NONLINEAR $\omega^*$-STABILIZATION OF THE $m=1$ MODE IN TOKAMAKS

BY

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AUGUST 1995
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Nonlinear $\omega_*$-stabilization of the 
$m = 1$ mode in tokamaks

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Abstract

Earlier studies of sawtooth oscillations in Tokamak Fusion Test Reactor supershots (Levinton, et al, Phys. Rev. Lett. 72, 2895 (1994); Zakharov, et al, Plasma Phys. and Contr. Nucl. Fus. Res., Proc. 15th Int. Conf., Seville 1994, Vienna) have found an apparent contradiction between conventional linear theory and experiment: even in sawtooth-free discharges, the theory typically predicts instability due to a nearly ideal $m = 1$ mode. Here, the nonlinear evolution of such mode is analyzed using numerical simulations of a two-fluid magnetohydrodynamic (MHD) model. We find the mode saturates nonlinearly at a small amplitude provided the ion and electron drift-frequencies $\omega_\text{i,e}$ are somewhat above the linear stability threshold of the collisionless $m = 1$ reconnecting mode. The comparison of the simulation results to $m = 1$ mode activity in TFTR suggests additional, stabilizing
effects outside the present model are also important.

52.35.Py, 52.30.Fb, 52.55.Fa
I. INTRODUCTION

Recent comparisons\textsuperscript{1,2} of sawtooth behavior in TFTR\textsuperscript{3} (Tokamak Fusion Test Reactor) to theoretical models have revealed an apparent inconsistency: in sawtooth-free, higher-$\beta$ discharges, conventional linear MHD (Magnetohydrodynamic) stability analysis indicates the presence of an unstable, predominantly ideal $m = 1$ mode.

To determine if this “anomalous stability” will apply to future tokamaks, its origin must be understood. Important progress on this was made in Ref. 1, in which a simple condition was found that reliably determines the sawtooth-free operational space of TFTR in all presently known regimes. This condition is in fact the linear, $m = 1$ mode $\omega_{\ast}$-stability condition of the two-fluid model in a certain limit. The limit corresponds to the collisionless reconnecting mode and may be defined in terms of three essential scale-lengths in the two-fluid model: the ideal mode layer-width\textsuperscript{4} $\lambda_h \simeq \sqrt{3}\pi r_1^2 (\beta_p^2 - (0.3)^2)/(q_f R^2)$ (related to the ideal MHD $m = 1$ growth rate by $\lambda_h = \gamma \tau_A/\langle r_1 q_f \rangle$ with $\tau_A = R/V_A|_{r_1}$, so $\lambda_h > 0$ indicates instability); the ion-sound Larmor radius $\rho_s = \sqrt{(T_e + T_i)/m_i \Omega_{ci}^2}$ (arising from finite parallel electron compressibility); and the collisionless skin depth $d_e = c/\omega_{pe}$ (due to electron inertia in Ohm’s law, which is dominant over resistivity in TFTR). The collisionless reconnecting mode limit is $|\lambda_h| \ll \rho_s, d_e$, i.e., the case in which the ideal MHD driving term $\lambda_h$ is negligible compared to the destabilizing collisionless effects associated with $\rho_s$ and $d_e$.

The contradiction with experimental analysis is that typical supershot discharges seem to correspond to the opposite case, $\lambda_h > \rho_s, d_e$, in which the linear mode is unstable and predominantly ideal. The results from linear theory in these two limits are now considered in detail.

For either ordering of $\lambda_h$, $\rho_s$, and $d_e$ the two-fluid model predicts the $m = 1$ mode can be linearly stabilized by sufficiently large values of $\omega_{\ast j} = -\eta_{j r}/(n q_J r B)|_{r_1}, j = i, e$. For example, taking the case of $T_i \gg T_e$ and isothermal ions, the stability condition is $\omega_{\ast i}/2 > \gamma_0$, where $\gamma_0$ is the growth rate of the mode for $\omega_{\ast j} = 0$. For typical TFTR supershot values, $\lambda_h \sim 1 - 2$ cm, $\rho_s \sim 0.5$ cm, $d_e \sim 0.08$ cm, one obtains (as discussed later)
\[ \gamma_0 \tau_A \simeq q_1 \sqrt{\frac{\lambda_h^2 + (\rho_s^2 + 5d_e^2)/2}{2}} > q_1 \lambda_h. \] For \( \lambda_h = 0 \) (and \( \rho_s > d_e \)), on the other hand, \( \gamma_0 \) is given by\(^5\) \[ \gamma_0 \tau_A \simeq q_1^2 \rho_s [2d_e/\rho_s]^{1/3} < q_1 \rho_s. \] Since \( \lambda_h > \rho_s \), a range of \( \omega_* \) values therefore exists in which the former expression for \( \gamma_0 \) leads to a prediction instability while the latter does not, and it is in such a range that the apparent violation of linear theory cited above occurs.

In summary, when \( \lambda_h \) is calculated for typical TFTR supershots (such as those discussed in Ref. 1), it is so large that linear MHD stability analysis cannot account for the observed periods of sawtooth-free operation with \( q(0) < 1 \). In particular, it is so large that the stability condition found in Ref. 1, which is based on \( \lambda_h = 0 \), cannot be justified in the context of linear theory. The agreement of that condition with experiment therefore needs further explanation.

A possible resolution of this problem is that effects outside the two-fluid model thus far considered effectively reduce the value of \( \lambda_h \). For example, trapped energetic\(^6\) and thermal\(^7\) particles can have such an influence. Studies\(^1^8\) of the former effect, however, have found it alone to be insufficient to explain the experimental results. Similarly, linear stabilization by the latter is problematic\(^7\) due to the usual disparity between \( T_i \) and \( T_e \) (e.g., \( T_i > 2T_e \)) in TFTR. At a more basic level, these effects, while apparently important quantitatively, seem too small to account for the observed stability in at least the more extreme cases.

Here, we follow the assumption that such stabilizing effects are absent or insufficient, and address the proposal of Ref. 2 that nonlinear \( \omega_* \) effects lead to the observed sawtooth-stability condition. Such an explanation is in fact consistent with\(^2\) the common observation in TFTR of a saturated, nonlinear \( m = 1 \) mode. Considering a linearly unstable parameter regime like that of TFTR described above, we present numerical simulations of a reduced two-fluid model that determine the early nonlinear evolution of the mode for various values of \( \omega_{*i,e} \).

The application of a reduced MHD model to a cylindrical geometry, like that previously considered in studies of the \( m = 1 \) mode,\(^9,10\) cannot address this problem, since the ideal
$m = 1$ mode in such a system is neutrally stable. The early nonlinear stage of interest here, though, may be analyzed using a more efficient approach. Following the standard boundary layer decomposition of linear and early-nonlinear$^{11,12}$ $m = 1$ mode theory, the reduced two-fluid equations are solved numerically in a narrow "layer region" ($|r - r_1| \ll r_1$) surrounding the $q = 1$ surface which includes the reconnection layer and $m = 1$ island only – see Fig. 1; the external regions are represented by the usual boundary condition appropriate to an arbitrary, specified $\Delta'_m$-spectrum. Curvature is neglected in the layer region, which is described in a Cartesian geometry. The parameter $\lambda_h$, which is related to $\Delta'_1$ by $\lambda_h = -\pi/\Delta'_1$, thus enters through the boundary condition.

To summarize the result, we find the linearly unstable mode indeed saturates, aside from a very slow diffusive evolution on a timescale $\tau > 100/\gamma_{\text{lin}}$ (possibly outside the model’s scope of physical validity), at a small, $\omega_*$-dependent amplitude, provided $\omega_{*,e}$ are somewhat above the reconnecting mode stability threshold. The main stabilizing effects are nonlinear/quasilinear enhancements near $r = r_1$ of $\omega_*$ (i.e., in the present isothermal model, $p' \propto n'$) and the magnetic shear $q'$. The latter change is also found to weaken the ideal mode contributions relative to those of the reconnecting mode. A nonlinearly/quasilinearly generated, sheared flow layer along $r = r_1$ is also observed. This flow, caused primarily by ion-gyroviscosity, is in the ion diamagnetic drift direction at the center of the layer, but reverses in the layer’s outer regions so that the net flow is zero. Its effect is stabilizing, but at a generally weaker level.

At least at smaller saturation amplitudes, these effects are well described by a quasilinear analysis of the $|m| = 1$ and $m = 0$ poloidal harmonics. In the case of the density, for example, the nonlinear steepening is then a consequence of $Dn/Dt \simeq 0$, or equivalently, the density continuity equation $\dot{n} = -\nabla \cdot n\nabla$ with $\nabla \cdot \nabla \simeq 0$. Averaging the latter form over $\theta$ in polar coordinates gives an evolution equation for $n_0$, the $m = 0$ component of $n$: $\dot{n}_0 = -(1/r)\partial_r (r \langle nV_r \rangle_\theta) \simeq -\partial_x \langle nV_r \rangle_\theta$, $x = r - r_1 \ll r_1$. Expanding $n$ and $V_r$ as $n = n_0 + n_1e^{i\theta} + n_2^*e^{-i\theta}$, $V = V_{r1}e^{i\theta} + V_{r1}^*e^{-i\theta}$, this becomes $\dot{n}_0 \simeq -2\text{Re}[\partial_x (n_1V_{r1}^*)]$. The perturbations $n_1$ and $V_{r1}$, with $\dot{n}_1 \simeq \gamma n_1$, are related by the $m = 1$ component of $Dn/Dt = 0$
as \( n_1 = -n_{0,x} V_r / \gamma \) (where \( n_{0,x} \equiv \partial_x n_0 \)). Thus, \( \dot{n}_0 \simeq 2 \gamma \partial_x (n_{0,x} |V_r / | \gamma^2 |) \), or with \( n_0 \simeq n^{(0)} + \tilde{n}_0 \) for some equilibrium profile \( n^{(0)}(x) \), \( \tilde{n}_0 \simeq \partial_x (n_{0,x} |V_r / | \gamma^2 |) \). Given the ideal MHD result (valid even for \( \omega_* \neq 0 \) – see the Appendix) \(|V_r / | (\xi_0^2 / (4 \pi^2)) (\tan^{-1}(x / \lambda_h))^2 \), where \( \xi_0 \) is shift of the plasma core, it is seen that quasilinear effects steepen the equilibrium profile near \( x = 0 \) as \( n_{0,x}(0) = n_{0,x}^{(0)}(0)[1 + \xi_0^2/(2 \lambda_h^2 \pi^2)] \). Similar results for the helical flux \( \psi \) and the stream function \( \phi \) are obtained in the ideal case from Ohm’s law and the vorticity equation: \( \dot{\psi}_0 \simeq \partial_x (\psi_x^{(0)} |V_r / | \gamma^2 |) \) and \( \phi_0 = (\gamma / 2) \tilde{n}_0 \).

We emphasize that the stabilizing contribution from \( \omega_* \) (i.e., \( n_x^{(0)} \) in the above argument) is essential for the nonlinear saturation of the mode. This is not the case in the one-fluid, ideal MHD limit \( d_\tau = \eta = \rho_s = 0, \omega_* = 0 \), in which the ideal \( m = 1 \) mode saturates at a finite displacement \( \xi_0 \sim 4.4 \lambda_h \) due to nonlinear changes in \( \psi \) alone.\(^{11}\) This stability is lost, however, in the presence non-ideal effects such as \( d_\tau \), which allow magnetic reconnection to take place and thus permit continued growth of the mode, e.g., \( \xi_0 \sim 54 \omega \) for a given \( m = 1 \) island width \( w \gg \lambda_h \). As described above, in the presence of \( \omega_* \) the nonlinear saturation can be regained at an amplitude which, depending on \( \omega_* \), can be smaller or larger than the ideal MHD value. In contrast to the ideal case, furthermore, the current layer near \( r = r_1 \) in the saturated state is nonsingular.

II. TWO-FLUID MODEL

We consider a reduced two-fluid model describing the evolution of the (normalized) velocity stream function \( \phi \), helical flux \( \psi \), and electron pressure \( p \):

\[
\dot{W} + \dot{V} \cdot \nabla W - \tau \nabla \cdot [p, \nabla \phi] = -B_x \cdot \nabla J + \mu \nabla^2 W, \tag{1}
\]

\[
\dot{\psi} + B_x \cdot \nabla (\phi - (1 + \tau) p) = \eta J + d_\tau \left( J + [\phi - \tau p, J] - \mu_e \nabla^2 J \right), \tag{2}
\]

\[
\dot{p} + \dot{V} \cdot \nabla p = - \left( \rho^2_e / (1 + \tau) \right) B_x \cdot \nabla J + \kappa \nabla^2 p, \tag{3}
\]

Definitions (with toroidal unit vector \( \hat{\gamma} = -\hat{\zeta} \)):
Here, length and time scales are normalized to $r_1$ and $\tau_\lambda^* = \tau_A/(r_1 q'_1)$, respectively, where the constants $q'_1, \omega_{ae}, p_1, p'_1$, etc., refer to the unperturbed tokamak equilibrium. (In TRTR, $r_1 \sim 20 - 30$ cm, $\tau_\lambda^* \sim 5 \times 10^{-7}$ sec.) This is a simplified version of the four-field model of Hazeltine et al.\cite{14}. Since our main goal here is to demonstrate the basic phenomena, we have omitted for simplicity (as in Ref. 15) various effects that we expect to have a quantitative but secondary role: the toroidal curvature terms, some finite ion Larmor radius terms $\propto \rho_i^2 \nabla^2$, and the parallel ion velocity terms. In the linear $m = 1$ mode limit, these equations correctly reproduce the $\omega_e$ factors obtained from more detailed two-fluid and kinetic treatments (e.g., Refs. 5 and 16), given the implicit assumption of constant $T_i, T_e$.

The diffusion terms $\propto \mu, \mu_e, \kappa$ represent\cite{9,10,17} effective ion and electron perpendicular viscosity ($\mu, \mu_e$) and anomalous particle diffusion ($\kappa$). The role of $\mu_e$ in the simulations deserves further comment. For $\mu_e = 0$ and $d_e > \eta^{1/3}$, two-dimensional models like the above\cite{17,18} generate exponentially small, sub-$d_e$ structure in the nonlinear current layer. This behavior is probably unphysical, as three-dimensional studies\cite{19} have found such structure to be broadened by current density gradient driven instabilities. In any event, the total
current associated with this structure is negligible, as is its effect on the global dynamics of our system. Here, the $\mu_e$ term acts to suppress such sub-$d_e$ scale-length features.

The code used here to analyze this model is a suitably modified version of that described in Ref. 17. The equations are solved on a Cartesian grid $n_x \times n_z$, $n_x = 205$, $128 \leq n_z \leq 256$, with a variable grid-spacing in the $x$-direction.

A. Asymptotic Matching to Linear Ideal Solution

The layer region can be described using a Cartesian, layer-centered coordinate system $(x, z)$, with radial coordinate $x$, helical coordinate $z$ (i.e., $z \simeq \theta - \zeta$ with $\theta$ and $\zeta$ being the poloidal and toroidal angles, respectively), and periodic boundary conditions in $z$. Some explanation is needed regarding the definition of $x$. As the $m = 1$ mode evolves nonlinearly, flux surfaces near the central core are shifted rigidly some finite distance $\xi_0(t)$. To maintain force balance in the early nonlinear stage ($\xi_0 \ll r_1$), the effective $q = 1$ surface – i.e., the axis of the reconnection layer and island – is shifted half this distance,\textsuperscript{20,13} and may be defined for example by $|r| \simeq \tilde{r}_1$, $\tilde{r}_1 = r_1 + \xi_0 \cos(\theta - \zeta)/2$. This shift also preserves the (approximate) symmetry of the helical flux and pressure gradient across the layer region. Therefore, defining $x$ as the radial distance from the shifted $q = 1$ surface as $x \simeq (r - \tilde{r}_1)/r_1$, one may apply even $x$-parity boundary conditions on $\psi$, $p'$ and $\phi'$ at the edge of the layer region. The resulting definite parity solutions can be obtained from a simulation of only the half-space $x \geq 0$, with appropriate “reflecting” boundary conditions at $x = 0$.

The $x$-boundary condition on the potentials can be applied at an arbitrary value $|x| = x_l$ satisfying $w/r_1 << x_l << 1$ (here, typically, $x_l \sim 0.3$). It is based on linear theory and the $x \to 0$ forms of the equilibrium profiles, taken as

$$\phi^{(0)} = 0$$

$$\psi^{(0)} = (\cos(k_xx) - 1)/k_x^2 \simeq -x^2/2 + k_x^2x^4/4! + \ldots$$

$$p^{(0)} = \omega_{ae}x$$
for given constants $\omega_{\infty} (= \tau^* \omega_{\infty})$, $k_x$. As expected, the simulations are insensitive to the choice of $k_x$ for $k_x \sim 1$ or smaller (e.g., $k_x \to 0$ and $k_x = 1$ yield nearly equivalent results). Note the electric potential is related to $\phi$ by $\phi_{\text{elec}} = \phi - \tau p$, so for $\tau \neq 0$ the above implies an equilibrium, uniform radial electric field $E_x$ of magnitude $\tau \omega_{\infty} = -\omega_{\infty}$. This results in no loss of generality, as other values of $E_x \to E_x - V_0$ are obtained with the transformation $y \to y - V_0 t$.

Neglecting the inertia terms in Eq. (1) for $x \sim x_l$, $\psi$ obeys $\mathbf{B}_* \cdot \nabla J \approx 0$, which leads to the conventional “exterior” linear solution for $\tilde{\psi} = \psi - \psi^{(0)}$:

$$\tilde{\psi} = \sum_m \tilde{\psi}_m e^{i m z}, \quad \tilde{\psi}_m \approx c_m (|x| + 2/\Delta'_m), \quad \Delta'_m = \left[ \frac{\tilde{\psi}_m(x)}{\tilde{\psi}_m(x_l)} \right]_{x \to 0}$$

for some $c_m$ and a specified set $\{\Delta'_m\}$. In detail,

$$\mathbf{B}_* \cdot \nabla J = [J, \psi] \approx \psi^{(0)} \sum_m i m (\tilde{\psi}_{m,xx} + k_m^2 \tilde{\psi}_m) = 0, \quad k_m^2 = k_x^2 - m^2$$

so $\tilde{\psi}_m$ may be written as

$$\tilde{\psi}_m(x) = \tilde{\psi}_m(x_l) \frac{f_m(x)}{f_m(x_l)}$$

where

$$f_m(x) = \frac{1}{k_m} \sinh(|k_m x|) + \frac{2}{\Delta'_m} \cosh(|k_m x|) \quad (k_m^2 > 0)$$

$$= \frac{1}{|k_m|} \sin(|k_m x|) + \frac{2}{\Delta'_m} \cos(|k_m x|) \quad (k_m^2 < 0)$$

$$= |x| + 2/\Delta'_m \quad (k_m x \to 0)$$

As a result,

$$\nabla^2 \psi = -k_x^2 \psi - 1, \quad [J, \psi] = 0$$

are satisfied exactly. Similar asymptotic forms for $\phi$ and $p$ follow from the linearized versions of Eqs. (2) and (3) (neglecting the small diffusion, $\rho_s$, and $d_x$ terms):

$$\phi_m(x) = \phi_m(x_l) \frac{\sin(k_x x_l)}{\sin(k_x x_l)} \frac{f_m(x)}{f_m(x_l)}$$

$$\tilde{\rho}_m(x) = \tilde{\rho}_m(x_l) \frac{\sin(k_x x_l)}{\sin(k_x x_l)} \frac{f_m(x)}{f_m(x_l)}$$
Linear theory clearly does not determine the amplitudes of the potentials at $x = x_t$. Rather, these amplitudes are determined by Fourier analysis at a given time in the simulation. The above asymptotic forms are then used to compute the various $x$-derivatives that are needed to advance the potentials to the next timestep near $x = x_t$.

We consider a $\Delta_m$-spectrum appropriate to a tokamak, e.g., $\Delta_1' = -\pi/\lambda_h \sim (R/r_1)^2$ and, for $|m| \neq 1$, $\Delta_m' \sim -1$. The initial perturbation is chosen to have an $|m| = 1$ component only. Near the edge of simulation where the boundary condition is applied, the $|m| = 1$ component remains much larger than the other components throughout the entire simulation. Consequently, the results are insensitive to the values of $\Delta_m'$ for $|m| \neq 1$. For $|m| \gg 1$, however, to avoid numerical instability, it is necessary to choose $\Delta_m' = -2k_m$, which eliminates the growing exponential in Eq. (10).

III. SIMULATION RESULTS

Simulations have been carried out in a parameter range relevant to future and present tokamaks (TFTR in particular) which is linearly unstable to a predominantly ideal $m = 1$ mode; a typical case, referred to below, is

$$\lambda_h = 0.03, \quad \rho_s = \frac{1}{2}\lambda_h = 0.015, \quad d_e = \frac{1}{3}\lambda_h = 0.01, \quad \tau = 1$$

$\eta = 10^{-9}, \quad \mu = \mu_e = \kappa = 5 \times 10^{-7}$

That is, $\lambda_h (= \lambda_h/r_1)$ is positive and larger than the collisionless scale lengths $\rho_s$ and $d_e$. (Here, $d_e$ is 2–3 times larger than realistic TFTR values.) These effects are dominant over the diffusive terms $\propto \eta, \mu, \mu_e, \kappa$, which modify the linear eigenfrequency by only a few percent. Various values of $\omega_*$ are considered, all of which are insufficient to stabilize the mode linearly (ignoring, that is, the diffusive terms, which can typically lead to weak instabilities for any $\omega_*$).

A useful diagnostic of the mode's evolution is the maximum displacement of a flux surface, $\xi(t)$, relative to its initial $x$-position far from the singular layer, $x \gg \lambda_h$. Referring to the
lab frame $\xi_0(t) = 2\xi(t)$, where $\xi_0$ is the core displacement discussed earlier. Shown in Fig. 2 are the results of two simulations, which differ only in the values of $\omega_e$: in one $\omega_{ei} = \omega_{ee} = 0$, while $\omega_{ei} = -\omega_{ee} = 0.054 = 1.8\lambda_h$ in the other. Both cases are linearly unstable, although the growth rate in the second ($\gamma_r (= \tau^*_r \gamma_r) \approx 0.017$) is reduced relative to the $\omega_e = 0$ value ($\gamma \approx 0.036$) by roughly 50%.

The mode in the $\omega_e \neq 0$ case does not completely saturate nonlinearly; rather, the amplitude continues to grow slowly after it reaches an initial maximum. Such residual, slow evolution results from the small diffusive terms $\propto \mu_e, \kappa, \mu$ which are necessarily included in the model at some level for numerical reasons. This is illustrated in Fig. 3, which shows $\xi(t)$ for the same parameters as the $\omega_e \neq 0$ case in Fig. 2 except $\mu_e = \kappa = \mu = 3 \times 10^{-6}$ (a factor of six increase). The resulting increase in the slow growth is consistent with a scaling, e.g., $\tau_{\text{slow}} \propto \mu^{-p}$, $1/3 < p < 1/2$ (further simulations indicate $\mu$ here has a stabilizing effect). Such behavior may be related to the claim of Ref. 9 that the resistive mode does not completely saturate for $|\omega_e| < \gamma_{\text{lin}}$.

The approximate saturation amplitudes obtained from the simulations for various $\omega_e$ values are plotted in Fig. 4. In the simulations marked by asterisk symbols, the parameters, aside from $\omega_e$ and $\mu = \mu_e = \kappa = 10^{-6}$, are the same as in Fig. 2. The triangles represent an increase in $\tau = T_e / T_i$ by a factor of three, i.e., $T_e = T_i / 3$. The squares, also with $T_e = T_i / 3$, indicate the result of doubling $\rho_s$, i.e., $\rho_s = 0.03 = \lambda_h$. In all cases, the symbols correspond to the $\xi$-values at which $\ddot{\xi}$ first passes through zero. These values are insensitive to changes in the small diffusion terms, and are typically about 25% smaller than the first maximum of $\xi$. In Figs. 2 and 3, for example, $\ddot{\xi} = 0$ at $t = 355$ and $t = 378$, respectively, where $\xi (= \xi / r_1) \approx 0.039$ in both cases.

The amplitudes for the larger $\omega_{ei}$ values fall below that expected in the ideal MHD case ($\xi \approx 2.2\lambda_h$), with larger, unshown $\omega_{ei}$ values leading to smaller amplitudes until linear stability is reached. The linear, reconnecting mode stability boundary discussed in Ref. 1 can be represented in Fig. 4, for each of the three cases shown, as a vertical line at a particular $\omega_e$ value. (For example, for $T_i = T_e$ and $\rho_s \gg d_e$, the marginal stability condition
is\(^5\) \(\omega_{*i} (= \omega_{*i} \tau_{A*}) = (\rho_s/r_1)[2d_e/(\rho_e \pi)]^{1/3}\), with larger \(\omega_\ast\) being stable.) Normalizing these marginal \(\omega_\ast\) values to the value of \(\lambda_h\) in the simulations (\(\lambda_h/\tau_1 = 0.03\)), one finds \(\omega_{*i}/\lambda_h \simeq 0.5\) (asterisks), 0.7 (triangles), 0.9 (squares). It is seen that the amplitudes increase more rapidly as these limits are approached. Due to the restriction \(\xi/r_1 \ll 1\), the largest amplitudes shown are about 0.12\(r_1\), which occurs for \(\omega_{*i}/\lambda_h\) about \(40-50\%\) above the respective reconnecting mode threshold. Thus, for the considered parameters, there is a substantial “gap” between the latter threshold and the region of early-nonlinear stability. This may indicate additional stabilizing factors, such as the trapped particle effects mentioned earlier, are important.

The contributions of \(\omega_{*i}\) and \(\omega_{*e}\) to the linear, \(\lambda_h = 0\) stability boundary are comparable (see, e.g., Ref. 16). As can be seen from the scaling of the nonlinear amplitudes with \(\tau\), the same is generally true nonlinearly. At the larger \(\omega_{*i}\) values with \(\rho_s = \lambda_h/2\) (asterisks, triangles), however, the contribution of \(\omega_{*e}\), relative to \(\omega_{*i}\), weakens. As discussed below, \(\omega_{*e}\) enters the linear and quasilinear theory only though the non-ideal terms, which (for the \(\rho_s < \lambda_h\) cases) are overwhelmed by ideal effects at larger \(\omega_{*i}\). Also, the larger amplitudes of the two \(T_e = T_i/3\) cases (triangles: \(\rho_s = 0.015\), squares: \(\rho_s = 0.03\)) tend to converge, indicating the influence of \(\rho_s\) becomes weaker.

Referring to the \(\omega_\ast \neq 0\) simulation of Fig. 2, Figs. 5a,b show equally spaced contours of the helical flux \(\psi\) and current \(J\) at the time \(t = 400\) of the first maximum of \(\xi\) (\(\xi \simeq -0.054 = 1.8\lambda_h\) - a typical value). The saturated configuration contains an \(m = 1\) island of half-width \(w/2 \sim \xi\) and angular extent (excluding the current layer) \(\Delta \theta \sim \pi\). At larger \(\xi/\lambda_h\) values, the island proportions increase, gradually approaching the \(\lambda_h = 0\) prediction\(^1^3\) \(w/2 \sim 2\xi\), \(\Delta \theta \sim 4\pi/3\).

At this same time \((t = 400)\), Figs. 6a-c show Fourier analyses of \(\phi, p-p^{(0)}, \psi - \psi^{(0)}\). The \(m = \pm 1\) components are dominant, followed in magnitude by the perturbation in the \(m = 0\) component, followed by \(m = \pm 2\), etc. (Since the potentials are real, \(\psi_m = \psi_{m}^*\), etc., so for \(m \neq 0\) we plot \(|\psi_m| + |\psi_{-m}| = 2|\psi_m|\) rather than \(|\psi_m|\).) To clarify the saturation mechanism we therefore consider the quasilinear analysis of the \(|m| = 1\) mode, retaining quasilinear
corrections from the $m = 0$ mode only. We show below these $m = 0$ corrections alone—particularly those in $p$ and $\psi$—can account for the saturation in the simulations. They represent nonlinear enhancements of the pressure gradient, magnetic shear, and poloidal flow, near $r = r_1$. Due to the weakness of the mode from the MHD viewpoint, these changes are small compared to the equilibrium profiles. This is demonstrated by Figs. 7a-c (solid lines), which show the full $m = 0$ harmonic of $\phi'$, $p'$, and $\psi'$ at $t = 400$, compared to the initial, unperturbed profiles (dashed lines). In physical units, $\phi' \rightarrow -(\tau_A^*/r_1)V_z$, $p' \rightarrow (\omega_e\tau_A^*/p_1^*)p'$, $\psi' \rightarrow (1-q)/(q_1^*r_1)$. Figure 7a therefore implies $V_z(x = 0) \sim 0.03r_1/\tau_A^* \sim \omega_e r_1$, i.e., a localized flow approaching the $\omega_e$-velocity.

A. Quasilinear Theory

Following a standard approach described in the Appendix, the $m = 1$ components of Eqs. (1)-(3), neglecting all other harmonics except $m = 0$, may be combined to yield a single differential equation obeyed by $\phi_1$. We take $\phi_1$, $\psi_1$, $p_1 \propto \exp(f(t)\gamma(t)dt)$, and consider $\phi_0$, $\psi_0$, $p_0$ for now to be arbitrary functions of $x$ which vary slowly in time relative to $\gamma^{-1}$. Also, for simplicity, we neglect the diffusive terms $\propto \mu$, $\mu_e$, $\kappa$, and some terms $\propto (\rho_s/\lambda_h)^4$ that vanish in the linear theory limit (i.e., arise from quasilinear corrections only)—see the Appendix. Defining

$$Z = \partial_x (\phi_1/\Gamma), \quad \Gamma = \gamma - i\phi_0$$

one finds $Z$ obeys

$$Z = \left(\frac{2\lambda_h}{\pi} \int_0^\infty Z dx\right) \frac{1}{D} + \frac{\psi_{0,x}^2}{D} \partial_x \left(\Lambda^2 \partial_x (\lambda^2 Z)\right),$$

where

$$D = \psi_{0,x}^2 + \lambda^2, \quad \lambda^2 = \Gamma - i\Omega_{ei}, \quad \Omega_{ei} = -\tau \Omega_{se} = -\tau p_0^*,$$

$$\Lambda^2 = \frac{\lambda^2}{\psi_{0,x}^2} + \frac{\lambda^2}{\lambda^2}, \quad \lambda_e^2 = \frac{\rho_s^2(\Gamma - i\Omega_{se})}{\Gamma + i(\Omega_{se} - \Omega_{ei})}, \quad \lambda_e^2 = \frac{d^2(\Gamma - i\Omega_{ei}) + \eta}{\Gamma + i(\Omega_{se} - \Omega_{ei})}.$$
Assuming $\psi_0 = \psi^{(0)}$, $p_0 = p^{(0)}$, etc. the above reduces to usual linear theory result.

In the case $\lambda_h^2 \gg \rho_s^2$, $d_e^2$, one can obtain an iterative solution to Eq. (19) in powers of $(\rho_s/\lambda_h)^2$, $(d_e/\lambda_h)^2$:

$$Z = Z^{(0)} + Z^{(1)} + Z^{(2)} + \ldots, \quad Z^{(0)} = \left( \frac{2\lambda_h}{\pi} \int_0^\infty Z dx \right) \frac{1}{D}.$$  \hspace{1cm} (22)

To first order in $\rho_s^2/\lambda_h^2$, $d_e^2/\lambda_h^2$, $Z$ can be replaced by $Z^{(0)}$ in $\Lambda^2$-term of Eq. (19):

$$Z = \left( \frac{2\lambda_h}{\pi} \int_0^\infty Z dx \right) \left( \frac{1}{D} + \frac{\psi_0^2}{D} \frac{\sqrt{\lambda^2}}{D} \right) + \ldots,$$  \hspace{1cm} (23)

which, integrating from $(0, \infty)$, leads to the approximate dispersion relation:

$$1 = \frac{2\lambda_h}{\pi} \int_0^\infty \left[ \frac{1}{D} + \Lambda^2 \left( \frac{\lambda^2}{D} \right) \right] dx.$$  \hspace{1cm} (24)

Neglecting quasilinear corrections (and taking $k_x = 0$), this equation leads the linear theory result $\lambda^2 = \lambda_h^2 + (\lambda_s^2 + 5\lambda_e^2)/2$, which agrees with Ref. 23 as it should for $\rho_s = 0$ and yields (complex) values of $\gamma$ that agree well with the simulation results in the linear phase.

We now turn to the calculation of the perturbation in the $m = 0$ harmonic. This perturbation, like the $|m| \geq 2$ harmonics, is generated by nonlinear interactions of the $m = \pm 1$ modes. Taking the $m = 0$ moment of Eqs. (1)-(3), for example, neglecting $|m| \geq 2$, one obtains (with $\nabla^2 \simeq \partial_x^2$):

$$\dot{\phi}_0 = 2\text{Im} \left[ (\phi_1 - \tau p_1) \phi_{1,x}^* + \psi_{1,x} \psi_{1}^* \right],$$  \hspace{1cm} (25)

$$\dot{\psi}_0 = 2\partial_x \text{Im} \left[ (\phi_1 - \tau p_1) \psi_{1}^* + p_1 \psi_{1}^* \right], \quad \psi_e = \psi - d_e^2 \Phi_{xx},$$  \hspace{1cm} (26)

$$\dot{p}_0 = 2\partial_x \text{Im} \left[ \phi_{1,p1}^* + \rho_s^2 \frac{p_1}{(1 + \tau)} \psi_{1,xx} \right].$$  \hspace{1cm} (27)

To leading (i.e., quadratic) order in the $m = 1$ amplitude $\sim \xi$, the quasilinear corrections in the $m = 1$ eigenfunctions on the right-hand side (RHS) may be neglected. For $\lambda_h^2 \gg \rho_s^2, d_e^2$, therefore, the leading order results for $\phi_0, \psi_0, \tilde{p}_0$ are obtained by substituting the ideal, linear $m = 1$ eigenfunctions, e.g., $Z^{(0)}$ with $\psi_{0,xx} = -x, p_{0,xx} = \omega_e$, etc., into the RHS, which gives

$$\phi_0 = -\left( \omega_e/2 \right) \partial_x |\phi_1/\Gamma|^2, \quad \tilde{p}_0 = \omega_e \partial_x |\phi_1/\Gamma|^2, \quad \tilde{\psi}_0 = -\partial_x \left( x |\phi_1/\Gamma|^2 \right),$$  \hspace{1cm} (28)
where
\[ |\phi_1/\Gamma|^2 = \left(\frac{\xi^2}{\pi^2}\right) \left(\tan^{-1}(x/\lambda_h)\right)^2. \]  (29)

It is seen from Figs. (8a-c) that these are in reasonable agreement with the simulation results.

Returning to the dispersion relation, it is now possible to compute the quasilinear eigenvalue \( \gamma \). We consider here only the simplest case in which \( \eta \) (in addition to \( \mu_z, \kappa \) and \( \mu \)) is neglected. As in the linear theory, in that case (and only in that case) real solutions for \( \omega = -i\gamma \) can exist, and one can calculate a stability boundary \( \xi_*(\omega_*) \), which is the value of \( \xi \) for a given \( \omega_* \) at which a real solution for \( \omega \) first appears. Larger values of \( \xi \) are stable.

The result obtained from Eq. (24) for \( \tau = 1 \) is shown as the dashed line in Fig. (4). Instead of the lowest order expressions of Eq. (28), slightly more refined versions of \( \phi_0, \tilde{\psi}_0, p_0 \) were used that account for the \( \rho_*, d_e \) corrections to the \( m = 1 \) linear eigenfunctions. This modification, however, makes only a small change in the final answer. The deviation of the analytic result from the simulation values at smaller \( \omega_* \) probably indicates the progressive breakdown of the perturbative treatment underlying Eqs. (23) and (24).

Due to the localization of \( Z \propto \phi'_1 \) near \( x = 0 \), a qualitative understanding of the saturation behavior may be obtained from Eq. (19) with the estimates
\[ \phi_{0,x}(x) \simeq \phi_{0,x}(0) = \chi^2 \omega_{*e}, \quad \chi = \xi/(\pi \lambda_h), \]  (30)
\[ p_{0,x}(x) \simeq p_{0,x}(0) = \alpha \omega_{*e}, \quad \alpha = 1 + 2\chi^2, \]  (31)
\[ \psi_{0,x}(x) \simeq \psi_{0,x}(0)x = -\bar{q}'x, \quad \bar{q}' = 1 + 6\chi^2. \]  (32)

Also taking \( \lambda^2 \simeq \lambda^2(0) \) for simplicity, Eq. (19) can then be written in a form which is identical to the linear theory result with "renormalized" parameters. For example, the quasilinear corrections associated with \( \phi_0 \) and \( p_0 \) lead to \( \gamma \to \Gamma(0) = \gamma - i\chi^2 \omega_{*e} \) and \( \omega_{*e} \to \Omega_{*e}(0) = \alpha \omega_{*e} \). Similarly, the quasilinear shear parameter \( \bar{q}' \) (which is related to the physical, perturbed shear as \( (q'_1)_{\xi\neq0} = \bar{q}'(q'_1)_{\xi=0} \)) is accounted for by the additional substitutions:
\[ \lambda^2 \to \lambda^2/\bar{q}'^2, \quad \lambda_h \to \lambda_h/\bar{q}'^2. \]  (33)
The linear dispersion relation mentioned above, for example, is therefore modified as \( \frac{\omega^2}{q^2} = \frac{\lambda_h^2}{q^4} + \frac{\lambda_s^2 + 5\lambda_e^2}{2} \), which, in the simplest case \( \omega_{si} \gg \omega_{se} \), leads to the stability condition

\[
(\alpha/q)\omega_{si} > 2\sqrt{\frac{\lambda_h^2}{q^4} + \left(\rho_s^2 + 5d_e^2\right)/2}.
\]  

(34)

Note the quasilinear enhancement of the shear \( (q'/q > 1) \) has a net stabilizing effect on the ideal mode contribution \( (\lambda_h) \), but a destabilizing effect with respect to the \( \rho_s \) and \( d_e \) terms. The quasilinear enhancement of \( \omega_{si} (\alpha > 1) \), on the other hand, is of course stabilizing for both. Thus, the two effects are additive (and stabilizing) with respect to the ideal mode but tend to cancel in the case of the reconnecting mode terms. This may account for the rough correlation of nonlinear stability in the simulations with the linear condition with \( \lambda_h = 0 \).

In the above approximation, \( \phi_0 \) introduces only a shift in the real frequency of the mode, and so does not effect the stability condition. Indeed, \( \phi_0 \) is generally found to make only a weak stabilizing contribution to the stability boundary discussed above. Also, note from Eq. (19) that \( \omega_{se} \) can enter the stability condition only through the \( \rho_s, d_e \) terms (that is, the ideal mode is sensitive to \( \omega_{si} \) only). As a result, the contribution of \( \omega_{se} \) to the saturation amplitude is small for \( \omega_{si} \) well above the collisionless stability threshold.

IV. CONCLUSION

On the basis of two-fluid MHD simulations, we have confirmed the theoretical prediction that a linearly unstable \( m = 1 \) mode, in a TFTR-like parameter regime, can saturate in the early nonlinear stage for sufficiently large values of the diamagnetic frequencies \( \omega_{si,e} \). Although this linear mode is predominantly ideal, the saturation process is essentially different from that of the purely ideal \( m = 1 \) mode,\(^1\) which is inoperative here due to effects that allow magnetic reconnection (e.g., finite electron inertia). Rather, as in the case of the resistive mode,\(^9\) the saturation depends on the presence of \( \omega_{si,e} \) to halt the reconnection.

We find nonlinear effects act in two main ways: (1) they weaken the ideal mode driving term \( (\propto \lambda_h) \) relative to those of the reconnecting mode \( (\propto \rho_s, d_e) \), and (2) they enhance...
the effective values of $\omega_{*i,e}$. These changes lead to a rough correlation between the early-nonlinear stability and the linear $\omega_{*i,e}$-stability condition with $\lambda_h = 0$; $\rho_s, d_e \neq 0 - i.e.,$ the collisionless reconnection mode stability condition.

The nonlinear saturation amplitude reached by the mode in the simulations is $\omega_{*i,e}$-dependent: it becomes finite as $\omega_s$ falls below the linear stability threshold, and continues to increase monotonically as $\omega_s$ is further reduced. The saturation level of the central core displacement $\xi_0$ can therefore be smaller or larger than, but is typically comparable to, the ideal MHD value$^{11}$ $\xi_0 \sim 4.4\lambda_h$. Because the validity of our approach is restricted to the early nonlinear stage ($\xi_0 \ll r_1$), the largest saturation amplitudes considered here are about $\xi_0 \sim (0.25)r_1 \sim 8\lambda_h$, which occur (for the simulated parameters) at $\omega_{*i,e}$ values roughly 1.5 - 2 times above the linear, $\lambda_h = 0$ stability limit. Thus, although the nonlinear amplitudes at these extreme $\omega_s$ values increase quite rapidly for decreasing $\omega_s$, no particular value of $\omega_s$ can be clearly identified which defines a transition from nonlinear stability to instability. This is in obvious contrast to linear theory, and also possibly experiments, as discussed further below.

The saturated configuration contains an $m = 1$ island of length $l \sim \pi r_1$ and width $w \sim c_0\xi_0$, where $c_0 \sim 1$ for $\xi_0 \sim 4\lambda_h$ or smaller, and $c_0 \sim 2$ for $\xi_0 \gg \lambda_h$. Due to small diffusion effects present in the model, this island can continue to evolve after saturation on a very slow timescale $\tau > 100/\gamma_{lin}$. The behavior of the saturated mode in TFTR over such timescales is erratic and complex. In view of the many effects not considered in this analysis, and the physical uncertainties in the diffusion terms, here we will leave this long-term behavior as an unresolved issue.

Fourier analyses of perturbations in the simulated potentials $p, \psi$ and $\phi$ at saturation show the $m = n = \pm 1$ and $m = n = 0$ harmonics are dominant. The latter, $m = n = 0$ perturbations were shown to arise from quadratic interactions of the $m = n = 1$ and $m = n = -1$ modes, and represent localized enhancements of the pressure gradient, magnetic shear, and poloidal flow in the singular layer. A quasilinear analysis indicates these enhancements alone can stabilize the $m = n = \pm 1$ modes, and lead to saturation amplitudes in reasonable
agreement with the simulations. As shown in Ref. 1, sawtooth stability in TFTR is reasonably well described by the linear, reconnecting mode (i.e., \( \lambda_h = 0; \rho_e, d_e \neq 0 \)) \( \omega_\omega \)-stability condition. As explained above, however, saturation in the early nonlinear stage is found here only for \( \omega_\omega \)-values somewhat above this level. Also, the saturation amplitudes predicted by the simulations are typically larger than those observed in experiments (where, e.g., \( \xi_0 \sim 1 - 2 \text{ cm} \)). This failure of the simulations to more closely reproduce the observed stability condition and \( m = 1 \) mode saturation amplitudes in TFTR seems to indicate stabilizing factors outside the present, simplified model are important. A possible explanation is that trapped particle effects, or some other stabilizing effects, act to reduce the effective value of \( \lambda_h \), thereby reducing the gap between the observed stability and linear theory. Further work is needed to determine the contribution of these and other physically realistic factors.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the useful comments of T. Antonsen, J. Drake and R. Kleva, who also helped in the development of the simulation code used in this study.

This work was supported by the U.S. Department of Energy Contract No. DE-AC02-76-CHO-3073.

APPENDIX: \( M = 1 \) QUASILINEAR EQUATIONS

With \( \nabla^2 \approx \partial_x^2 \), the vorticity equation (1) may be integrated once over \( x \) with the result

\[
\phi_x + [\phi - \tau_p, \phi_{,x}] = [\psi, \psi_{,x}] - [\psi, \psi_{,x}]_{x_\infty}
\]

(A1)

for some \( x_\infty \sim x_i \gg \lambda_h \). The \( x_\infty \) term may be written as

\[
[\psi, \psi_{,x}] = -i \psi_{0,xx} \partial_x (\psi_1/\psi_{0,x})
\]

(A2)

\[
\approx -\frac{2\lambda_h}{\pi} \left( \frac{\psi_1}{\psi_{0,x}} \right)_{x_\infty}, \quad x \to x_\infty
\]

(A3)
\[
\frac{p_1}{\psi_0,\omega} = -\frac{2\lambda_h}{\pi} (\phi_1/\Gamma)_{x_0},
\]

where the second line follows from the definition of \(\lambda_h = -\pi/\Delta'\). The third line follows from the pressure equation (3) and Ohm’s law (2) neglecting \(\eta, d, \rho_e\) (valid for large \(x\)), which yield \(p_1 \approx -i\Omega_e \phi_1/\Gamma\) and \(\psi_1 \approx -i\psi_0,\omega \phi_1/\Gamma\). In detail, the pressure equation, eliminating \([\psi, J]\) with Eq. (1), can be written as

\[
p_1 = -i\Omega_e \phi_1/\Gamma + \frac{\rho_e^2}{(1 + \tau)\Gamma} \partial_x \left( (\Gamma - i\Omega_e) \phi_{1,x} + i\phi_0,\omega (\phi_1 - \tau p_1) \right)
\]

\[
\simeq -i\Omega_e \phi_1/\Gamma + \frac{\rho_e^2}{(1 + \tau)\Gamma} \partial_x \left( (\Gamma - i\Omega_e) (\phi_{1,x} + i\phi_0,\omega \phi_1/\Gamma) \right)
\]

\[
= -i\Omega_e \phi_1/\Gamma + \frac{\rho_e^2}{(1 + \tau)\Gamma} \partial_x \left( \lambda^2 \partial_x (\phi_1/\Gamma) \right),
\]

where \(p_1 \approx -i\Omega_e \phi_1/\Gamma + O(\rho_e^2/\lambda_h^2)\) was used in the quasilinear correction term \(\phi_0,xx p_1\) in the second line, leading to an error of \(O(\rho_e^2/\lambda_h^2)\). Substituting this result for \(p_1\) into Ohm’s law (2) gives

\[
\frac{\psi_1}{\psi_0,\omega} = -i \frac{\phi_1}{\Gamma} + i \frac{\lambda^2}{\lambda^2} \partial_x \left( \lambda^2 \partial_x \left( \frac{\phi_1}{\Gamma} \right) \right) + \frac{\eta \psi_{1,xx}/\psi_{0,x}}{\Gamma + i(\Omega_e - \Omega_0)}
\]

\[
+ \frac{d_e^2 (\Gamma - i\Omega_e)}{\Gamma + i(\Omega_e - \Omega_0)} \left( \frac{\psi_{1,xx}}{\psi_{0,x}} + i \frac{\phi_1}{\Gamma} \psi_{0,xxx}/\psi_{0,x} \right).
\]

Neglecting quasilinear correction terms of order \(d_e^2 \rho_e^2/\lambda_h^2\), we may take \(i\phi_1/\Gamma \simeq -\psi_1/\psi_0,\omega\) in the \(d_e^2\) term above, in which case

\[
\frac{\psi_{1,xx}}{\psi_{0,x}} + i \frac{\phi_1}{\Gamma} \frac{\psi_{0,xxx}}{\psi_{0,x}} \simeq \frac{\psi_{1,xx}}{\psi_{0,x}} - \frac{\psi_1}{\psi_0,\omega} = \frac{1}{\psi_0,\omega} \partial_x \left( \psi_0,\omega \partial_x \left( \frac{\psi_1}{\psi_0,\omega} \right) \right).
\]

Indeed, it is seen to be consistent with the earlier assumption \(\nabla^2 \simeq \partial_x^2\) to completely neglect the \(\phi_1\) term in the above compared to \(\psi_{1,xx}/\psi_{0,x}\), in which case that expression may also be used in the \(\eta\) term. As a result, Eq. (8) becomes:

\[
\frac{\psi_1}{\psi_0,\omega} = -i \frac{\phi_1}{\Gamma} + i \frac{\lambda^2}{\lambda^2} \partial_x \left( \lambda^2 \partial_x \left( \frac{\phi_1}{\Gamma} \right) \right) + \frac{\lambda^2}{\psi_0,\omega} \partial_x \left( \psi_0,\omega \partial_x \left( \frac{\psi_1}{\psi_0,\omega} \right) \right).
\]

The final result, Eq. (19), is obtained by operating on the above with \(\partial_x\), and using the (integrated) vorticity equation (A1) to eliminate \(\partial_x (\psi_1/\psi_0,\omega)\). The latter equation, replacing \(p_1\) with Eq. (A7), can be written as
\[ \lambda^2 \partial_z \left( \frac{\phi_1}{\Gamma} \right) = -i \psi_{0,x}^2 \partial_x \left( \frac{\psi_1}{\psi_0,x} \right) + \frac{2\lambda_h}{\pi} \left( \frac{\phi_1}{\Gamma} \right)_{x=0} \tag{A11} \]

or equivalently

\[ \partial_z \left( \frac{\psi_1}{\psi_0,x} \right) = i \frac{1}{\psi_{0,x}^2} \left( \lambda^2 Z - \frac{2\lambda_h}{\pi} \int_0^\infty Z dx \right). \tag{A12} \]
REFERENCES


8 Y. Zhao, R. B. White, private communication (1994).


FIGURES

FIG. 1. A sketch of the simulated “layer region” containing the $m = 1$ island and reconnection layer.

FIG. 2. $\xi (= \xi/r_1)$ vs $t (= t/\tau^*_A)$ for $\omega_\ast = 0$ (upper curve) and $\omega_{\ast i}(= \omega_{\ast i}/\tau^*_A) = -\omega_{\ast e} = 0.054$ (lower curve).

FIG. 3. Same as the $\omega_\ast \neq 0$ case in Fig. 2 (shown as dashed line) except $\mu = \mu_e = \kappa = 3 \times 10^{-6}$.

FIG. 4. Approximate saturation amplitudes obtained from simulations: (1) asterisks: $\rho = 1$, $\rho_\ast(= \rho_\ast/r_1) = 0.015$; (2) triangles: $r = 3$, $\rho_\ast = 0.015$; (3) squares: $r = 3$, $\rho_\ast = 0.03$. Also, dashed line is analytic result for $r = 1$, $\rho_\ast = 0.015$ from Eq. (24).

FIG. 5. (a) Contours of $\psi$ at $t = 400$ of $\omega_\ast \neq 0$ case in Fig. 2 ($x = x/r_1$, $z = z/r_1$). (b) Contours of $J = \nabla^2 \psi$ at $t = 400$.

FIG. 6. (a) Fourier transform of $\phi$ for $|m| = 1, 0, 2, 3$ at $t = 400$ (descending order, $x < 0.01$). Curves show $|\phi_m| + |\phi_{-m}| = 2|\phi_m|$ for $m \neq 0$ and $|\phi_0|$ for $m = 0$. (b) Fourier transform of $p - p(0)$ for $|m| = 1, 0, 2, 3$ at $t = 400$, displayed as in Fig. 6a. (c) Fourier transform of $\psi - \psi(0)$ for $|m| = 1, 0, 2, 3$ at $t = 400$, displayed as in Fig. 6a.

FIG. 7. (a) The $m = 0$ harmonic of $\phi'$ vs $x$ at $t = 400$ from simulation (solid line) and unperturbed, $t = 0$ profile (dashed line). (b) The $m = 0$ harmonic of $p'$ vs $x$ at $t = 400$ from simulation (solid line) and unperturbed, $t = 0$ profile (dashed line). (c) The $m = 0$ harmonic of $\psi'$ vs $x$ at $t = 400$ from simulation (solid line) and unperturbed, $t = 0$ profile (dashed line).

FIG. 8. (a) $\phi_0$ vs $x (= x/r_1)$ at $t = 400$ from simulation (solid line) and from analytic expression in Eq. (28) (dashed line). (b) $p_0 - p_0(0)$ vs $x$ at $t = 400$ from simulation (solid line) and from analytic expression in Eq. (28) (dashed line). (c) $\psi_0 - \psi_0(0)$ vs $x$ at $t = 400$ from simulation (solid line) and from analytic expression in Eq. (28) (dashed line).
Simulation region

$\xi_0 = 2\xi$
Fig. 6a

Fig. 6b

Fig. 6c