp^3} \frac{\tilde{B}_\parallel}{c} \left(\mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot (\nabla \mathbf{V} + c_\parallel \nabla \hat{\mathbf{n}}) \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}} - \mathbf{k}_\perp \cdot (\nabla \mathbf{V} + c_\parallel \nabla \hat{\mathbf{n}}) \cdot \mathbf{k}_\perp \right). \end{aligned}$$

Hence, for the terms with $\mathbf{A}_\perp$; i.e., $B_\parallel$, we obtain:

$$B \left(\frac{\partial}{\partial t} + (c_\parallel \hat{\mathbf{n}} + \mathbf{V} + \mathbf{c}_D) \cdot \nabla \right) \left[\frac{J_1 c_\perp}{k_\perp B} \frac{\tilde{B}_\parallel}{c} \right] - J_0 \frac{i c_\parallel}{k_\perp^2} \frac{\tilde{B}_\parallel}{c} \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \mathbf{k}_\perp \times \hat{\mathbf{n}}. \quad (\text{C25})$$

Thus, we obtain the final result:

$$\begin{aligned} \langle G_\mu^1 \rangle_\alpha &= B \left(\frac{\partial}{\partial t} + (c_\parallel \hat{\mathbf{n}} + \mathbf{V} + \mathbf{c}_D) \cdot \frac{\partial}{\partial \mathbf{R}} \right) \left[\frac{1}{B} \left\langle \Phi - \frac{\mathbf{A} \cdot \mathbf{V}}{c} - \frac{\mathbf{A} \cdot \mathbf{c}}{c} \right\rangle_\alpha \right] \\ &\quad + \hat{\mathbf{n}} \cdot \nabla \mathbf{V} \cdot \left\langle \frac{A_\parallel c_\perp}{c} - \frac{\mathbf{A} c_\parallel}{c} \right\rangle_\alpha \end{aligned} \quad (\text{C26})$$

With respect to $\langle \mathbf{G}_R^1 \rangle_\alpha$, we again follow similar procedures (i.e., making use of the expressions for gyrophase averages given throughout this Appendix) to obtain:

$$\langle \mathbf{G}_R^1 \rangle_\alpha = \frac{c}{B} \hat{\mathbf{n}} \times \nabla_\perp \left[J_0 \left(\tilde{\Phi} - \frac{\tilde{A}_\parallel c_\parallel}{c} - \frac{\tilde{\mathbf{A}} \cdot \mathbf{V}}{c} \right) + \frac{c_\perp}{k_\perp} J_1 \frac{\tilde{B}_\parallel}{c} \right]. \quad (\text{C27})$$

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